ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

DEPARTMENT OF STATISTICS

POSTGRADUATE PROGRAM

RISK PROCESSES WITH DELAYED CLAIM SETTLEMENT

By

Fotini N. Fanara

A THESIS
Submitted to the Department of Statistics
of the Athens University of Economics and Business
in partial fulfilment of the requirements for
the degree of Master of Science in Statistics

Athens, Greece
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ΤΜΗΜΑ ΣΤΑΤΙΣΤΙΚΗΣ

ΔΙΑΔΙΚΑΣΙΕΣ ΚΙΝΔΥΝΟΥ ΜΕ ΚΑΘΥΣΤΕΡΗΜΕΝΕΣ ΑΣΦΑΛΙΣΤΙΚΕΣ ΖΗΜΙΕΣ

Φωτεινή Νικήτα Φανάρα

ΔΙΑΤΡΙΒΗ

Που υποβλήθηκε στο Τμήμα Στατιστικής
του Οικονομικού Πανεπιστημίου Αθηνών
ως μέρος των απαιτήσεων για την απόκτηση
Μεταπτυχιακού Διπλώματος Ειδίκευσης στη Στατιστική

Αθήνα
Οκτώβριος 1999

ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

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A Thesis submitted in partial fulfilment of

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RISK PROCESSES WITH DELAYED CLAIM SETTLEMENT

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Approved by the Graduate Committee

Professor J. Panaretos
Director of the Graduate Program
June 2000

DEDICATION

To
my parents
and
brother.

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I would like to thank my supervisor, Michael Zazanis, for his great help, ideas and persistence. Many special thanks are due to the libraries of the Athens University of Economics and Business and the Department of Mathematics of the National Capodistrian University of Athens for all the papers that I obtained.

VITA

I was born in Athens in 1975. In 1993 I became a student in the department of Mathematics in the National Capodistrian University of Athens and graduated in 1997. As a post-graduate student in the department of Statistics in A.U.E.B. My interests focus on stochastic models and risk theory.

ABSTRACT

Fotini Fanara

Risk processes with Delayed Claim settlement.

October 1999

The usual model of a risk business is based on a point process N which describes the times that a claim occurs and a sequence $\{Z_k\}$ of independent and identically distributed random variables having the same distribution function F that describes the size of the claims. By studying this model, typically one derives asymptotic results about the probability of ruin such as the classical exponential approximations of Lundberg's.

In this thesis we introduce a modified risk process, which attempts to capture the effect of delays in claim settlement as well as claim settlement in installments. As in the classical risk model, our goals are very much the same. We want to approximate the sum of the delayed claims by an appropriate probability function, to study the probability function of ruin, find a possible exponential bound, and calculate Lundberg's exponent. More specifically we introduce three different models: a single contract (ON-OFF) process characterizing the delayed claims of our model, a model of M such processes, and a superposition model of the process as M tends to infinity. The first model, while of limited practical significance, constitutes a building block for the analysis of superposition models which are closer to reality.

For the analysis of the single contract model we use renewal theoretic tools, which, together with an asymptotic analysis of Laplace transformations via Tauberian theorems gives expressions for the asymptotic mean and variance of the process. These enable us to develop a diffusion approximation using Brownian motion for our processes. Here we make use of the theory of weak convergence of probability measures on metric spaces. By using this result we can calculate the upper bound for the ruin probability and give a crude Lundberg exponent.

For calculating Lundberg's exponent we resort to Large Deviation techniques. We consider the moment generating function of the sum of claims up to time t, say $f(t,\theta)=Ee^{\theta A(t)}$ (where $A(t)$ is the sum of claims up to time t) and, using once more renewal theoretic techniques we evaluate $a(\theta)=\lim_{t\to\infty}\frac{1}{t}\log f(t,\theta)$. This in turn, with the help of Large Deviation techniques allows us to obtain asymptotic expressions for the probability of ruin. An application is made of the above when claims are exponentially distributed.

Finally, we examine the asymptotic properties of the superposition model when the number of contracts becomes large. In particular, we obtain conditions under which the claim arrival process converges to a Poisson process.

ΠΕΡΙΛΗΨΗ

Φωτεινή Φανάρα

Διαδικασίες κινδύνου με καθυστερημένες ασφαλιστικές ζημίες.

Οκτώβριος 1999

Το σύνηθες μοντέλο μιάς διαδικασίας κινδύνου σε μια ασφαλιστική εταιρεία βασίζεται σε μια σημειακή διαδικασία N η οποία περιγράφει τους χρόνους κατά τους οποίους δημιουργούνται οι απαιτήσεις-ζημίες και μια σειρά $\{Z_k\}$ από ανεξάρτητες και ισόνομες τυχαίες μεταβλητές, έχοντας την ίδια συνάρτηση κατανομής F η οποία περιγράφει την αποτίμηση-κόστος των ζημιών. Μελετώντας αυτό το μοντέλο κάποιος μπορεί να βγάλει αποτελέσματα για την πιθανότητα καταστροφής της ασφαλιστικής εταιρείας, όπως η κλασσική προσέγγιση του Lundberg.

Σε αυτή τη διπλωματική εισαγάγουμε μια τροποποιημένη διαδικασία κινδύνου, η οποία συλλαμβάνει την έννοια της καθυστερημένης ζημίας τόσο καλά όσο στο κλασσικό μοντέλο της στιγμιαίας πληρωμής των ζημιών. Θέλουμε να προσεγγίσουμε το άθροισμα των καθυστερημένων ζημίων από μια κατάλληλη συνάρτηση πιθανότητας, να μελετήσουμε την πιθανότητα καταστροφής, να βρούμε ένα πιθανό εκθετικό φράγμα και να υπολογίσουμε τον εκθέτη του Lundberg. Πιο συγκεκριμένα εισαγάγουμε τρία διαφορετικά μοντέλα: μια διδικασία ενός συμβολαίου (ON-OFF) η οποία χαρακτηρίζει τις καθυστερημένες απαιτήσεις του μοντέλου μας, ένα μοντέλο M τέτοιων διαδικασίων και ένα συνδυασμό ,δεδομένων M τέτοιων διαδικασιών, των απαιτήσεων όλων αυτών των n διαδικασιών σε μια διαδικασία κινδύνου καθώς το n τείνει στο άπειρο (superposition model). Το πρώτο μοντέλο, αν και περιορισμένης πρακτικής σημασίας, είναι πολύ βασικό καθώς τα άλλα δυο είναι επεκτάσεις αυτού.

Για την ανάλυση του πρώτου μοντέλου χρησιμοποιούμε ανανεωτικά εργαλεία, τα οποία, μαζί την ασυμπτωτική ανάλυση μετασχηματισμών Laplace μέσω των Tauberian θεωρημάτων δίνουν εκφράσεις για την ασυμπτωτική μέση τιμή και τη διασπορά της διαδικασίας. Έτσι είμαστε ικανοί να κάνουμε μια προσέγγιση διάχυσης χρησιμοποιώντας κίνηση Brown για τις διαδικασίες μας. Εδώ χρησιμοποιούμε τη θεωρία ασθενούς σύγλισης μέτρων πιθανότητας σε μετρικούς χώρους. Κάνοντας χρήση αυτών των αποτελεσμάτων εκτιμάμε ένα άνω φράγμα της πιθανότητας καταστροφής και ένα μη ακριβή εκθέτη του Lundberg.

Για έναν πιο ακριβή υπολογισμό του εκθέτη του Lundberg μια άλλη μέθοδος περιλαμβάνεται: αυτή των Large deviation (Ευρείας διασποράς) τεχνικών. Θεωρούμε την πρώτη ροπή του αθροίσματος των απαιτήσεων μέχρι τη στιγμή t, $f_k(t,\theta)=Ee^{\theta A_k(t)}$ (όπου $A_k(t)$ είναι το άθροισμα των

απαιτήσεων μέχρι τη χρονική στιγμή *t*) και χρησιμοποιώντας ανανεωτικές τεχνικές υπολογίζουμε το $a_k(\theta) = \lim_{t\to\infty} \frac{1}{t} \log f_k(t,\theta)$. Αυτό με τη βοήθεια των Large deviation τεχνικών μας επιτρέπει να πάρουμε ασυμπτωτικές εκφράσεις για την πιθανότητα καταστροφής. Υπάρχει ενδεικτικά μια εφαρμογή όταν η κατανομή των απαιτήσεων είναι η εκθετική.

Τέλος,μελετάμε ασυμπτωτικές ιδιότητες του superposition μοντέλου όταν ο αριθμός των συμβολαίων γίνει πολύ μεγάλος. Συγκεκριμένα παίρνουμε κάποιες συνθήκες κάτω από τις οποίες η διαδικασία κάτω από την οποία φθάνουν οι ζημίες ακολουθεί την κατανομή Poisson.

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Chapter 1

Insurance Risk Processes with Delayed Claims

Collective risk theory is concerned with the random fluctuations of the total assets, the risk reserve, of an insurance company. Consider a company which only writes ordinary insurance policies such as accident, disability, fire, health, and whole life. The policyholders pay premiums regularly and at certain random times make claims to the company. A policyholder's premium, the gross risk premium, is a positive amount composed of two components. The net risk premium is the component calculated to cover the payments of claims on the average, while the security risk premium, or safety loading, is the component which protects the company against large deviations of claims from the average and allows an accumulation of capital.

We are dealing with a risk model with delayed settlement. Below we will introduce some models which describe the risk reserves and the way claims are settled by the insurer. In practice claim settlement, particularly for large claims, frequently occurs in installments often spread out over significant periods of time. Therefore, we will consider not only the time each claim occurs, but also the time interval necessary in order for the insurer to settle the claim. One typically assumes this time to be a random variable with a given distribution. Models that incorporate the time period between the epoch a claim occurs and the epoch the final installment has been payed are known in the insurance risk literature as *delayed settlement* models.

Several authors have considered models that capture some aspects of delayed settlement. Among them we mention Boogaert and Haezendonck (1989), Neuhaus (1992), Klüppelberg and Mikosch (1995), and Brémaud (1998).

In the section that follows we give more details regarding delayed claims as well as a brief review of some recent papers on this topic.

1.1 Delayed claims

In the classical insurance risk model with initial reserve or capital u and premium rate c, the basic stochastic elements are

- the epochs when claims occur, denoted by $T_0 = 0, T_1, T_2, \ldots$,
- the number of claims up to time t, defined by $N(t)$,
- the claim sizes, X_i, $i = 1, 2, \ldots$,
- the total claim amount by time t, $S(t) = \sum_{i=1}^{N(t)} X_i$, and
- the risk reserve at time t, i.e. $Y(t) = u + ct - S(t)$.

In the classical risk theoretic models it is assumed that the claims are settled by the insurer at the time they occur. In reality this is rarely the case. Very often a claim is unknown to the insurer when it occurs and is reported after a cerain time of delay. There is always a delay between the time a claim occurs and the time of the settlement, i.e. payment of that claim by the insurance company. The period of time between these two events is called the settling delay. Another kind of delay is caused by the claims whose existence is known but their cost development is incomplete as e.g. in rehabilitation following accidents. If the delay degenerates at zero the process reduces to the classical risk model, where it is assumed that the claims are settled by the insurer immediately at the time when they occur. Of course, in the classical risk model the settling delay has no real impact on the insurer's surplus. On the contrary when interest and inflation are taken into consideration, the settling delay affects this surplus.

Many authors have discussed the problem of delayed settlement and have given models that capture various aspects of this problem. A frequently used model is the "transient integrated shot noise process". In this model we assume that when a claim occurs, the company is not required to pay immediately the full amount in one lump sum. Instead the occurrence of a claim generates a stream of payments that may extend for months or years into the future. In this case the risk process becomes

$$R(t) = u + ct - \sum_{k=1}^{N(t)} h(Z_k, t - T_k),$$

where $\{T_n\}$ are the claim occurrence epochs and $h(r, s)$ is a function that describes the payment rate at time s as a result of a claim of total size r occuring at time 0. Because of the delayed claims the sample paths of the resulting risk process $R(t)$ are smoother than those of the classical model.

A more general risk model considered by a number of authors is the following :

$$S(t) = \sum_{n \geq 1} X_n(t - T_n), \quad t \geq 0 \tag{1.1}$$

The above is the explosive shot noise process where X, $X_1, X_2, \ldots$, are i.i.d. random non-null measures with support on R^+, $X_n(t) = X_n([0,t])$, $t \geq 0$, and $(T_n)_n \in N$ are random variables such that $N(t) := \#\{n : T_n \leq t\}$ is a homogeneous Poisson process with intensity $a > 0$.

We assume that for $n \in N$ the functions $(X_n(t))_{t\geq 0}$ are non-decreasing and cadlag. So the realizations of $(X_n(t))_{t\geq 0}$ are measure-defining functions. The mean and the variance of the stochastic process $(S(t))_{t\geq 0}$ are versions of Campbell's theorem.

One way for finding the moments and covariances is by calculating the Laplace functional of the random measure S. We have

$$E[e^{-aS(t)}] = \exp\{-\lambda t(1 - t^{-1} \int_0^t Ee^{-aX(u)} du)\}, \tag{1.2}$$

where λ is the rate of the Poisson process.

Now from (1.2) by differentiating at $a = 0$ we get the following

$$\mu(t) = E[S(t)] = \lambda \int_0^t E[X(u)] du$$

$$\sigma^2(t) = \mathsf{Var}[S(t)] = \lambda \int_0^t E[X^2(u)] du$$

$$\mathrm{Cov}[S(s), S(t)] = \sigma^2(s) + \lambda \int_0^s E[X(u)X(u, u+t-s)]du$$

by assuming that $\mu(t) < \infty$ and $\sigma^2(t) < \infty$.

The moment functions of S are not invariant under shifts except for the Compound Poisson process, that is why $(S(t))_{t\geq 0}$ is not stationary of any order.

The Poisson shot noise as in (1.1) has been investigated for i.i.d. stochastic processes $(X_n(t))_{t\geq 0}$, $n \in N$, whose sample paths decrease to zero. It is also a natural realization of the classical compound Poisson process.

This can also be applied to some specific insurance problems. The explosive shot noise process can be viewed as a model for delay in claim settlement. The T_n, $n \in N$, are considered as the claim arrival times, and the measure $X_n(\cdot - T_n)$ describes the evolution of hte pay-off process for the n^{th} claim. Since every realization of the process $X_n(t)$ is a non-decreasing function of t, the limit $\lim_{t\to\infty} X_n(t) = X_n(\infty)$ exists (possibly infinite) and is the total pay-off caused by the n^{th} claim. Then $(S(t))t \geq 0$ as defined in (1.1) is the total claim amount process.

The model in (1.1) is introduced in Boogaert and Haezendonck (1989) and also Klüppelberg and Mikosch (1995) and Bremaud (1998).

The limit distributions of the process

$$\frac{S(t) - \mu(t)}{\sigma(t)} \qquad \text{for } t \to \infty$$

have been characterized by Lane (1984) as infinitely divisible laws. For studying the asymptotic normality we can start by a Berry-Essen estimate under the Lyapunov-type condition.

We can also use a functional version of Central Limit Theorem. Now we must recall that we work in $D[0,1]$ which is the space of cadlag functions on the unit interval. We suppose that $D[0,1]$ is equipped with the supremum-norm topology and with projection σ-algebra.

We define

$$S_x(t) = \frac{S(xt) - \mu(xt)}{\sigma(t)}, \quad 0 \leq x \leq 1, \quad t \geq 0$$

Now we can establish the conditions for the convergence of the finite-dimensional distribution of the process $S(t)$. From Lane (1984) we have the following theorem, which resembles closely the Lindeberg condition for sums of independent random variables, that will help us establish those conditions.

Theorem 1 *When the following conditions hold :*

$$\mathsf{Var}[S(t)] < \infty, \quad \textit{for all } t,$$

$$\mu_t = E[S(t)] \quad \textit{and} \quad \sigma_t^2 \sim \mathsf{Var}[S(t)] \to \infty \quad \textit{as } t \to \infty$$

then the condition

$$\sigma_t^2 \int_{x\sigma_1}^{\infty} y\mathcal{F}_t(y)dy \to 0, \quad \textit{for all } x > 0$$

is nesessary and sufficient for convergence in distribution of $Z(t)$ *to a normal* $N(0,1)$ *limit, as* $t \to \infty$.

Where it is

$$\mathcal{F}_t(y) = \int_{-\infty}^{\infty} \overline{F}_H[(-y,y);t,s]\Lambda(ds),$$

where we denote the distribution of $X(t,s)$ by $F_X(x;t,s) = P\{X(t,s) \leq x\}$and $\overline{F}_X$is the tail distribution.

Now for creating the conditions requested the following theorem (see Kluppelberg and Mikosch 1995) holds. We need some regularity conditions on the process $(X(t))_{t\geq 0}$ involving some regular variation property in R^2. A measurable function $f : R_+ \times R_+ \to R_+$ is regularly varying in R^2 if for all $x, y > 0$ the limit

$$C(x,y) = \lim_{t\to\infty} \frac{f(xt,yt)}{f(t,t)}$$

exists and is positive. In this case $C(x,y)$ is homogeneous, i.e.

$$C(kx,ky) = k^{\rho}C(x,y)$$

holds for all $x, y, k > 0$ and some fixed number ρ, and $C(1,1) = 1$. ρ is called the index the index of regular variation and we write $f \in RV(\rho)$.

Theorem 2 *Suppose that* $E[X(s)X(t)] \in RV(\rho - 1), \rho \geq 1$. *Then the limits*

$$C(x,y) = \lim_{t\to\infty} \mathsf{Cov}[S_x(t), S_y(t)],$$

where $x, y \in [0,1]$, exist and are finite. Moreover, there exist a Gaussian process $(B_x)_{0<x<1}$ with zero mean and covariance function $C(x,y)$, $0 \leq x, y \leq 1$, and with almost surely continuous paths. The finite-dimensional distributions of the process $S(t)$ converge to those of B if and only if

$$\frac{1}{\sigma^2(t)} \int_{\epsilon\sigma(t)}^{\infty} y \int_0^t P\{X(u) > y\} du dy \to 0, \qquad \textit{for every } \epsilon > 0.$$

Yet another model is the following: A loss may be unpaid because it has not yet been reported or because its size has not yet been settled. "IBNR" Incurred But Not yet Reported must be understood as "IBNP" (Incurred But Not Paid) in most cases. IBNR-models are mathematical models allowing to estimate the future losses akin to casualties which have already occured.

We introduce a sequence of non-negative, iid delay variables $(D_n)_{n\in N}$ which are independent of $(T_n)_{n\in N}$ and of $(X_n)_{n\in N}$ and modify model (1.1) as follows

$$S^D(t) = \sum_{n=1}^{N(t)} X_n(t - T_n - D_n), \quad t \geq 0.$$

Then the n^{th} claim occurs at time T_n, but it is only reported after a random delay D_n.This can be considered as a replacement of the claims arrival process $(T_n)_{n\in N}$ by $(T_n + D_n)_{n\in N}$. Using the properties for the probability generating functional of the Poisson process $(N(t))_{t\geq 0}$ we see that the process $(S^D(t))_{t\geq 0}$ generates an infinitely divisible random measure S^D with Laplace functional

$$\begin{aligned} Lf &= E\exp\{-\int_{-\infty}^{\infty} f(x) dS^D(x)\} \\ &= \exp\{-\lambda Ex(\int_0^{\infty}\int_0^{\infty}(1 - \exp\{-\int_0^{\infty} f(x) dX(x-u-y)\}) du dF_D(y))\}, \end{aligned}$$

where F_D is the distribution of D.

Last but not least, one can consider claims arriving according to an inhomogeneous Poisson processes. Here $(N(t))_{t\geq 0}$ is an inhomogeneous Poisson process with intensity measure $\lambda(t)$, $t \geq 0$, i.e. λ is a continuous non–decreasing function satisfying $\lambda(0) = 0$ and $\lambda(t) < \infty$ for all $t \geq 0$.

For each of the previous models Campbell's Theorem (see Appendix) can be applied for calcualating moments of order k.

1.2 Stable Lévy motion approximation in collective risk theory

In this section, quite independent from the previous section, we are dealing with a different approximation of our risk process. This analysis was carried out in Furrer, Michna and Weron (1997). A risk process can also be approximated by an α–stable Lévy motion ($1 < \alpha < 2$) with drift. Here especially relevant are weak approximations whenever we have heavy-tailed claims.

Here our mathematical model is the following

$$R(t) = u + ct - \sum_{k=1}^{N(t)} Y_k,$$

where we assume that claims occur at jumps of the point process $N = (N(t); t \geq 0)$. Here the distribution of N (maybe a Poisson maybe not) plays no important role in our analysis.The successive claims $(Y_k : k \in N)$ are assumed to form a sequence of independent, identically distributed random variables with $E[Y_k] = \mu > 0$. The initial risk reserve of the company is $u > 0$ and the gross risk premium is $c > 0$ per unit time.

Remark 1 *The notation $X \sim S_\alpha(\sigma, \beta, \mu)$ indicates that the random variable X has a stable distribution, characterized by four parameters*

i) the index of stability, $0 < \alpha \leq 2$,

ii) the scale parameter, $\sigma > 0$,

iii) the skewness parameter, $-1 \leq \beta \leq 1$ *and*

iv) the shift parameter, $\mu \in R$

The definition of an univariate stable distribution derives from the stability property that the family of stable distributions is preserved under convolution.

Definition 1 *A stochastic process $Z_\alpha = (Z_\alpha(t) : t > 0)$ is called (standard) α–stable Lévy motion if* *(i)* $Z_\alpha(0) = 0$ *almost surely,*

(ii) Z_α *has independent increments,*

(iii) $Z_\alpha(t) - Z_\alpha(s) \sim S_a((t-s)^{\frac{1}{\alpha}}, \beta, 0)$

for any $0 \leq s < t < \infty$ *and for some* $0 < \alpha \leq 2$, $|\beta| \leq 1$.

The sequence $(Q^{(n)} : n \in N)$ of risk processes are given as follows

$$Q^{(n)} = u^{(n)} + c^{(n)}t - \sum_{k=1}^{N^{(n)}(t)} Y_k^{(n)}, \quad t \geq 0.$$

We now assume that the claims are of the form

$$Y_k^{(n)} = \frac{Y_k}{\varphi(n)},$$

where $(Y_k : k \in N)$ is a sequence of iid random variables with common distribution function F and mean μ such that

$$\frac{1}{\varphi(n)} \sum_{k=1}^{n} (Y_k - \mu) \Rightarrow Z_\alpha(1),$$

$n \to \infty$. The function φ is given by:

$$\varphi(n) = n^{1/\alpha} L(n),$$

where L is slowly varying at infinity.

The condition $a > 1$ is needed to guarantee a finite mean of the variable $Z_\alpha(t)$.

Theorem 3 *Let the sequence* $(Y_k : k \in N)$ *be as above and let* $(N^{(n)} : n \in N)$ *be a sequence of point processes such that :*

$$\frac{N^{(n)}(t) - \lambda n t}{\varphi(n)} \to 0, n \to \infty$$

in probability in the Skorokhod topology for some positive constant λ. *Assume also that*

$$\lim_{n \to \infty} \left(c^{(n)} - \lambda n \frac{\mu}{\varphi(n)} \right) = c,$$

$$\lim_{n\to\infty} u^{(n)} = u.$$

Then

$$u^{(n)} + c^{(n)}t - \frac{1}{\varphi(n)} \sum_{k=1}^{N^{(n)}(t)} Y_k \to u + ct - \lambda^{1/\alpha} Z_\alpha(t),$$

$n \to \infty$ *in the Skorokhod topology.*

Proposition 4 *Let* $(N(t) : t \geq 0)$ *be a renewal process with inter-occurance times* $(T_k : k \in N)$ *and assume that there exist a positive constant* λ *and a slowly varying function* L *such that*

$$\frac{1}{\varphi(n)} \sum_{k=1}^{[nt]} \left(T_k - \frac{1}{\lambda}\right) \to B(t), \quad n \to \infty$$

in the Skorokhod topology where $\varphi(n) = \sqrt{n}L(n)$. *Then, for* $1 < \alpha < 2$ *we have*

$$\frac{N(nt) - \lambda nt}{n^{1/\alpha}} \to 0, \quad n \to \infty$$

in probability in the Skorokhod topology.

Chapter 2

The Single Contract Model

Here we examine the free reserve process, i.e. reserve for fluctuations in the technical results, assuming that there is a *single* contract which provides a constant rate c of premium income and generates claims at times T_n, $n = 1, 2, 3, \ldots$. These claims are settled *at constant rate* which we will assume equal to 1 without loss of generality. While this model is far from being realistic and has little practical significance in itself, it will provide a basic building block from which multicontract models of practical interest will be constructed.

To specify the statistics of the model, let $\{X_n\}$, $\{Y_n\}$, $n = 1, 2, \ldots$ be two independent sequences of non–negative i.i.d. random variables. We assume that $P(X_n \leq x) = F(x)$, $P(Y_n \leq y) = 1 - e^{-\lambda y}$, $y \geq 0$. Define the point processes $\{T_n; n = 1, 2, 3, \ldots\}$, $\{S_n; n = 0, 1, 2, \ldots\}$ via the relationships $S_0 = 0$, $S_n = S_{n-1} + Y_n + X_n$, and $T_n = S_{n-1} + Y_n$, $n = 1, 2, \ldots$. Thus, X_n is the amount of time it takes to settle the n'th claim which occurs at time T_n, while Y_n is the amount of time that elapses between the settlement of the n'th claim (at time S_n) and the occurrence of the $n+1$'th claim (see figure 2.1, below). It is further assumed in this model that during the time one claim is being settled, another one cannot occur.

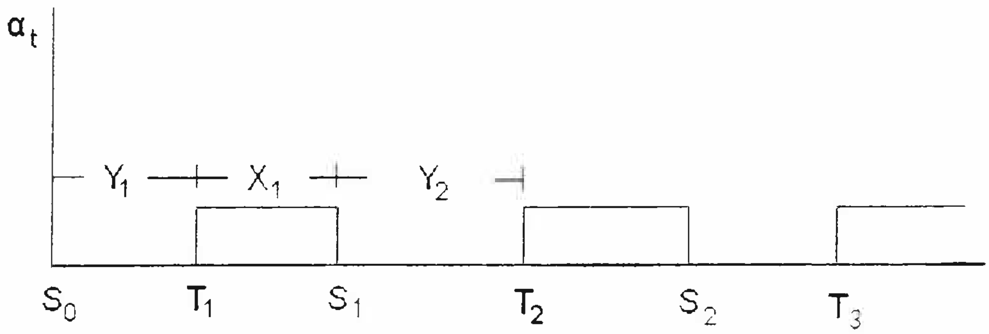

figure2.1

We assume that, initially, the insurer has free reserves u and that $c > \frac{\lambda EX}{1+\lambda EX}$ ensuring profitability for the insurer on the average (positive safety loading). If we define the process $\{\alpha_t; t \geq 0\}$ as

$$\alpha_t := \sum_{n=1}^{\infty} \mathbf{1}(T_n \leq t < S_n)$$

then

$$A(t) := \int_0^t \alpha_s ds$$

is the total amount that has been paid by the insurer by time t, while $Z_t := u + ct - A(t)$ is the free reserves process. This model resembles the classical risk model with Poisson times, where an asymptotic expression for the probability of ruin is obtained using the Cramér–Lundberg asymptotic formula. Instead we will use a diffusion approximation approach. This will allow us to extend our analysis to multi–contract portfolios.

2.1 Asymptotic Mean and Variance Calculations for the Single Contract Model

As a first step towards this goal we will compute the expected amount of money paid by the insurer by time t, $EA(t)$ as well as the variance $\mathsf{Var}[A(t)]$ using standard renewal–theoretic arguments (see Çinlar, 1975, Asmussen, 1987). Indeed, $\{\alpha_t; t \geq 0\}$ is a regenerative process with respect to the ordinary renewal process $\{T_n; n = 1, 2, \ldots\}$ and thus, using standard arguments, we can obtain the following renewal equation for $\tilde{f}(t) := EA(t)$.

$$
\begin{aligned}
\tilde{f}(t) &= E[A(t); T_1 > t] + E[A(t); T_1 \leq t] \\
&= E[A(t); T_1 > t] + \int_0^t E[A(t) \mid T_1 = u] G(du) \\
&= E[A(t); T_1 > t] + \int_0^t E[A(T_1) \mid T_1 = u] G(du) + \int_0^t E[A(T_1, t] \mid T_1 = u] G(du)
\end{aligned}
$$

where, $A(s,t] := \int_{(s,t]} \alpha_u du = A(t) - A(s)$ and $G(t) = P(X_1 + Y_1 \leq t)$. We thus have

$$
\begin{aligned}
\tilde{f}(t) &= E[A(t); T_1 > t] + E[A(T_1); T_1 \leq t] + \int_0^t \tilde{f}(t-u) G(du) \\
&= E[A(T_1 \wedge t)] + \int_0^t \tilde{f}(t-u) G(du) \\
&= b(t) + \int_0^t \tilde{f}(t-u) G(du)
\end{aligned}
$$

with

$$b(t) := E[A(T_1 \wedge t)],$$

whence we obtain the following renewal equation for $\tilde{f}(t)$:

$$\tilde{f}(t) = b(t) + \int_0^t \tilde{f}(t-u) G(du). \tag{2.1}$$

Taking Laplace transforms in this last equation we have

$$s \int_0^\infty e^{-st} \tilde{f}(t) dt = s \int_0^\infty e^{-st} b(t) dt + s \int_0^\infty e^{-st} \int_0^t \tilde{f}(t-u) G(du) dt.$$

Denote by $\hat{f}(s) := s \int_0^\infty e^{-st} \tilde{f}(t) dt$ the Laplace transform of $\tilde{f}$ (and similarly by $\hat{b}(s)$ the Laplace transform of $b(t)$) and $\hat{G}(s) := \int_0^\infty e^{-st} dG(t)$ the above equation gives

$$\hat{f}(s) = \hat{b}(s) + \hat{f}(s)\hat{G}(s), \tag{2.2}$$

(Note the difference in definition of the Laplace transform for functions such as $f(t)$ and measures such as the distribution function G cf. section 3.2.1)

Clearly, since $G(t) = P(X_1 + Y_1 \leq t)$, $G(s) = \hat{X}(s)\frac{\lambda}{\lambda+s}$, in view of the independence of X_1 and Y_1, where $\hat{X}(s)$ and $\frac{\lambda}{\lambda+s}$ are the Laplace transforms of X_1 and the exponential random variable Y_1 respectively. Next we obtain an expression for $\hat{b}(s)$ as follows

$$\begin{aligned}
\hat{b}(s) &= s\int_0^\infty e^{-st}E[A(T_1\wedge t)]dt \\
&= E[\int_0^{T_1} se^{-st}A(t)dt + \int_{T_1}^\infty se^{-st}A(T_1)dt] \\
&= E[\int_{Y_1}^{X_1+Y_1} se^{-st}(t-Y_1)dt] + E[A(T_1)(-(e^{-st})_{X_1+Y_1}^\infty)] \\
&= E[\int_{Y_1}^{X_1+Y_1} ste^{-st}dt] - E[\int_{Y_1}^{X_1+Y_1} Y_1se^{-st}dt] - E[A(T_1)(e^{-st})_{X_1+Y_1}^\infty].
\end{aligned}$$

Taking into account the fact that $A(T_1) = \int_0^{T_1}\alpha_u du = X_1$, the above expression becomes

$$\begin{aligned}
\hat{b}(s) &= E\left[-X_1e^{-s(X_1+Y_1)} - Y_1e^{-s(X_1+Y_1)} + Y_1e^{-sY_1} - \frac{1}{s}e^{-s(X_1+Y_1)} + \frac{1}{s}e^{-sY_1}\right] \\
&\quad -E\left[-Y_1e^{-s(X_1+Y_1)} + Y_1e^{-sY_1}\right] - E[-X_1Y_1e^{-s(X_1+Y_1)}].
\end{aligned}$$

Now, use the independence of X_1, Y_1, as well as the fact that $E[X_1e^{-sX_1}] = -X'(s)$ (where prime denotes differentiation with respect to s) to obtain

$$\begin{aligned}
\hat{b}(s) &= \hat{X}'(s)Y(s) + \hat{Y}'(s)\hat{X}(s) - \hat{Y}'(s) - \frac{1}{s}\hat{X}(s)\hat{Y}(s) + \frac{1}{s}\hat{Y}(s) - \hat{Y}'(s)\hat{X}(s) \\
&\quad +Y'(s) - X'(s)Y(s)
\end{aligned}$$

or

$$\begin{aligned}
\hat{X}(s) &= -\frac{1}{s}\hat{X}(s)\frac{\lambda}{\lambda+s} + \frac{1}{s}\frac{\lambda}{\lambda+s} \\
&= \frac{\lambda}{\lambda+s}\frac{1-\hat{X}(s)}{s}. \qquad (2.3)
\end{aligned}$$

Denoting by $F_I(x) := \frac{1}{\mu_1}\int_0^x[1-F(y)]dy$ the *integrated tail distribution* that corresponds to F and using the fact that its Laplace transform is related to the Laplace transform of F via the relationship $\hat{F}_I(s) := \frac{1-\hat{F}(s)}{\mu_1 s}$, we have from (2.2) and (2.3)

$$\hat{f}(s) = \frac{\lambda\mu_1\hat{X}_I(s)}{s(1+\lambda\mu_1X_I(s))} \qquad (2.4)$$

2.2 Laplace Transform for the Second Moment $E[A(t)^2]$

In this section we will use a renewal–theoretic argument in order to obtain the Laplace transform of $h(t) := E[A(t)^2]$. Combined with the results of the previous section, and the fact that $Var[A(t)] = E[A(t)^2] - E[A(t)]^2$ this will provide the asymptotic behavior of the variance.

$$\begin{aligned} h(t) &= E[A^2(t); T_1 > t] + \int_0^t E[A^2(t) \mid T_1 = u]G(du) \\ &= E[A^2(t); T_1 > t] + \int_0^t E\left[\left(\int_0^{T_1} a_u du + \int_{T_1}^t a_d u\right)^2 \mid T_1 = u\right] G(du) \end{aligned}$$

The integral in the last term above can be rewritten as the following sum of three terms:

$$\int_0^t E\left[\left(\int_0^{T_1} a_u du\right)^2 \mid T_1 = u\right] G(du) \; + \; \int_0^t E[A^2(t-u)]G(du)$$
$$+ 2\int_0^t E\left[\int_0^{T_1} a_u du \mid T_1 = u\right] \; E\left[\int_{T_1}^t a_u du \mid T_1 = u\right] G(du)$$

or

$$\int_0^t E[A^2(T_1) \mid T_1 = u]G(du) + \int_0^t E[A^2(t-u)]G(du)$$
$$+ 2\int_0^t E[(A(T) \mid T_1 = u]E[A(t-u)]G(du),$$

and hence

$$\begin{aligned} h(t) &= E[A^2(t); T_1 > t] + \int_0^t E[A^2(T_1); T_1 \le t]G(du) + \int_0^t E[A^2(t-u)]G(du) \\ &\quad + 2\int_0^t E[(X_1 \mid T_1 = u]E[A(t-u)]G(du), \end{aligned}$$

which gives

$$h(t) = E[A(T_1 \wedge t)^2] + \int_0^t h(t-u)G(du) + 2\int_0^t E[(X_1 \mid T_1 = u]f(t-u)G(du). \quad (2.5)$$

We now compute the Laplace transform $\hat{h}(s) := s\int_0^\infty e^{-st}h(t)dt$. The transform of the first term on the right hand side of (2.5) is

$$\begin{aligned} s\int_0^\infty e^{-st}E[A^2(T_1 \wedge t)]dt &= E\left[\int_0^{T_1} se^{-st}A^2(t)dt + \int_{T_1}^\infty se^{-st}A^2(T_1)dt\right] \\ &= E\left[\int_{Y_1}^{X_1+Y_1} se^{-st}(t - Y_1)^2 dt\right] + E\left[A^2(T_1)e^{-sT_1}\right] \end{aligned}$$

where, in the first equality we have used Fubini's theorem. Taking into account that $A(T_1) = X_1$, and the independence of X_1 and Y_1 this last expression can be rewriten in terms of the Laplace transforms of X_1, Y_1, and their derivatives as

$$\frac{2}{s}\frac{\lambda}{\lambda+s}\left(\frac{1-\hat{X}(s)}{s}+\hat{X}'(s)\right). \tag{2.6}$$

The Laplace transform of the third term on the right hand side of (2.5) is

$$\begin{aligned} s\int_0^\infty \int_0^t e^{-st} E[X_1 \mid T_1 = u]\, \tilde{f}\,(t-u)G(du) \\ &= \int_0^\infty e^{-su} E[X_1 \mid T_1 = u] G(du) \int_u^\infty s e^{-s(t-u)}\, \tilde{f}\,(t-u)dt \\ &= \hat{f}(s) \int_0^\infty E[X_1 e^{-sT_1} \mid T_1 = u] G(du) \\ &= \hat{f}(s) E[X_1 e^{-s(X_1+Y_1)}] = -\frac{\lambda}{\lambda+s}\hat{X}'(s)\hat{f}(s), \end{aligned} \tag{2.7}$$

where in the last step we have used the independence of X_1 and Y_1 and $\hat{f}(s)$ is the Laplace transform of $\tilde{f}\,(t)$ given in (2.4). Hence, taking into account (2.6), (2.7), we obtain (after considerable rearranging) the following expression for the Laplace transform of $h(t) = E[A(t)^2]$

$$\hat{h}(s) = \frac{2}{s^2}\left(\frac{\lambda\mu_1\hat{X}_I(s)}{1+\lambda\mu_1\hat{X}_I(s)} + \frac{\lambda\hat{X}'(s)}{(1+\lambda\mu_1\hat{X}_I(s))^2}\right). \tag{2.8}$$

2.3 Tauberian Theorems and Asymptotics

A key result in deducing the asymptotic behaviour of a real function, $f(x)$ as $x \to \infty$ in terms of the behaviour of its Laplace transform is Karamata's Tauberian theorem. Standard references for this topic are Widder [33], Bingham, Goldie, and Teugels, [4], and Feller [13]. We begin with the following

Definition 2 [Functions of Slow Variation] *A measurable function* $l : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ *is* slowly varying *at infinity if, for every* $x > 0$,

$$\lim_{y\to\infty} \frac{l(xy)}{l(y)} = 1.$$

We say that l *is slowly varying at 0 if* $l(1/x)$ *is slowly varying at infinity.*

Examples of familiar functions that are slowly varying at infinity include functions $l(x)$ that converge to a strictly positive, finite limit, as $x \to \infty$, and $l(x) = \log(x)$.

Following Feller [13], let U be the measure concetrated on $[0, \infty]$ such that its Laplace transform

$$\hat{U}(s) = \int_0^\infty e^{-sx} U\{dx\} \tag{2.9}$$

exists for $s > 0$. It will be convinient to describe the measure U in terms of its improper distribution function defined for $x \geq 0$ by $[U\{0, x\}]$. Under fairly general conditions the behaviour of $\hat{U}$ near the origin uniquely determines the asymptotic behaviour of $U(x)$ as $x \to \infty$ and vice versa. Any relation describing the asymptotic behaviour of U in terms of $\hat{U}$ is called a Tauberian theorem, whereas theorems desribing the behaviour of $\hat{U}$ in terms of U are usually called Abelian.

If $f : R \to R$ is locally integrable, and vanishes on $(-\infty, 0)$, it is convenient to define its Laplace transform as

$$f(t) := s \int_{-\infty}^\infty e^{-sx} f(x) dx = s \int_0^\infty e^{-sx} f(x) dx, \tag{2.10}$$

again for all s for which the integral converges absolutely. It is importrant to note the slightly different definition of the Laplace transform for measures and functions respectively. We tacitly followed this convention in the previous sections.

The following fundamental theorem due to Karamata (see [13], [4], [33]) will enable us to obtain the asymptotic behaviour of $\tilde{f}(t) = EA(t)$ and $h(t) = E[A(t)^2]$ (and hence that of $\mathsf{Var}[A(t)]$) as $t \to \infty$ from the corresponding behaviour of their Laplace transforms near zero. (For two real functions f, g, write $f(x) \sim g(x)$ as $x \to \infty$ (respectively $x \to 0$) if $\frac{f(x)}{g(x)} \to 1$.)

Theorem 5 (Karamata's **Tauberian Theorem**) *Let U be a non-decreasing right-continuous function on R with $U(x) = 0$ for all $x < 0$. If l varies slowly and $c \geq 0$, $\rho \geq 0$, the following are equivalent*

$$U(x) \sim cx^\rho l(x) \frac{1}{\Gamma(\rho + 1)} \tag{2.11}$$

where $x \to \infty$,

$$\hat{U}(s) \sim cs^{-\rho} l(\frac{1}{s}) \tag{2.12}$$

where $s \downarrow 0$.

When $c = 0$, *(2.9) is to be interpreted as* $U(x) = o(x^\rho l(x))$; *similarly for (2.10).*

It is important to note that the slowly varying function $l(\cdot)$ which appears in (2.11) and (2.12) is the same.

Indeed, from (2.4) we readily see that

$$\lim_{s \downarrow 0} sF(s) = \lim_{s \downarrow 0} \frac{\lambda\mu_1 X_I(s)}{1 + \lambda\mu_1 X_I(s)} = \frac{\lambda\mu_1}{1 + \lambda\mu_1}$$

since $X_I(0) = 1$. Therefore Karamata's theorem applies (with $\rho = 1$, $l(x) = 1$, and $c = \frac{\lambda\mu_1}{1+\lambda\mu_1}$) and allows us to conlude that

$$\lim_{t \to \infty} \frac{\tilde{f}(t)}{t} = \frac{\lambda\mu_1}{1 + \lambda\mu_1} \frac{1}{\Gamma(2)}$$

or, equivalently, that

$$\tilde{f}(t) \sim \frac{\lambda\mu_1}{1 + \lambda\mu_1} t.$$

Taking this analysis one step further, consider the function $\tilde{f}(t) - \frac{\lambda\mu_1}{1+\lambda\mu_1} t$ which has Laplace transform $\hat{f}(s) - \frac{1}{s}\frac{\lambda\mu_1}{1+\lambda\mu_1}$ and consider its behaviour near the origin. We have

$$\begin{aligned} \lim_{s \downarrow 0} \left(\hat{f}(s) - \frac{1}{s} \frac{\lambda\mu_1}{1 + \lambda\mu_1} \right) &= \lim_{s \downarrow 0} \frac{\lambda\mu_1}{1 + \lambda\mu_1} \frac{\frac{\hat{X}_I(s) - 1}{s}}{1 + \lambda\mu_1 \hat{X}_I(s)} \\ &= \frac{\lambda\mu_1}{1 + \lambda\mu_1} \frac{\hat{X}_I'(0)}{1 + \lambda\mu_1 \hat{X}_I(0)} = -\frac{\lambda\mu_2}{2(1 + \lambda\mu_1)^2}. \end{aligned}$$

(In the above computation we have used the fact that the mean of the integrated tail distribution F_I is $-\hat{X}_I'(0) = \frac{\mu_2}{2\mu_1}$ (see, e.g. Çinlar, 1975).) Therefore, Karamata's theorem shows that $\tilde{f}(t) - \frac{\lambda\mu_1}{1+\lambda\mu_1} t \sim -\frac{\lambda\mu_2}{2(1+\lambda\mu_1)^2}$ which we rewrite as

$$\tilde{f}(t) = \frac{\lambda\mu_1}{1 + \lambda\mu_1} t - \frac{\lambda\mu_2}{2(1 + \lambda\mu_1)^2} + o(1). \tag{2.13}$$

Let us now repeat this process for $\hat{h}(s)$ given in (2.8). It is immediate that

$$\begin{aligned}\lim_{s\downarrow 0} s^2\hat{h}(s) &= 2\lim_{s\downarrow 0}\left(\frac{\lambda\mu_1\hat{X}_I(s)}{1+\lambda\mu_1\hat{X}_I(s)}+\frac{\lambda\hat{X}'(s)}{(1+\lambda\mu_1\hat{X}_I(s))^2}\right)\\ &= 2\left(\frac{\lambda\mu_1}{1+\lambda\mu_1}\right)^2 \qquad (2.14)\end{aligned}$$

and hence, from Karamata's theorem it follows that

$$h(t)\sim t^2\left(\frac{\lambda\mu_1}{1+\lambda\mu_1}\right)^2.$$

Consider now the Laplace transform of $h(t)-t^2\left(\frac{\lambda\mu_1}{1+\lambda\mu_1}\right)^2$ which can be written as

$$\begin{aligned}&\hat{h}(s)-\frac{2}{s^2}\left(\frac{\lambda\mu_1}{1+\lambda\mu_1}\right)^2\\ &\quad=\frac{2}{s^2}\left(\left(\frac{\lambda\mu_1\hat{X}_I(s)}{1+\lambda\mu_1\hat{X}_I(s)}\right)^2-\left(\frac{\lambda\mu_1}{1+\lambda\mu_1}\right)^2+\frac{\lambda}{s}\frac{1-\hat{X}(s)+s\hat{X}'(s)}{(1+\lambda\mu_1\hat{X}_I(s))^2}\right).\end{aligned}$$

Multiplying the above expression by s and taking the limit as $s\downarrow 0$ gives, with the help of de l'Hôpital's rule,

$$\lim_{s\downarrow 0} s\left(\hat{h}(s)-\frac{2}{s^2}\left(\frac{\lambda\mu_1}{1+\lambda\mu_1}\right)^2\right)=\frac{\mu_2}{2\mu_1}\left(\frac{1}{(1+\lambda\mu_1)^2}-\frac{2(\lambda\mu_1)^2}{(1+\lambda\mu_1)^3}\right).$$

Therefore, appealing once more to Karamata's theorem, we have

$$h(t)=t^2\left(\frac{\lambda\mu_1}{1+\lambda\mu_1}\right)^2+t\frac{\mu_2}{2\mu_1}\left(\frac{1}{(1+\lambda\mu_1)^2}-\frac{2(\lambda\mu_1)^2}{(1+\lambda\mu_1)^3}\right)+O(1). \qquad (2.15)$$

We are now in position to combine (2.13) and (2.15) in order to obtain the following asymptotic expansion for the variance

$$\mathsf{Var}[A(t)]=h(t)-\tilde{f}(t)^2\sim t\frac{\lambda\mu_2}{(1+\lambda\mu_1)^3}. \qquad (2.16)$$

2.4 Asymptotic Mean and Variance for M Contracts

We now examine a model consisting of M independent contracts, each behaving like the model introduced in the beginning of this chapter. The contracts are not assumed to have necessarily identical parameters. In particular we assume that the k'th ($k = 1, 2, \ldots, M$) contract has exponential rate λ_k and claim duration with mean μ_{1k} and second moment μ_{2k}.

$A_k(t)$ is the sum of claims up to time t of the k th process, i.e.

$$A_k(t) = \int_0^t a_k(s)ds$$

By using the renewal arguments and Laplace transformations we want to obtain a formula for the mean of $A_k(t)$ and also the variation of $A_k(t)$.

Second model We consider now a more complex problem. We will denote by

$$A(t) := \sum_{k=1}^{M} A_k(t),$$

the sum of claims up to time t the superposition of of n independent replicates of $A_k(t)$. This model is closer to a real model and can have different claim X distributions each time. The Y distribution continues being an exponential with the same or different parameters. We now examine this kind of problem, but now the formulas for the first and second order moments are easier to obtain.

Here we should make a parenthesis.

Two other possible models

In this parentheses we discuss two other possible models that can derive, we call them the third and fourth model respectively.

Third model By taking the limit for M in the previous model, $M \to \infty$, we examine another kind of model very similar with the previous, but now we work mostly asymptotically. Because of the way this model is defined we are able to use Limit Theorems for superpositions.

Fourth and general model Finally, a very interesting model is obtained by assuming that the X and Y random variables have unknown distributions and they are not necessary independent. This last model is a generalized form of all the models that were previously described. In other words we can obtain from this one all the previous as well as their properties.

Returning to our second model we have:

Now we can expand our model by having a new $A(t) = \sum_{i=1}^{M} A_i(t)$ with $A_i(t)$ defined as previously.This model is closer to a real risk model with delayed claims that has different claim distributions each time. We suppose that Y has the same exponential distribution with the same parameter λ for every i process of the M.

For the mean

$$E[A(t)] = \sum_{i=1}^{M} EA_i(t) = \sum_{i=1}^{M} \left(t\frac{\lambda\mu_{1i}}{1+\lambda\mu_{1i}} - \frac{\lambda\mu_{2i}}{2(1+\lambda\mu_{1i})^2} \right).$$

In patricular, when all the contracts have the same characterisics, we have

$$E[A(t)] = M \left(t\frac{\lambda\mu_1}{1+\lambda\mu_1} - \frac{\lambda\mu_2}{2(1+\lambda\mu_1)^2} \right).$$

For the variance

In the case where the model has different claim distributions each time

$$\mathsf{Var}[A(t)] = \mathsf{Var}[\sum_{i=1}^{M} A_i(t)] = \sum_{i=1}^{M} \mathsf{Var}[A_i(t)] = \sum_{i=1}^{M} t\frac{\lambda\mu_{2i}}{(1+\lambda\mu_{1i})^3}.$$

In the case where the claims have the same distribution with the same parameters

$$\mathsf{Var}[A(t)] = Mt\frac{\lambda\mu_2}{(1+\lambda\mu_1)^3}.$$

We can see and easily prove that the above results fulfill Campbell's Theorem (see Appendix section 6.5 and Daley, D.J. and D. Vere–Jones 1988).

Here we can introduce a case of great importance for understanding our model in the long run that seems to simplify some things.

So we define

$$\lambda_M = \frac{\lambda}{M}$$

and also assume that the claims have the same distribution with the same parameters.

In this case, for the mean we have

$$\begin{aligned} E[A(t)] &= M\left(t\frac{\lambda_M \mu_1}{1+\lambda_M \mu_1} - \frac{\lambda_M \mu_2}{2(1+\lambda_M \mu_1)^2}\right) \\ &= M\left(t\frac{\frac{\lambda}{M}\mu_1}{1+\frac{\lambda}{M}\mu_1} - \frac{\frac{\lambda}{M}\mu_2}{2(1+\frac{\lambda}{M}\mu_1)^2}\right) \rightarrow_{M\rightarrow\infty} t\lambda\mu_1 - \lambda\frac{\mu_2}{2} \end{aligned}$$

$$\Rightarrow \frac{E[A(t)]}{t} \rightarrow \lambda\mu_1 \tag{2.17}$$

While for the variance we have

$$\mathsf{Var}[A(t)] = Mt\frac{\lambda_M \mu_2}{(1+\lambda_M \mu_1)^3} = Mt\frac{\frac{\lambda}{M}\mu_2}{(1+\frac{\lambda}{M}\mu_1)^3} \rightarrow_{M\rightarrow\infty} t\lambda\mu_2$$

$$\Rightarrow \frac{\mathsf{Var}[A(t)]}{t} \rightarrow_{M\rightarrow\infty} \lambda\mu_2 \tag{2.18}$$

It is obvious that from (2.17) and (2.18) Campbell's Theorem is fully applied. What is more (2.17) and (2.18) is in fact our already defined third model.

2.5 Details and properties for the third model and approximation of the claim number process by the Poisson distribution.

By taking the limit for M in the previous model, $M \rightarrow \infty$, we examine another kind of model very similar with the previous, but now we work mostly asymptotically.

In fact, if $\lambda_M = \frac{\lambda}{M}$ and also assume that the claims have the same distribution with the same parameters we get (2.17) and (2.18).

By the above we can understand that asymptotically for our model (and more specifically the second and third one) the amount $A(t)$ of the sum of delayed claims is the sum of individually negligible number of claims (of each point process). The negligible number of claims is due to the way the distribution of the silent times is defined. It is for the parameter of the exponential distribution of the silent times that $\lambda_M = \frac{\lambda}{M}$ and so for $M \to \infty$ it is obvious that $\lambda_M \to 0$ and so the time periods, that the claims come, tend to be very few.

Because of the way this model is defined we are able to use Limit Theorems for superpositions.

Our goal is to approximate our model by a model that the arrival of the claims follow the Poisson distribution. Having achieved this, we are able to examine many properties and asymptotic results.

Chapter 3

Portfolios with a Large Number of Contracts

3.1 Superpositions of point processes

We begin our analysis with some facts regarding superpositions of point processes. The formal setting for studying the sum or superposition of a large number of point processes (or random measures) is a triangular array $\{\xi_{ni} : i = 1, \ldots, m_n; n = 1, 2, \ldots\}$ and its associated row sums

$$\xi_n = \sum_{i=1}^{m_n} \xi_{ni},$$

$n = 1, 2, \ldots$. If for each n the processes $\{\xi_{ni} : i = 1, \ldots, m_n\}$ are mutually independent, we speak of an independent array.

Definition 3 *When an independent array satisfies the condition that for all $\varepsilon > 0$ and all bounded $A \in \mathcal{B}_{\mathcal{X}}$*

$$\lim_{n \to \infty} \sup_i P\{\xi_{ni}(A) > \varepsilon\} = 0$$

the array is uniformly asymptotically negligible.

When we are dealing with a triangular array the previous condition reduces to

$$\lim_{n\to\infty} \sup_i P\{N_{ni}(A) > 0\} = 0.$$

Proposition 6 *The triangular uniformly asymptotically negligible array* $\{N_{ni} : i = 1,\ldots,m_n; n = 1,2,\ldots\}$ *converges weakly to a Poisson process with mean measure* μ *if and only if for all bounded Borel sets* A *with* $\mu(\partial A) = 0$,

$$\sum_{i=1}^{m_n} P\{N_{ni}(A) \geq 2\} \to 0 \tag{3.1}$$

where $n \to \infty$ *and*

$$\sum_{i=1}^{m_n} P\{N_{ni}(A) \geq 1\} \to \mu(A) \tag{3.2}$$

where $n \to \infty$.

3.2 Portfolios consisting of a large number of contracts

In this section we will examine the behavior of portfolios consisting of a large number of contracts of the type considered in chapter 2. In order to obtain an asymptotic result we will suppose that we are given a double array of such single contract models. The n'th row of the array consists of n independent ON/OFF processes with exponential silent periods with rates λ_{ni}, $i = 1, 2, \ldots, n$, and active periods with distribution F. N_{ni} is the point process of the beginnings of active periods of the i'th process in the n'th row. We will assume that the rates λ_{ni} satisfy the following two conditions:

Condition C1: $\max_{i\leq n} \lambda_{ni} \to 0$ as $n \to \infty$.

Condition C2: $\sum_{i\leq n} \lambda_{ni} \to \lambda \in (0, \infty)$ as $n \to \infty$.

We will show that as $n \to \infty$, the claims occur according to a Poisson process with rate λ and hence the resulting claim process is

$$S(t) = \sum_{n=1}^{\infty} \int_0^t \mathbf{1}(T_n \leq s < T_n + X_n)ds$$

where $\{T_n\}$ are the points of the Poisson process with rate λ when claims occur, and $\{X_i\}$ are i.i.d. random variables with distribution F that correspond to the duration of payments. Hence this is an integrated Poisson shot noise process as described and analyzed in Klüppelberg and Mikosch (1995) and Brémaud (1998).

In our model we remark that the random measure ξ_{ni} is denoted by N_{ni}, since we now have a triangular array of point processes. So by $N_{ni}(A)$ we denote the number of times we get a claim in a Borel set A.

In the sequel we shall need the following

Lemma 7 *Let* $\{\lambda_{ni}\}$, $i = 1, 2, \ldots, n$, $n = 1, 2, \ldots$, *be a double array of nonnegative real numbers satisfying C1 and C2 above. Then*

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 - e^{-\lambda_{ni}x}\right) = \lambda x$$

for any $x > 0$.

Proof: From the inequality

$$\frac{t}{1+t} \leq 1 - e^{-t} \leq t$$

which holds for any $t \geq 0$ it follows that

$$\sum_{i=1}^{n} \frac{\lambda_{ni}x}{1+\lambda_{ni}x} \leq \sum_{i=1}^{n} \left(1 - e^{-\lambda_{ni}x}\right) \leq x \sum_{i=1}^{n} \lambda_{ni}.$$

This inequality can be reinforced to obtain

$$\frac{1}{1 + x \max_{i\leq n} \lambda_{ni}}\, x \sum_{i=1}^{n} \lambda_{ni} \leq \sum_{i=1}^{n} \left(1 - e^{-\lambda_{ni}x}\right) \leq x \sum_{i=1}^{n} \lambda_{ni}.$$

Letting $n \to \infty$ above and using conditions C1 and C2 establishes the lemma.

In order to use Proposition 1 we first need to show that we have a uniformly asymptotically negligible array. Setting $N_{ni}(x) := N_{ni}(0, x]$, it suffices to show that

$$\lim_{n\to\infty} \sup_{i} P\{N_{ni}(x) > 0\} = 0$$

Proof: Starting with the inequality

$$E[N_{ni}(x)] \geq P\{N_{ni}(x) = 0\} + P\{N_{ni}(x) > 0\}$$

it follows that

$$P\{N_{ni}(x) > 0\} \leq E[N_{ni}(x)] = \sum_{k=0}^{\infty} G_{ni}^{*(k+1)} * F^{*k}(x) \tag{3.3}$$

where

$$G_{ni}(x) = 1 - e^{-\lambda_{ni} x}.$$

However, $G_{ni}^{*(k+1)} * F^{*k}(x) \leq G_{ni}^{*(k+1)}(x)$ and thus from (3.3) we have that

$$P\{N_{ni}(x) > 0\} \leq \sum_{k=0}^{\infty} G_{ni}^{*(k+1)}(x) = \lambda_{ni} x$$

Letting $n \to \infty$ in the above inequality and taking into account C1 we have

$$\lim_{n\to\infty} \sup_{i\leq n} P\{N_{ni}(x) > 0\} \leq x \lim_{n\to\infty} \sup_{i\leq n} \lambda_{ni} = 0.$$

This establishes that under condition C1 we have indeed a uniformly asymptotically negligible array.

Now we should prove that the two conditions (3.1) and (3.2) hold. We should show that

$$\sum_{i=1}^{n} P\{N_{ni}(x) \geq 2\} \to 0$$

where $n \to \infty$ and

$$\sum_{i=1}^{n} P\{N_{ni}(x) \geq 1\} \to \mu(x)$$

where $n \to \infty$.

We first check (3.1):

Proof: We have

$$\sum_{i=1}^{n} P(N_{ni}(x) \geq 2) \leq \sum_{i=1}^{n} G_{ni}^{*2} * F(x) \leq \sum_{i=1}^{n} G_{ni}^{*2}(x).$$

However $G_{ni}^{*2}(x) \leq (G_{ni}(x))^2$ and thus the right hand side of the above inequality is less than

$$\max_{i\leq n} G_{ni}(x) \sum_{i=1}^{n} G_{ni}(x) = \left(1 - e^{-(\max_{i\leq n} \lambda_{ni})x}\right) \to 0$$

as $n \to \infty$ in view of Lemma 1 and conditions C1 and C2. This establishes (3.1).

We will establish now that (3.2) holds as well

Proof: Indeed we have

$$\sum_{i=1}^{n} P(N_{ni}(x) \geq 1) = \sum_{i=1}^{n} G_{ni}(x) = \sum_{i=1}^{n} \left(1 - e^{-\lambda_{ni}x}\right) \lambda x$$

as a consequence of Lemma 1. Hence (3.2) holds with $\mu(x) = \lambda x$.

Chapter 4

Diffusion Aproximation

We shall consider approximation of a risk reserve process for our first model by a Wiener process (or Brownian motion) using a functional central limit theorem. We follow Grandell (1977).

Our main mathematical tool is the theory of weak convergence of probability measures on metric spaces.

4.1 Theory of weak convergence

Let $\mathcal{D} = \mathcal{D}[0,\infty)$ be the space of cadlag functions on $[0,\infty)$. Those are functions which are right continuous at each point of $[0,\infty)$ and which have a left limit at each point of $[0,\infty)$. $\mathcal{D}$ is separable and metrisable with a complete metric ,endowed with the Skorokhod J_1 topology.

Let Z, X_1, $X_2, \ldots$, be stochastic processes. It is possible to give a precise meaning of $X_n \xrightarrow{d} Z$, i.e. X_n tends in distribution to Z. This is the same as to say that the distributions of X_n converge weakly to the distribution of Z.

We consider the function $S_T X = \sup\limits_{0 \le t \le T} X(t)$ from $\mathcal{D}$ into the real line.

If $X_n \xrightarrow{d} Z$ then $S_T X_n \xrightarrow{d} S_T Z$ for all finite T such that $P\{Z(T-) = Z(T)\} =$

1.

$S_T X_n \xrightarrow{d} S_T Z$ means ordinary convergence in distribution for random variables.

The situation is more difficult for an infinite period since $X_n \xrightarrow{d} Z$ does in general not imply that $\sup_{t\geq 0} X_n(t) \xrightarrow{d} \sup_{t\geq 0} Z(t)$.

If $S_T X \geq u$ the risk business is said to be ruined before time T.

Definition 4 *A standard Wiener process $W = \{W(t); t \in [0,\infty\}\}$ is a process with stationary and independent increments such that $W(1)$ is normally distributed with $E[W(t)] = 0$ and $Var[W(1)] = 1$. The distribution function $W(1)$ will be denoted by Φ.*

Our risk model is $X(t) = Y(t) - ct$. Generally the gross risk premium is choosen so large that X has a negative drift. Therefore it is not surprising that it will turn out to be natural to approximate X by a Wiener process with drift.

Skorohod has shown that

$$P\{\sup_{0\leq t\leq T}(W(t) - \gamma t) > x\} = 1 - \Phi(\frac{\gamma T + K}{\sqrt{T}}) + e^{-2\gamma x}\Phi(\frac{\gamma T - K}{\sqrt{T}})$$

for $x > 0$ and $\gamma > 0$.

This probability is the basis for many results in the section for estimating a Normal limit for the first model.

$P\{S_T Z > x\}$ is a continuous function for all $x > 0$. Therefore $P\{S_T X_n > x\} \to P\{S_T Z\}$ for all $x > 0$ and all finite T.

Definition 5 *A sequence of functions $(y_n)_{n\in N}$ in $\mathcal{D}$ converges uniformly on compacta to a function y if*

$$\sup_{0\leq t\leq k} |y_n(t) - y(t)| \to 0 \text{ as } n \to \infty \text{ for each fixed } k \in N.$$

Definition 6 *A stochastic process $Y = (Y(t))t \geq 0$ is said to be in $\mathcal{D}$ if all its paths are in $\mathcal{D}$. Let Y, $Y_1, Y_2, \ldots,$ be stochastic processes in $\mathcal{D}$ with Y in*

$\mathcal{C}[0,\infty)$; *i.e.* Y *belongs to* $\mathcal{D}$ *and has continuous paths. We say that* Y_n *converges in distribution to* Y*, and we write* $Y_n \xrightarrow{d} Y$ *if, for each fixed* $k \in N$, $E[f(Y_n)] \to E[f(Y)]$ *for all bounded and continuous functionals* f *on* $\mathcal{D}[0,\infty]$ *equipped with the uniform metric .*

Definition 7 (Polish space) *Let* S *be a separable and complete metric space and let* $\mathcal{B}(S)$ *be the* σ*-algebra generated by open sets. A space with these topological properties is called Polish.*

We may note that the metric ρ which we work with do not need to be the metric that makes S complete. Let $\xi_1, \xi_2, \ldots,$ be S-valued random variables with distributions $P, P_1, P_2, \ldots, .$ If $\int_S f dP_n \to \int_S f dP$ for all bounded and continuous functions $f : S \to R$ we say that ξ_n converges in distribution to ξ and use the notation $\xi_n \xrightarrow{d} \xi$ or that P_n converges weakly to P.

The main motivation for the study of weak convergence is the following theorem given by Billingsley (1968, pp.30-31).

Theorem 8 (main theorem of weak convergence) *If* h *is a measurable function from* S *into some metric space* S' *and if*

$$\xi_n \xrightarrow{d} \xi$$

then also,

$$h(\xi_n) \xrightarrow{d} h(\xi),$$

provided that $P\{\xi \in \text{the set of discontinuity points of } h\} = 0$.

We shall be interested in a function h which is not continuous. The following theorem is a special case of a result given by Billingsley (1968, p.25).

Theorem 9 *Let* $\{h_n\}$ *be a sequence of continuous functions from* S *into some separable metric space* S' *with metric* ρ'*. If*

(i) $h(x) = \lim_{k\to\infty} h_k(x)$ *is a well-defined function* $S \to S'$,

(ii) $\xi_n \xrightarrow{d} \xi$,

(iii) $\lim_{k\to\infty}\limsup_{n\to\infty} P\{\rho'(h_k(\xi_n), h(\xi_n)) > \varepsilon\} = 0$ *for each* $\varepsilon > 0$,

then

$$h(\xi_n) \xrightarrow{d} h(\xi).$$

Let Λ_t denote the class of strictly increasing continuous functions from $[0,t]$ onto itself. For x and y in $\mathcal{D}[0,t]$ we define a metric ρ by

$$\rho(x,y) = \inf_{\lambda\in\Lambda_t}(\max(\| x\circ\lambda - y \|, \| \lambda - e_t \|)),$$

where $x\circ\lambda(s) = x(\lambda(s))$, $e_t(s) = s$ and $\| x \| = \sup_{0\le s\le t} | x(s) |$.

With this metric $\mathcal{D}[0,t]$ is *Polish*. The space $\mathcal{D}[0,t]$ has the same properties as $\mathcal{D}[0,1]$ for which Billingsley (1968) is the standard reference.

Consider the function $S_t : \mathcal{D}[0,t] \to R$ defined by $S_t x = \sup_{0\le s\le t} x(s)$ as previously. We shall prove that this function is continuous .

Proof: Choose $\varepsilon > 0$ and x and $y \in \mathcal{D}[0,t]$ such that $\rho(x,y) \le \varepsilon$. There exists $\lambda \in \Lambda_t$ such that $\| x\circ\lambda - y \| \le \varepsilon$ and $\| \lambda - e_t \| \le \varepsilon$. Since $S_t x = S_t x\circ\lambda$ it follows that $y(s) \le S_t x + \varepsilon$ for all s and thus $S_t y \le S_t x + \varepsilon$ and thus $| S_t x - S_t y | \le \varepsilon$ which was to be proved.

Let Λ be the set of strictly increasing continuous functions from $[0, \infty)$ onto itself and let $e \in \Lambda$ be defined by $e(s) = s$. Take x, x_1, x_2, $\ldots \in \mathcal{D}$.

Let $x_n \to x$ mean that there exist $\lambda_1, \lambda_2, \ldots$, such that

$$\sup_{0\le s\le t} | x_n\circ\lambda_n(s) - x(s) | \to 0$$

as $n \to \infty$ for all finite $t \in (0, \infty)$ and that:

$$\sup_{0\le s\le t} | \lambda_n(s) - s | \to 0$$

as $n \to \infty$. With this definition of convergence $\mathcal{D}$ is *Polish.*

The following theorem brings the question of convergence in distribution of $\mathcal{D}$-valued stochastic processes back to convergence in distribution of $\mathcal{D}[0,t]$-valued processes.

Theorem 10 *Let X, $X_1, X_2, \ldots$ be $\mathcal{D}$-valued stochastic processes and let t_1, $t_2, \ldots \in (0, \infty)$ be such that $t_k \to \infty$ as $k \to \infty$ and such that $P\{X(t_k^-) = X(t_k)\} = 1$ for all k. Then*

$$S_{t_k} X_n \xrightarrow{d} S_{t_k} X$$

as $n \to \infty$ if and only if

$$X_n \xrightarrow{d} X$$

as $n \to \infty$.

Remark 2 *From now on all processes are assumed to be $\mathcal{D}$-valued.*

Suppose that $X_n \xrightarrow{d} X$ and assume that X is continuous in probability, i.e. $P\{X(t^-) = X(t)\} = 1$ for all t. It follows that :

$$S_t X_n \xrightarrow{d} S_t X.$$

Consider now the function s defined by

$$sx = \sup_{t \geq 0} x(t)$$

for all $x \in D$, where s is a function from D into $(-\infty, \infty]$.

If $X_n \xrightarrow{d} X$ it does not in general follow that $sX_n \xrightarrow{d} sX$. In order to realize this we consider $X_n = x_n$ with probability one where

$$x_n(t) = \begin{cases} 1 & \text{if } t \geq n \\ 0 & \text{in } t < n \end{cases}$$

The following theorem may be helpful in some cases.

Theorem 11 *Assume that $X_n(0) = 0$ for all n that X is continuous in probability. If $X_n \xrightarrow{d} X$ and if $\lim_{k\to\infty} \limsup_{n\to\infty} P\{\sup_{t\geq k} X_n(t) > 0\} = 0$ then $sX_n \xrightarrow{d} sX$.*

The following theorem is useful for proving weak convergence.

Theorem 12 *Let $X_1, X_2, \ldots,$ be either summation processes, i.e. sums of independent random variables, or stochastic processes with stationary and independent increments. Define ξ_n by $\xi_n(t) = X_n(nt)$ and let ξ be a stochastic process with stationary and independent increments.*

If $\xi_n(t) \xrightarrow{d} \xi(t)$ for all $t \in [0, \infty)$ then

$$\xi_n \xrightarrow{d} \xi.$$

4.2 Theorems useful in diffusion approximation

Now we shall prove two theorems that depend on the theory of weak convergence that is already discussed. Those two theorems are useful for obtaining a Normal or Brownian limit.

Theorem 13 *Let $Y_n \xrightarrow{d} W$ and define X_n and Z by*

$$X_n(t) = [Y(nt) - c_n nt]/b\sqrt{n}$$

and

$$Z(t) = W(t) - \gamma t.$$

Then

$$X_n \xrightarrow{d} Z$$

iff

$$(c_n - a_n)\sqrt{n} \to \gamma b.$$

Proof: We have $X_n(t) = \frac{[Y(nt)-ant]}{b\sqrt{n}} - \frac{n(c_n-a)t}{b\sqrt{n}}$.

Define h_n by $h_n(t) = \frac{n(c_n-a)t}{b\sqrt{n}}$ and h by $h(t) = \gamma t$ and thus $X_n = Y_n - h_n$. For all t we have

$$\sup_{0 \le s \le t} \mid h_n(s) - h(s) \mid = \mid \frac{\sqrt{n}(c_n - a)t}{b} - \frac{\gamma b}{b} \mid t \to 0$$

as $n \to \infty$ and thus $h_n \to h$ in D.

Since h is continuous the result follows from the theorem of weak convergence.

For proving the other part of the theorem we consider $t = 1$. Then we have

$$X_n(1) = Y_n(1) - \frac{\sqrt{n}(c_n - a)t}{b} \xrightarrow{d} W(1) - \gamma.$$

Since $Y_n(1) \xrightarrow{d} W(1)$ it follows that

$$\frac{\sqrt{n}(c_n - a)t}{b} \to \gamma.$$

Alternatively we can replace b_n instead of b, i.e a term that depends on the n^{th} sequence of a point process according to our first model.

We define $\psi(r, u, t)$ as the ruin probabitity accepted by the company where r is the safety loading, u is the initial capital and t time.

Theorem 14 *Assume that $Y_n \xrightarrow{d} W$ and let δ, x and t be constants. Then*

$$\lim_{n\to\infty} \psi\left(\frac{\delta}{\sqrt{n}}, x\sqrt{n}, tn\right) = 1 - \Phi\left(\frac{\delta t + x}{b\sqrt{t}}\right) + e^{-\frac{2\delta x}{b^2}} \Phi\left(\frac{\delta t - x}{b\sqrt{t}}\right).$$

Proof: *We have*

$$\begin{aligned}
\psi\left(\frac{\delta}{\sqrt{n}}, x\sqrt{n}, tn\right) &= P\left\{\sup_{0\le s\le nt} (Y(s) - (a + \frac{\delta}{\sqrt{n}})s) > x\sqrt{n}\right\} \\
&= P\left\{\sup_{0\le s\le nt} (Y(ns) - (a + \frac{\delta}{\sqrt{n}})ns) > x\sqrt{n}\right\} \\
&= P\left\{\sup_{0\le s\le nt} (Y(ns) - (a + \frac{\delta}{\sqrt{n}})ns) > x\sqrt{n}\right\} \\
&= P\left\{\sup_{0\le s\le nt} \frac{Y(ns) - (a + \frac{\delta}{\sqrt{n}})ns}{b\sqrt{n}} > \frac{x}{b}\right\}.
\end{aligned}$$

It follows from the previous Theorem that

$$\psi\left(\frac{\delta}{\sqrt{n}}, x\sqrt{n}, tn\right) \to P\left\{s_t Z > \frac{x}{b}\right\},$$

where $Z(t) = W(t) - (\frac{\delta}{b})t$.

4.2.1 Building approximation for our first model by a Wiener process

All stochastic processes considered have their realizations in the space D of functions that are right continuous and have left hand limits. Our first model is already defined previously. We consider the function $S_T X = \sup_{0\leq t\leq T} X(t)$, from D into the real line, and we define our risk model in an inverse kind of way.We have

$$X(t) = A(t) - ct \tag{4.1}$$

We will investigate through our risk model the approximation of ruin probabilities for a finite period of time.

The idea is to approximate X by a Wiener process. Generally the gross risk premium is choosen so large that X has negative drift. Therefore it is not surprising that it will turn out to be natural to approximate X by a Wiener process with drift. If there shall be any hope to approximate X by a Wiener process, we must in some way be able to use the central limit theorem. We consider $A(t)$ because it is the only part of $X(t)$ that involves random variables. One, and probably the only natural, way of doing that is to compress time. If we only compress time everything will explode and therefore we have to normalize in some way.

We assume that everything depends on n.

We consider A_n defined by

$$A_n(t) = \frac{A(nt) - a_n nt}{b_n\sqrt{n}} \tag{4.2}$$

and assume that

$$A_n \xrightarrow{d} W \tag{4.3}$$

for some choice of a_n and b_n.

Let us now consider the risk process $X(t)$ and the ruin probability $P\{S_T X > u\}$.If we shall be able to use the assumptions that $A_n \xrightarrow{d} W$ in some way. Let a'_n and b'_n be constants and consider

$$X_n(t) \quad = \quad \frac{X(nt) - a'_n tn}{b'_n\sqrt{n}} =$$

$$= \frac{A(nt) - c_n nt - a'_n nt}{b'_n \sqrt{n}} =$$

$$= \frac{b_n}{b'_n} \frac{A(nt) - a_n nt}{b_n \sqrt{n}} + \frac{(a_n - c_n - a'_n)tn}{b'_n \sqrt{n}}.$$

By choosing $a'_n = a_n - c_n$ and $b'_n = b_n$ we have

$$X_n \xrightarrow{d} W$$

and so

$$P\{S_T X_n > u\} = P\{S_T W > u\}.$$

For obtaining a limit that is Brownian motion but has also a (negative) drift we introduce the following theorems and propositions.

Here we should introduce one result from the first section of this chapter.

Firstly

$$P\left\{\sup_{0 \le t \le T} (W(t) - \gamma t) > x\right\} = 1 - \Phi\left(\frac{\gamma T + K}{\sqrt{T}}\right) + e^{-2\gamma x}\Phi\left(\frac{\gamma T - K}{\sqrt{T}}\right)$$

for $x > 0$ and $\gamma > 0$, Φ the distribution function of $W(1)$.

Theorem 15 *Let $A_n \to W$ and define X_n and Z by*

$$X_n(t) = [A(nt) - c_n nt]/b_n \sqrt{n}$$

and

$$Z(t) = W(t) - \gamma t.$$

Then

$$X_n \xrightarrow{d} Z$$

iff

$$\frac{(c_n - a_n)\sqrt{n}}{b_n} \to \gamma,$$

where $a_n = \frac{\mu_1}{\mu_1 + \frac{1}{\lambda_n}}$ and $b_n = \frac{\mathrm{Var}[A(t)]}{t}$.

Here from (4.2) we have

$$A_n(t) = \frac{A(nt) - \frac{\mu_1}{\mu_1 + \frac{1}{\lambda_n}} nt}{\sqrt{n}} \to W(0, \sigma^2 t)$$

where σ^2 is defined as $\frac{\mathsf{Var}(A(t))}{t}$.

So while already having estimated $a'_n = \frac{\mu_1}{\mu_1+\frac{1}{\lambda_n}} - c_n$ it should be $\frac{a'_n\sqrt{n}}{b_n} \to -\gamma$.

Now from the theorem we have $X_n(t) = \frac{A(nt)-a_n nt}{b_n\sqrt{n}} - \frac{c_n nt-a_n nt}{b_n\sqrt{n}}$ and $Z(t) = W(t) - \gamma t$. Thus $X_n \xrightarrow{d} Z$ iff $\frac{(c_n-a_n)}{b_n}\sqrt{n} \to \gamma$.

So it is

$$\frac{\left(\frac{\mu 1}{\mu_1+\frac{1}{\lambda_n}} - a'_n - \frac{\mu_1}{\mu_1+\frac{1}{\lambda_n}}\right)}{b_n}\sqrt{n} \to \gamma$$

or

$$\frac{a'_n\sqrt{n}}{b_n} \to -\gamma$$

or

$$\frac{\left(\frac{\mu_1}{\mu_1+\frac{1}{\lambda_n}} - c_n\right)}{b_n}\sqrt{n} \to -\gamma.$$

As we expected by defining an inverse kind of risk process the drift here is negative. Now for $X_n(t) = [A(nt) - c_n nt]/\sqrt{n}$ and $Z(t) = W(0, \sigma^2 t) - t$ i.e. for $\gamma = 1$ and $b_n = \frac{\sigma^2}{t}$ we have

$$\frac{a'_n\sqrt{n}}{\frac{\sigma^2}{t}} \to -1. \tag{4.4}$$

1. or

$$\frac{\left(\frac{\mu_1}{\mu_1+\frac{1}{\lambda_n}} - c_n\right)}{\frac{\sigma^2}{t}}\sqrt{n} \to -1 \tag{4.5}$$

Other cases

Now we can distinguish two cases in the above general way of dealing with our problem.

In the first case we can assume that the counting process does not depend on n, so $N_n := N$ and as a result $c_n = c$ or c_n does not depend on n and this goes for λ_n as well.

So the (4.5) result becomes

$$\left(\frac{\mu_1}{\mu_1 + \frac{1}{\lambda}} - c\right)\sqrt{n} \to -1.$$

In the second case we assume that it is only λ_n constant. So we have that (4.5) becomes

$$\left(\frac{\mu_1}{\mu_1 + \frac{1}{\lambda}} - c_n\right)\sqrt{n} \to -1$$

and it is also

$$c_n = \frac{\mu_1}{\mu_1 + \frac{1}{\lambda}} + \frac{1}{\sqrt{n}}$$

and

$$\frac{1}{\sqrt{n}} \int_0^{nt} (a(nt) - \frac{\mu_1}{\mu_1 + \frac{1}{\lambda}})dt \to W(0, \sigma^2 t).$$

4.2.2 Calculating the upper bound for a ruin probability and a crude Lundberg exponent

We shall now try to derive approximate ruin probabilities by using the previous Theorem. The constants a and b can be regarded as parameters characterising the portfolio. we shall interpret a as the net risk premium and b as a measure of the dangerousness of the risk business. A strategy for the company can be formulated as follows.The company decides to accept a certain ruin probability under a certain time.We have $r = c - m$, the safety loading, where m is the mean of $A(t)$. The safety loading and the amount u can be regarded as decision variables. Let $\psi(r, u, t)$ be the ruin probability.If the company accepts a ruin probability p up to time T it is natural to choose r and u such that $\psi(r, u, t) = p$. Using the second theorem from the second section of this chapter, we can get the appropriate approximation for the ruin probability.

Theorem 16 *Assume that $A_n \xrightarrow{d} \sigma W$ and let x, t be constants,*

$$\lim_{n\to\infty} \psi\left(\frac{\delta}{\sqrt{n}}, x\sqrt{n}, tn\right) = 1 - \Phi\left(\frac{\delta t + x}{b\sqrt{t}}\right) + e^{-2\delta x/b^2}\Phi\left(\frac{\delta t - x}{b\sqrt{t}}\right),$$

where $b^2 = \frac{\mathrm{Var}(A(t))}{t}$.

Consider now $\psi(\lambda, u, T)$. Choose n and put $\delta = \lambda\sqrt{n}$, $x = u/\sqrt{n}$ and $t = T/n$. According to the main theorem the approximation $\psi(\lambda, u, T) = 1 - \Phi(\frac{\lambda T + u}{b\sqrt{T}}) + e^{-2\lambda u/b^2}\Phi(\frac{\lambda T - u}{b\sqrt{T}})$ seems reasonable if λ is small, u is large and T is very large in the sense that u, λ^{-1}, and $\sqrt{T}$ are all of the same order. Further this approximation shall not be used when very small ruin probabilities are of interest. The reason for this belief is an association with the question of Large Deviations in connection with the ordinary central limit theorem.

If we allow T to tend to infinity then we get :

$$\psi(r, u) = e^{-2ru/b^2}.$$

This last result is the probability of ruin for an infinite period of time.

From previous section it is proven that for the variance we have

$$b^2 = \frac{\mathsf{Var}(A(t))}{t} = \frac{\lambda\mu_2}{(1+\lambda\mu_1)^3}.$$

The problem is that $X_n \xrightarrow{d} Z$ does in general not imply that $\sup_{t\geq 0} X_n(t) \xrightarrow{d} \sup_{t\geq 0} Z(t)$. In other words the problem is that the time-point of ruin may tend to infinity during the limit procedure.

In Grandell's paper it is shown that under some regularity conditions in the Poisson case we can obtain such a limit.

Now we can expand our model by having a new $A(t) = \sum_{i=1}^{N} A_i(t)$ with $A_i(t)$ defined as previously.This model is closer to a real risk model with delayed claims that has different claim distributions each time. Here we can again apply a functional central limit theorem for obtaining a Normal limit.

Chapter 5

Large Deviations Heuristics – Calculating Lundberg's exponent.

Suppose that X_i, $i = 1, 2, 3, \ldots$, are i.i.d. with distribution function F, corresponding mean $m = \int_{\mathbf{R}} xF(dx)$, and moment generating function $M(\theta) := \int_{\mathbf{R}} e^{\theta x} F(dx)$. Set $S_n = X_1 + \cdots + X_n$. The weak law of large numbers guarrantees that

$$\lim_{n\to\infty} P(S_n \geq nx) = 0 \qquad \text{for } x > m \tag{5.1}$$

and similarly that

$$\lim_{n\to\infty} P(S_n \leq nx) = 0 \qquad \text{for } x < m \tag{5.2}$$

One important question is *how fast do the above probabilities go to zero.* It turns out that they go to zero exponentially fast (we always assume having light tails - exponentially fast convergence doesnot always hold when having heavy tails), i.e. that

$$P(S_n \geq nx) \asymp e^{-nI(x)} \qquad \text{for } x > m. \tag{5.3}$$

In the above formula note that the exponential rate of decay $I(x)$ is a function of x. The meaning of (5.3) is made precise if we state it as

$$\lim_{n\to\infty} \frac{1}{n} \log P(S_n \geq nx) = -I(x) \qquad \text{for } x > m. \tag{5.4}$$

5.1 Chernoff bounds

In the same framework as before X_i, $i = 1, 2, \ldots$ are assumed to be i.i.d. r.v.'s with moment generating function $M(\theta)$. We start with the obvious inequality

$$\mathbf{1}(S_n \geq nx)e^{nx\theta} \leq e^{\theta S_n}$$

which holds for all real θ since the exponential is non–negative. Taking expectations in the above inequality we obtain

$$P(S_n \geq nx) \leq e^{-nx\theta} E[e^{\theta X_1 + X_2 + \cdots + X_n}] = e^{-nx\theta} M(\theta)^n$$

The above inequality provides an upper bound for $P(S_n \geq nx)$ for each $\theta \in \mathbf{R}$. Since the left hand side in the above inequality does not depend on θ we can obtain the best possible bound by setting

$$P(S_n \geq nx) \leq \inf_\theta e^{-n\{x\theta - \log M(\theta)\}} = e^{-n \sup_\theta \{x\theta - \log M(\theta)\}}$$

Define now the *rate* function

$$I(x) := \sup_\theta \{x\theta - \log M(\theta)\}. \tag{5.5}$$

With this definition the Chernoff bound becomes

$$P(S_n \geq nx) \leq e^{-nI(x)} \tag{5.6}$$

In many cases this upper bound can be turned into an asymptotic inequality. This is the content of Cramér's theorem.

5.2 Applications in Risk Theory

Large Deviation Theory has been applied to sophisticated models in risk theory. Assume that an insurance company settles a fixed number of claims in a fixed period of time, assume also that it receives a steady income p form premium payments The sizes of the claims are random and there is therefore the risk that, at the end of some planning period of length T, the total amount paid in settlements of claims will exceed the total income from premium payments over the period. This risk is inevitable, but the company will want to ensure that it is small (in the interest of its reinsurers or some regulatory agency). So we are interested in the small probabilities concerning the sum of a large number of random variables.

If the sizes X_t of claims are independent and identically distributed, then we can apply Cramér's Theorem to approximate the probability of ruin, the probability that the amount $\sum_{t=1}^{T} X_t$ paid out during the planning period T exceeds the premium income pT received in that period:

$$P(\sum_{t=1}^{T} X_t > pT) \approx e^{-TI(p)}.$$

If we require that the risk of ruin is small, for example e^{-r} for some large positive number r, then we can use the rate-function I to choose an appropriate value of p

$$P(\frac{1}{T}\sum_{t=1}^{T} X_t > p) \approx e^{-r}$$

$$e^{-TI(p)} \approx e^{-r}$$

$$I(p) \approx \frac{r}{T}.$$

Since $I(x)$ is convex, it is monotonically increasing for x greater than the mean of X_t and so the equation

$$I(p) = \frac{r}{T}$$

has a unique solution for p.

In general, whenever the rate-function can be approximated near its maximum by a quadratic form, we can expect the Central Limit Theorem to hold.

The name "Large Deviations" arises from the contrast between the Central Limit Theorem and Large Deviation Theory. The Central Limit Theorem governs random fluctuations only near the mean - deviations from the mean of the order of $\frac{\sigma}{\sqrt{n}}$. Fluctuations which are of the order of σ are, relative to typical fluctuations, much bigger : they are large deviations from the mean. They happen only rarely, and so Large Deviation Theory is often described as the theory of rare events, events which take place away from the mean, out in the tails of the distribution.

5.2.1 Theory for infinite horizon used for estimating Lundberg's exponent

Recalling the model of risk theory we discussed above we have that since the sizes of the claims are random, there is the risk that at the end of the planning period T, the total amount paid in settlement of claims will exceed the total assets of the company. When we discussed this model before, we assumed that the only asset of the company with whom the company has to cover the claims is the income from premium payments; it is however likely that the company would have some other assets, say a fixed value u at the begining. We want to evaluate the risk of the amount $\sum_{t=1}^{T} X_t$ paid out over the planning period T exceeding the total assets $pT + u$ of the company. This risk-theory model is similar to a single-server queue: the claims are like customers, the premium income is like service capacity and the initial assets u are like a buffer, guarding temporarily against large claims which exceed the premium income.

We assume that the sizes X_t of the claims are independent and identically distributed so that we can apply Cramér's Theorem to approximate the probability of ruin.

$$P(\frac{1}{T}\sum_{t=1}^{T} X_t > x) \approx e^{-TI(x)},$$

where $x = p + \frac{u}{T}$.

Our problem is finding the right exponent so we can approximate the ruin probability for our first model.

With

$$X(t) = ct - A(t)$$

and

$$S_n = cT_n - A(T_n),$$

where the T_n's are as in Chapter 2, we want to compute

$$P\{\inf_{t>0} X(t) < -u\} = P\{\inf_{n\in N} S_n < -u\} = P\{u < \sup_{n\in N}(-S_n)\}.$$

Since

$$P\{\frac{S_n}{n} > x\} \approx e^{-tI(x)},$$

we have

$$P\{S_n > u\} = P\{\frac{S_n}{n} > \frac{u}{n}\} \approx e^{-tI(\frac{u}{n})} = e^{-u\frac{I(\frac{u}{n})}{\frac{u}{n}}}$$

so that

$$P\{\inf_{t>0} X(t) < -u\} \approx e^{-u\frac{I(u)}{u}} + e^{-u\frac{I(\frac{u}{2})}{\frac{u}{2}}} + \cdots + e^{-u\frac{I(\frac{u}{n})}{\frac{u}{n}}} + \cdots$$

and the term which dominates when u is large is the one for which $\frac{I(\frac{u}{n})}{\frac{u}{n}}$ is the smallest, that is the one for which $\frac{I(x)}{x}$ is a minimum

$$P\{\sup_{n\in N} S_n > u\} \approx e^{-u \min_x \frac{I(x)}{x}} = e^{-u\delta}.$$

We note that we can also characterise δ in the following terms

$$\theta \leq \min_x \frac{I(x)}{x} \quad \text{if and only if} \quad \theta \leq \frac{I(x)}{x} \quad \text{for all } x.$$

$\Rightarrow$ if and only if $\theta x \leq I(x)$ for all x

$\Rightarrow$ if and only if $\max_x\{\theta x - I(x)\} \leq 0$;

thus

$$\theta \leq \delta \text{ if and only if } \lambda(\theta) \leq 0$$

and so

$$\delta = \max\{\theta : \lambda(\theta) \leq 0\}$$

which is our final result.

For the same problem, that is calculating Lundberg's exponent, another method is the following by calculating the mean $E(e^{\theta A_k(t)})$, where $A_k(t)$ is the sum of claims up to time t of a k process and is defined as our first model.

Defining

$$f_k(t,\theta) = E(e^{\theta A_k(t)})$$

we take $\frac{1}{t}\log f_k(t,\theta) = a_k(\theta)$ then

$$f_k(t,\theta) \approx e^{a_k(\theta)t}. \tag{5.7}$$

where the $a_k(\theta)$ is what we are looking for. Hopefully in the above steps we conclude to an equation the solution of which gives us the expected result. For this result we make use of Laplace theorems.

5.3 Calculating Lundberg's exponent - conditions

We assume of course that $c < 1$ because otherwise there would be no risk of ruin.

A first result, but an asymptotic one, that helps us realize what we should expect is the following:

By assuming the result in (5.7) it is

$$\frac{E[e^{\theta A(t)}]}{e^{l(\theta)t}} \longrightarrow c$$

we can make use of it for obtaining a quick result for $F(s, \theta)$. And then

$$F(s,\theta) = \int_0^\infty e^{-st} f(t,\theta)dt = \int_0^\infty e^{-st} E[e^{\theta A(t)}]dt = \int_0^\infty e^{-st} c e^{l(\theta)t} dt = \frac{c}{s - l(\theta)}.$$

So it is

$$F(s,\theta) = \frac{c}{s - l(\theta)}. \tag{5.8}$$

By applying $s = 0$ we get the Laplace transform while $s = 0$, that is

$$F(0,\theta) = \frac{c}{-l(\theta)}.$$

In order to simplify the notation we will drop the subscript k in $A_k(t)$. We thus have

$$\begin{aligned}
f(t,\theta) &= E[e^{\theta[A(t)-ct]}] = E[e^{\theta[A(t)-ct]}; T_1 > t] + E[e^{\theta[A(t)-ct]}; T_1 \le t] \\
&= E[e^{\theta[A(t)-ct]}; T_1 > t] + \int_0^t E[e^{\theta[A(t)-ct]} \mid T_1 = u] G(du) \\
&= E[e^{\theta[A(t)-ct]}; T_1 > t] + \int_0^t E[e^{\theta[\int_0^t a(s)ds - ct]} \mid T_1 = u] G(du) \\
&= E[e^{\theta[A(t)-ct]}; T_1 > t] + \int_0^t E[e^{\theta[X_1 - cX_1 - cY_1]} \mid T_1 = u] f(t-u,\theta) G(du) \\
&= b(t) + \int_0^t E[e^{\theta[X_1 - cX_1 - cY_1]} \mid T_1 = u] f(t-u,\theta) G(du)
\end{aligned}$$

Here I apply the Laplace transform and it is

$$\begin{aligned}
\int_0^\infty e^{-st} f(t,\theta)dt &= \int_0^\infty e^{-st} b(t)dt \\
&\quad + \int_0^\infty e^{-st} \int_0^t E[e^{\theta[X_1 - cX_1 - cY_1]} \mid T_1 = u] f(t-u,\theta) G(du)dt
\end{aligned}$$

$$\int_0^\infty e^{-st}f(t,\theta)dt = \int_0^\infty e^{-st}b(t)dt + \int_0^\infty e^{-st}e^{su}\int_0^t E[e^{\theta[X_1-cX_1-cY_1]} \mid T_1=u]e^{-su}f(t-u,\theta)G(du)dt$$

$$F(s,\theta) = B(s) + \int_0^\infty e^{-s(t-u)}f(t-u,\theta)d(t-u)\int_0^\infty e^{-su}E[e^{\theta[X_1-cX_1-cY_1]} \mid T_1=u]G(du)$$

$$F(s,\theta) = B(s) + F(s,\theta)\int_0^\infty e^{-su}E[e^{\theta[X_1-cX_1-cY_1]} \mid T_1=u]G(du)$$

A last form for the solution of $F(s,\theta)$ is the following

$$F(s,\theta) = \frac{B(s)}{1-\int_0^\infty e^{-su}E[e^{\theta[X_1-cX_1-cY_1]} \mid T_1=u]G(du)}. \tag{5.9}$$

For $B(s)$ in (5.9) we work as follows

$$\begin{aligned}
B(s) &= \int_0^\infty e^{-st}b(t)dt \\
&= \int_0^\infty e^{-st}E[e^{\theta[A(t)-ct]}; T_1>t]dt \\
&= \int_0^\infty e^{-st}E[e^{\theta[\int_0^t a(s)ds-ct]}; T_1>t]dt \\
&= E[\int_0^{X_1+Y_1} e^{-st}e^{\theta[\int_0^t a(s)ds-ct]}dt] \\
&= E[\int_0^{Y_1} e^{-st}e^{\theta[\int_0^t a(s)ds-ct]}dt + \int_{Y_1}^{X_1+Y_1} e^{-st}e^{\theta[\int_0^t a(s)ds-ct]}dt] \\
&= E[\int_0^{Y_1} e^{-st}e^{-\theta ct}dt] + E[\int_{Y_1}^{X_1+Y_1} e^{-st}e^{\theta[(t-Y_1)-ct]}dt] \\
&= E[-\frac{1}{s+c\theta}[e^{-(s+c\theta)t}]_0^{Y_1}] + E[e^{-\theta Y_1}\int_{Y_1}^{X_1+Y_1} e^{[\theta-(s+c\theta)]t}dt \\
&= E[-\frac{1}{s+c\theta}(e^{-(s+c\theta)Y_1}-1)] + E[e^{-\theta Y_1}\int_{Y_1}^{X_1+Y_1} e^{-[(s+c\theta)-\theta]t}dt] \\
&= E[-\frac{1}{s+c\theta}(e^{-(s+c\theta)Y_1}-1)] + E[-e^{-\theta Y_1}\frac{1}{s+c\theta-\theta}[e^{-(s+c\theta)+\theta}]_{Y_1}^{X_1+Y_1}] \\
&= -\frac{1}{s+c\theta}E[e^{-(s+c\theta)Y_1}] + \frac{1}{s+c\theta} + \frac{1}{s+c\theta-\theta}E[e^{-(s+c\theta)Y_1}] - \\
&\quad -\frac{1}{s+c\theta-\theta}E[e^{-(s+c\theta-\theta)X_1}]E[e^{-(s+c\theta)Y_1}]
\end{aligned}$$

$$
\begin{aligned}
B(s) &= \int_0^\infty e^{-st} b(t) dt \\
&= -\frac{1}{s+c\theta} Y(s+c\theta) + \frac{1}{s+c\theta} + \frac{1}{s+c\theta-\theta} Y(s+c\theta) - \\
&\quad - \frac{1}{s+c\theta-\theta} X(s+c\theta-\theta) Y(s+c\theta)
\end{aligned}
$$

where $Y(s+c\theta)$ and $X(s+c\theta-\theta)$ are the Laplace transforms of the exponential and the F distribution at $s+c\theta$ and $s+c\theta-\theta$ respectively.

For the denominator of (5.9) we work in the same way as previously. We define the denominator as

$$1 - C(s) = 1 - \int_0^\infty e^{-su} E[e^{\theta[X_1 - cX_1 - cY_1]} \mid T_1 = u] G(du).$$

So for $C(s)$ it is

$$
\begin{aligned}
C(s) &= \int_0^\infty e^{-su} E[e^{\theta[X_1 - cX_1 - cY_1]} \mid T_1 = u] G(du) \\
&= \int_0^\infty E[e^{-su} e^{\theta[X_1 - cX_1 - cY_1]} \mid T_1 = u] G(du) \\
&= \int_0^\infty E[e^{-s(X_1+Y_1)} e^{\theta(X_1 - cX_1 - cY_1)} \mid_1 = u] G(du) \\
&= E[e^{(\theta - s - c\theta)X_1}] E[e^{-(s+c\theta)Y_1)}]
\end{aligned}
$$

$$C(s) = X^*(\theta - s - c\theta) Y(s + c\theta)$$

where $X^*(\theta - s - c\theta)$ and $Y(s+c\theta)$ are the moment generating function of F distribution evaluated at $\theta - s - c\theta$ and the Laplace transform of the exponential distribution evaluated at $s + c0$ respectively.

So (5.9) becomes

$$F(s,\theta) = \frac{-\frac{1}{s+c\theta} Y(s+c\theta) + \frac{1}{s+c\theta} + \frac{1}{s+c\theta-\theta} Y(s+c\theta)[1 - X(s+c\theta-\theta)]}{1 - X^*(\theta - s - c\theta) Y(s+c\theta)} \qquad (5.10)$$

Thus (5.10) gives us a final formula for the solution of $F(s,\theta)$. We want to find those solutions for s that makes $F(s,\theta)$ tend to infinity. In other words we are looking for the poles of (5.10). In order to eliminate poles that are not real ones we try to change the final form of $F(s,\theta)$.

It is

$$F(s,\theta) = \frac{\frac{1}{s+c\theta}[1-Y(s+c\theta)] + \frac{1}{s+c\theta-\theta}[1-X(s+c\theta-\theta)]Y(s+c\theta)}{1-X^*(\theta-s-c\theta)Y(s+c\theta)}$$

$$F(s,\theta) = \frac{\frac{1}{s+c\theta}[1-\frac{\lambda}{\lambda+s+c\theta}] + \frac{1}{s+c\theta-\theta}[1-X(s+c\theta-\theta)]\frac{\lambda}{\lambda+s+c\theta}}{1-X^*(\theta-s-c\theta)\frac{\lambda}{\lambda+s+c\theta}}$$

$$F(s,\theta) = \frac{\frac{1}{s+c\theta}\frac{s+c\theta}{\lambda+s+c\theta} + \frac{1}{s+c\theta-\theta}[1-X(s+c\theta-\theta)]\frac{\lambda}{\lambda+s+c\theta}}{1-X^*(\theta-s-c\theta)\frac{\lambda}{\lambda+s+c\theta}}$$

$$F(s,\theta) = \frac{\frac{1}{\lambda+s+c\theta} + \frac{1}{s+c\theta-\theta}[1-X(s+c\theta-\theta)]\frac{\lambda}{\lambda+s+c\theta}}{1-X^*(\theta-s-c\theta)\frac{\lambda}{\lambda+s+c\theta}}$$

$$F(s,\theta) = \frac{1+\frac{\lambda}{s+c\theta-\theta}[1-X(s+c\theta-\theta)]}{\lambda+s+c\theta-\lambda X^*(\theta-s-c\theta)} \tag{5.11}$$

The numerator is finite provided that

$$s+c\theta-\theta \geq -\gamma,$$

where $-\gamma$ is the abscissa of convergence for the Laplace transform of F. (Clearly $\gamma \geq 0$.)

So we must check what happens in case we have a distribution for which

$$s+c\theta-\theta \to -\gamma.$$

From $s+c\theta-\theta \to -\gamma \Rightarrow s \to -\gamma - c\theta + \theta$. For the above limits we get that (5.11) has a zero to zero limit. That is why we apply the de l'Hôpital rule and we have that

$$\lim_{s\to -\gamma-c\theta+\theta} F(s,\theta) = -\frac{1}{\gamma\lambda}.$$

Because we get a limit other than infinity it is obvious that the value $s = c\theta - \theta - \gamma$ could not be the solution we are looking for.

Now we concetrate on the denominator. We believe that the denominator is going to give the exponent we are looking for or in other words some assumptions that should be fullfilled.The appropriate θ will make the denominator a zero, so θ will a pole of the $F(s,\theta)$ function. In the case where we have one or more poles θ^* of the $F(s,\theta)$ function it is proven in the chapter-theory that we choose the θ that is greater than the others.

The denominator is:

$$1 - X^*(\theta - s - c\theta)Y(s + c\theta).$$

So our equation is

$$1 - X^*(\theta - s - c\theta)Y(s + c\theta) = 0 \Rightarrow 1 - X^*(\theta - s - c\theta)\frac{\lambda}{\lambda + s + c\theta} = 0$$

$$1 = X^*(\theta - s - c\theta)\frac{\lambda}{\lambda + s + c\theta}$$

$$\frac{\lambda + s + c\theta}{\lambda} = X^*(\theta - s - c\theta)$$

or equivalently

$$X(s - \theta(1 - c)) = \frac{\lambda + s + c\theta}{\lambda}. \tag{5.12}$$

First of all we want the greater s for which $F(s,\theta) \to \infty$. We call this $s^* = s(\theta)$, i.e. it depends on θ. Then by using the Large Deviation Principle we want the maximum θ, call it θ^*, of the $s(\theta) = 0$ equation. This argument is equivalent to set $s = 0$ in the 5.12 and try to find θ^*.

From (5.12) setting $s = 0$ we obtain the condition

$$X(-\theta(1 - c)) = 1 + \frac{c\theta}{\lambda}. \tag{5.13}$$

This is a first condition from which we can derive the maximum θ for which (5.7) exists. We are aware that in order for the insurance business to be profitable on the average we need the following contition to hold (positive safety loading)

$$c(E[X] + \frac{1}{\lambda}) > E[X] \Rightarrow$$

$$E[X(1 - c)] < \frac{c}{\lambda} \tag{5.14}$$

The above condition should be verified also by our model. From (5.13) we can make a diagram versus θ of two functions $f_1(\theta) = X(-\theta(1-c))$ and $f_2(\theta) = 1 + \frac{c\theta}{\lambda}$.

These two function are both increasing and so we can have one or two solutions for θ. The case where we have one solution is a trivial one, because obviously then $\theta = 0$.

Here we either have the $\theta = 0$ and $\theta < 0$ case or the $\theta = 0$ and $\theta > 0$ case. So we should choose between these two solutions and because we want the greater one we choose the $\theta = 0$ and $\theta > 0$ case. So our θ^* is the one for which $\theta^* > 0$. For the second case it should be that the slope of $f_2(\theta)$ should be larger than the slope of $f_1(\theta)$.

That is

$$\frac{c}{\lambda} > \frac{d}{d\theta} X(-\theta(1-c)) \Rightarrow$$

$$E[X(1-c)] < \frac{c}{\lambda}$$

This last argument verifies (5.14).

So the final two conditions required for estimating $l(\theta)$ and θ are the following (5.13) and (5.14), respectively

$$X(-\theta(1-c)) = 1 + \frac{c\theta}{\lambda},$$

or

$$E[X(1-c)] < \frac{c}{\lambda}.$$

5.3.1 An application

Now we can make a certain application of the above when the F-distribution is the exponential distribution.

We work with the denominator, since all the other possible solution for getting the fraction to infinity have already been eliminated according to the general problem.

Now (5.12) becomes

$$\frac{\mu}{\mu + s - \theta(1-c)} = \frac{\lambda + s + c\theta}{\lambda}.$$

Now we solve this equation with s as the unknown parameter and it is

$$s^2 + s(\lambda + \mu - \theta + 2c\theta) + (\lambda c\theta - \lambda\theta + c\theta\mu - c\theta^2 + c^2\theta^2) = 0$$

from this ordered to s equation we expect that according to the general situation for $s = 0$ we obtain the greater θ. So the of the last equation should be zero this will only happen if $-4(\lambda c\theta - \lambda\theta + c\theta\mu - c\theta^2 + c^2\theta^2) = 0$. From this equality we have two solutions for θ. The first solution $\theta = 0$ is a trivial one, but the second one refers to a positive solution, which was what we intended for, and is

$$\theta = \frac{\mu}{1-c} - \frac{\lambda}{c}.$$

Here the condition (5.14) should be fulfilled and because it is $\theta > 0$ fullfilled.

Chapter 6

Appendix

6.1 Characteristic functions.

A simple solution of an extremely wide range of problems of probability theory, especially those associated with the summation of independent random variables, is obtained by means of characteristic functions, the theory of which has been developed in Analysis and it is known by the name of Fourier transformations.

Definition 8 *The expectation of a random variable $e^{it\xi}$ is called of the random variable ξ, where t is a real parameter. If $F(x)$ is the distribution function of the variable ξ, then its characteristic function is*

$$\varphi(t) = \int e^{itx} dF(x). \tag{6.1}$$

From the fact that $| e^{itx} | = 1$ for all real values of t, it follows that the integral (6.1) exists for all distribution functions, in other words a characteristic function may be defined for every random variable.

Theorem 17 *A distribution function is uniquely defined by its characteristic function.*

$$F(x) = \frac{1}{2\pi} \lim_{y \to \infty} \lim_{c \to \infty} \int_{-c}^{c} \frac{e^{-ity} - e^{-itx}}{it} \varphi(t) dt$$

Characteristic functions play a major role in risk theory. Here, the two theorems of Helly are of great importance and useful both computationally and for establishing existence in limiting arguments.

6.1.1 Helly's Theorems

Theorem 18 (Helly's first Theorem) *Any sequence of uniformly bounded nondecreasing functions $F_1(x), F_2(x), \ldots, F_n(x), \ldots$, contains at least one subsequence $F_{n_1}(x), F_{n_2}(x), \ldots, F_{n_k}(x), \ldots$, that converges weakly to some nondecreasing function $F(x)$.*

Theorem 19 (Helly's second Theorem) *Let $f(x)$ be a continuous function and let the sequence of non-decreasing uniformly bounded functions : $F_1(x), F_2(x), \ldots, F_n(x), \ldots$ converge weakly to the function $F(x)$ on some finite interval $a \leq x \leq b$, where a and b are continuity points of the function $F(x)$; then*

$$\lim_{n\to\infty} \int_a^b f(x)dF_n(x) = \int_a^b f(x)dF(x).$$

The Generalized Second Theorem of Helly follows.

Theorem 20 (The Generalized Second Theorem) *If the function $f(x)$ is continuous and bounded over the entire line $-\infty < x < \infty$, the sequence of uniformly bounded nondecreasing functions $F_1(x), F_2(x), \ldots, F_n(x), \ldots$, converges weakly to the function $F(x)$ and $\lim_{n\to\infty} F_n(-\infty) = F(-\infty)$, $\lim_{n\to\infty} F_n(+\infty) = F(+\infty)$, it follows that:*

$$\lim_{n\to\infty} \int f(x)dF_n(x) = \int f(x)dF(x).$$

Now, we give two other limit theorems. Limit theorems for characteristic functions i.e. the Direct Limit theorem and the Converse Limit theorem which are proven with the help of the Generalized Helly Theorem and Helly's First theorem respectively, state that the correspondence existing between distribution function and characteristic function is not only one-to-one, but also continuous.

6.1.2 Limit Theorems

Theorem 21 **(The Direct Limit Theorem)** *If a sequence of distribution functions $F_1(x), F_2(x), \ldots, F_n(x), \ldots$, converges weakly to the distribution function $F(x)$, then the sequence of characteristic functions $f_1(t). f_2(t), \ldots, f_n(t), \ldots$, converges to the characteristic function $f(t)$. This convergence is uniform in each finite interval of t.*

Theorem 22 (The Convergence Limit Theorem) *If a sequence of characteristic functions $f_1(t), f_2(t), \ldots, f_n(t), \ldots$ converges to the continuous function $f(t)$, then the sequence of distribution functions $F_1(x), F_2(x), \ldots. F_n(x), \ldots$ converges weakly to some distribution $F(x)$.*

Classical Central Limit theorems such as those requiring the Lindeberg conditions use characteristic functions for their proof.

6.1.3 Lindeberg condition

For any $\tau > 0$

$$\lim_{n\to\infty} \frac{1}{B_n^2} \sum_{k=1}^{n} \int_{|x-a_k|>\tau B_n} (x-a_k)^2 dF_k(x) = 0 \tag{6.2}$$

where $F_k(x)$ is the distribution function of the variable ξ_k and $a_k = E[\xi_k]$, $b_k^2 = \mathsf{Var}[\xi_k]$, $B_n^2 = \sum_{k=1}^{n} b_k^2 = \mathsf{Var}[\sum_{k=1}^{n} \xi_k]$. Roughly speaking, this condition requires that the variance b_k^2 is due mainly to masses in an interval whose length is small in comparison with B_n^2. It is clear that $\frac{b_k^2}{B_n^2}$ is less than t^2 plus the left side in (6.2) and, t being arbitrary, (6.2) implies that for arbitrary $\varepsilon > 0$ and n sufficiently large

$$\frac{b_k}{B_n} \leq \varepsilon, \tag{6.3}$$

where $k = 1, \ldots, n$.

This, of course, implies that $B_n \to \infty$.

The ratio $\frac{b_n}{B_n}$ may be taken as a measure for the contribution of the component X_n to the weighted sum $\frac{S_n}{B_n}$ and so (6.3) may be described as stating that asymptotically $\frac{S_n}{B_n}$ is the sum of many individually negligible components.

When the Lindeberg condition holds then the distribution functions of the sums : $\frac{1}{B_n}\sum_{k=1}^{n}(\xi_k - a_k)$ converge to the Normal distribution law, where: $a_k = E[\xi_k]$, $b_k^2 = \mathsf{Var}[\xi_k]$, $B_n^2 = \sum_{k=1}^{n} b_k^2 = \mathsf{Var}[\sum_{k=1}^{n}\xi_k]$.

6.1.4 Other theorems and definitions

Theorem 23 (Lyapunov's Theorem(1)) *If a sequence of mutually independent random variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$, for any constant $\tau > 0$ satisfies the Lindberg condition*

$$\lim_{n\to\infty}\frac{1}{B_n^2}\sum_{k=1}^{n}\int_{|x-a_k|>\tau B_n}(x-a_k)^2 dF_k(x) = 0$$

then, as $n \to \infty$,

$$P(\frac{1}{B_n^2}\sum_{k=1}^{n}(\xi_k - a_k) < x) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz$$

uniformly in x.

Corollary 24 *If the independent random variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$, are identically distributed and have a finite variance different from zero, then as n tends to infinity,*

$$P(\frac{1}{B_n}\sum_{k=1}^{n}(\xi_k - E[\xi_k]) < x) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz$$

uniformly in x.

Theorem 25 (Lyapunov's Theorem(2)) *If for a sequence of mutually independent random variables ξ_1, ξ_2, $\ldots$, ξ_n, $\ldots$ it is possible to choose a positive number $\delta > 0$ such that as $n \to \infty$*

$$\frac{1}{B_n^{2+\delta}}\sum_{k=1}^{n} E[\mid \xi_k - a_k \mid^{2+\delta}] < x) \to 0$$

then as n tends to infinity

$$P(\frac{1}{B_n}\sum_{k=1}^{n}(\xi_k - a_k) < x) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz$$

uniformly in x.

Definition 9 (lattice distribution) *A discrete random variable ξ has a lattice distribution if there exist numbers a and $h > 0$ such that all possible values of ξ may be represented in the form $a + kh$, where the parameters k can assume any integral values ($-\infty < k < \infty$) and h is called the span of the distribution.*

The Poisson, Bernoulli and other distributions are lattice distributions.

Lemma 26 *For a random variable ξ to have a lattice distribution it is necessary and sufficient that for some $t \neq 0$ the absolute value of its characteristic function be equal to unity.*

6.2 Infinitely Divisible Distributions

All the above can be very useful in the theory of Infinitely Divisible Distributions.

6.2.1 Definitions, theorems and properties

Definition 10 *A distribution law $F(x)$ is called infinitely divisible if, for any n its characteristic function is the nth power of some other characteristic function. It is clear that this definition is equivalent to the following : the law $F(x)$ is called infinitely divisible if, no matter what natural number n is taken, the random variables distributed in accordance with the $F(x)$ law is the sum of n independent random variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ with one and the same distribution law $F_n(x)$ (dependent on number of summands n).*

Some of the most important properties of the infinitely divisible distributions (Lévy distributions) which are given more in detail below are :

(i) Sums of independent infinitely divisible r.v.'s are also infinitely divisible.

(ii) If a sequence of infinitely divisible r.v.'s converges in distribution to a finite limit then this limit is also infinitely divisible.

(iii) All infinitely divisible distributions can be represented in terms of a compound Poisson distribution. This is the celebrated canonical representation of

Lévy and Khinchine.

Theorem 27 (Canonical Representation) *For a distribution function $F(x)$ with finite variance to be infinitely divisible, it is nesessary and sufficient that the logarithm of its characteristic function have the form:*

$$\log \varphi(t) = i\gamma t + \int \{e^{itx} - 1 - itx\} \frac{1}{x^2} dG(x) \tag{6.4}$$

where γ is a real constant and $G(x)$ is a nondecreasing function of bounded variation.

Any infinitely divisible law is either a convolution of a finite number of Poisson laws and the normal law or the limit of a uniformly converging sequence of such laws. We thus see that the Normal and Poisson laws are the basic elements that comprise every infinitely divisible law.

The Theorems that follow give conditions that suffice for a given sequence of infinitely divisible distribution functions to converge to the limit distribution function (also infinitely divisible function).

Theorem 28 (Limit Theorem) *In order for a sequence $\{F_n(x)\}$ of infinitely divisible distributions functions to converge, as $n \to \infty$, to some distribution function $F(x)$ and for their variances to converge to the variance of the limit law, it is necessary and sufficient that there exist a constant γ and the function $G(x)$, for which, as $n \to \infty$,*

(i)$G_n(x)$ converges weakly to $G(x)$

(ii)$G_n(\infty) - G(-\infty) \to G(\infty) - G(-\infty)$,

(iii)$\gamma_n \to \gamma$,

where γ_n and $G_n(x)$ are defined by formula (6.4), for the law $F_n(x)$, and the constant γ and the function $G(x)$ define, by the same formula, the limit law $F(x)$.

Now some Limit Theorems for Sums follow :

Theorem 29 *The distribution functions of the sequence of sums*

$$\varsigma_n = \xi_{n_1} + \xi_{n_2} + \cdots + \xi_{n_{k_n}}$$

converge to a limit distribution function as $n \to \infty$ if and only if the sequence of infinetely divisible laws whose characteristic function have logarithms given by the formula:

$$\psi_n(t) = \sum_{k=1}^{k_n} \{itE[\xi_{n_k}] + \int (e^{itx} - 1) dF_{n_k}(x)$$

to converge to a limit law.

Definition 11 *An elementary system is a double sequence satisfying the following conditions:*

(1) The variables ξ_{n_k} have finite variances

(2) The variances of the sums ς_n are bounded from above by a constant C which is independent of n

(3) $\beta_n = \max_{1 \leq k \leq k_n} Var(\xi_{n_k}) \to 0$ as $n \to \infty$. The last requirement means that the effect of the individual terms in the sum becomes smaller and smaller with increasing n.

Theorem 30 *Every distribution law that is a limit law for the distribution functions of sums in an elementary system is infinitely divisible with finite variance and, conversely, every infinitely divisible law with finite variance is a limit law for the distributions functions of the sums of some elementary systems.*

Then there are two theorems for convergence to the Normal and Poisson Laws.

Theorem 31 *If an elementary system is normalized by the relations*

$$\sum_{k=1}^{k_n} \int x^2 dF_{n_k}(x) = 1$$

and

$$\int x dF_{n_k}(x) = 0.$$

where $1 \leq k \leq k_n$, $n = 1, 2, \ldots$.

then for the convergence of the distribution functions of the sums $\xi_n = \xi_{n_1} + \xi_{n_2} + \cdots + \xi_{n_{k_n}}$ *to the normal law it is necessary and sufficient that for all* $\tau > 0$, *as* n *tends to infinity,*

$$\sum_{k=1}^{k_n} \int_{|x|>\tau} x^2 dF_{n_k}(x) \to 0.$$

Theorem 32 *Let an elementary system that obeys the conditions*

$$\sum_{k=1}^{k_n} E[\xi_{n_k}] \to \lambda$$

and

$$\sum_{k=1}^{k_n} \mathsf{Var}[\xi_{n_k}] \to \lambda.$$

The distribution function of the sums $\xi_n = \xi_{n_1} + \xi_{n_2} + \cdots + \xi_{n_{k_n}}$ *converge to the law*

$$P(x) = 0, \ \textit{for } x \leq 0$$

or

$$P(x) = \sum_{0 \leq k < x} e^{-\lambda} \frac{\lambda^{\kappa}}{k!}, \qquad \textit{for } x > 0$$

if and only if for any $\tau > 0$

$$\sum_{k=1}^{k_n} \int_{|x-1|>\tau} x^2 dF_{n_k}(x + E[\xi_{n_k}]) \to 0$$

for $n \to \infty$.

6.3 Stable Distributions

A class of infinitely divisible distributions which will play an important role in our analysis of risk processes with delayed claims are stable distributions.

Definition 12 *A distribution R is stable if for each n there exist constants $c_n > 0$, γ_n such that*

$$X_1 + X_2 + \cdots + X_n = S_n \stackrel{d}{=} c_n X + \gamma_n.$$

R is not concetrated at one point and is stable in a strict sense if $\gamma_n = 0$.

It can in fact be shown that the norming constants are of the form $c_n = n^{1/a}$ with $0 < a \leq 2$. The constant a will be called the characteristic exponent of R.

Theorem 33 *If R is stable with an exponent $\alpha \neq 1$ the centering constant b may be chosen so that $R(x+b)$ is strictly stable.*

It is

$$s^{1/\alpha} X_1 + t^{1/\alpha} X_2 = (s+t)^{1/\alpha} X.$$

From here we can see the importance of the normal distribution that is due largely to the Central Limit Theorem. The Central Limit Theorem proves that the normal distribution or Wiener process is the only stable distribution with finite variance (any stable distribution with finite variance corresponds to $\alpha = 2$).

For distributions with infinite variance similar limit theorems may be formulated, but the norming constants must be chosen differently. The interesting point is that all stable distributions and no others occur as such limits. This is something that we are going to use in approximating the sum of the delayed claims.

6.4 Other Definitions

6.4.1 Compound Poisson Process

A stochastic process $\{X(t); t \geq 0\}$ is said to be a Compound Poisson Process if it can be represented, for $t \geq 0$, by

$$X(t) = \sum_{i=1}^{N(t)} X_i,$$

where $\{N(t); t \geq 0\}$ is a Poisson process and $\{X_i, i = 1, 2, \ldots\}$ is a family of independent and identically distributed random variables that is independent of the process $\{N(t); t \geq 0\}$. Thus, if $\{X(t); t \geq 0\}$ is a compound Poisson process then $X(t)$ is a compound Poisson random variable.

6.4.2 Diffusion Process

A continuous time parameter stochastic process which posseses the (strong) Markov property and for which the sample paths $X(t)$ are (almost always) continuous functions of t is called a diffusion process.

Every diffusion process satisfies the following condition.

For every $\varepsilon > 0$,

$$\lim_{h \downarrow 0} \frac{1}{h} P\{\mid X(t+h) - x / X(t) = x\} = 0 \tag{6.5}$$

for all x in I.

A Markov process for which (6.5) holds in an appropriate uniform sense is a diffusion process.

Definition 13 *A stochastic process is continuous in probability if for any $\varepsilon > 0$ and $s > 0$*

$$\lim_{t \to s} P\{\mid X(t) - X(s) > \varepsilon\} = 0.$$

A criterion frequently used to check that a one–dimensional stochastic process $X(t)$ (not necessarily possessing the Markov property) has continuous path realization is the condition of Kolmogorov now stated.

Let $\{X(t), t \geq 0\}$ be a stochastic process obeying the bound

$E[\mid X(t) - X(s) \mid^{\gamma}] \leq c \mid \varphi(t) - \varphi(s) \mid^{1+a}$, for all s, $t > 0$, where a, γ and c are positive constants independent of s and t and φ is a continuous non decreasing function. Then there exist an equivalent version $\tilde{X}(t)$ possessing continuous paths.

6.5 The Generalized Campbell's Theoerm

Equations of the form (6.7)have been included under the name Campbell's Theorem. The following come from Daley, D.J. and D. Vere–Jones (1988).

For any random measure ξ on the c.s.m.s. $\mathcal{X}$ and any Borel set A, consider the expectation

$$M(A) = E[\xi(A)] \quad \text{finite or infinite).} \tag{6.6}$$

Clearly, M inherits the property of finite additivity from the underlying random measure ξ. Moreover, if the sequence $\{A_n\}$ of Borel sets is monotonic increasing to A, then by monotone convergence $M(A_n) \uparrow M(A)$. Thus, $M(\cdot)$ is continuous from below and therefore a measure. In general, it need not take finite values, even on bounded sets. When it does so, we say that the expectation measure of ξ exists and is given by (6.6).

When it does exist, the above argument can readily be extended to the random integrals $\int f d\xi$ for $f \in BM(\mathcal{X})$. Thus, if f is the indicator function of the bounded Borel set A, $E[\int f d\xi] = M(A)$. Extending in the usual way through linear combinations and monotone limits we find

$$E[\int f d\xi] = \int f dM f \in BM(\mathcal{X}) \tag{6.7}$$

The expectation measure $M(\cdot)$ may also be called the first moment measure of ξ.

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