# ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

## DEPARTMENT OF STATISTICS

## POSTGRADUATE PROGRAM

# CHARACTERIZATIONS OF LIFETIME DISTRIBUTIONS BASED ON RELIABILITY MEASURES

By

**Maria Theodosia P. Benia**

A THESIS
Submitted to the Department of Statistics
of the Athens University of Economics and Business
in partial fulfilment of the requirements for
the degree of Master of Science in Statistics

Athens, Greece
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# ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ

## ΤΜΗΜΑ ΣΤΑΤΙΣΤΙΚΗΣ


# ΧΑΡΑΚΤΗΡΙΣΜΟΙ ΚΑΤΑΝΟΜΩΝ ΖΩΗΣ ΒΑΣΙΣΜΕΝΟΙ ΣΕ ΜΕΤΡΑ ΑΞΙΟΠΙΣΤΙΑΣ


Μαρία Θεοδοσία Π. Μπενία


ΔΙΑΤΡΙΒΗ

Που υποβλήθηκε στο Τμήμα Στατιστικής

του Οικονομικού Πανεπιστημίου Αθηνών

ως μέρος των απαιτήσεων για την απόκτηση

Μεταπτυχιακού Διπλώματος Ειδίκευσης στη Στατιστική


Αθήνα

Μάιος, 2006

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Maria Theodosia P. Benia

**Approved by the Graduate Committee**

| | | |
|---|---|---|
| C. Dimaki | A. Kostaki | St. Psarakis |
| Associate Professor | Assistant Professor | Assistant Professor |
| Thesis Supervisor | Members of the Committee | |

**Athens, September 2006**

**Epameinondas Panas, Professor**
**Director of the Graduate Program**

# DEDICATION

Dedicated to my father's memory.

## ACKNOWLEDGEMENTS

I would like to thank my supervisor Associate Professor Caterina Dimaki for her guidance, help and understanding throughout this work. Also I would like to thank my colleague and friend Evgenia Tsobanaki for her valuable help and encouragement during the last years. Finally, I'm grateful to my family for their support and patience.

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# VITA

I was born in Sydney, Australia in March 9, 1974 and graduated from the 1th High school of Galatsi in 1991. In 1993 I entered the Department of Mathematics of the University of Ioannina and four years later, in September 1997 received my degree in Mathematics. From 1998 to 2000 I attended the MSc program in Statistics at the University of Economics and Business.

At present I am an employee at the Ministry of Economics.

# ABSTRACT

Maria - Theodosia Benia

**Characterizations of Lifetime Distributions Based on Reliability Measures**

May 2006

The aim of this dissertation is to provide several characterization theorems that can be used to identify lifetime distributions by their reliability measures. We deal with the most important lifetime distributions such as the Exponential Distribution and the Pareto Distribution, as well as discrete distributions such as the Geometric and the Yule.

A sort description of the most common used reliability measures that are helpful in describing the evolution of the risks to which an item is subjected over time is given in the first part. The relationship between the parent distribution and its weighted counterpart in the context of reliability is examined.

A collection of characterization theorems concerning each distribution seperately is provided. Characterizations that arise not only from the simple form of the distribution, but also from the size-biased form have been studied. Particular emphasis is given on the Weibull distribution, because of the particularity of this disdribution.

The Weibull model can be used in many analyses relating to health sciences, for example, the time of occurrence of cancer in a tissue follow a Weibull distribution.

# ΠΕΡΙΛΗΨΗ

Μαρία – Θεοδοσία Μπενία

**Χαρακτηρισμοί κατανομών Ζωής**
**Βασισμένοι σε Μέτρα Αξιοπιστίας**

Μάιος 2006

Σκοπός της εργασίας αυτής είναι να προσφέρει μερικούς χαρακτηρισμούς με τους οποίους μπορούμε να αναγνωρίσουμε κάποια κατανομή χρόνου από τα μέτρα αξιοπιστίας της. Η εργασία αυτή ασχολείται με τις βασικότερες κατανομές ζωής όπως είναι η Εκθετική κατανομή και η κατανομή Pareto, καθώς επίσης και με διακριτές κατανομές όπως η Γεωμετρική και η Yule.

Στην πρώτη ενότητα δίνεται μια σύντομη περιγραφή των μέτρων αξιοπιστίας που χρησιμοποιούνται πιο συχνά και χρησιμεύουν στην περιγραφή των πιθανών κινδύνων στους οποίους υπόκειται ένα υποκείμενο μέσα στο χρόνο. Επίσης μελετάτε η σχέση μεταξύ της αρχικής κατανομής και της ζυγισμένης συμπληρωματικής της.

Δίνεται μια συλλογή από θεωρήματα (χαρακτηρισμούς) για κάθε κατανομή ξεχωριστά. Επίσης υπάρχουν χαρακτηρισμοί των μεροληπτικών ως προς το μέγεθος, εκδοχών των κατανομών αυτών. Έμφαση δίνεται στη κατανομή Weibull εξαιτίας της ιδιομορφίας της. Το μοντέλο της Weibull μπορεί να χρησιμοποιηθεί σε πολλές αναλύσεις στις επιστήμες της υγείας, για παράδειγμα ο χρόνος έως την εμφάνιση καρκίνου κάποιου ιστού ακολουθεί την κατανομή Weibull.

# TABLE OF CONTENTS

LIST OF TABLES

## LIST OF FIGURES

# Chapter 1

# Introduction

Over the past 10 years there has been a heightened interest in improving quality, productivity and reliability of manufactured products. Global competition and higher customer expectations for safe, reliable products are driving this interest. To meet this need, many companies have trained their design engineers and manufacturing engineers in the appropriate use of designed experiments and statistical process control. Now reliability is being viewed as the product feature that has the potential to provide an important competitive edge. A current industry concern is in developing better processes to move rapidly from product conceptualization to a cost- efficient highly reliable product. A reputation for unreliability can doom a product, if not the manufacturing company.

But reliability theory can be applied to other areas than those related to engineering that gave the theory its first impetus as for example in health, economic and environmental studies. There are many books related with the subject, for example Mann et al (1974), Kalfleisch et al (1980), Lowless (1982).

Chapter 2 provides a review of the most common used reliability measures that are helpful in describing the evolution of the risks to which an item is subjected over time. These measures can be applied to both continuous and discrete lifetimes.

The concept of weighted distribution is considered. This concept is widely applied in reliability, biometry, survival analysis and several other fields. A number of papers has appeared during the last fifteen years using the concepts of weighted and size-biased sampling, see for example Gupta and Keating (1986), Jain et al (1989), Patil (1991).

Even though the main interest of this thesis concerns the characterizations of lifetime distributions it is necessary to consider briefly lifetime distributions. Chapter 3 is devoted to the presentation of these distributions. Throughout the literature on life data certain parametric models have been used repeatedly, exponential and Weibull models, for example, are often used. These distributions admit closed form expressions for tail area probabilities and thereby simple formulas for survival and hazard functions. The properties and the theoretical bases of these distributions are considered only briefly. These have been discussed in some detail by Johnson and Kotz (1993, 1994) and Mann et al (1974) for many of the models introduced.

Mann et al (1974) summarized from an industrial life testing point of view, estimation procedures for these as well as other distributions, both for single sample and two sample problems, with censoring. Gross and Clark (1975) give similar result from the biometrical point of view. In Chapter 3 we also examine the effect of weighting not only upon the distributions but also upon their reliability measures.

Chapter 4 provides a collection of characterization theorems concerning each distribution separately. Characterizations that arise not only from the simple form of the distribution, but also from the size-biased form have been studied. Particular emphasis is given on the Weibull distribution.

The Weibull model was utilized in many analysises relating to health sciences. For example, Pike (1966) suggested a model to describe the process underlying the phenomena of carcinogenesis, that is the time of occurrence of canser in a tissue follow a weibull distribution and Berry (1975) discussed the design of carcinogenesis experiments using this model. Chapter 5 includes an analytical presentation of these papers as well as a collection of other applications, in different fields of this distribution.

# Chapter 2

# Reliability Measures

## 2.1 Introduction

This chapter includes a definition of reliability as being mentioned by Leemis (1995) and Kales (1998) and illustrates some examples for further understanding this meaning. Various reliability measures that are helpful in describing the evolution of the risks to which an item is subjected over time are introduced. In particular, five reliability measures are presented: the reliability function, the hazard rate function, the failure rate function, the mean residual life function and the vitality function. These five measures apply to both continuous (for example, a light bulb) and discrete (for example, a computer program that is run weekly) lifetimes.

## 2.2 Reliability

**Definition 1.** *The reliability of an item is the probability that the item will perform a specific function under specified operational and environmental conditions, at and throughout a specified time.*

The first thing to notice in this definition is that reliability is a probability, so we are dealing with the lows of random chance as they appear in nature. Indeed, occurrences of inopportune interruptions in functionability or service in a system are random events.

The next thing to notice is that the definition depends on a specified function, operating conditions, environment and time. So before we deal with reliability, the producer (or provider) and the user must reach formal agreements on what the product or service is to do, how the user is to use the product or service (that is, how he or she will operate the product or receive or apply the service). The environmental conditions must be specified. Conditions such as temperature, humidity and turning speed all affect the lifetime of a machine tool. The 20.000-mile reliability of a subcompact car is different if it is used for highway driving or to tow a trailer down city streets. Environmental conditions associated with the lifetime of a person might be the city in which they live and whether they smoke. Also the instant or duration in time that the performance of the product or service is demanded must be defined.

The definition of reliability allows for the specification of demand time to be either an instant in time or a time interval. In actuality, the demand time may be a sequence of instances or it may be a series of intervals. That depends, of course, on the type of system or service. How we apply the definition of reliability to an actual product or service depends heavily upon the nature of the demand time.

If the demand time of an item's performance is a time interval or is continuous we describe the performance as time dependent. A time dependent performance may be for a specified mission or may be continuously operating. Examples of a specified mission operation are the launching of a satellite or a haircut. Examples of a continuously operating item are a refrigerator. A power-generating station or the telephone company's directory assistance service. Time-dependent items are expected to operate throughout their demand intervals without interruption or, in case of continuously operating items, all the time.

Reliability is often misunderstood. A single grenade, for example, that explodes when it should might be called to have 100% reliable. This is inaccurate since a reliability of 100% implies that each grenade of this type will explode when it should. The true reliability of these grenades might be 99.99% and just happened to toss one that worked.

Also there are differences between reliability and quality. The primary difference between these two terms is that reliability incorporates the passage of time, whereas quality does not, since it is a static descriptor for an item. Two transistors of equal quality sit side by side on a shelf. One of these transistors will be used in a television set, the other in a common launch environment. Both transistors are of identical quality, but the first one has a higher reliability since it will operate in a less stressful environment.

High reliability implies high quality, although the converse is not necessarily true. Consider two automobile tires, each of high quality. One has produced in 1957, the other in 1995. Although each was produced with the most stringent quality control procedures available, their reliabilities will be different due to technology changes introduced
between 1957 and 1995, such as steel-belted redials.

The 60.000-mile reliability of the tire produced in 1995 will be higher than the reliability of the 1957 tire. Technology advances in the 38 years between the manufacture of the two tires may come in the form of improved design (for example, tread or steel belts), components (for example, rubber), or processes (for example, manufacturing advances). Some quality improvements (for example, improved tread design) improve the reliability of the tire, while others will not.

## 2.3 Reliability Measures

This section introduces five reliability measures that define the distribution of a continuous nonnegative random variable t, the lifetime of an item. The five reliability measures are not the only ways to define the distribution of T. Other methods include the moment generating function $E[e^{sT}]$, the characteristic function $E[e^{isT}]$ and the Mellin transformation $e[T^s]$. The five reliability measures used here have been chosen because of their intuitive appeal and their usefulness in problem solving.

### 2.3.1 The Reliability Function

The first reliability measure is the reliability function (or survivor function) $\overline{F}$(t). The reliability function is a generalization of reliability. Whereas reliability is defined as the probability that an item is functioning at one particular time, the reliability function is the probability that an item is functioning at any time t:

$\overline{F}(t) = P(T \geq t), t \geq 0.$

It is assumed that $\overline{F}(t) = 1$ for all t<0. All reliability functions must satisfy three conditions :

$\overline{F}(t) = 1$ for all t<0

$\lim_{t \to \infty} \overline{F}(t) = 0$

$\overline{F}(t)$ is nonincreasing.

Since reliability function is a probability function the following hold:

1. $0 \leq \overline{F}(t) \leq 1$
2. $\overline{F}(t) = 1$ implies certainty of success.
3. $\overline{F}(t) = 0$ implies certainty of failure.
4. If $F(t)$ is the unreliability at the time t (i.e., the probability of a failure period to the time t), because success and failure are mutually exclusive and exhaustive events at any time t , $F(t) + \overline{F}(t) = 1$ for all values of t.
5. If A and B operate independently $\overline{F}_{AB}(t) = \overline{F}_A(t) \times \overline{F}_B(t)$, otherwise,

$$\overline{F}_{AB}(t) = \overline{F}_{A/B}(t) \times \overline{F}_B(t) = \overline{F}_{AB}(t) = \overline{F}_{B/A}(t) \times \overline{F}_A(t)$$

There are two interpretations of the reliability function. First, $\overline{F}(t)$ is the probability that an individual item is functioning at time t. This important in determine the lifetime distribution of a system from the distribution of the lifetimes of its individual components. Second, if there is a large population of items with identically distributed lifetimes, $\overline{F}(t)$ is the expected fraction of the population that is functioning at time t.

The reliability function is useful for comparing the reliability patterns of several populations of items. The graph in figure 2.1 is a plot of $\overline{F}_1(t)$ and $\overline{F}_2(t)$ where $\overline{F}_1(t)$ corresponds to population 1 and $\overline{F}_2(t)$ corresponds to population 2. Since

$\overline{F_1}$ (t) $\geq \overline{F_2}$(t) for all t values, it can be concluded that the items in population 1 are superior to those in population 2 with regard to reliability.

The *failure density function* is defined by f(t) = - $\overline{F}$ ′(t) where the derivative exists and has the probabilistic interpretation
f(t)Δ(t) = P(t ≤T ≤ t+ Δ(t)) for small Δ(t) values.

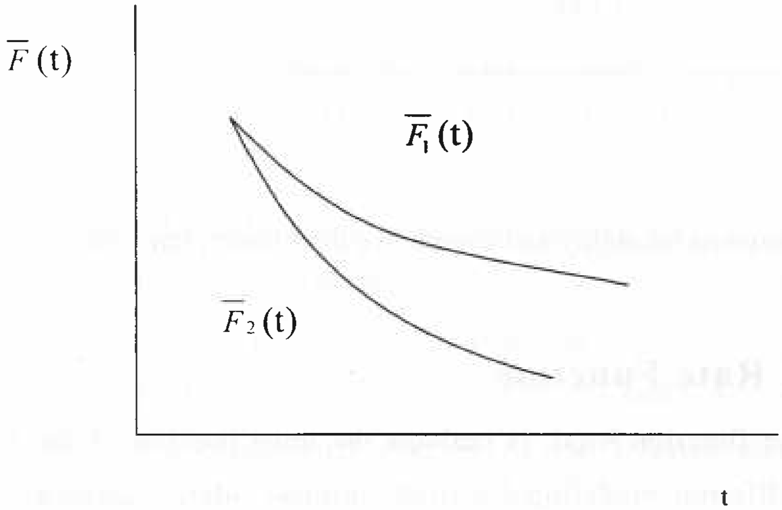


**Figure 2.1** Two Reliability Functions

All failure density functions for lifetimes must satisfy two conditions

- $\int_0^{\infty} f(t)dt = 1$
- f(t) ≥ 0 for all t ≥ 0.

It is assumed that f(t) = 0 for all t < 0. The failure density function shown in figure 2.2 illustrates the relationship between comulative distribution function F(t) and the reliability function $\overline{F}$ (t). The area to the left of time $t_0$ is $F(t_0)$ and the area to the right of $t_0$ is $\overline{F}(t_0)$.

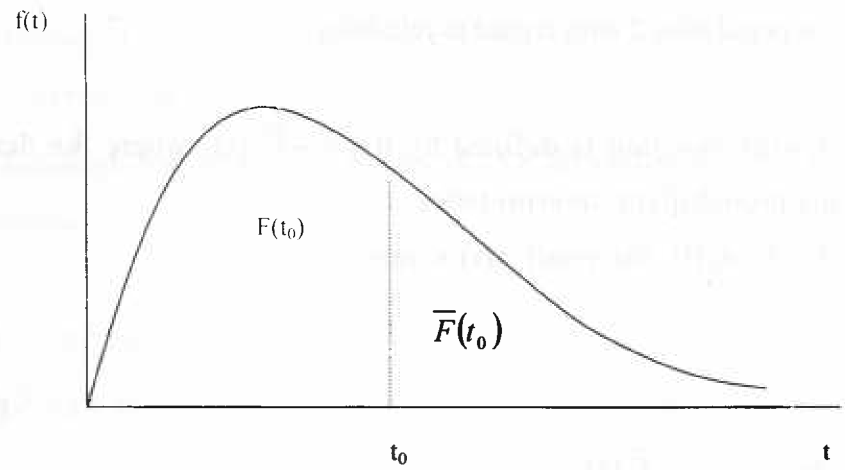


**Figure 2.2** Relationship between reliability and comulative distribution function

## 2.3.2 The Hazard Rate Function

The hazard rate function $h_T(t)$, is perhaps the most popular of the five reliability measures for lifetime modeling due to its intuitive interpretation as the amount of risk associated with an item at time t. A second reason for its popularity is it usefulness in comparing the way risk change over time for several populations of items by plotting their hazard rate functions on a single axis. A third reason is that the hazard rate function is a special case of the intensity function for a non homogeneous Poisson process. A hazard rate function models the occurrence of one event, a failure, whereas the intensity function models the occurrence of a sequence of events over time.

The hazard rate function can be derived using conditional probability. First, consider the probability of failure between t and t + Δ(t):

$$P(t \le T \le t + \Delta(t)) = \int_{t}^{t+\Delta(t)} f(\tau)\, d\tau = \overline{F}(t) - \overline{F}(t + \Delta(t))$$

Conditioning on the event that the item is working at time t yields

$$P(t \le T \le t + \Delta(t) \;/\; T \ge t) = \frac{P(t \le T \le t + \Delta(t))}{P(T \ge t)} = \frac{\overline{F}(t) - \overline{F}(t + \Delta t)}{\overline{F}(t)}$$

If this conditional probability is averaged over the interval [t, t+Δt] by dividing by Δt, an average rate of failure is obtained

$$\frac{\overline{F}(t)-\overline{F}(t+\Delta t)}{\overline{F}(t)\Delta(t)}$$

As $\Delta(t) \rightarrow 0$ this becomes the instantaneous failure rate, which is the hazard rate function

$$h_T(t) = \lim_{\Delta t \to 0} \frac{\overline{F}(t)-\overline{F}(t+\Delta t)}{\overline{f}(t)\Delta(t)} = -\frac{\overline{F}'(t)}{\overline{F}(t)} = \frac{f(t)}{\overline{F}(t)} \quad , t > 0$$

Thus, the hazard rate function is the ratio of the probability density function to the reliability function. Using the previous derivation, a probabilistic interpretation of the hazard rate function is

$h_T(t)\,\Delta(t) = P(t \leq T \leq t + \Delta(t) \;/\; T \geq t)$ for small $\Delta(t)$ values,

which is a conditional version of the interpretation for the failure density function.

All hazard rate functions must satisfy two conditions:

- $\int_0^\infty h_x(t)dt = \infty$
- $h_T(t) \geq 0$ for all $t \geq 0$.

The shape of the hazard rate function is of great interest because it gives information about how a system ages. Gaver and Acar (1979) and Leemis (1995) are dealing with the different hazard rate function shapes.

It is plausible to think that the time series of failures in a system may involve these stages.

1. *Early failure*. There may be a relatively large number of failures soon after a system is introduced because of design defects, production errors, or errors stemming from maintenance personnel inexperience. This situation is characterized by a hazard rate function that is initially large, but that decreases with time. "Infant mortality" is an evidence.

2. *Random failures*. Following the early failure period there may be a period during which failures occur at an essentially constant rate for a rather prolonged time. During this period the hazard rate function is nearly constant, so the times between failures are close to being exponentially distributed. The effect of age or wearout is not yet apparent.

3. *Wearout failures*. Eventually following the period during which a constant hazard is evident there is likely to be a period of ever-increasing failure rate caused by wearout of system components.

The term failure may refer to an event that is analogous to human death, after which the entire system is replaced. On the other hand, repair or component replacement may occur after failure, the system is only repaired, not entirely replaced.

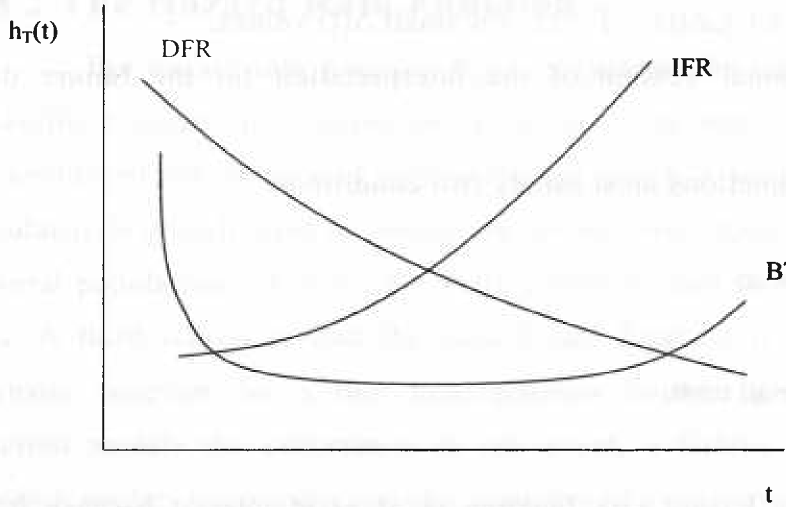


**Figure 2.3** Common hazard rate function shapes.

The slope of the hazard rate function indicates how an item ages. The intuitive interpretation as the amount of risk an item is subjected to at time t indicates that when the hazard rate function is large the item is under greater risk, and when the hazard function plotted in figure 2.3 correspond to an increasing hazard rate function (labeled IFR for increasing failure rate), a decreasing hazard rate function (labeled DFR for decreasing failure rate) and a bathtub-shaped hazard rate function (labeled BT for bathtub-shaped failure rate).

The increasing hazard function is probably the most likely situation of the three. In this case, items are more likely to fail as time passes. In other words, items wear out or degrade with time. This is almost certainly the case with mechanical items that undergo wear or fatigue. It can also be the case in certain biomedical experiments. If T is the time until a tumor appears after the injection of a substance into a laboratory animal and the substance makes the tumor more likely to appear as time passes, the hazard rate function
associated with T is increasing.

The second situation, the decreasing hazard rate function is less common. In this case, the item is less likely to fail as time passes. Items with this type of hazard rate function improve with time. Some metals work-harden through use and thus have increased strength as time passes. Another situation for which a decreasing hazard rate function might be appropriate is in working the bugs out of computer programs. Bugs are more likely to appear initially, but the likelihood of them appearing decreases as time passes.

The third situation, a bathtub-shaped hazard rate function, occurs when the hazard rate function decreases initially and then increases as items age.
Items improve initially and then degrade as time passes. One situation where the bathtub-shaped hazard rate function arises is in the lifetimes of manufactured items. Often, manufacturing design, or component defects cause early failures. The period in which these failures occur is sometime called the burn-in period. The time value during which early failures have been eliminated may be valuable to a producer who is determining an appropriate warranty period. Once items pass through this early part of their lifetime, they have a fairly constant hazard function, and failures are equally likely to occur at any point in time. Finally, as items continue to age, the hazard rate function increases without limit, resulting in wear-out failures.

The bathtub-shaped hazard rate function also arises in the lifetimes of people. In this case, the early failures are known as infant mortality deaths and occur during the first few years of life. After this time, the hazard rate function has a very gentle increase through the tee-age years and into adulthood. Finally, old age deaths occur during the later years of life. The magnitude of the hazard rate function depends on factors such as the standard of living and medical services available. Also, occupation (for example, flower arranger versus stunt man) and life style (for example, eating habits, sleeping habits, smoking habits, stress level) affect a lifetime distribution. The hazard rate function is used in actuarial science, the appropriate premium for a life insurance policy is based on probabilities associated with the lifetime distribution. The lowest life insurance premiums are usually for children who have survived the infant mortality pert of their lifetimes.

Care must be taken to differentiate between the hazard rate function for a population and the hazard fate function for an individual item under consideration. To use human lifetimes as an illustration, consider the following question: do two healthy 7- year - old boys living in the same town necessarily have the same hazard rate function? The answer is no. The reason is that all people are born with genetic predispositions that will influence their risk as they age. So, although a hazard rate function could be drown for all 7 - year - old boys living in that particular town, it could be an aggregate hazard rate function representing the population, and individual boys may be at increased or decreased risk. This is why life insurance companies typically require a medical exam to determine weather an individual is at higher risk than the rest of the population. The common assumption in most probabilistic models and statistical analyses is that of independent and identically distributed random variables, which in this case are lifetimes. This assumption is not always valid in reliability since items are typically manufactured in diverse conditions (for example, humidity, temperature and raw materials).

At this point, it will be appropriate to mention the *comulative hazard rate function* $H_T(t)$, which can defined by

$$H_T(t) = \int_0^t h_T(\tau)d\tau \ , \ t > 0$$

The comulative hazard rate function is also known as the integrated hazard rate function. All comulative hazard rate functions must satisfy three conditions:

- $H_T(0) = 0$
- $\lim_{t\to\infty} H_T(t) = 0$
- $H_T(t)$ is nondecreasing

The cumulative hazard rate function is valuable for generation in Monte Carlo simulation, implementing certain procedures in statistical inference, and defining certain distribution classes.

### 2.3.3 The Mean Residual Life Function

The mean residual life function $\mu^T(t)$ is defined by,

$\mu^T(t) = E[T - t / T \geq t]$ , $t \geq 0$.

The mean residual life function is the expected remaining life, T - t given that the item has survived to time t. the unconditional mean of the distribution, E(T) is a special case given by $\mu^T(0)$. To determine a formula for this expectation, the conditional probability density function is needed

$$f_{T/T\geq t}(\tau) = \frac{f(\tau)}{\bar{F}(\tau)} \quad , \quad \tau \geq t.$$

The conditional probability density function is actually a family of probability density functions (one of each value of t ) each of which has an associated mean.

$$E[T / T \geq t] = \int_t^{\infty} \tau \, f_{T/T\geq t}(\tau) \, d\tau = \int_t^{\infty} \tau \frac{f(\tau)}{\bar{F}(\tau)} d\tau .$$

Since the mean residual life function is the expected remaining life, t must subtracted yielding

$$\mu^T(t) = E[T - t / T \geq t] = \int_t^{\infty} (\tau - t) \frac{f(\tau)}{\bar{F}(\tau)} d\tau = \int_t^{\infty} \tau \frac{f(\tau)}{\bar{F}(\tau)} d\tau - t = \frac{1}{\bar{F}(t)} \int_t^{\infty} \tau f(\tau) d\tau - t$$

All mean residual life functions associated with distributions having a finite mean must satisfy three conditions

- $\mu^T(t) \geq 0$

- $\mu'^T(t) \geq -1$

- $\displaystyle\int_0^{\infty} \frac{dt}{\mu^T(t)} = \infty .$

The distribution function F(t),its reliability function $\overline{F}(t)$ or its corresponding random variable T,is said to have the following properties (Abouammoh, 1988):

**1.Increasing (decreasing) failure rate** IFR (DFR) : if $h_T(t)$ is increasing (decreasing) in t > 0 .

**2.Increasing (decreasing) failure rate average** IFRA (DFRA) : if $t^{-1}\int_0^t h_T(x)dx$ is increasing (decreasing) in t > 0.

**3.New better (worse) than used** NBU (NWU): if $\overline{F}(t+s)\leq(\geq)\overline{F}(t)\overline{F}(s)$, for all s, t > 0.

**4.New better (worse) than used in expectation** NBUE (NWUE): if

$\int_t^\infty \overline{F}(x)dx \leq (\geq)\mu\overline{F}(t)$ for t >0,

where $\mu=\int_0^\infty \overline{F}(x)dx<\infty$.

**5.Increasing (decreasing) mean residual life** DMRL (IMRL): if

$\frac{1}{\overline{F}(t)}\int_t^\infty \overline{F}(x)dx$ is increasing (decreasing) in t > 0.

**6.Harmonic new better (worse) than used in expectation** HNBUE (HNWUE) if: $\int_t^\infty \overline{f}(x)dx \leq (\geq)\mu\exp\left(-\frac{t}{\mu}\right)$ for t > 0.

**7.New better (worse) than average used in failure rate** NBAFR (NWAFR) if:

L = $\lim_{s\to 0} s^{-1}\log\overline{F}(s)$ exists and $t^{-1}\log\overline{F}(t)\leq L$, t > 0.

**8.New better (worse) than used in failure rate** NBUFR (NWUFR) if:

L exists and $\frac{d}{dt}\log\overline{F}(t)\leq L$ , t > 0.

We summarize the implications between these classes in figure 2.3

$$\begin{array}{c}
\qquad\qquad \text{DRML} \\
\nearrow \qquad\qquad \searrow \\
\text{IFR} \Rightarrow \text{IFRA} \Rightarrow \text{NBU} \Rightarrow \text{NBUE} \Rightarrow \text{HNBUE} \\
\Downarrow \\
\text{NBUFR} \Rightarrow \text{NBAFR}
\end{array}$$

**Figure 2.3** Implications between classes 1-8

Now, we introduce the following classes of life distributions that express some criteria of aging in terms of the mean residual life.

Let T de a nonnegative random variable with reliability function $\overline{F}(t)$ then

1. $\overline{F}$ is said to have DMRL if

   $\mu(t) = \left[\overline{F}(t)\right]^{-1} \int_t^{\infty} \overline{F}(x)dx$ is decreasing in t > 0.

2. $\overline{F}$ is said to have specific interval decreasing mean residual life average (SIDMRLA) if

   D(t , s) = $t^{-1} \int_s^{s+t} \mu(x)dx$ is decreasing in t for all s, t ≥ 0.

3. $\overline{F}$ is said to have decreasing mean remaining life average (DMRLA) if

   $t^{-1} \int_0^t \mu(y)dy$ is decreasing in t ≥ 0,

   i.e., D(t , 0) = $t^{-1} \int_0^t \left[\overline{F}(y)\right]^{-1} \int_y^{\infty} \overline{F}(x)dxdy$ is decreasing in t > 0.

4. $\overline{F}$ is said to have new better than average mean residual life (NBUMRL) property if $\mu(0) \geq t^{-1} \int_0^t \mu(y) dy$

i.e., $t^{-1} \int_0^t \left[\overline{F}(y)\right]^{-1} \int_y^\infty \overline{F}(x) dx dy \leq \mu$

where $\mu = \int_0^\infty \overline{F}(x) dx$.

5. $\overline{F}$ is said to belong to the class of new better than used mean remaining life (NBUMRL) if $\mu(t) \leq \mu$, that is

$\left[\overline{F}(t)\right]^{-1} \int_s^\infty \overline{F}(x) dx \leq \mu$

6. $\overline{F}$ is said to belong to the class of decreasing harmonic mean residual life average (DHMRLA) if

$t^{-1} \int_0^t \mu^{-1}(x) dx$ is increasing in t > 0.

i.e., $\left[\mu^{-1} \int_t^\infty \overline{F}(x)\right]^{1/t} dt$ is decreasing in t > 0.

7. $\overline{F}$ is said to have new better than used harmonic mean residual life property (NBUHMRL) if

$t^{-1} \int_0^t \mu^{-1}(x) dx \leq \mu^{-1}$

i.e., $\mu^{-1} \int_t^\infty \overline{F}(x) dx \leq \exp\left(-\frac{t}{\mu}\right)$, t > 0.

Abouammoh (1988) has proved the following theorems

1. SIDMRLA and DMRL are equivalent
2. DMRL implies DMRLA
3. DMRLA implies NBAMRL

4. DMRL implies NBUMRL
5. NBUMRL implies NBAMRL
6. DMRL implies DHMRLA
7. DHMRLA implies NBUHMRL
8. NBUMRL implies NBUHRML

These implications are summarized in figure 2.4

$$
\begin{array}{ccccc}
 & & \text{DMRLA} \Rightarrow \text{NBAMRL} & & \\
 & \nearrow & & \nearrow & \\
 & \text{DMRL} & \Rightarrow & \text{NBUMRL} & \\
 & \Updownarrow & \searrow & & \searrow \\
\text{SIDMRLA} & & \text{DHMRLA} & \Rightarrow & \text{NBUHMRL}
\end{array}
$$

**Figure 2.5** Implications between properties given in theorems 1-8.

## 2.3.4 Other Reliability Measures

For a given distribution function $F_X(t) = P(X \leq t)$ the *residual life distribution* at time t $F^X(t)$ can be defined as

$$F^X(t) = P\,(x < X \leq x + t \,/\, X > x)\;,\; t \geq 0.$$

The residual life distribution is defined for those x that P (X > x) >0.
A direct consequence of this definition is the following expression of the residual life function at time t,

$$F^X(t) = \frac{P(x < X \leq x+t)}{P(X > x)} = \frac{F_X(x+t) - F_X(x)}{\overline{F}_X(t)}$$

Thus, the residual life function can be written equivalently as

$$F^X(t) = 1 - \frac{\overline{F}_X(x+t)}{\overline{F}_X(t)}, \ t \geq 0.$$

The ratio

$$r^x(t) = \frac{P(X > x+t)}{P(X > x)} = \frac{\overline{F}_X(x+t)}{\overline{F}_X(t)}, \ t \geq 0$$

express the possibility that a system survive during t given that it had survived until the time instant x , and it is known as the *failure rate function.*

Two meanings are introduced here the *multiplicative failure rate function* which is defined by the ratio

$$r_x(t) = \frac{P(X > x \cdot t)}{P(X > x)}, \ t \geq 1$$

and the *additive failure rate function* which is defined by the ratio

$$r^x(t) = \frac{P(X > x+t)}{P(X > x)}, \ t \geq 0.$$

An other reliability measure is the *vitality function* $u_X(t)$ that measures the vitality of a time period in terms of the increase in average lifespan, which results from surviving that time period.

It is defined as

$$u_X(t) = E\,[X / X>t]\ .$$

Obviously,

$$u_X(t) = \mu^X(t) + t.$$

Finally,

$$\sigma^2{}_X(t) = \mathrm{Var}\,[X - t / x > t]$$

is the *residual life variance function.*

## 2.4 Reliability Measures of Discrete Distributions

Many of the concepts that apply to continuous distributions also apply to discrete distributions. Discrete failure time distributions are applied less frequently than continuous distributions since there are fewer situations for which failures can only occur at discrete points in time.

A situation for which time might be modeled discretely is software reliability. The modeler must determine whether time should be modeled continuously (for example, an operating system) or discretely (for example, a monthly payroll program). If time is measured discretely and the time values correspond to the run number, the run number when failure occurs is the failure time. It can be argued that there is no such thing as a "bug" in a program, and programs just do what they are instructed to do. This philosophy indicates that there is not a problem with the program when it fails but rather there is a problem in the data that caused the failure to occur. In either case, the program or the data should be modified so that the program will not fail. The failure rate for a new computer program is usually decreasing over time, since bugs generally become less likely with subsequent runs.

It is not always clear whether a discrete or continuous model should be used. The modeler should consider whether failure can occur at any moment in time (for example, fuse failure, machine breakdown or fatigue failure for systems operating continuously in time) or only upon demand (for example, a motor that doesn't start, a switch that fails to open, or automobile brakes that fail).

All reliability measures apply to discrete distributions as well as continuous distributions. The probability density function will be replaced by the probability mass function and the names of the other measures remain the same. Assume that the nonnegative discrete random variable T may assume the values $t_1$, $t_2$,... where $0 \leq t_1 < t_2 < \ldots$. The probability mass function is

$f(t_j) = P(T = t_j)$ $j = 1, 2, \ldots$

so the reliability function is the left-continuous ( that is , for all t and $\varepsilon > 0$ $\lim_{\varepsilon \to 0}(\overline{F}_T(t-\varepsilon) - \overline{F}_T(t)) = 0$ ) nonincreasing step function

$$\overline{F}_T(t) = P(T \geq t) = \sum_{j/t_j \geq t} f(t_j) \qquad t \geq 0$$

The probability mass function has nonzero mass at the time values $t_1$, $t_2$,...while the reliability function is defined for all nonnegative t value. The hazard rate function is also defined at the discrete points in time $t_1$, $t_2$,... and the magnitude of the hazard rate function is still interpreted as the risk at time $t_j$.

Also, since time is discrete, the hazard rate function is no longer derived as a limit.

$$h(t_j) = P(T = t_j / T \geq t_j) = \frac{P(T = t_j)}{P(T \geq t_j)} = \frac{f(t_j)}{\overline{F}_T(t_j)}$$

A dilemma is encountered when attempting to define the comulative hazard rate function when the time is discrete. Two possible but different choices for the definition are

$$H(t) = -\log \overline{F}_T(t) \quad t \geq 0.$$

$$H(t) = \sum_{j/t_j \geq t} h(t_j) \quad t \geq 0,$$

the first definition parallels the relationship establishes in the continuous case and the second definition accumulates the hazard rate function as it evolves over time. Both definitions are very close when the probability mass function values are small. Thus, the comulative hazard rate function

$H(t) = -\log \overline{F}_T(t)$ is a left-continuous nondecreasing step function. The mean residual life function is defined as before

$$\mu^T(t) = E[\, T - t / T \geq t] \quad , \; t \geq 0$$

which is calculated by

$$\mu^T(t) = \frac{1}{\overline{F}_T(t)}\left[\sum_{j/t_j \geq t} t_j f(t_j)\right] - t .$$

Since the mean residual life function is defined for all t ≥ 0 and everything in the expression except t is constant between mass function values, the mean residual life function decreases with a slop of -1 at all values for which there is no mass.

## 2.5 Relationships Among Reliability Measures

All the above reliability measures are equivalent in the sense that each completely satisfies a lifetime distribution. Any one reliability measure of a lifetime distribution implies the other. Algebra and calculus can be used to find on one reliability measure of a lifetime distribution give that another is known. The matrixes in table 2.1 and 2.2 shows the relationships among some of the most important reliability measures of continuous and discrete lifetime distributions respectively.

| | $f_X(t)$ | $\bar{F}_X(t)$ | $h_X(t)$ | $\mu^X(t)$ |
|---|---|---|---|---|
| $f_X(t)$ | | $-\dfrac{d\bar{F}_X(t)}{dt}$ | $h_X(t)\exp[-\int_0^t h_X(x)dx]$ | $\exp[-\int_0^t \dfrac{1+\mu^{X'}(x)}{\mu^X(x)}dx]$ |
| $\bar{F}_X(t)$ | $\int_t^\infty f_X(x)dx$ | | $\exp[-\int_0^t h_X(x)dx]$ | $\dfrac{1+\mu^{X'}(x)}{\mu^X(x)}\exp[-\int_0^t \dfrac{1+\mu^{X'}(x)}{\mu^X(x)}dx]$ |
| $h_X(t)$ | $\dfrac{f_X(t)}{\int_t^\infty f_X(x)dx}$ | $-\dfrac{d\ln\bar{F}_X(t)}{dt}$ | | $\dfrac{1+\mu^{X'}(x)}{\mu^X(x)}$ |
| $\mu^X(t)$ | $\dfrac{\int_t^\infty x f_X(x)dx}{\int_t^\infty f_X(x)dx}-t$ | $\dfrac{1}{\bar{F}_X(t)}\int_t^\infty \bar{F}_X(x)dx$ | $\dfrac{\int_t^\infty \exp\left[-\int_0^x h_X(y)dy\right]dx}{\exp\left[-\int_0^t h_X(x)dx\right]}$ | |

**Table 2.1** Relationships among reliability measures of continuous lifetime distributions

| | $f_X(t)$ | $\overline{F}_X(t)$ | $h_X(t)$ | $\mu^X(t)$ |
|---|---|---|---|---|
| $f_X(t)$ | | $-\dfrac{d\overline{F}_X(t)}{dt}$ | $h_X(t)\exp[\,-\sum_{k=0}^{t} h_X(k)\,]$ | $\exp[\,-\sum_{k=0}^{t}\dfrac{1+\mu^{X'}(k)}{\mu^X(k)}\,]$ |
| $\overline{F}_X(t)$ | $\sum_{k=t}^{\infty} f_X(k)$ | | $\exp[\,-\sum_{k=0}^{t} h_X(k)\,]$ | $\dfrac{1+\mu^{X'}(x)}{\mu^X(x)}\exp[\,-\sum_{k=0}^{t}\dfrac{1+\mu^{X'}(k)}{\mu^X(k)}\,]$ |
| $h_X(t)$ | $\dfrac{f_X(t)}{\sum_{k=t}^{\infty} f_X(k)}$ | $-\dfrac{d\ln\overline{F}_X(t)}{dt}$ | | $\dfrac{1+\mu^{X'}(x)}{\mu^X(x)}$ |
| $\mu^X(t)$ | $\dfrac{\sum_{k=t}^{\infty} k f_X(k)}{\sum_{k=t}^{\infty} f_X(k)}-t$ | $\dfrac{1}{\overline{F}_X(t)}\sum_{k=t}^{\infty}\overline{F}_X(k)$ | $\dfrac{\sum_{k=t}^{\infty}\exp\left[-\sum_{i=0}^{k} h_X(i)\right]}{\exp\left[-\sum_{k=0}^{t} h_X(k)\right]}$ | |

**Table 2.2** Relationships among reliability measures of discrete lifetime distributions

## 2.6 Reliability Measures of Weighted Distributions

In this section the definition of the weighted distributions is given (Patil and Rao (1977)) and some of their reliability measures are studied.

Consider a natural mechanism generating a random variable X with probability density function f(x;θ) where Θ∈Ω, the parameter space. For drawing a random sample of observations on X, we have to use a method of selection, which gives the same chance of including in the sample any observation produced by the original mechanism. But in practice it may so happen that the relative chances of inclusion of two observations x and y are w(x):w(y) where w(.) is nonnegative valued function. Then the recorded X to be

Denoted by $X_w$ has the probability density function

$$f^{w}_{X}(x\,;\,\theta) = \frac{w(x)f_X(x;\theta)}{E[w(X)]} \tag{2.1}$$

where

$E[w(X)] = \int w(x)f(x;\theta)dx$ or $\sum w(x)f(x;\theta)$ depending on whether X is continuous or discrete. Further, if $0 \le w(x) \le 1$, E[w(X)] is the probability of including an observed

value in the sample.

The distribution defined by (2.1) is called a weighted distribution with weight function

w (x), which can be arbitrary.

When an investigator collects a sample of observations produced by nature, according to a certain model, the original distribution may not thus be reproduced. The main interest in any investigation is, however, to determine the characteristics of the original distribution. Further, it also becomes important to assess the nature and amount of distortion caused in the determination of these characteristics in case the change in the underlying distribution due to sampling bias is ignored. A general situations involving"non-response" responsible for generating weighted distributions are: Truncation, missing data and damaged observations.

The assumptions of the implications of the relationship between the original distribution of X and the weighted distribution obtained using some weight function w(x) can generate interesting and useful characterization results.

Table 2.3 gives various weight functions, which are commonly used in statistical work. Note that the weight functions in the table are all monotone functions, either increasing or decreasing.

| $X \geq 0$ | $W(x)$ |
|---|---|
| General | $x$ |
| Discrete | $x^a,\ a > 0$ |
| Continuous | $x^a,\ a > 0$ |
| Discrete | $1-(1-\beta)^x,\ 0 < \beta < 1$ |
| Discrete | $x+1$ |
| Discrete | $x(x-1)\ldots(x-r+1)$ |
| Discrete | $\varphi^x,\ 0 < \varphi < 1$ |
| Continuous | $e^{wx}$ |

**Table 2.3** Some weighted functions

The following theorems by Jain et al. (1989) are indicating the relationships between the reliability measures of the weighted distribution and the reliability measures of the original distribution.

**Theorem 1.** *Let X be a nonnegative continuous random variable, denoting the life time of a component with probability density function $f_X(x)$ and distribution function $F_X(x)$. let the weight function $w(x)$ be a positive function with $0 \neq E[w(x)] < \infty$ and $E(X^2) < \infty$. The corresponding weighted random variable $X^w$ with probability density function and distribution function denoted by $f^w{}_X(x)$ and $F_X{}^w(x)$ respectively. Let also $A(x) = E[w(x) / X>x]$.*

*Then the reliability function $\overline{F}_{X^w}(x)$ can be expressed as follows*

$$\overline{F}_{X^w}(x) = \frac{1}{E[w(x)]}\overline{F}_X(x)A(x)$$

**Theorem 2.** *Let X and $X^w$ be defined as in Theorem 1. Let also $h_X(t)$ be the hazard rate function of X. Then the hazard rate function $h_X^w(t)$ of $X^w$ can be expressed as follows*

$$h_X^w(t) = \frac{w(x)h_X(x)}{A(x)} \quad \text{where } A(x) = E[w(x) / X > x]$$

**Theorem 3.** *Let X and $X^w$ be defined as in Theorem 1. Let also $\mu^X(t)$ be the mean residual life function of X. Then the mean residual life function $\mu^{X^w}(x)$ of $X^w$ can be expressed as follows*

$$\mu^{X^w}(x) = \frac{\mu^X(x)}{A(x)} \int_x^{\infty} \frac{A(t)}{\mu^X(t)} \exp\left[-\int_x^t \frac{du}{\mu^X(u)}\right] dt.$$

Moreover in the following theorem taken by the paper of Jain et al.(1989) the hazard rate function of the parent distribution is expressed in terms of the corresponding measure of the weighted distribution.

**Theorem 4.** *Let X and $X^w$ be defined as in Theorem 1. Let also $h_X(t)$ be the hazard rate function of X. Then the hazard rate function $h_X(t)$ of X can be expressed as follows*

1. $$h_X(x) = \frac{h_{X^w}(x) \big/ w(x)}{\int_x^{\infty} \frac{h_{X^w}(t)}{w(t)} \exp\left[-\int_x^t h_{X^w}(u)du\right] dt}$$

2. $$A(x) = \frac{\exp\left[-\int_0^x \frac{du}{\mu^{X^w}(u)}\right]}{\mu^{X^w}(x) \int_x^{\infty} \frac{1+\mu'^{X^w}(t)}{w(t)\left(\mu^{X^w}(t)\right)} \exp\left[-\int_0^t \frac{du}{\mu^{X^w}(u)}\right] dt}$$

Dimaki et al.,(1998) have examined the discrete case. As before, the following theorems indicate the relationship between the reliability measures of the weighted distribution and the reliability measures of the parent distribution.

**Theorem 5.** *Let X be a nonnegative integer valued random variable with probability function $p_t$. let also w(t) be a nonnegative strctly monotonic weight function and assume that E[w(x)] exists. Denote by $X^w$ the new random variable with probability density function $p_t^w$. Then, the reliability functions $\overline{F}_X(t)$ and $\overline{F}_X{}^w(t)$ of X and $X^w$ respectively, satisfy the condition*

$$\overline{F}_{X^w}(t)=\overline{F}_X(t)\frac{E[w(x)/X>t]}{E[w(x)]}.$$

**Theorem 6.** *Let X and $X^w$ be defined as in Theorem 5. Then the hazard rate functions $h_X(t)$ and $h_X{}^w(t)$ of X and $X^w$ respectively satisfy the condition*

$$h_X{}^w(t)=\frac{h_X(t)}{\dfrac{E[w(x)/X>t]}{w(t)}+\left[1-\dfrac{E[w(X)/X>t]}{w(t)}\right]h_X(t)}$$

**Theorem 7.** *Let X and $X^w$ be defined as in Theorem 5. Then the mean residual life functions $\mu^X(t)$ and $\mu^{X^w}(t)$ of X and $X^w$ respectively satisfy the condition*

$$\mu^{X^w}(t)=\frac{1}{\overline{F}_X(t)E[w(x)/X>t]}\sum_{x=t}^{\infty}\overline{F}_X(x)E[w(x)/X>x].$$

# Chapter 3

# Specific Distributions

## 3.1 Introduction

This chapter includes some general information about some specific distributions (continuous and discrete) which are the Geometric Distribution, the Yule Distribution, the Exponential Distribution, the Pareto Distribution and the Weibull Distribution. Also the corresponding sized-based distribution of them has been found.

## 3.2 Discrete Distributions

Suppose that X is a random vector for a random experiment, taking values in a subset of $R^n$. If S is countable, X is said to have a discrete distribution. The (discete) density function of X is the function f from S to R defined by

F(x) = P(X = x) for x $\in$ S.

f satisfies the following properties:

a. f(x) $\geq$ 0 for x $\in$S

b. $\sum_{x \in S} f(x) = 1$

c. $\sum_{x \in A} f(x) =$ P( X $\in$A) for A $\subseteq$ S

Property (c) is particularly important since it shows that the probability distribution of a discrete random variable is completely determined by its density function. Conversely, any function that satisfies properties (a) and (b) is a density, and then property (c) can be used to construct a discrete probability distribution on S.

We can extend f, if we want, to all of $R^n$ by defining f(x) = 0 for $x \notin S$. Sometimes this extension simplifies formulas and notation.

A vector $x \in S$ that maximizes the density f is called a mode of the distribution. When there is only one mode, it is sometimes used as a measure of the center of the distribution.

A discrete probability distribution is equivalent to a discrete mass distribution, with total mass 1. In this analogy, S is the (countable) set of point masses, and f (x) is the mass of the point x in S. Property (c) simply means that the mass of a set a can be found by adding the masses of the points in A.

For probabilistic interpretation, suppose that we replicate the underlying experiment repeatedly. For each x in S, let $f_n(x)$ denote the relative frequency of x in the first n runs (the number of times that x occurred, divided by n). Note that for each x , $f_n(x)$ is a random variable for the compound experiment, but by the low of large numbers, $f_n(x)$ should converge to f (x) as n increases. The function $f_n$ is called the empirical density function, these functions are displayed in most of the simulation applets that deal with discrete variables.

The density function of a random vector X is based, of course, on the underlying probability measure P for the experiment. This measure could be a conditional probability measure, conditioned on a given event B in the experiment with P(B) > 0. The usual notation is f(x / B) = P(X = x / B) for $x \in S$.

Suppose that X is a discrete random variable taking values in a subset S, and that B be an event in the experiment (that is, a subset of the underlying sample space).

Then P(B) = $\sum_{x \in S} P(B / X = x) P(X = x)$ is the law of total probability.

This result is useful, naturally, when the distribution of X and the conditional probability of B given the values of X are known.

And $P(X = x / B) = \dfrac{P(B / X = x)P(X = x)}{\sum_{y \in S} P(B / X = y)P(X = y)}$ is the Bayes' Theorem .

Bayes' theorem is a formula for the conditional density of X given B, as with the law of total probability, it is useful, when the quantities on the right are known. The (unconditional) distribution of X is referred to as the prior distribution and the conditional density as the posterior density.

## 3.3 Continuous Distributions

Suppose that X is a random vector for a random experiment, taking values in a subset S of $R^n$. Then X is said to have a continuous distribution if
$P(X = x) = 0$ for each x in S.
Moreover, a real-valued function f defined on S is said to be a (continuous) probability function for X if f satisfies the following properties

a. $f(x) \geq 0$ for $x \in S$.
b. $\int_S f(x)dx = 1$.
c. $\int_A f(x)dx = P(X \in A)$ for $A \subseteq S$.

Property (c) is particularly important since it implies that the probability distribution of X is completely determined by the density function. Conversely, any function that satisfies properties (a) and (b) is a probability density function, and then property (c) can be used to define a continuous distribution on S.
A vector $x \in S$ that maximizes the density f is called a mode of the distribution. If there is only one mode, it is sometimes used as a measure of the center of the distribution.
Unlike the discrete case, the density function of a continuous distribution is not unique. Note that the values of f on a finite (or even countable) set of points could be changed to other nonnegative values, and properties (a), (b) and (c) would still hold. The key fact is that only integrals of f are important.

The fact that X takes any particular value with probability 0 might seem paradoxical at first, but conceptually it is the same as the fact that an interval of r can have positive length even though it is composed of points, each of which has 0 length. Similarly, an region of $R^2$ can have positive area even though it is composed of points (or lines) each of which has area 0.

Suppose that X is a continuous random vector taking values in a subset S of $R^n$. Suppose that the underlying experiment has sample space T, a subset of $R^k$. The density function of X, of course, is based on the underlying probability measure P for the experiment. This measure could be a conditional probability measure, conditioned on a given event B. The usual notation is f (x / B), x $\in$ S, this function is a continuous density function. That is, satisfies properties (a) and (b) while property (c) becomes $\int_A f(x/B)dx = P(X \in a/B)$ for A $\subseteq$ S.

Unlike the discrete case, the existence of a density function for a continuous distribution is an assumption that we are making. It is possible to have a continuous distribution without a density.

First, suppose that X is a random vector taking values in a subset S of $R^n$ whose n-dimensional volume is 0. It is possible for X to have a continuous distribution, P(X = x) = 0 for each x in S.

But X could not have an n-dimensional density function in the sense of the definition above. In particular, property (c) could not hold since the integral on the left could be 0 for any subset A of S. However, we may be able to find a random vector Y taking values in a subset t of $R^k$ (where k<n) such that T does have a density and X = r (Y) for some function r from T into S. In this case, any probability problem involving X can be changed into a problem involving Y.

It is also possible to have a continuous random vector X that takes values in a subset S of $R^n$ with positive n-dimensional volume, yet X still does not have a density function. Such distributions are said to be singular, and are rare in applied probability.

## 3.4 Specific Discrete Distributions

A number of discrete distributions is examined in the sequel. We also examine the effect of weighting on these distributions. Many properties and characterizations are mentioned.

### 3.4.1 The Geometric Distribution

Consider a sequence of independent trials with probability of success p at each trial. Then the number of failures encountered in order to obtain the first success has the geometric distribution. A random variable X that follows the geometric distribution is often referred as discrete waiting-time random variable. It represents how long (in the terms of the number of failures) one has to wait for a success.

**Definition 1.** *A random variable X has a geometric distribution with parameter p if its probability function is given by,*

$$f_X(t) = pq^t \text{ , } t = 0, 1, 2, \ldots \text{ and } 0 < p < 1, q = 1 - p$$

*Symbolically $X \sim G(p)$*

The Geometric distribution has an important property the lack of augmentative memory. When the lifetime of a component follows the Geometric distribution with parameter p>0 it can be proved that the conditional probability of the time until failure X to exceed t+y given that it has already exceeded t equals to the probability that X exceed y.

**Theorem 8.** *Let X be a discrete random variable taking values in {0, 1, ..} the equation $P(X>t+y \,/\, X>t) = P(X\geq y)$ , t,y=0, 1,...* (3.1)
*defines univocally the distribution of X as Geometric with parameter p>0*

**Proof.**

**Necessity:** Let X ~ G(p). Then,

P(X>t+y / X>t)

$$= \frac{P(X>t+y, X>t)}{P(X>t)} = \frac{P(X>t+y)}{P(X>t)} = \frac{q^{t+y}}{q^t} = q^y =$$

$$= \overline{F}_X(y-1) = P(X>y-1) = P(X \geq y)$$

**Sufficiency:** Let X be a random variable with reliability function $\overline{F}_X(t)$ and the equation (3.1) holds for t=k-1 and t+y=k, meaning that t and t+y are successive. The equation (3.1) can be written equivalently

$$\frac{P(X>k, X>k-1)}{P(X>k-1)} = P(X \geq 1) \Rightarrow$$

$$\frac{P(X>k)}{P(X>k-1)} = 1 - P(X=0) \qquad (3.2)$$

If we denote $q_t$ = P(X>1) and $p_t$ = P(X=1), i=0,1,…,n then the equation (3.2) can be written as

$$\frac{q_k}{q_{k-1}} = 1 - p_0$$

Consequently,

$q_k = (1-p_0)^{k+1}$

That is, $q_k$ = P (X>t) = $\overline{F}_X(t) = q_0^{t+1}$.

Therefore X~G ($p_0$)

If X follows a geometric distribution with parameter p>0 then the formulas of the most commonly used reliability measures are,

the *probability distribution function* of X :

$F_X(t)=P(X \geq t)=1-q^{t+1}$,

the *reliability function* :

$\overline{F}_X(t) = q^{t+1}$,

the *hazard rate function* :

$h_X(t)=p$,

the *mean residual life function* :

$$\mu^X(t)=\frac{1}{p},$$

the *additive failure rate function* :

$r^X(t)=q^t$,

the *vitality function* :

$$u_X(t)=\frac{1}{p}+t.$$

The weighted form of the geometric distribution is

$$f_X^{*(r)}(t)=\binom{t}{t-r}q^{t-r}p^{r+1}$$ which is a negative binomial distribution.

In the special case of $w(x)=x$ the weighted form is

$f_X^*(t)=tp^2q^{t-1}$.

Moreover, the probability generating function of a geometric random variable is given by,

$$G_X(t)=\frac{p}{1-qt}$$

So the probability generating function of the size-biased version is given by,

$$G_{X^*}(t)=t\left(\frac{p}{1-qt}\right)^2,$$

The probability generating function of the size-biased version of factorial order r is

$$G_{X^{*(r)}}(t)=t^r\left(\frac{p}{1-qt}\right)^{r+1}.$$

The reliability measures of X* the size-biased version of X when X follows a Geometric distribution with parameter p are

the *probability distribution function* of X* :

$F_{X^*}(t)=P(X^*\geq t)=1-q^{t+1}$,

the *reliability function* :

$\overline{F}_{X^*}(t)=q^t(pt+1)$,

the *hazard rate function* :

$$h_{X^*}(t) = \frac{p^2 t}{tp+q},$$

the *mean residual life function* :

$$\mu^{X*}(t) = \frac{tp+q+1}{p(pt+1)},$$

the *vitality function* :

$$u_{X^*}(t) = \frac{tp+q+1}{p(pt+1)} + t.$$

## 3.4.2 The Yule Distribution

This section contains some general information about the Yule distribution as well as the Univariate Generalized Waring distribution and the Waring distribution since they are an extension of the Yule distribution. Johnson et al (1993) and Dimaki et al (1998) have studied thoroughly these distributions.

**Definition 2.** *A non-negative integer valued random variable X is said to have the Univariate Generalized Waring distribution (UGWD) with parameters α, k and p if its probability function is given by.*

$$p_x = P(X=x) = \frac{p_{(k)}}{(a+p)_{(k)}} \frac{a_{(x)}k_{(x)}}{(a+k+p)_{(x)}} \frac{1}{x!},\ x = 0, 1, 2, \ldots,\ \alpha>0,\ k>0,\ p>0$$

$$\text{where } \alpha(r) = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)},\ r = 0, 1, 2, \ldots$$

**Definition 3.** *A non-negative integer valued random variable X is said to have the Waring distribution with parameters α and p if its probability function is given by,*

$$p_x = P(X=x) = \frac{pa_{(x)}}{(a+p)_{(x+1)}},\ x = 0, 1, 2, \ldots,\ \alpha>0,\ p>0$$

Clearly, X~Waring(α, p) ⇔ X~UGWD(α, 1;p)

**Definition 4.** *A non-negative integer valued random variable X is said to have the Yule distribution with parameter p if its probability function is given by,*

$$p_x = P(X=x) = \frac{px!}{(p+1)_{(x+1)}}, \ x = 0, 1, 2, \ldots, \quad p>0$$

Obviously, X~Yule(p) $\Leftrightarrow$ X~Waring(1, p)

If X follows a Yule distribution with parameter p>0 then,

the *reliability function* is

$$\overline{F}_X(t) = \frac{t+1}{p} P(X=t), \ t=0, 1, 2, \ldots$$

where $$P(X=t) = \frac{pt!}{(p+1)(p+2)\ldots.(p+t+1)}, \ t=0, 1, 2, \ldots$$

the *hazard rate function* is

$$h_X(t) = \frac{p}{p+t+1}, \ t=0, 1, 2, \ldots$$

the *mean residual life function* is

$$\mu^X(t) = \frac{p+t+1}{p-1}, \ t=0, 1, 2, \ldots$$

and the *vitality function* is

$$u_X(t) = \frac{p(t+1)+1}{p-1}, \ t=0, 1, 2, \ldots$$

Regarding the size-biased version X* of X when X follows a UGWD, a Waring distribution or a Yule distribution the following holds

- If X is distributed according to a variant of the generalized Waring distribution UGWD(α, k;p) denoted like:

$$f_X(x) = \begin{cases} d_x, x = 0,1,2,\ldots,r-1 \\ c_r \dfrac{p_{(k)}}{(a+p)_{(k)}} \dfrac{a_{(x)}k_{(x)}}{(a+k+p)_{(x)}} \dfrac{1}{x!}, x = r, r+1, \ldots \end{cases}$$

where $c_r$ and $d_x$ are arbitrary constants in order to make $\sum_{x=0}^{\infty} f_X(x) = 1$, then the random variable X* defined as $f_{X}^{*(r)}=\frac{x^{(r)}f_{X}(x)}{E\left[X^{(r)}\right]}$, x=r,r+1,..., given that

$E[X^{(r)}] \equiv \mu^{(r)} <\infty$, follows a shifted r units to the right generalized Waring distribution UGWD(α+r, k+r; p-r).

- If X is a random variable taking values on {1, 2, 3,...} with E(X)<∞ and the distribution of X is the zero truncated univariate generalized Waring (α, k; p) distribution then the distribution of the random variable X* that also takes values on {1, 2, 3,...} with E(X*)<∞ is the shifted Waring(α +1, k+1; p-1).

- If X is a random variable taking values on {1, 2, 3,...} with E(X)<∞ and the distribution of X is the shifted Yule with parameter p+1then the distribution of the random variable X* that also takes values on {1, 2, 3,...} with E(X*)<∞ is the shifted Waring distribution with parameters(1, 2; p).

- If X is a random variable taking values on {1, 2, 3,...} with E(X)<∞ and the distribution of X is the Yule with parameter p then the distribution of the random variable X* that also takes values on {1, 2, 3,...} with E(X*)<∞ is the univariate generalized Waring distribution UGWD(2, 2; p-1).

- If X is a random variable taking values on {1, 2, 3,...} with E(X)<∞ and the distribution of X is the Yule with parameter p+1 then the distribution of the random variable X* that also takes values on {1, 2, 3,...} with E(X*)<∞ is the shifted
UGWD(1, 2; p-1).

The reliability measures of X* the size-biased version of X when X follows a shifted Yule distribution with parameter p+1, p>0 are
the *reliability function* :

$\overline{F}_{X^*}(t)=\frac{2+t-1}{p}P(X^*=t)$, t= 1, 2,...

where $P(X^*=t) = \frac{pt!}{(p+1)(p+2)....(p+t+1)}$,

the *hazard rate function* :

$$h_{X^*}(t) = \frac{p}{t+p+1},$$

the *mean residual life function* :

$$\mu^{X^*}(t)=\frac{tp+p+1}{p-1},$$

the *vitality function* :

$$u_{X^*}(t) = \frac{tp+p+1}{p-1}+t.$$

## 3.5 Specific Continuous Distributions

A number of continuous distributions are examined in the sequel. We also examine the effect of weighting on these distributions. Many properties and characterizations are mentioned.

### 3.5.1 The Exponential Distribution

Consider a device subject to shocks following a Poisson process with parameter λ. Then X, the time interval between successive occurrences of shocks has the exponential distribution eith parametr λ. In general, the time between two successive events of a Poisson process follows the exponential disribution.

**Definition 5.** *Let X be a positive random variable. Then x follows the Exponential distribution with parameter λ>0 if its probability density function is given by,*

$$f_X(x;\lambda) = \lambda e^{-\lambda x}, \ x>0$$

where λ>0.

Symbolicall, X~Exp(λ).

The Exponential distribution has an important property the lack of augmentative memory. When the lifetime of a component follows the

Exponential distribution with parameter $\lambda>0$ it can be proved that the conditional probability of the time until failure X to exceed t+y given that it has already exceeded t equals to the probability that X exceed y.

**Theorem 9.** *Let X be a continuous random variable with values $t>0$ the equation*

$$P(X>t+y \,/\, X>t) = P(X>y)\ ,\ t,y>0 \tag{3.2}$$

*defines univocally the distribution of X as Exponential with parameter $\lambda>0$*

**Proof.**

**Necessity:** Let $X \sim Exp(\lambda)$. Then,

$P(X>t+y \,/\, X>t)$

$$= \frac{P(X>t+y, X>t)}{P(X>t)} = \frac{P(X>t+y)}{P(X>t)} = \frac{e^{-\lambda(t+y)}}{e^{-\lambda t}}$$

$= e^{-\lambda y} = P(X>y)$

**Sufficiency: Let X be a random variable with reliability function $\overline{F}_X(t)$ and the equation (3.2) holds .The equation (3.2) can be written equivalently**

$$\frac{P(X>t+y, X>t)}{P(X>t)} = P(X \geq y) \Rightarrow$$

$$P(X>t+y) = P(X>y)P(X>t) \tag{3.3}$$

**But $\overline{F}_X(t)$= P(X>t), so the equation (3.3) can be written as**

$\overline{F}_X(t+y) = \overline{F}_X(y)\,\overline{F}_X(t)$

**The general solution of this functional equation is**

$\overline{F}_X(t) = e^{ct}$

**Since $\lim_{t\to\infty} \overline{F}_X(t) = 0 \Rightarrow c = -\lambda$ where $\lambda>0$.**

Thus, $\overline{F}_X(t) = e^{-\lambda t}$

Therefore , $X \sim Exp(\lambda)$

**If X follows a Exponential distribution with parameter $\lambda>0$ then the formulas of the most commonly used reliability measures are,**

the *probability distribution function* of X :

$F_X(t)=1-e^{-\lambda t}$,

the *reliability function* :

$\overline{F}_X(t)=e^{-\lambda t}$

the *hazard rate function* :

$h_X(t)=\lambda$,

the *mean residual life function* :

$\mu^X(t)=\dfrac{1}{\lambda}$,

the *additive failure rate function* :

$r^X(t)=e^{-\lambda t}$

the *vitality function* :

$u_X(t)=\dfrac{1}{\lambda}+t$.

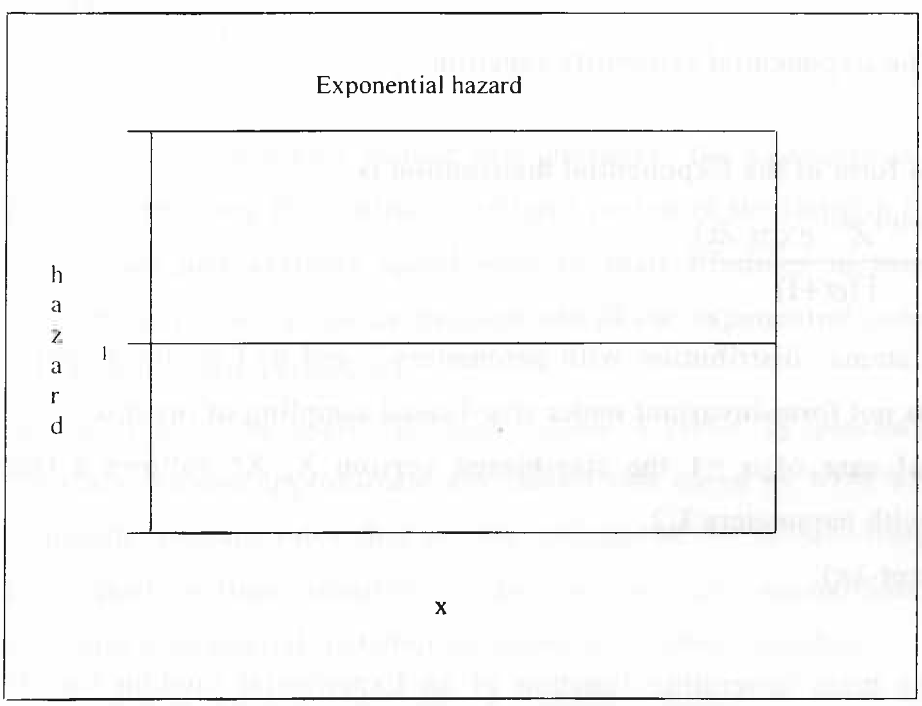


**Figure 3.1** The Exponential hazard function

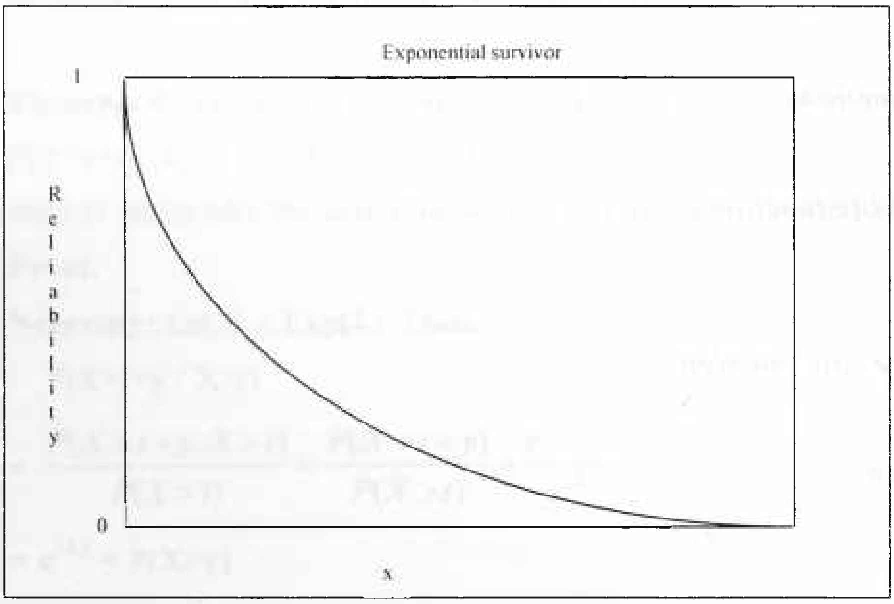


**Figure 3.2** The Exponential reliability function

The weighted form of the Exponential distribution is

$$f_X{}^{*\alpha}(t) = \frac{x^{(\alpha+1)-1}\lambda^{\alpha+1}\exp(-\lambda x)}{\Gamma(\alpha+1)}$$

which is a Gamma distribution with parameters λ and α+1.So the Exponential distribution is not form-invariant under size-biased sampling of order α.

In the special case of α =1 the size-biased version X, X* follows a Gamma distribution with parameters λ,2.

$f_{X*}(t) = \lambda^2 x\exp(-\lambda x)$.

Moreover, the mass generating function of an Exponential random variable is given by,

$$M_X(t)= \frac{\lambda}{\lambda - t}$$

So the mass generating function of X* is given by,

$$M_{X*}(t) = \left(\frac{\lambda}{\lambda - t}\right)^2.$$

The reliability measures of X* the size-biased version of X when X follows an Exponential distribution with parameter $\lambda$ are

the *probability distribution function* of X* :

$F_{X^*}(t) = 1 - \lambda t\exp(-\lambda t) - \exp(-\lambda t)$,

the *reliability function* :

$\overline{F}_{X^*}(t) = \exp(-\lambda t)(\lambda t+1)$,

the *hazard rate function* :

$$h_{X^*}(t) = \frac{\lambda^2 t}{\lambda t + 1},$$

the *mean residual life function* :

$$\mu^{X^*}(t) = \frac{\lambda t + 2}{\lambda(\lambda t + 1)},$$

the *vitality function* :

$$u_{X^*}(t) = \frac{\lambda t + 2}{\lambda(\lambda t + 1)} + t.$$

Because of its constant failure rate property, the exponential is an excellent model for the long flat "intrinsic failure" potion of the Bathtub Curve since most components and systems spend most of their lifetimes in this portion of the Bathtub Curve, this justifies frequent use of the exponential (when early failures or wear out is not a concern).

Just as it is often useful to approximate a curve by piecewise straight line segments, we can approximate any failure rate curve by week by week or month by month constant rates that are the average of the actual changing rate during the respective time duration's. That way we can approximate any model by piecewise exponential distribution segments patched together.

Some natural phenomenon has a constant failure rate (or occurrence rate) property; for example, the arrival rate of cosmic ray alpha particles or geiger counter tics. The exponential model works well for inter arrival times (while the Poisson distribution describes the total number of events in a given period). When these events trigger failures, the exponential life distribution model will naturally apply.

### 3.5.2 The Pareto Distribution

Let X(t) be the individual income at time t. assume that:

1. The population is closed (no births or deaths)
2. There is a minimum income level $x_0$
3. The incrementary random variable Y = [dlogX(t)] is independent of all past increments
4. The random variable Y, given X(t) = α, has the Normal distribution with parameters -η, η>0 and $\sigma^2$

Then the limiting distribution of X(t) is Pareto with parameter $\lambda = x_0$ and

$\theta = \frac{2\eta}{o^2}$.

**Definition 6.** *Let a random variable X. then X follows the Pareto distribution with parameters θ and λ if its probability density function is given by,*

$$f_X(x) = \lambda\theta^{\lambda}x^{-(\lambda+1)},$$

$$\text{where } \theta>0,\ \lambda>0,\ x\geq\theta.$$

*Symbolically, X~Pareto(θ, λ)*

The Pareto distribution has an important property, the lack of multiplicative memory. When the lifetime of a component follows the Pareto distribution with parameter λ>0 and θ=1 it can be proved that the conditional probability of the time until failure X to exceed ty given that it has already exceeded t equals to the probability that X exceed y.

**Theorem 9.** *Let X be a continuous random variable with values t≥1 the equation*

$$P(X>ty\ /\ X>t) = P(X>y)\ ,\ t\geq 1, y>0 \tag{3.4}$$

*defines univocally the distribution of X as Pareto with parametes 1,λ, λ >0*

**Proof.**

**Necessity:** Let X ~ Pareto($1,\lambda$). Then,

P(X>ty / X>t)

$$= \frac{P(X>ty, X>t)}{P(X>t)} = \frac{P(X>ty)}{P(X>t)} = \frac{(ty)^{-l}}{t^{-l}}$$

$= y^{-l}$ = P(X>y)

**Sufficiency:** Let X be a random variable with reliability function $\overline{F}_X(t)$ and the equation (3.4) holds .The equation (3.4) can be written equivalently

$$\frac{P(X>ty, X>t)}{P(X>t)} = P(X>y) \Rightarrow$$

$$P(X>ty) = P(X>y)P(X>t)$$

(3.5)

But $\overline{F}_X(t)$= P(X>t), so the equation (3.5)can be written as

$$\overline{F}_X(ty) = \overline{F}_X(y)\,\overline{F}_X(t)$$

The general solution of this functional equation is

$$\overline{F}_X(t) = x^c$$

Since $\lim_{t\to\infty} \overline{F}_X(t)=0 \Rightarrow c = -\lambda$ where $\lambda>0$.

Thus, $\overline{F}_X(t) = t^{-\lambda}$, $t\geq 1$, $\lambda>0$

Therefore, X~Pareto (1, λ).

If X follows a Pareto distribution with parameters θ,λ >0 then the formulas of the most commonly used reliability measures are,

the *probability distribution function* of X :

$$F_X(t) = 1-\lambda^{\vartheta} t^{-\lambda} \,,$$

the *reliability function* :

$$\overline{F}_X(t) = \lambda^{\vartheta} t^{-\lambda} \,,$$

the *hazard rate function* :

$$h_X(t) = \frac{\lambda}{t},$$

the *mean residual life function* :

$$\mu^X(t) = \frac{t}{\lambda-1},\ \lambda>1,$$

the *multiplicative failure rate function* :

$$r_X(t) = t^{-\lambda},$$

the *vitality function* :

$$u_X(t) = \frac{\lambda t}{\lambda-1},\ \lambda>1 \,.$$

The weighted form of the Pareto distribution is

$f_X^{*\alpha}(t) = (\lambda-\alpha)\theta^{\lambda-\alpha}\, x^{-(\lambda-\alpha+1)}$

which is a Pareto distribution with parameters θ, λ-α .So the Pareto distribution is form-invariant under size-biased sampling of order α.

In the special case of α =1 the size-biased version X, X* follows a Pareto distribution with parameters θ, λ-1

$f_X^{*}(t) = (\lambda-1)\theta^{\lambda-1}\, x^{-\lambda}$

The reliability measures of X* the size-biased version of X when X follows a Pareto distribution with parameters θ,λ >0 are

the *probability distribution function* of X* :

$F_{X^*}(t) = P(X^* \geq t) = 1 - \theta^{\lambda-1}\, t^{-(\lambda-1)}$,

the *reliability function* :

$\overline{F}_{X^*}(t) = \theta^{\lambda-1}\, t^{-(\lambda-1)}$

the *hazard rate function* :

$h_{X^*}(t) = \frac{\lambda-1}{t}, \lambda>1,$

the *mean residual life function* :

$\mu^{X^*}(t) = \frac{t}{(\lambda-2)}, \lambda>2,$

the *vitality function* :

$u_{X^*}(t) = \frac{t(\lambda-1)}{\lambda-2}, \lambda>2.$

### 3.5.3 The Weibull Distribution

**Definition 7.** *Let a random variable X. then X follows the Weibull distribution with parameters α,β and γ if its probability density function is given by,*

$$f_X(x) = \frac{\beta}{\alpha}\left(\frac{x-\gamma}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{x-\gamma}{\alpha}\right)\right]^{\beta},$$

*where α>0, β>0 and x≥ γ ,*

*Symbolically, X~Weibull(α, β ;γ )*

If X follows a Weibull distribution with parameters α=1, γ=0, β>0 then the formulas of the most commonly used reliability measures are,

the *probability distribution function* of X:

$$F_X(t) = 1 - \exp\left[-\left(\frac{t-\gamma}{\alpha}\right)\right]^{\beta},$$

the *reliability function*:

$$\overline{F}_X(t) = \exp\left[-\left(\frac{t-\gamma}{\alpha}\right)\right]^{\beta},$$

the *hazard rate function*:

$$h_X(t) = \frac{\beta}{\alpha}\left(\frac{t-\gamma}{\alpha}\right)^{\beta-1},$$

the *mean residual life function*:

$$\mu^X(t) = \exp\left[-\left(\frac{t-\gamma}{\alpha}\right)\right]^{\beta} \int_t^{\infty} \exp\left[-\left(\frac{x-\gamma}{\alpha}\right)\right]^{\beta} dx.$$

Because of its flexible share and ability to model a wide range of failure rates, the Weibull distribution has been used successfully in many applications as a purely empirical model.

The Weibull model can be derived theoretically as a form of Extreme Value Distribution, governing the time to occurrence of the "weakest link" of many competing failure processes. This may explain why it has been so successful in applications such as capacitor, ball bearing, relay and material strength failures.

Another special case of the Weibull occurs when the shape parameter is 2. the distribution is called the Rayleigh Distribution and it turns out to be the theoretical probability model for the magnitude of radial error when the x and y coordinate errors are independent normals with 0 mean and the same standard deviation.

*with par*

*Symbolically*

# Chapter 4

# Characterizations of Specific Lifetime Distributions

## 4.1 Introduction

This chapter is the most important since it includes many proved characterizations of the lifetime distributions that are studied in this dissertation. These characterizations are based on the reliability measures of the distributions with the exception of the Weibull distribution, which unfortunately has only a few characterizations related with its reliability measures. Considering the importance of the Weibull distribution in the reliability theory some other characterizations are mentioned. Characterizations that arise not only from the simple form of the distribution, but also from the size-biased form have been studied

## 4.2 Some Basic Theorems

The following two theorems have been proved by Dimaki and Xekalaki (1996) and shows that for every strictly monotonic function w(.), E[w(X) / X > t] can be expressed in terms of w(x) and $\overline{F}_X(\cdot)$ only.

**Theorem 1.** *Let X be a continuous random variable with reliability function $\overline{F}_X(t)$ for all t≥0. Let also z(t) be a differentiable function such that E[z(t)]<∞. Then,*

$$E[z(t)/X>t] = z(t) + \frac{\int_t^{\infty}\overline{F}_X(x)dz(x)}{\overline{F}_X(t)} \tag{4.1}$$

**Proof.**

$$E[z(t)/X>t] = \frac{\int_t^{\infty} z(t)dF_X(x)}{1-F_X(t)} = \frac{-\int_t^{\infty} z(t)d(-F_X(x))}{1-F_X(t)} = \frac{-\int_t^{\infty} z(t)d\overline{F}_X(x)}{\overline{F}_X(t)}$$

$$= \frac{1}{\overline{F}_X(t)}\left\{\left[z(x)\overline{F}_X(x)\right]_t^{\infty} + \int_t^{\infty}\overline{F}_X(x)dz(x)\right\}$$

$$= \frac{z(t)\overline{F}_X(t) + \int_t^{\infty}\overline{F}_X(x)dz(x)}{\overline{F}_X(t)}$$

$$= z(t) + \frac{\int_t^{\infty}\overline{F}_X(x)dz(x)}{\overline{F}_X(t)}$$

**Theorem 2.** *Let X be a discrete random variable with reliability function $\overline{F}_X(t)$*

*t=0, 1,2, . Let also z(t) be a differentiable function such that E[z(t)]<∞.*

*Then,*

$$E[z(t)/X>t] = z(t+1) + \frac{\sum_{x=t+1}^{\infty}[z(x+1)-z(x)]\overline{F}_X(x)}{\overline{F}_X(t)} \tag{4.2}$$

**Proof.**

$$E[z(t)/X>t] = z(t+1) + \sum_{x=t+1}^{\infty} z(x)P(X=x/X>t) = \sum_{x=t+1}^{\infty} z(x)\frac{P(X=x)}{P(X>t)}$$

$$= \sum_{x=t+1}^{\infty} z(x) \frac{P(X \geq x) - P(X \geq x+1)}{P(X > t)}$$

$$= \frac{1}{P(X > t)} \sum_{x=t+1}^{\infty} [z(x)P(X \geq x) - z(x)P(X \geq x+1)]$$

$$= \frac{1}{P(X > t)} \left[ \sum_{x=t+1}^{\infty} z(x)P(X \geq x) - \sum_{x=t+1}^{\infty} z(x)P(X \geq x+1) \right]$$

$$= \frac{1}{P(X > t)} \left[ z(t+1)P(X \geq t+1) + \sum_{x=t+2}^{\infty} z(x)P(X \geq x) - \sum_{x=t+1}^{\infty} z(x)P(X \geq x+1) \right]$$

$$= \frac{1}{P(X > t)} \left[ z(t+1)P(X > t) + \sum_{x=t+1}^{\infty} z(x+1)P(X \geq x+1) - \sum_{x=t+1}^{\infty} z(x)P(X \geq x+1) \right]$$

$$= \frac{1}{P(X > t)} \left[ z(t+1)P(X > t) + \sum_{x=t+1}^{\infty} z(x+1)P(X > x) - \sum_{x=t+1}^{\infty} z(x)P(X > x) \right]$$

$$= \text{z(t+1)} + \frac{1}{P(X>t)} \left[ \sum_{x=t+1}^{\infty} z(x+1)P(X>x) - \sum_{x=t+1}^{\infty} z(x)P(X>x) \right]$$

$$= \text{z(t+1)} + \frac{1}{P(X>t)} \sum_{x=t+1}^{\infty} [z(x+1) - z(x)] P(X>x)$$

$$= \text{z(t+1)} + \frac{\sum_{x=t+1}^{\infty} [z(x+1+-z(x)] \overline{F}_X(x)}{\overline{F}_X(t)}.$$

The following theorems give us a useful equation which expresses the mean residual life function in terms of the reliability function.

**Theorem3.** *Let X be a continuous random variable with reliability function $\overline{F}_X(t)$ for all t≥0. Then,*

$$\mu^X(t) = \frac{1}{\overline{F}_X(t)} \int_t^{\infty} \overline{F}_X(x)dx, \quad t \geq 0 \tag{4.3}$$

**Proof.**

If in equation (4.1) we replace z(x) = x-t we get

$$E[X-t/X>t] = \frac{1}{\overline{F}_X(t)} \int_t^{\infty} \overline{F}_X(x)dx,$$

But, by definition $\mu^X(t) = E[X-t/X>t]$.

Thus, $\mu^X(t) = \dfrac{1}{\overline{F}_X(t)} \displaystyle\int_t^{\infty} \overline{F}_X(x)dx$.

**Theorem 4.** *Let X be a discrete random variable with reliability function* $\overline{F}_X(t)$

*$t=0,1,2,\ldots$ . Then,*

$$\mu^X(t) = \frac{1}{\overline{F}_X(t)} \sum_{x=t}^{\infty} \overline{F}_X(x), \quad t=0,1,2,\ldots \tag{4.4}$$

**Proof.**

If in equation (4.2) we replace z(x) = x-t we get

$$E[X-t/X>t] = \frac{1}{\overline{F}_X(t)} \sum_{x=t}^{\infty} \overline{F}_X(x)$$

But, by definition $\mu^X(t) = E[X-t/X>t]$.

Thus, $\mu^X(t) = \dfrac{1}{\overline{F}_X(t)} \displaystyle\sum_{x=t}^{\infty} \overline{F}_X(x)$.

**Theorem 5.** *Let X be a continuous random variable with reliability function $\overline{F}_X(t)$ for all $t \geq 0$. The form of anyone of the following functions determines uniquely the distribution of X*

*i) the reliability function*

*ii) the hazard rate function $h_X(t)$*

*iii) the mean residual life function $\mu^X(t)$*

**Proof.**

By definition, $h_X(t) = \dfrac{f_X(t)}{\overline{F}_X(t)}$

and $f_X(t) = -\dfrac{d\overline{F}_X(t)}{dt}$

thus, $h_X(t) = \frac{-d\overline{F}_X(t)}{\overline{F}_X(t)}\frac{1}{dt} = -\frac{d\ln\overline{F}_X(t)}{dt} \Rightarrow$

$$\int_0^t d\ln\overline{F}_X(x) = -\int_0^t h_X(x)dx \Rightarrow \ln\overline{F}_X(t) - \ln\overline{F}_X(0) = -\int_0^t h_X(x)dx$$

but $\overline{F}_X(0) = 1 \Rightarrow \ln\overline{F}_X(0) = 0$

consequently, $\overline{F}_X(t) = \exp\left[-\int_0^t h_X(x)dx\right]$ (4.5)

We have already proved that the mean residual life of a continuous random variable X can be written in terms of its reliability function as following,

$$\mu^X(t) = \frac{1}{\overline{F}_X(t)}\int_t^\infty \overline{F}_X(x)dx \Rightarrow \mu^X(t) = \frac{1}{\overline{F}_X(t)}\left[\int_0^\infty \overline{F}_X(x)dx - \int_0^t \overline{F}_X(x)dx\right] \quad (4.6)$$

but, $\mu^X(0) = \frac{1}{\overline{F}_X(0)}\int_0^\infty \overline{F}_X(x)dx = \int_0^\infty \overline{F}_X(x)dx = \left[x\overline{F}_X(x)\right]_0^\infty - \int_0^\infty xd\overline{F}_X(x) ==$

$$-\int_0^\infty xd\overline{F}_X(x) = \int_0^\infty xdF_X(x) = E(X) = \mu \Rightarrow \mu^X(0) = \mu$$

The equation (4.6) can be written as

$$\mu^X(t) = \frac{1}{\overline{F}_X(t)}\left[\mu - \int_0^t \overline{F}_X(x)dx\right] \Rightarrow$$

$$\mu^X(t)\overline{F}_X(t) = \mu - \int_0^t \overline{F}_X(x)dx \Rightarrow$$

$$\frac{d}{dt}(\mu^X(t)\overline{F}_X(t)) = \frac{d}{dt}\left(\mu - \int_0^t \overline{F}_X(x)dx\right) \Rightarrow$$

$$(\mu^X(t))'\overline{F}_X(t) + \mu^X(t)(\overline{F}_X(t))' = -\overline{F}_X(t) \Rightarrow$$

$$\left[1 + (\mu^X(t))'\right]\overline{F}_X(t) = -\mu^X(t)(\overline{F}_X(t))' \Rightarrow$$

$$\frac{1 + (\mu^X(t))'}{\mu^X(t)} = h_X(t).$$

But, $\overline{F}_X(t) = \exp\left[-\int_0^t h_X(x)dx\right] \Rightarrow$

$$\overline{F}_X(t) = \exp\left[-\int_0^t \frac{1+(\mu^X(x))'}{\mu^X(x)}dx\right] \tag{4.7}$$

$$\int_0^t \frac{1+\left(\mu^X(x)\right)'}{\mu^X(x)}dx = \int_0^t \frac{dx}{\mu^X(x)} + \int_0^t \frac{\left(\mu^X(x)\right)'}{\mu^X(x)}dx =$$

$$\int_0^t \frac{dx}{\mu^X(x)} + \int_0^t d\ln\mu^X(x) =$$

$$\int_0^t \frac{dx}{\mu^X(x)} + \left[\ln\mu^X(x)\right]_0^t =$$

$$\int_0^t \frac{dx}{\mu^X(x)} + \ln\mu^X(t) - \ln\mu^X(0) =$$

$$\int_0^t \frac{dx}{\mu^X(x)} + \ln\frac{\mu^X(t)}{\mu^X(0)}$$

The equation (4.7) can be written as,

$$\overline{F}_X(t) = \exp\left[-\int_0^t \frac{dx}{\mu^X(x)} - \ln\frac{\mu^X(t)}{\mu^X(0)}\right] \Rightarrow$$

$$\overline{F}_X(t) = \frac{\mu^X(0)}{\mu^X(t)}\exp\left[-\int_0^t \frac{dx}{\mu^X(x)}\right] \tag{4.8}$$

**Theorem 6.** *Let X be a discrete nonnegative random variable with reliability function $\overline{F}_X(t)$ for all t≥0. The form of anyone of the following functions determines uniquely the distribution of X*

*i) the reliability function*

*ii) the hazard rate function $h_X(t)$*

*iii) the mean residual life function $\mu^X(t)$*

**Proof.**

By definition, $h_X(t) = \frac{P(x=t)}{P(X \geq t)} \Rightarrow P(X \geq t) = \frac{P(X=t)}{h_X(t)}$ (4.9)

Specializing (4.9) for t=r and t=r+1 we get,

$$P(X \geq r) = \frac{P(X=r)}{h_X(r)}, \qquad P(X \geq r+1) = \frac{P(X=r+1)}{h_X(r+1)}$$

by subtracting the resulting equations we obtain,

$$P(X=r) = \frac{P(X=r)}{h_X(r)} - \frac{P(X=r+1)}{h_X(r+1)} \Rightarrow$$

$$P(X \geq r) - P(X \geq r+1) = \frac{P(X=r)}{h_X(r)} - \frac{P(X=r+1)}{h_X(r+1)} \Rightarrow$$

$$\frac{P(X=r+1)}{h_X(r+1)} - \frac{P(X=r)}{h_X(r)} + P(X=r) = 0 \Rightarrow$$

$$\frac{P(X=r+1)}{h_X(r+1)} - \frac{1-h_X(r)}{h_X(r)} P(X=r) = 0 \Rightarrow$$

$$P(x=r+1) - \frac{(1-h_X(r))h_X(r+1)}{h_X(r)} P(X=r) = 0$$

The unique solution of this difference equation is given by,

$$P(X=r) = P(X=0) \prod_{i=0}^{r-1} \frac{(1-h(i))h(i+1)}{h(i)}, \quad r=0,1,2,\ldots \tag{4.10}$$

We have already proved that the mean residual life of a discrete random variable X can be written in terms of its reliability function as following,

$$\mu^X(t) = \frac{1}{\overline{F}_X(t)} \sum_{x=t}^{\infty} \overline{F}_X(x) \Rightarrow$$

$$\mu^X(t) = \frac{1}{\overline{F}_X(t)} \left[ \sum_{x=0}^{\infty} \overline{F}_X(x) - \sum_{x=0}^{t-1} \overline{F}_X(x) \right] \Rightarrow$$

$$\mu^X(t)\ \overline{F}_X(t)=\sum_{x=0}^{\infty}\overline{F}_X(x)-\sum_{x=0}^{t-1}\overline{F}_X(x) \tag{4.11}$$

But, $\mu^X(0)=\frac{1}{\overline{F}_X(0)}\sum_{x=0}^{\infty}\overline{F}_X(x)=\sum_{x=0}^{\infty}\overline{F}_X(x)=\sum_{x=0}^{\infty}P(X>x)=$

$$=\sum_{x=0}^{\infty}xP(X=x)=E(X)=\mu \Rightarrow \mu^X(0)=\mu$$

Now the **equation (4.11)** can be written as,

$$\mu^X(t)\ \overline{F}_X(t)=\mu-\sum_{x=0}^{t-1}\overline{F}_X(x) \tag{4.12}$$

Specializing (4.12) for t=r and t=r+1 we get,

$$\mu^X(r)\ \overline{F}_X(r)=\mu-\sum_{x=0}^{r-1}\overline{F}_X(x)$$

$$\mu^X(r+1)\ \overline{F}_X(r+1)=\mu-\sum_{x=0}^{r}\overline{F}_X(x)$$

by subtracting the resulting equations we obtain,

$$\mu^X(r+1)\ \overline{F}_X(r+1)-\ \mu^X(r)\ \overline{F}_X(r)=-\overline{F}_X(r)\Rightarrow$$

$$\overline{F}_X(r+1)-\frac{\mu^X(r)-1}{\mu^X(r+1)}\overline{F}_X(r)=0$$

The unique solution of this difference equation is,

$$P(X>r)=\prod_{i=0}^{r}\frac{\mu^X(i)-1}{\mu^X(i+1)},\ \ r=0,1,2,\ldots \tag{4.13}$$

## 4.3 Characterizations

### 4.3.1 The Exponential Distribution

**Theorem 7.** *Let X be a continuous random variable defined in $[0, \infty)$. X follows the Exponential Distribution with parameter $\lambda>0$ if and only if the hazard rate function at the time t is constant, equal to $\lambda$.*

*Symbolically,* $X \sim Exp(\lambda) \Leftrightarrow h_x(t) = \lambda$.

**Proof.**

**Necessity:** Let X follows an Exponential distribution with parameter λ, then the probability density function of X is $f_X(t) = \lambda e^{-\lambda t}$, $t>0$

and the reliability function of X is $\overline{F}_X(t) = e^{-\lambda t}$

Consequently, $h_X(t) = \dfrac{f_X(t)}{\overline{F}_X(t)} = \dfrac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$.

**Sufficiency:** Let X be a continuous random variable with a constant hazard rate function $h_x(t)=\lambda, \lambda>0$.

Then we can calculate the reliability function $\overline{F}_X(t)$ of X by using equation (4.5)

$$\overline{F}_X(t) = \exp\left[-\int_0^t h_X(x)dx\right] \Rightarrow$$

$$\overline{F}_X(t) = \exp\left[-\int_0^t \lambda dx\right] = e^{-\lambda t}$$

Since the form of the reliability function of a random variable determines uniquely the distribution of the random variable (theorem 5) it can be concluded that X follows an Exponential distribution with parameter λ.

**Theorem 8.** *Let X be a continuous random variable defined in $[0, \infty)$. X follows the Exponential Distribution with parameter $\lambda>0$ if and only if the mean residual life function at the time t is constant, equal to* $1/\lambda$.

*Symbolically,* $X \sim Exp(\lambda) \Leftrightarrow \mu^X(t) = \dfrac{1}{\lambda}$.

**Proof.**

**Necessity:** Let X follows an Exponential distribution with parameter λ, then the reliability function of X is $\overline{F}_X(t)=e^{-\lambda t}$

Consequently,

$$\mu^X(t) = \frac{1}{\overline{F}_X(t)}\int_t^\infty \overline{F}_X(x)dx = \frac{1}{e^{-\lambda t}}\int_t^\infty e^{-\lambda x}dx = \frac{1}{-\lambda e^{-\lambda t}}(-e^{-\lambda t}) = \frac{1}{\lambda}.$$

**Sufficiency:** Let X be a continuous random variable with a constant mean residual life function $\mu^X(t)=c=\frac{1}{\lambda}$, c,λ>0.

Then we can calculate the reliability function $\overline{F}_X(t)$ of X by using equation (4.8)

$$\overline{F}_X(t) = \frac{\mu^X(0)}{\mu^X(t)}\exp\left[-\int_0^t \frac{dx}{\mu^X(x)}\right] \Rightarrow$$

$$\overline{F}_X(t) = \frac{c}{c}\exp\left[-\int_0^t \frac{1}{c}dx\right] = e^{-\frac{1}{c}t} = e^{-\lambda t}$$

Since the form of the reliability function of a random variable determines uniquely the distribution of the random variable (theorem 5) it can be concluded that X follows an Exponential distribution with parameter λ.

From theorems 7 and 8 can be observed that if X is a continuous random variable that describes the life time of a component and X~Exp(λ), λ>0, then the product of the hazard rate function and the mean residual life function is constant, in particular it is equal to 1.Next will be proved that this property leads to a unique determination of the distribution of X as Exponential.

**Theorem 9.** *Let X be a continuous random variable defined in [0, ∞).*
*The equation $h_X(t)\mu^X(t)=1$*
*determines uniquely the distribution of X as Exponential with parameter λ,λ>0.*

*Symbolically,* $X\sim Exp(\lambda) \Leftrightarrow h_X(t)\,\mu^X(t)=1$.

**Proof.**

**Necessity:** Let X follows an Exponential distribution with parameter λ, then the hazard rate function of X is $h_X(t) = \lambda$ and the mean residual life function is $\mu^X(t) = \frac{1}{\lambda}$

Consequently, $h_X(t)\, \mu^X(t) = 1$.

**Sufficiency:** Supose that the equation $h_X(t)\, \mu^X(t) = 1$ holds.

The hazard rate function can be relared to the mean residual life function through

the equation

$$\frac{1+(\mu^X(t))'}{\mu^X(t)} = h_X(t) \Rightarrow$$

$$1+(\mu^X(t))' = h_X(t)\mu^X(t) \Rightarrow$$

$$1+(\mu^X(t))' = 1 \Rightarrow$$

$$(\mu^X(t))' = 0 \Rightarrow$$

$\mu^X(t) = c$, where c is a constant

Consequently, X is following an Exponential distribution with parameter $\frac{1}{c} = \lambda$, λ>0.

**Theorem 10.** *Let X be a continuous random variable defined in [0, ∞). X follows the Exponential Distribution with parameter λ>0 if and only if the additive failure rate function is independent from x for each t≥0.*

*Symbolically,* $X \sim Exp(\lambda) \Leftrightarrow r^X(t) = g(t),\ t \geq 0.$

**Proof.**

**Necessity:** Let X follows an Exponential distribution with parameter λ, then the reliability function of X is $\overline{F}_X(t) = e^{-\lambda t}$

Consequently,

$$r^X(t) = \frac{\overline{F}_X(x+t)}{\overline{F}_X(x)} = \frac{e^{-\lambda(x+t)}}{e^{-\lambda x}} = e^{-\lambda t} = g(t), t \geq 0.$$

**Sufficiency:** Let X be a continuous random variable with additive failure rate function

$r^X(t)$ independent from x for each t≥0. Then,

$r^X(t) = g(t)$, t≥0

Thus, $\frac{\overline{F}_X(x+t)}{\overline{F}_X(x)} = g(t), t \geq 0.$

Consequently, $\overline{F}_X(x+t) = g(t)\overline{F}_X(x).$

The general solution of this functional equation is given by the following equations,

$\overline{F}_X(t) = be^{ct}$ and $g(t) = e^{ct}$

But, $\overline{F}_X(0) = 1$ and $\overline{F}_X(\infty) = 0$

So it can be concluded that, $c = -\lambda$ ,$\lambda>0$

Consequently, $\overline{F}_X(t) = e^{-\lambda t}$

Since the form of the reliability function of a random variable determines uniquely the distribution of the random variable (theorem 5) it can be concluded that X follows an Exponential distribution with parameter λ.

**Theorem 11.** *Let X be a continuous random variable defined in [0, ∞). X follows the Exponential Distribution with parameter λ>0 if and only if the vitality function of X is equal to* $\frac{1+\lambda t}{\lambda}$, *λ>0, t≥0*

*Symbolically,* $X \sim Exp(\lambda) \Leftrightarrow u_X(t) = \frac{1+\lambda t}{\lambda}$, *λ>0, t≥0.*

**Proof.**

Let X follows an Exponential distribution with parameter λ

$$X \sim \exp(\lambda) \Leftrightarrow \mu^X(t) = \frac{1}{\lambda} \Leftrightarrow \mu^X(t) + t = \frac{1}{\lambda} + t \Leftrightarrow u_X(t) = \frac{1}{\lambda} + t \Leftrightarrow$$

$$u_X(t) = \frac{1+\lambda t}{\lambda}, \lambda>0, t\geq 0.$$

**Theorem 12.** *Let $X^{*a}$ be the size-biased version of order $\alpha = 1$ of a random variable X defined in $(\theta, +\infty)$. The reliability function $\overline{F}_{X^*}(x) = (\lambda t+1)e^{-\lambda t}$, $\theta>0$, $\lambda>\alpha$, $x\geq\theta$ if and only if X following a Exponential($\lambda$), $\lambda>0$ distribution.*

*Symbolically, $X\sim Exp(\lambda) \Leftrightarrow \overline{F}_{X^*}(x) = (\lambda t+1)e^{-\lambda t}$, $\theta>0$, $\lambda>\alpha$, $x\geq\theta$.*

**Proof.**

Let X be a continuous, non-negative random variable following an Exponential($\lambda$), $x\geq\theta$, distribution and $\theta$, $\lambda>0$. By direct calculation we find that the reliability function of this distribution is:

$$\overline{F}_X(x) = e^{-\lambda t} \tag{4.14}$$

The reliability function of the size-biased of order $\alpha$ version of a continuous distribution is given by,

$$\overline{F}_{X^{*a}}(x) = \frac{1}{E(X^\alpha)}\overline{F}_X(x)A(x)$$

where $A(x) = E(X^\alpha / X>x)$.

For the Exp($\lambda$) and for $\alpha = 1$

$$E(X) = \frac{1}{\lambda}, \lambda>0.$$

Also, Dimaki and Xekalaki (1996) showed that for every strictly monotonic function $w(\cdot)$, $E[w(X)/X>t]$ can be expressed in terms of $w(\cdot)$ and $\overline{F}_X(\cdot)$ only, namely:

$$E[w(X)/X>t] = w(t) + \frac{\int_t^\infty \overline{F}_X(x)dw(x)}{\overline{F}_X(t)}, t\geq 0.$$

Applying the above result for w(x) = x we have that,

$$E\left(X \;/ X > x\right) = \frac{\lambda t + 1}{\lambda}. \tag{4.15}$$

Consequently, by substitution back to (4.14) we obtain that,

$$\overline{F}_{X^*}(x) = \lambda e^{-\lambda t} \frac{\lambda t + 1}{\lambda}$$

i.e. $\overline{F}_{X^*}(x) = (\lambda t + 1)e^{-\lambda t}$

**Theorem 13.** *Let $X^{*\alpha}$ be the size-biased version of order $\alpha = 1$ of a random variable X defined in $(\theta, +\infty)$. The hazard rate function is $h_{X^*}(x) = \dfrac{\lambda^2 x}{\lambda x + 1}$, $\theta > 0$, $\lambda > \alpha$, $x \geq \theta$ if and only if X follows a Exp(λ), λ>0 distribution.*

*Symbolically, $X \sim Exp(\lambda) \Leftrightarrow h_{X^*}(x) = \dfrac{\lambda^2 x}{\lambda x + 1}$, $\theta > 0$, $\lambda > \alpha$, $x \geq \theta$.*

**Proof.**

Let X be a continuous, non-negative random variable following an Exp(λ), $x \geq 0$, distribution and $\lambda > 0$. By direct calculation we find that the hazard rate function of this distribution is:

$$h_X(x) = \lambda, \quad \lambda > 0 \tag{4.16}$$

The hazard rate function of the size-biased of order $\alpha = 1$ version of a continuous distribution is given by,

$$h_{X^{*\alpha}}(x) = \frac{x^{\alpha} h_X(x)}{A(x)} \tag{4.17}$$

where $A(x) = E(X^{\alpha} / X > x)$.

For the Exp(λ) and for $\alpha = 1$

$$A(x) = \frac{\lambda t + 1}{\lambda}, \quad \lambda > 0. \tag{4.18}$$

Substituting (4.16) and (4.18) in (4.17) it follows that,

$$h_{X^*}(x) = \frac{\lambda x}{\dfrac{\lambda x + 1}{\lambda}}$$

Consequently,

$$h_{X^*}(x) = \frac{\lambda^2 x}{\lambda x + 1}, \lambda > \alpha$$

**Theorem 14.** *Let $X^{*\alpha}$ be the size-biased version of order $\alpha = 1$ of a random variable X defined in $(\theta, +\infty)$. The mean residual life function is* $\mu^{X^*}(x) = \dfrac{\lambda x + 2}{\lambda(\lambda x + 1)}$ *$\theta>0$, $\lambda>\alpha$, $x\geq\theta$ if and only if X follows a Pareto$(\theta, \lambda)$, $\lambda>0$ distribution.*

*Symbolically, $X \sim Exp(\lambda) \Leftrightarrow \mu^{X^*}(x) = \dfrac{\lambda t + 2}{\lambda(\lambda x + 1)}$, $\theta>0$, $\lambda>\alpha$, $x\geq\theta$.*

**Proof.**

Let X be a continuous, non-negative random variable following an Exp(λ), $x\geq 0$, distribution and $\lambda > 0$. By direct calculation, since $\overline{F}_X(x) = e^{-\lambda t}$ and $f_X(x) = \lambda e^{-\lambda x}$ we find that the maen residual life function of this distribution is:

$$\mu^X(x) = \int_x^\infty \frac{\overline{F}_X(t)}{\overline{F}_X(x)} dt = \frac{1}{\lambda}, \lambda>0 \tag{4.19}$$

The mean residual life function of the size-biased of order α version of a continuous distribution is given by,

$$\mu^{X^{*\alpha}}(x) = \frac{\mu^X(x)}{A(x)} \int_x^\infty \frac{A(t)}{\mu^X(t)} \exp\left[ - \int_x^t \frac{du}{\mu^X(u)} \right] dt. \tag{4.20}$$

where $A(x) = E(X^\alpha / X>x)$.

For the Exp(λ) and for $\alpha = 1$

$$A(x) = \frac{\lambda x + 1}{\lambda}, \lambda>0. \tag{4.21}$$

Substituting (4.19) and (4.21) in (4.20) it follows that,

$$\mu^{X^*}(x)=\frac{\frac{1}{\lambda}}{\frac{\lambda x+1}{\lambda}}\int_x^{\infty}\frac{\frac{\lambda t+1}{\lambda}}{\frac{1}{\lambda}}\exp\left[-\int_x^t\frac{du}{\frac{1}{\lambda}}\right]dt=$$

$$=\frac{1}{\lambda x+1}\int_x^{\infty}(\lambda t+1)\exp[-\lambda(t-x)]dt$$

$$=\frac{1}{\lambda x+1}\lambda\exp(\lambda x)\int_x^{\infty}t\exp(-\lambda t)dt+\frac{1}{\lambda x+1}\exp(\lambda x)\int_x^{\infty}\exp(-\lambda t)dt=$$

$$=\frac{1}{\lambda x+1}x+\frac{1}{\lambda}\frac{1}{\lambda x+1}+\frac{1}{\lambda}\frac{1}{\lambda x+1}$$

i.e.

$$\mu^{X^*}(x)=\frac{\lambda x+2}{\lambda(\lambda x+1)},\ \lambda>1.$$

**Theorem 15.** *Let a random variable X be defined in (θ, +∞) with hazard rate function* $h_X(x)$, $x\geq\theta$, $\theta>0$. *The hazard rate function of the corresponding size-biased of order* $\alpha, \alpha>0$ *distribution is* $h_{X^{*\alpha}}(x)$ *the ratio is* $\frac{h_X(x)}{h_{X^*}(x)}$ *is equal to* $\frac{\lambda x+1}{\lambda x}$ *if and only if the original random variable X follows an Exp(λ), λ>0 distribution.*

*Symbolically,* $X\sim Exp(\lambda) \Leftrightarrow \frac{h_X(x)}{h_{X^*}(x)}=\frac{\lambda x+1}{\lambda x}$

**Proof.**

**Necessity:** Let X be a continuous, non-negative random variable following an **Exp(λ)** distribution and λ >0. Then,

$h_X(x)=\lambda$,

the hazard rate function of the size-biased of order α = 1 version of a continuous distribution is given by,

$$h_{X^*}(x) = \frac{xh_X(x)}{E(X/X>x)}$$

for the Exp(λ)

$$h_{X^*}(x) = \frac{\lambda^2 x}{\lambda x+1}.$$

Thus,

$$\frac{h_X(x)}{h_{X^*}(x)} = \frac{\lambda x+1}{\lambda x}.$$

**Sufficiency:** Soppose that $\frac{h_X(x)}{h_{X^*}(x)} = \frac{\lambda x+1}{\lambda x}$

From the definition of the hazard rate function it follows that:

$$\frac{h_X(x)}{h_{X^*}(x)} = \frac{h_X(x)}{\frac{xh_X(x)}{E(X/X>x)}} = \frac{E(X/X>x)}{x}.$$

Dimaki and Xekalaki (1996) showed that for every strictly monotonic function w(.), E[w(X) / X>t] can be expressed in terms of w(.) and $\overline{F}_X(\cdot)$ only, namely:

$$E[w(X) / X>t] = w(t) + \frac{\int_t^\infty \overline{F}_X(x)dw(x)}{\overline{F}_X(t)}, t \geq 0.$$

Applying the above result for w(x) = x we have that,

$$E[X/X>x] = x + \frac{1}{\overline{F}_X(x)}\int_x^\infty \overline{F}_X(t)dt$$

Then,

$$\frac{h_X(x)}{h_{X^*}(x)} = 1 + \frac{1}{x}\frac{1}{\overline{F}_X(x)}\int_x^\infty \overline{F}_X(t)dt = \frac{\lambda x+1}{\lambda x} \Leftrightarrow 1 + \frac{1}{x}\mu^X(x) = \frac{\lambda x+1}{\lambda x} \Leftrightarrow$$

$$\mu^X(x) = \frac{1}{\lambda} \Leftrightarrow X \sim Exp(\lambda).$$

### 4.3.2 The Pareto Distribution

**Theorem 16.** *Let X be a continuous random variable defined in $[0, \infty)$. X follows the Pareto Distribution with parameters $\theta$, $\lambda>0$ if and only if the hazard rate function at the time t is inversely proportional to t, equal to $\lambda/t$.*

*Symbolically,* $X\sim Pareto(\theta,\lambda) \Leftrightarrow h_x(t) = \dfrac{\lambda}{t}$.

**Proof.**

**Necessity:** Let X follows a Pareto distribution with parameters θ, λ, then the probability density function of X is $f_X(t) = \lambda\theta^{\lambda} t^{-\lambda-1}$, $t>0$

and the reliability function of X is $\overline{F}_X(t)=\theta^{\lambda}t^{-\lambda}$

Consequently, $h_X(t) = \dfrac{f_X(t)}{\overline{F}_X(t)} = \dfrac{\lambda\theta^{\lambda}t^{-a-1}}{\theta^{\lambda}t^{-a}} = \dfrac{\lambda}{t}$.

**Sufficiency:** Let X be a continuous random variable such that the hazard rate function $h_X(t)$ is inversely proportional to t ,that is, $h_X(t)=\dfrac{\lambda}{t}$,$\lambda>0$.

Then we can calculate the reliability function $\overline{F}_X(t)$ of X by using equation (4.5)

$$\overline{F}_X(t) = \exp\left[-\int_{\theta}^{t} h_X(x)dx\right] \Rightarrow$$

$$\overline{F}_X(t) = \exp\left[-\int_{\theta}^{t}\frac{\lambda}{x}dx\right] = \exp(-\lambda\ln t+\lambda\ln\theta) = \exp\left(\ln\left(\frac{t}{\theta}\right)^{-\lambda}\right) = \theta^{\lambda}t^{-\lambda}$$

Since the form of the reliability function of a random variable determines uniquely the distribution of the random variable (theorem 5) it can be concluded that X follows a Pareto distribution with parameters θ, λ.

**Theorem 17.** *Let X be a continuous random variable defined in $[0, \infty)$. X follows the Pareto Distribution with parameters $\theta$, $\lambda>0$ if and only if the mean residual life function at the time t is a linear function of t, equal to* $t/_{\lambda-1}, \lambda>1$.

*Symbolically,* $X\sim Pareto(\lambda,\theta) \Leftrightarrow \mu^X(t) = \dfrac{t}{\lambda-1}$.

**Proof.**

**Necessity:** Let X follows a Pareto distribution with parameters θ, λ, then the reliability function of X is $\overline{F}_X(t)=\theta^{\lambda}t^{-\lambda}$

Consequently,

$$\mu^X(t) = \frac{1}{\overline{F}_X(t)}\int_t^{\infty}\overline{F}_X(x)dx = \frac{1}{\theta^{\lambda}t^{-\lambda}}\int_t^{\infty}\theta^{\lambda}x^{-\lambda}dx = \frac{1}{t^{-\lambda}}\int_t^{\infty}x^{-\lambda}dx = \frac{t}{\lambda-1}.$$

**Sufficiency:** Let X be a continuous random variable such that the mean residual life function is a linear function of t, that is, $\mu^X(t)=\dfrac{t}{\lambda-1}, \lambda>1$.

Then we can calculate the reliability function $\overline{F}_X(t)$ of X by using equation (4.8)

$$\overline{F}_X(t)=\frac{\mu^X(\theta)}{\mu^X(t)}\exp\left[-\int_{\theta}^{t}\frac{dx}{\mu^X(x)}\right]\Rightarrow$$

$$\overline{F}_X(t)=\frac{\dfrac{\theta}{\lambda-1}}{\dfrac{t}{\lambda-1}}\exp\left[-(\lambda-1)\int_{\theta}^{t}\frac{dx}{x}\right]=\frac{\theta}{t}\exp\left[-(\lambda-1)\ln\frac{t}{\theta}\right]=\frac{\theta}{t}\frac{t^{-\lambda+1}}{\theta^{-\lambda+1}}=\theta^{\lambda}t^{-\lambda}$$

Since the form of the reliability function of a random variable determines uniquely the distribution of the random variable (theorem 5) it can be concluded that X follows a Pareto distribution with parameters λ, θ.

From theorems 16 and 17 can be observed that if X is a continuous random variable that describes the life time of a component and X~Pareto(λ, θ), λ, θ>0, then the product of the hazard rate function and the mean residual life function is a constant c = $\frac{\lambda}{\lambda-1}$, c>1. Next will be proved that this property leads to a unique determination of the distribution of X as Pareto.

**Theorem 18.** *Let X be a continuous random variable defined in $[0, \infty)$.*

*The equation $h_X(t)\mu^X(t)=c$, $c>1$*

*determines uniquely the distribution of X as Pareto with parameters $\lambda$, $\theta, \lambda, \theta>0$.*

*Symbolically, X~Pereto($\lambda$) $\Leftrightarrow$ $h_X(t)\ \mu^X(t) = c, c>1$.*

**Proof.**

**Necessity:** Let X follows a Pareto distribution with parameters λ,θ then the hazard rate function of X is $h_X(t) = \frac{\lambda}{t}$ and the mean residual life function is

$\mu^X(t) = \frac{t}{\lambda-1}$

Consequently, $h_X(t)\ \mu^X(t) = \frac{\lambda}{\lambda-1} = c$, c>1.

**Sufficiency:** Supose that the equation $h_X(t)\ \mu^X(t) = c, c>1$ holds.

The hazard rate function can be relared to the mean residual life function through the equation

$$\frac{1+(\mu^X(t))'}{\mu^X(t)} = h_X(t) \Rightarrow$$

$$1+(\mu^X(t))' = h_X(t)\mu^X(t) \Rightarrow$$

$1+(\mu^X(t))' = k$, where k is a constant $\Rightarrow$

$$(\mu^X(t))' = 1 \Rightarrow$$

$\mu^X(t) = (c-1)t$, a linear function of t

Consequently, X is following a Pareto distribution with parameters

$\frac{1}{c-1} = \lambda$ and θ,

λ, θ>0.

**Theorem 19.** *Let X be a continuous random variable defined in $[0, \infty)$. X follows the Pareto Distribution with parameters λ, θ>0 if and only if the multiplicative failure rate function is independent from x for each t≥0.*

*Symbolically, X~Pareto(λ, θ) ⇔ $r_X(t) = g(t)$, t≥0.*

**Proof.**

**Necessity:** Let X follows a Pareto distribution with parameters λ,θ, then the reliability function of X is $\overline{F}_X(t) = \theta^{\lambda} t^{-\lambda}$

Consequently,

$$r_X(t) = \frac{\overline{F}_X(x \cdot t)}{\overline{F}_X(x)} = \frac{\theta^{\lambda}(xt)^{-\lambda}}{\theta^{\lambda}x^{-\lambda}} = t^{-\lambda} = g(t), t \geq 0.$$

**Sufficiency:** Let X be a continuous random variable with multiplicative failure rate function $r_X(t)$ independent from x for each t≥0. Then,

$r_X(t) = g(t)$, t≥0

Thus, $\frac{\overline{F}_X(x \cdot t)}{\overline{F}_X(x)} = g(t), t \geq 0.$

Consequently, $\overline{F}_X(x \cdot t) = g(t)\overline{F}_X(x)$.

The general solution of this functional equation is given by the following equations,

$\overline{F}_X(t) = bt^c$ and $g(t) = t^c$

But, $\overline{F}_X(1) = 1$ and $\overline{F}_X(\infty) = 0$

So it can be concluded that, c = -λ ,λ>0

Consequently, $\overline{F}_X(t)=t^{-\lambda}$

Since the form of the reliability function of a random variable determines uniquely the distribution of the random variable (theorem 5) it can be concluded that X follows a Pareto distribution with parameters λ,1.

**Theorem 20.** *Let X be a continuous random variable defined in [0, ∞). X follows the Paretol Distribution with parameters λ,θ>0 if and only if the vitality function of X is equal to* $\frac{\lambda t}{\lambda-1}$*,λ>0,t≥0*

*Symbolically, X~Pareto(λ,θ)* $\Leftrightarrow u_X(t)=\frac{\lambda t}{\lambda-1}$*, λ>0, t≥0.*

**Proof.**

Let X follows a Pareto distribution with parameters λ,θ

$$X\sim \text{Pareto}(\lambda,\theta) \Leftrightarrow \mu^X(t)=\frac{t}{\lambda-1} \Leftrightarrow \mu^X(t)+t=\frac{\lambda t}{\lambda-1}$$

$$\Leftrightarrow u_X(t)=\frac{\lambda t}{\lambda-1}, \quad \lambda>0,\ t\geq 0.$$

**Theorem 21.** *Let a random variable X be defined in (θ, +∞) with probability density function $f_X(x)$, x ≥ θ, θ>0. Then the corresponding size-biased of order α,α >0 distribution is the Pareto(θ, λ-α), if and only if the original random variable X follows a Pareto(θ, λ), λ>0 distribution.*

*Symbolically, X~Pareto(θ,λ)* $\Leftrightarrow X^{*\alpha}\sim$ *Pareto(θ,λ-α)*

**Proof.**

**Necessity:** Applying the definition of the weighted distribution with $w(x)=x^{\alpha}$ for the case of the Pareto distribution we get,

$$f^{*\alpha}{}_X(x)=\frac{x^{\alpha}f_X(x)}{E(X^{\alpha})}=\frac{x^{\alpha}\lambda\theta^{\lambda}x^{-(\lambda+1)}}{E(X^{\alpha})},$$

where $E(X^{\alpha})=\frac{\lambda\theta^{\alpha}}{\lambda-\alpha}$.

So, $f_X^{*\alpha}(x) = \dfrac{x^{\alpha}\lambda\theta^{\lambda}x^{-(\lambda+1)}(\lambda-\alpha)}{\lambda\theta^{\alpha}} = (\lambda-\alpha)\theta^{\lambda-\alpha}x^{-(\lambda-\alpha+1)}$

where $f^{*\alpha}_X(x)$ is the Pareto distribution with parameters $\theta,\lambda-\alpha$..

**Sufficiency:** Let $f^{*\alpha}_X(x) = \dfrac{x^{\alpha} f_X(x)}{C}, x>1$

$$f^{*\alpha}_X(x) = \frac{x^{\alpha} f_X(x)}{C}, x>1$$

where $f_X^{*\alpha}(x) = (\lambda-\alpha)\theta^{\lambda-\alpha}x^{-(\lambda-\alpha+1)}$, $x \geq \theta$.

Then, $f_X(x) = Cx^{-\alpha}f^{*\alpha}_X(\chi)$.

It is obvious that $\int_{\theta}^{\infty} f_X(x)dx = 1$.

Therefore,

$$C\int_{\theta}^{\infty} x^{-\alpha} f_X^{*\alpha}(x)dx = 1.$$

Then, $C = \dfrac{\lambda\theta^{\alpha}}{\lambda-\alpha}$, so $f_X(x) = \lambda\theta^{\lambda}x^{-(\lambda+1)}$

i.e. $X \sim \text{Pareto}(\theta,\lambda)$.

**Theorem 22.** *Let $X^{*\alpha}$ be the size-biased version of order $\alpha, \alpha>0$ of a random variable X defined in $(\theta, +\infty)$. The reliability function $\overline{F}_{X^{*\alpha}}(x) = \theta^{\lambda-\alpha}x^{-(\lambda-\alpha)}$, $\theta>0$, $\lambda>\alpha$, $x\geq\theta$ if and only if X follows a Pareto$(\theta, \lambda)$, $\lambda>0$ distribution.*

*Symbolically,* $X \sim \text{Pareto}(\theta, \lambda) \Leftrightarrow \overline{F}_{X^{*\alpha}}(x) = \theta^{\lambda-\alpha}x^{-(\lambda-\alpha)}$, $\theta>0$, $\lambda>\alpha$, $x\geq\theta$.

**Proof.**

Let X be a continuous, non-negative random variable following a Pareto$(\theta, \lambda)$, $x\geq\theta$, distribution and $\theta, \lambda>0$. By direct calculation we find that the reliability function of this distribution is:

$$\overline{F}_X(x) = \theta^{\lambda}x^{-\lambda} \tag{4.22}$$

The reliability function of the size-biased of order α version of a continuous distribution is given by,

$$\overline{F}_{X^{*\alpha}}(x)=\frac{1}{E(X^{\alpha})}\overline{F}_X(x)A(x) \tag{4.23}$$

where $A(x) = E(X^{\alpha}/ X>x)$.

For the Parerto(θ,λ) it can easily been proved that:

$$E\left(X^{\alpha}\right)=\frac{\lambda\theta^{\alpha}}{\lambda-\alpha}, \lambda>\alpha.$$

Also, Dimaki and Xekalaki (1996) showed that for every strictly monotonic function $w(\cdot)$, $E\left[w(X)/X>t\right]$ can be expressed in terms of $w(\cdot)$ and $\overline{F}_X(\cdot)$ only, namely:

$$E\left[w(X)/X>t\right]=w(t)+\frac{\int_t^{\infty}\overline{F}_X(x)dw(x)}{\overline{F}_X(t)}, t\geq 0.$$

Applying the above result for $w(x) = x^{\alpha}$ we have that,

$$E\left(X^{\alpha}/X>x\right)=\frac{\lambda x^{\alpha}}{\lambda-\alpha}.$$

Consequently, by substitution back to (4.23) we obtain that,

$$\overline{F}_{X^{*\alpha}}(x)=\frac{1}{\frac{\lambda\theta^{\alpha}}{\lambda-\alpha}}\theta^{\lambda}x^{-\lambda}\frac{\lambda x^{\alpha}}{\lambda-\alpha}$$

i.e.

$\overline{F}_{X^{*\alpha}}(x)=\theta^{\lambda-\alpha}x^{-(\lambda-\alpha)}$ , x≥θ and θ>0, λ>α.

**Theorem 23.** *Let $X^{*\alpha}$ be the size-biased version of order $\alpha, \alpha>0$ of a random variable X defined in $(\theta, +\infty)$. The hazard rate function is $h_{X^{*\alpha}}(x)=\frac{\lambda-\alpha}{x}$, θ>0, λ>α, x≥θ if and only if X follows a Pareto(θ, λ), λ>0 distribution.*

*Symbolically, X~Pareto(θ, λ) $\Leftrightarrow h_{X^{*\alpha}}(x)=\frac{\lambda-\alpha}{x}$ , θ>0, λ>α, x≥θ.*

**Proof.**

Let X be a continuous, non-negative random variable following a Pareto($\theta$, $\lambda$) $x \geq \theta$, distribution and $\theta$, $\lambda > 0$. By direct calculation we find that the hazard rate function of this distribution is:

$$h_X(x) = \frac{\lambda}{x}, \ \lambda > 0 \tag{4.24}$$

The hazard rate function of the size-biased of order $\alpha$ version of a continuous distribution is given by,

$$h_{X^{*\alpha}}(x) = \frac{x^\alpha h_X(x)}{A(x)} \tag{4.25}$$

where $A(x) = E(X^\alpha / X > x)$.

For the Parerto($\theta$,$\lambda$)

$$A(x) = \frac{\lambda x^\alpha}{\lambda - \alpha}, \ \lambda > \alpha. \tag{4.26}$$

Substituting (4.24) and (4.26) in (4.25) it follows that,

$$h_{X^{*\alpha}}(x) = \frac{x^\alpha \frac{\lambda}{x}}{\frac{\lambda x^\alpha}{\lambda - \alpha}}$$

Consequently,

$$h_{X^{*\alpha}}(x) = \frac{\lambda - \alpha}{x}, \lambda > \alpha.$$

**Theorem 24.** *Let* $X^{*\alpha}$ *be the size-biased version of order* $\alpha, \alpha > 0$ *of a random variable* $X$ *defined in* $(\theta, +\infty)$. *The mean residual life function is*

$\mu^{X^{*\alpha}}(x) = \dfrac{x}{\lambda - \alpha - 1}$ $\theta > 0$, $\lambda > \alpha$, $x \geq \theta$ *if and only if X follows a Pareto*($\theta$, $\lambda$), $\lambda > 0$

*distribution*

*Symboli*

*cally,* $X \sim Pareto(\theta, \lambda) \Leftrightarrow \mu^{X^{*\alpha}}(x) = \dfrac{x}{\lambda - \alpha - 1}$, $\theta > 0$, $\lambda > \alpha$, $x \geq \theta$.

**Proof.**

Let X be a continuous, non-negative random variable following a Pareto($\theta$, $\lambda$) , $x \geq \theta$, distribution and $\theta$, $\lambda > 0$. By direct calculation, since $\overline{F}_X(x) = \theta^{\lambda} x^{-\lambda}$ and $f_X(x) = \lambda \theta^{\lambda} x^{-(\lambda+1)}$ we find that the maen residual life function of this distribution is:

$$\mu^X(x) = \int_x^{\infty} \frac{\overline{F}_X(t)}{\overline{F}_X(x)} dt = \frac{x}{\lambda - 1}, \ \lambda > 1 \tag{4.27}$$

The mean residual life function of the size-biased of order $\alpha$ version of a continuous distribution is given by,

$$\mu^{X^{*\alpha}}(x) = \frac{\mu^X(x)}{A(x)} \int_x^{\infty} \frac{A(t)}{\mu^X(t)} \exp\left[-\int_x^{t} \frac{du}{\mu^X(u)}\right] dt. \tag{4.28}$$

where $A(x) = E(X^{\alpha} / X > x)$.

For the Parerto($\theta$,$\lambda$)

$$A(x) = \frac{\lambda x^{\alpha}}{\lambda - \alpha}, \ \lambda > \alpha. \tag{4.29}$$

Substituting (4.27) and (4.29) in (4.28) it follows that,

$$\mu^{X^{*\alpha}}(x) = \frac{\dfrac{x}{\lambda - 1}}{\dfrac{\lambda x^{\alpha}}{\lambda - \alpha}} \int_x^{\infty} \frac{\dfrac{\lambda t^{\alpha}}{\lambda - \alpha}}{\dfrac{t}{\lambda - 1}} \exp\left[-\int_x^{t} \frac{du}{\dfrac{u}{\lambda - 1}}\right] dt =$$

$$= x^{\alpha - 1} \int_x^{\infty} t^{\alpha - 1} \left(\frac{t}{x}\right)^{-(\lambda - 1)} dt$$

$$= x^{\lambda - \alpha} \int_x^{\infty} t^{\alpha - \lambda} dt$$

i.e.

$$\mu^{X^{*\alpha}}(x) = \frac{x}{\lambda - \alpha - 1}, \ \lambda > \alpha + 1.$$

**Theorem 25.** *Let a random variable X be defined in (θ, +∞) with hazard rate function $h_X(x)$, $x \geq \theta$, $\theta>0$. The hazard rate function of the corresponding size-biased of order $\alpha, \alpha>0$ distribution is $h_{X^{*\alpha}}(x)$ the ratio is $\frac{h_{X^{*\alpha}}(x)}{h_X(x)}$ is equal to $\frac{\lambda-\alpha}{\alpha}$ if and only if the original random variable X follows a Pareto(θ, λ), λ>0 distribution.*

*Symbolically,* $X \sim Pareto(\theta,\lambda) \Leftrightarrow \frac{h_{X^{*\alpha}}(x)}{h_X(x)} = \frac{\lambda-\alpha}{\alpha}$

**Proof.**

**Necessity:** Let X be a continuous, non-negative random variable following a Pareto(θ, λ) , x≥θ, distribution and θ, λ >0. Then,

$h_X(x) = \frac{\lambda}{x}$.

The hazard rate function of the size-biased of order α = 1 version of a continuous distribution is given by,

$$h_{X^*} = \frac{x h_X(x)}{E(X/X>x)}$$

For thePareto(θ,λ)

$$h_{X^*}(x) = \frac{\lambda-1}{x}.$$

Thus,

$$\frac{h_X(x)}{h_{X^*}(x)} = \frac{\lambda}{\lambda-1}.$$

**Sufficiency:** Soppose that $\frac{h_X(x)}{h_{X^*}(x)} = \frac{\lambda}{\lambda-1}$.

From the definition of the hazard rate function it follows that:

$$\frac{h_X(x)}{h_{X^*}(x)} = \frac{h_X(x)}{\frac{xh_X(x)}{E(X/X>x)}} = \frac{E(X/X>x)}{x}.$$

Dimaki and Xekalaki (1996) showed that for every strictly monotonic function w(.), E[w(X) / X>t] can be expressed in terms of w(.) and $\overline{F}_X(\cdot)$ only, namely:

$$E[w(X) / X>t] = w(t) + \frac{\int_t^{\infty} \overline{F}_X(x)dw(x)}{\overline{F}_X(t)}, t \geq 0.$$

Applying the above result for w(x) = x we have that,

$$E[X/X>x] = x + \frac{1}{\overline{F}_X(x)} \int_x^{\infty} \overline{F}_X(t)dt$$

Then,

$$\frac{h_X(x)}{h_{X^*}(x)} = 1 + \frac{1}{x}\frac{1}{\overline{F}_X(x)} \int_x^{\infty} \overline{F}_X(t)dt = \frac{\lambda-1}{\lambda} \Leftrightarrow 1 + \frac{1}{x}\mu^X(x) = \frac{\lambda}{\lambda-1} \Leftrightarrow$$

$$\mu^X(x) = \frac{x}{\lambda-1} \Leftrightarrow X\sim\text{Pareto}(\theta,\lambda).$$

### 4.3.3 The Geometric Distribution

**Theorem 26.** *Let X be a discrete random variable taking values in {0,1,2,...}. X follows the Geometric Distribution with parameter p>0 if and only if the hazard rate function at the time t is constant, equal to p.*

*Symbolically,* $X \sim G(p) \Leftrightarrow h_X(t) = p$.

**Proof.**

**Necessity:** Let X follows a Geometric distribution with parameter p, then the probability density function of X is $P(X=t) = pq^t$, t = 0,1,2,..., 0<p<1, q = 1-p and the reliability function of X is $\overline{F}_X(t) = q^{t+1}$, t = 0,1,2,..., 0<p<1, q = 1-p

Then,

P(X≥ t) = P(X>t) + P(X= t) = $q^{t+1}+(1-q)q^t=q^t$.

Consequently, $h_X(t)=\dfrac{P(X=t)}{P(X\geq t)}=\dfrac{pq^t}{q^t}=p.$

**Sufficiency:** Let X be a discrete random variable with a constant hazard rate function $h_x(t)=p$, $p>0$.

Then we can calculate the probability function P(X = t) of X by using equation (4.10)

$$P(X=t)=P(X=0)\prod_{i=0}^{t-1}\frac{(1-h(i))h(i+1)}{h(i)},\quad t=0,1,2,\ldots$$

$$P(X=t)=P(X=0)\prod_{i=0}^{t-1}\frac{(1-p)p}{p}\Rightarrow$$

$$P(X=t)=P(X=0)(1-p)^t\Rightarrow \tag{4.30}$$

$$\sum_{t=0}^{\infty}P(X=t)=\sum_{t=0}^{\infty}P(X=0)(1-p)^t\Rightarrow$$

$$1=P(X=0)\sum_{t=0}^{\infty}(1-p)^t \tag{4.31}$$

but, $0<p<1\Rightarrow|1-p|<1$

so $\sum_{t=0}^{\infty}(1-p)^t=\dfrac{1}{1-(1-p)}=\dfrac{1}{p}.$

The equation (4.31) can be written now as,

$1=P(X=0)\dfrac{1}{p}\Rightarrow P(X=0)=p$

Substituting this result in equation (4.30) we get,

$$P(X=t)=P(X=0)(1-p)^t\Rightarrow P(X=t)=p(1-p)^t\Rightarrow P(X=t)=pq^t. \tag{4.32}$$

Since the form of the probability function of a random variable determines uniquely the distribution of the random variable, it can be concluded that X follows a Geometric distribution with parameter p.

**Theorem 27.** *Let X be a discrete random variable taking values in {0,1,2,...}. X follows the Geometric Distribution with parameter p>0 if and only if the mean residual life function at the time t is constant, equal to* $1/p$.

*Symbolically,* $X \sim G(p) \Leftrightarrow \mu^X(t) = \dfrac{1}{p}$.

**Proof.**

**Necessity:** Let X follows a Geometric distribution with parameter p . Consequently,

$$\mu^X(t) = \frac{1}{\overline{F}_X(t)} \sum_{x=t}^{\infty} \overline{F}_X(x) \Rightarrow$$

$$\mu^X(t) = \frac{1}{\overline{F}_X(t)} \left[ \sum_{x=0}^{\infty} \overline{F}_X(x) - \sum_{x=0}^{t-1} \overline{F}_X(x) \right] \Rightarrow$$

$$\mu^X(t)\overline{F}_X(t) = \sum_{x=0}^{\infty} \overline{F}_X(x) - \sum_{x=0}^{t-1} \overline{F}_X(x) \Rightarrow$$

But,

$$\mu^X(0) = \frac{1}{\overline{F}_X(0)} \sum_{x=0}^{\infty} \overline{F}_X(x) = \sum_{x=0}^{\infty} \overline{F}_X(x) = \sum_{x=0}^{\infty} P(X > x) =$$

$$= \sum_{x=0}^{\infty} x P(X = x) = E(X) = \mu \ .$$

So,

$$\mu^X(t)\overline{F}_X(t) = \mu - \sum_{x=0}^{t-1} \overline{F}_X(x) \tag{4.33}$$

The mean of the Geometric distribution is $\mu = E(X) = \dfrac{q}{p}$

and the reliability function is $\overline{F}_X(t) = q^{t+1}$

The equation (4.33) can be written as,

$$\mu^X(t) q^{t+1} = \frac{q}{p} - \sum_{x=0}^{t-1} q^{x+1}$$

but, $\displaystyle\sum_{x=0}^{t-1} q^{x+1} = q + q^2 + q^3 + .... + q^t = 1 + q + q^2 + ... + q^t - 1 =$

$$=\frac{q^{t+1}-1}{q-1}-1=-\frac{q^{t+1}-1}{p}-1$$

Consequently,

$$\mu^X(t)=\left(\frac{q}{p}+p\frac{q^{t+1}-1}{}+1\right)\frac{1}{q^{t+1}}=\frac{q+q^{t+1}-1+p}{pq^{t+1}}=$$

$$=\frac{q+q^{t+1}-1+1-q}{pq^{t+1}}=\frac{q^{t+1}}{pq^{t+1}}=\frac{1}{p}\Rightarrow$$

$$\Rightarrow \mu^X(t)=\frac{1}{p}.$$

**Sufficiency:** Let X be a discrete random variable with a constant mean residual life function $\mu^X(t)=c=\frac{1}{p}$, c,p>0.

Then we can calculate the reliability function $\overline{F}_X(t)$ of X by using equation (4.13)

$$\overline{F}_X(t)=P(X>t)=\prod_{i=0}^{t}\frac{\mu^X(t)-1}{\mu^X(t+1)}\Rightarrow$$

$$\overline{F}_X(t)=\prod_{i=0}^{t}\frac{\frac{1}{p}-1}{\frac{1}{p}}=\prod_{i=0}^{t}(1-p)=(1-p)^{t+1}=q^{t+1}$$

Since the form of the reliability function of a random variable determines uniquely the distribution of the random variable (theorem 6) it can be concluded that X follows a Geometric distribution with parameter p.

From theorems 26 and 27 can be observed that if X is a discrete random variable that describes the life time of a component and X~G(p), p>0, then the product of the hazard rate function and the mean residual life function is constant, in particular it is equal to 1.Next will be proved that this property leads to a unique determination of the distribution of X as geometric.

**Theorem 28.** *Let X be a discrete random variable taking values in{0,1,2,...}. The equation $h_X(t)\mu^X(t)=1$*

*determines uniquely the distribution of X as Geometric with parameter p,p>0.*

*Symbolically, $X \sim G(p) \Leftrightarrow h_X(t)\ \mu^X(t) = 1$.*

**Proof.**

**Necessity:** Let X follows an Geometric distribution with parameter p, then the hazard rate function of X is $h_X(t) = p$ and the mean residual life function is

$\mu^X(t) = \dfrac{1}{p}$

Consequently, $h_X(t)\ \mu^X(t) = 1$.

**Sufficiency:** Supose that the equation $h_X(t)\ \mu^X(t) = 1$ holds.

The mean residual life function can be relared to the reliability function through the equation

$$\mu^X(t) = \frac{1}{\overline{F}_X(t)} \sum_{x=t}^{\infty} \overline{F}_X(x) = \frac{1}{P(X>t)} \sum_{x=t}^{\infty} P(X>x) \Rightarrow$$

$$\mu^X(t) P(X>t) = \sum_{x=t}^{\infty} P(X>x) \tag{4.34}$$

Specializing (4.34) for t=r and t=r+1 we get,

$$\mu^X(r) P(X>r) = \sum_{x=r}^{\infty} P(X>x)$$

$$\mu^X(r+1) P(X>r+1) = \sum_{x=r+1}^{\infty} P(X>x)$$

by subtracting the resulting equations we obtain,

$$\mu^X(r+1) P(X>r+1) - \mu^X(r) P(X>r) = -P(X>r) \Rightarrow$$

$$\mu^X(r+1) P(X>r+1) - \left[\mu^X(r) - 1\right] P(X>r) = 0 \Rightarrow$$

$$\mu^X(r+1)[\ P(X \geq r+1) - P(X=r+1)\ ] - [\mu^X(r)-1]P(X>r) = 0 \Rightarrow$$

$$\mu^X(r+1)P(X \geq r+1) - \mu^X(r+1)P(X=r+1) - [\mu^X(r)-1]P(X \geq r+1) = 0 \Rightarrow$$

$$[\mu^X(r+1) - \mu^X(r) + 1]P(X \geq r+1) = \mu^X(r+1)P(X=r+1) \Rightarrow$$

$$\frac{\mu^X(r+1) - \mu^X(r) + 1}{\mu^X(r+1)} = \frac{P(X=r+1)}{P(X \geq r+1)} \Rightarrow$$

$$\frac{\mu^X(r+1) - \mu^X(r) + 1}{\mu^X(r+1)} = h_X(r+1) \Rightarrow$$

$$\mu^X(r+1) - \mu^X(r) + 1 = \mu^X(r+1)h_X(r+1) \Rightarrow$$

but, we supposed that $h_X(t)\ \mu^X(t) = 1$. So,

$$\mu^X(r+1) - \mu^X(r) + 1 = 1 \Rightarrow$$

$$\mu^X(r+1) - \mu^X(r) = 0 \Rightarrow$$

$\mu^X(r) = c$, where c is a constant

Consequently, X is following a Geometric distribution with parameter $\frac{1}{c} = p$, p>0.

**Theorem 29.** *Let X be a discrete random variable taking values in {0,1,2,...}. X follows the Geometric distribution with parameter p>0 if and only if the additive failure rate function is independent from x for each t∈{0,1,2,...}.*

*Symbolically, X~G(p) ⇔ $r^X(t) = g(t)$, t≥0.*

**Proof.**

**Necessity:** Let X follows a Geometric distribution with parameter p, then the reliability function of X is $\overline{F}_X(t)=q^{t+1}$

Consequently,

$$r^X(t) = \frac{\overline{F}_X(x+t)}{\overline{F}_X(x)} = \frac{q^{x+t+1}}{q^{x+t}} = q^t = g(t), t \geq 0.$$

**Sufficiency:** Let X be a discrete random variable with additive failure rate function

$r^X(t)$ independent from x for each t≥0. Then,

$r^X(t) = g(t)$, t≥0

Thus, $\dfrac{\overline{F}_X(x+t)}{\overline{F}_X(x)} = g(t), t \geq 0.$

Consequently,

$$\frac{\overline{F}_X(x+t)}{g(t)} = \overline{F}_X(x), t \geq 0. \qquad (4.35)$$

Specializing (4.35) for t=r and t=r+1 we get,

$$\frac{\overline{F}_X(x+r)}{g(r)} = \overline{F}_X(x)$$

$$\frac{\overline{F}_X(x+r+1)}{g(r+1)} = \overline{F}_X(x)$$

by subtracting the resulting equations we obtain,

$$\frac{\overline{F}_X(x+r)}{g(r)} - \frac{\overline{F}_X(x+r+1)}{g(r+1)} = 0 \Rightarrow$$

$$\overline{F}_X(x+r+1) - \frac{g(r+1)}{g(r)}\overline{F}_X(x+r) = 0$$

The general solution of this difference equation is,

$$\overline{F}_X(r)=\prod_{i=0}^{r}\frac{g(i+1)}{g(i)}\Rightarrow$$

$$\overline{F}_X(r)=\frac{g(r+1)}{g(0)}\Rightarrow\overline{F}_X(r)=\frac{q^{r+1}}{1}$$

Consequently,

$\overline{F}_X(r)=q^{r+1}$

Since the form of the reliability function of a random variable determines uniquely the distribution of the random variable (theorem 5) it can be concluded that X follows an Geometric distribution with parameter p =1-q.

**Theorem 30.** *Let X be a discrete random variable taking values in {0,1,...}. X follows the Geometric Distribution with parameter p>0 if and only if the vitality function of X is equal to* $\frac{1+pt}{p}$*,p>0,t≥0*

*Symbolically,* $X\sim G(p)\Leftrightarrow u_X(t)=\frac{1+pt}{p}$*, p>0, t≥0.*

**Proof.**

Let X follows a Geometric distribution with parameter p

$$X\sim G(p)\Leftrightarrow \mu^X(t)=\frac{1}{p}\Leftrightarrow \mu^X(t)+t=\frac{1}{p}+t\Leftrightarrow u_X(t)=\frac{1}{p}+t\Leftrightarrow$$

$$u_X(t)=\frac{1+pt}{p},\ p>0,\ t\geq 0.$$

**Theorem 31.** *Let* $X^*$ *be the size-biased version of order a=1 of a random variable X taking values on {1,2,3...}. The reliability function is* $\overline{F}_{X^*}(x)=q^t(pt+1)$*, if and only if X follows a Geometric(p) distribution.*

*Symbolically,* $X\sim G(p)\Leftrightarrow\overline{F}_{X^*}(x)=q^t(pt+1)$

**Proof.**

Let X be a non-negative integer-valued random variable distributed according to the G(p), p >0 distribution.

The reliability function of the size-biased of order α=1 version of a discrete distribution is given by,

$$\overline{F}^{*}_{X}(x)=\frac{1}{E(X)}\overline{F}_X(x)E[X/X>t]$$

Dimaki and Xekalaki (1996) showed that for every strictly monotonic function $w(\cdot)$, $E[w(X)/X>t]$ can be expressed in terms of $w(\cdot)$ and $\overline{F}_X(\cdot)$ only, namely:

$$E[w(X)/X>t]=w(t+1)+\left[\overline{F}_X(t)\right]^{-1}\cdot\sum_{x=t+1}^{\infty}[w(x+1)-w(x)]\overline{F}_X(x). \tag{4.36}$$

Applying the above result for w(x) = x we have that,

$$E(X\ /X>t)=(t+1)+q^{-(t+1)}\sum_{x=t+1}^{\infty}q^{x+1}=t+1+\frac{q}{1-q}\Rightarrow$$

$$E(X\ /X>t)=\frac{1+pt}{p} \tag{4.37}$$

Consequently, by substitution back to (4.14) we obtain that,

$$\overline{F}_{X^*}(t)=q^{t+1}\frac{\frac{1+pt}{p}}{\frac{q}{p}}=\frac{q^{t+1}}{q}(pt+1)=q^{t}(pt+1).$$

from which follows that X* is distributed according to the G(p) .

**Theorem 32.** *Let $X^*$ be the size-biased version of order α=1 of a random variable X taking values on {1,2,3…}. The hazard rate function is* $h_{X^*}(x)=\frac{tp^2}{tp+q}$, *if and only if X follows a G(p) distribution.*

*Smbolically, X~ G(p)* $\Leftrightarrow h_{X^*}(x)=\frac{tp^2}{tp+q}$

**Proof.**

Let X be a non-negative integer valued random variable distributed according to the G(p), p > 0.

It have been proved that,

If $X \sim G(p) \Leftrightarrow h_X(t) = p$ , t =0,1,2,...

Therefore, in the present case

$$X \sim G(p) \Leftrightarrow h_X(t) = \frac{p+1}{p+t+1} \quad , t = 0,1,2,... \tag{4.38}$$

The hazard rate function of a weighted distribution is given by,

$$h_{X^w}(t) = \frac{h_X(t)}{\frac{E[w(X)/X>t]}{w(t)} + \left[1 - \frac{E[w(X)/X>t}{w(t)}\right] h_X(t)} \tag{4.39}$$

Applying equation (4.15) for w(x)=x we have that for the G(p)

$$E[X / X > t] = \frac{1+pt}{p} \tag{4.40}$$

Substituting (4.38) and (4.40) in (4.37) it follows that,

$$h_{X^*}(x) = \frac{p}{\frac{1+tp}{pt} + \left[1 - \frac{1+pt}{pt}\right] p} =$$

$$= \frac{p}{\frac{1+pt}{pt} - \frac{p^2t - p(1+pt)}{pt}} =$$

$$= \frac{p^2t}{(1+pt)+p^2t-p(1+pt)} = \frac{p^2t}{tp(q+p)+q} = \frac{p^2t}{tp+q}.$$

Consequently,

$$h_{X^*}(x) = \frac{tp^2}{tp+q}.$$

**Theorem 33.** *Let $X^*$ be the size-biased version of order $a=1$ of a random variable X taking values on {0,1,2,...}. The mean residual life function is*

$\mu^{X^*}(x)=\dfrac{tp+q+1}{p(pt+1)}$, *t=0,1,2,... if and only if X follows a G(p) distribution.*

*Symbolically,* $X\sim G(p) \Leftrightarrow \mu^{X^*}(x)=\dfrac{tp+q+1}{p(pt+1)}$, *t=0,1,2,...*

**Proof.** Let X be a non-negative integer valued random variable distributed according to the G(p), p > 0.

It have been proved that,

If $X\sim G(p) \Leftrightarrow \overline{F}_X(t)=q^{t+1}$, t=0,1,2,... (4.41)

The mean residual life function of a weighted distribution is given by,

$$\mu^{X^w}(t)=\frac{1}{\overline{F}_X(t)E[w(X)/X>t]}\sum_{x=t+1}^{\infty}\overline{F}_X(x)E[w(X)/X>x] \tag{4.42}$$

Applying equation (4.15) for w(x)=x we have that for the G(p)

$$E[X/X>t]=\frac{1+pt}{p} \tag{4.43}$$

Substituting (4.41) and (4.43) in (4.42) it follows that,

$$\mu^{X^*}(t)=\frac{1}{q^{t+1}\left(\dfrac{pt+1}{p}\right)}\sum_{x=t+1}^{\infty}q^{x+1}\left(\frac{px+1}{p}\right)=$$

$$=\frac{1}{q^{t}(pt+1)}\sum_{x=t}^{\infty}(q^{x}px+q^{x})=$$

$$=\frac{1}{pt+1}\left(\frac{tp+q+1)}{p}\right)$$

Consequently,

$$\mu^X(x) = \frac{tp+p+1}{p(pt+1)}.$$

**Theorem 34.** *Let a random variable taking values in {0,1,2,...} with hazard rate function $h_X(t)$. The hazard rate function of the corresponding size-biased of order α=1 distribution is* $h_{X^*}(x)$ *the ratio is* $\frac{h_X(x)}{h_{X^*}(x)}$ *is equal to* $\frac{tp+q}{tp}$ *if and only if the original random variable X follows G(p) distribution.*

*Symbolically,* $X \sim G(p) \Leftrightarrow \frac{h_X(x)}{h_{X^*}(x)} = \frac{tp+q}{tp}$. (4.44)

**Proof.**

**Necessity:** Let X be a random variable distributed according to a G(p).

Obviously,

$$h_X(t) = p$$

and

$$h_{X^*}(t) = \frac{tp^2}{tp+q}$$

Thus, $\frac{h_X(x)}{h_{X^*}(x)} = \frac{tp+q}{tp}$.

**Sufficiency:** From the definition of the hazard rate function it follows that:

$$\frac{h_X(t)}{h_{X^*}(t)} = \frac{\frac{P(X=t)}{P(X \geq t)}}{\frac{P(X^*=t)}{P(X^* \geq t)}} = \frac{\frac{P(X=t)}{\sum_{x=t}^{\infty} P(X=x)}}{\frac{\frac{tP(X=t)}{E(X)}}{\frac{\sum_{x=t}^{\infty} xP(X=x)}{E(X)}}} \Rightarrow \frac{\sum_{x=t}^{\infty} xP(X=x)}{t\sum_{xc=t}^{\infty} P(X=x)} = \frac{tp+q}{tp} \Rightarrow$$

i.e.

$$p\sum_{x=t}^{\infty} xP(X=x)=(p+1)t\sum_{x=t}^{\infty} P(X=x). \qquad (4.45)$$

Specializing (4.45) for t = r and t = r+1 we get

$$p\sum_{x=t}^{\infty} xP(X=x)=(tp-p+1)\sum_{x=t}^{\infty} P(X=x)$$

$$p\sum_{x=t+1}^{\infty} xP(X=x)=(tp+1)\sum_{x=t+1}^{\infty} P(X=x).$$

By subtracting the resulting equations we obtain

$$ptP(X=t)=(tp+1)P(X=t)-p\sum_{x=t}^{\infty} P(x=x), t=r,r+1,.. \qquad (4.46)$$

$$P(X=t)=p\sum_{x=t}^{\infty} P(X=x)$$

Applying the same procedure in relation (4.24), yields

$P(X=t) - P(X=t+1) = pP(X=t)$

or, equivalently

$P(X=t+1)-(1-p)P(X=t)=0$

The unique solution of this difference equation is given by,

$$P(X=t)=P(X=1)\prod_{i=1}^{t-1}(1-p), i=1,2,..$$

$$= P(X=1)(1-p)^{t-1}$$

But $\sum_{t=1}^{\infty} P(X=t)=1$. Therefore,

$$\sum_{t=1}^{\infty} P(X=1)(1-p)^{t-1}=1$$

leading to

$P(X=1)=p(1-p)$

Consequently,

$P(X=t)= p(1-p)(1-p)^{t-1}= p(1-p)^{t}=pq^{t}, t=0,1,2,\ldots$ and $0< p <1$, $q=1-p$

i.e. $X\sim G(p)$.

### 4.3.4 The Yule Distribution

**Theorem 35.** *Let X be a non-negative integer valued random variable. Then X follows the Yule Distribution with parameter p, p>0 if and only if the hazard rate function at the time t is inversely proportional to time t, i.e. if and only if*

$$h_X(t) = \frac{p}{p+t+1}, t = 0,1,2,...$$

*Symbolically, X~Yule(p)* $\Leftrightarrow h_X(t) = \dfrac{p}{p+t+1}$.

**Proof.**

**Necessity:** Let X follows a Yule distribution with parameter p,p>0 then the probability density function of X is $P(X=t) = \dfrac{pt!}{(p+1)(p+2)...(p+t+1)}$

and the reliability function of X is $\overline{F}_X(t) = P(X > t) = \dfrac{t+1}{p} P(X = t)$

Consequently,

$$h_X(t) = \frac{P(X = t)}{P(X \geq t)} = \frac{P(X = t)}{P(X > t) + P(X = t)} = \frac{P(X = t)}{\frac{t+1}{p} P(X = t) + P(X = t)}$$

$$= \frac{1}{\frac{t+1}{p} + 1} = \frac{p}{t+p+1}.$$

**Sufficiency:** Let X be a discrete random variable such that the hazard rate function $h_X(t)$ is inversely proportional to t ,that is, $h_X(t) = \dfrac{p}{p+t+1}, t = 0,1,2,...$

Then we can calculate the probability function P(X = t) of X by using equation (4.10)

$$P(X=t)=P(X=0)\prod_{i=0}^{t-1}\frac{(1-h(i))h(i+1)}{h(i)}, \quad t=0,1,2,...$$

$$=P(X=0)\prod_{i=0}^{t-1}\frac{\left(1-\dfrac{p}{p+i+1}\right)\dfrac{p}{p+i+2}}{\dfrac{p}{p+i+1}}=$$

$$=P(X=0)\prod_{i=0}^{t-1}\frac{i+1}{p+i+2}=P(X=0)\left(\frac{1}{p+2}\frac{2}{p+3}\cdots\frac{t}{p+t+1}\right)=$$

$$=P(X=0)\frac{t!}{(p+2)....(p+t+1)}\Rightarrow$$

$$\Rightarrow\frac{P(X=t)}{P(X=0)}=\frac{t!}{(p+2)...(p+t+1)}\Rightarrow X\sim \text{Yule(p)}.$$

**Theorem 36.** *Let X be a non-negative integer valued random variable. Then, X follows the Yule Distribution with parameter p, p>0 if and only if its mean residual life function at the time t is a linear function of t, given by,*

$$\mu^X(t)=\frac{p+t+1}{p-1}, t=0,1,2,...$$

*Symbolically, X~Yule(p)* $\Leftrightarrow \mu^X(t)=\dfrac{p+t+1}{p-1}$.

**Proof.**

**Necessity:** Let X follows a Yule distribution with parameter p, then the reliability function of X is $\overline{F}_X(t)=$ P(X > t) $=\dfrac{t+1}{p}P(X=t)$

Consequently,

$$\mu^X(t)=\frac{1}{\overline{F}_X(t)}\sum_{x=t}^{\infty}\overline{F}_X(x)\Rightarrow$$

$$\mu^X(t)=\frac{1}{P(X>t)}\sum_{x=0}^{\infty}P(X>x+t)\Rightarrow$$

$$\mu^X(t) = \frac{1}{\frac{t+1}{p}P(X=t)} \sum_{x=0}^{\infty} \frac{x+t+1}{p} P(X=x+t) \Rightarrow$$

$$\mu^X(t) = \frac{1}{\frac{t+1}{p}\frac{pt!}{(p+1)..(p+t+1)}} \sum_{x=0}^{\infty} \frac{x+t+1}{p} \frac{p(x+t)!}{(p+1)...(p+x+t+1)} \Rightarrow$$

$$\mu^X(t) = \frac{\Gamma(p+t+2)\Gamma(p-1)}{\Gamma(p)\Gamma(p+t+1)} \Rightarrow \mu^X(t) = \frac{p+t+1}{p-1}.$$

**Sufficiency:** Let X be a discrete random variable such that the mean residual life function is a linear function of t, that is, $\mu^X(t)=\frac{p+t+1}{p-1}$,p>1

Then we can calculate the reliability function $\overline{F}_X(t)$ of X by using equation (4.13)

$$\overline{F}_X(t) = P(X>t) = \prod_{i=0}^{t} \frac{\mu^X(r)-1}{\mu^X(r+1)} \Rightarrow$$

$$\overline{F}_X(t) = \prod_{i=0}^{t} \frac{\frac{p+r+1}{p-1}-1}{\frac{p+r+2}{p-1}} = \prod_{i=0}^{t} \frac{r+2}{p+r+2} = \frac{2}{p+2}\frac{3}{p+3}\cdots\frac{t+2}{p+t+2} =$$

$$= \frac{(t+2)!}{(p+2)...(p+t+2)} \Rightarrow$$ X follows the truncated Yule distribution.

From theorems 35 and 36 can be observed that if X is a discrete random variable that describes the life time of a component and X~Yule(p), p>0, then the product of the hazard rate function and the mean residual life function is a constant c = $\frac{p}{p-1}$, c>1. Next will be proved that this property leads to a unique determination of the distribution of X as Yule.

**Theorem 37.** *Let X be a non-negative integer valued random variable. Then, X follows the Yule(p), p>0 distribution if and only if,* $h_X(t)\,\mu^X(t) = c$, *where c constant,* $t=0,1,2,\ldots$

*Symbolically,* $X \sim Yule(p) \Leftrightarrow h_X(t)\,\mu^X(t) = c, c>1$.

**Proof.**

**Necessity:** Let X follows a Yule distribution with parameter p then the hazard rate function of X is $h_X(t) = \dfrac{p}{p+t+1}$

and the mean residual life function is $\mu^X(t) = \dfrac{p+t+1}{p-1}$

Consequently, $h_X(t)\,\mu^X(t) = \dfrac{p}{p-1} = c$, $c>1$.

**Sufficiency:** Supose that the equation $h_X(t)\,\mu^X(t) = c, c>1$ holds. The hazard rate function can be relared to the mean residual life function through the equation

$$h_X(t+1) = \frac{\mu^X(t+1) - \mu^X(t) + 1}{\mu^X(t+1)}$$

or equivalently,

$$\mu^X(t+1) - \mu^X(t) + 1 = \mu^X(t+1)h_X(t+1)$$

but $h_X(t)\,\mu^X(t) = c$

So,

$$\mu^X(t+1) - \mu^X(t) + 1 = c,$$

which yields

$$\mu^X(t+1) - \mu^X(t) = c-1, t = 0,1,2,\ldots$$

The unique solution of this difference equation is given by

$\mu^X(t) = k + (c-1)t, t = o,1,2,\ldots$: where k, c are constsnts.

This is a necessary and sufficient condition for X to be distributed as Yule(p) with $p = \frac{c}{c-1} = \frac{k+1}{k-1} > 0.$

**Theorem 38.** *Let X be a non-negative integer valued random variable. Then, X follows the Yule Distribution with parameter p, p>0 if and only if its vitality function at the time t is a linear function of t, given by,*

$$u^X(t) = \frac{pt+p+1}{p-1}, t = 0,1,2,...$$

*Symbolically,* $X \sim Yule(p) \Leftrightarrow u^X(t) = \frac{pt+p+1}{p-1}.$

**Proof.**

**Necessity:** Let X follows a Yule distribution with parameter p

Then,

$$X \sim Yule(p) \Leftrightarrow \mu^X(t) = \frac{p+t+1}{p-1} \Rightarrow \mu^X(t) + t = \frac{p+t+1}{p-1} + t$$

$$\Rightarrow u_X(t) = \frac{pt+p+1}{p-1} = \alpha \cdot t + b,$$

where $\alpha = \frac{p}{p-1}, b = \frac{p+1}{p-1}$

**Sufficiency: :** Let X be a discrete random variable such that the vitality function is a linear function of t, that is, $u^X(t) = at + b$

Then we can calculate the mean residual life function $\mu^X(t)$ of X

$u^X(t) = \mu^X(t) + t = at + b \Rightarrow \mu^X(t) = (a-1)t + b$

This is a necessary and sufficient condition for X to be distributed as Yule(p) with $p = \frac{a}{a-2} = \frac{b+1}{b} > 0.$

**Theorem 39.** *Let the random variables X and X* taking values on {1,2,3,...}, and also that E(X) < ∞. Then X* is distributed according to the shifted Waring distribution with parameters (1,2;p) if and only if the distribution of X is the shifted Yule with parameter p+1.*

*Symbolically,* $X \sim$ *shifted Waring(1,2;p)* $\Leftrightarrow X^* \sim$ *shifted Yule(p+1).*

**Proof.**

**Necessity:** Let X be a random variable distributed according to a shifted Yule (p+1). Then,

$$f_X(x) = \frac{(p+1)(x-1)!}{(p+2)_{(x)}}, x = 1,2,...$$

and $\mu = E(X) = \frac{p+1}{p}$. Then X* is defined by,

$$f_X^*(x) = \frac{x\frac{(p+1)(x-1)!}{(p+2)_{(x)}}}{\frac{p+1}{p}} =$$

$$= \frac{p(p+1)x!}{(p+2)_{(x)}(p+1)} \Rightarrow$$

$$f_X^*(x) = \frac{p_{(2)}2_{(x-1)}1_{(x-1)}}{(p+1)_{(2)}(p+3)_{(x-1)}}\frac{1}{(x-1)!}.$$

Therefore the random variable X* follows a shifted generalized Waring distribution UGWD(1,2;p).

**Sufficiency:** Let X* be a random variable distributed according to a shifted UGWD(1,2;p). also X* is given by

$$f^*_X(x) = \frac{x\ f_X(x)}{E(X)}, x = 1,2,...$$

$$f_X(x) = \frac{\mu \cdot f_X^*(x)}{x} \Rightarrow$$

$$f_X(x) = \mu \frac{p_{(2)} 2_{(x-1)} 1_{(x-1)}}{x(p+1)_{(2)}(p+3)_{(x-1)}} \frac{1}{(x-1)!} \Rightarrow$$

$$f_X(x) = \mu \frac{p_{(2)} 2_{(x-1)}}{x(p+1)_{(2)}(p+3)_{(x-1)}} \Rightarrow$$

$$f_X(x) = \mu \frac{p \cdot 1_{(x-1)}}{(p+2)(p+3)_{(x-1)}}$$

But, it is obvious that $\sum_{x=1}^{\infty} f_X(x) = 1$.

Consequently,

$$\mu \frac{p}{p+2} \sum_{x=1}^{\infty} \frac{1_{(x-1)}}{(p+3)_{(x-1)}} = 1 \Rightarrow$$

$$\mu \frac{p}{p+2} \sum_{x=1}^{\infty} \frac{1_{(x-1)}(p+1)}{(p+2)_{(x)}} = 1.$$

The expression in the summation is the shifted Yule with parameter p+1

and obviously $\sum_{x=1}^{\infty} \frac{1_{(x-1)}(p+1)}{(p+2)_{(x)}} = 1$.

Consequently,

$$\mu \frac{p}{p+2} = 1 \Rightarrow \mu = \frac{p+1}{p}.$$

Therefore, X ~ shifted Yule (p+1).

**Theorem 40.** *Let $X^*$ be the size-biased version of order $\alpha=1$ of a random variable $X$ taking values on $\{1,2,3...\}$. The reliability function is* $\overline{F}_{X^*}(x)=\frac{t+1}{p}P(X^*=t)$, *if and only if $X$ follows a shifted Yule(p+1) distribution.*

*Symbolically, X~ shifted Yule( p+1)* $\Leftrightarrow \overline{F}_{X^*}(x)=\frac{t+1}{p}P(X^*=t)$

**Proof.**

Let X be a non-negative integer-valued random variable distributed according to the shifted Yule (p+1), p >0 distribution. Then,

$$\overline{F}_X(t)=\frac{t}{p+1}P(X=t), t=0,1,2,...$$

where $P(X=t)=\frac{(p+1)(t-1)!}{(p+2)_{(t)}}$

Consequently,

$$\overline{F}_X(t)=\frac{t}{p+1}\frac{(p+1)(t-1)!}{(p+2)_{(t)}}=\frac{t!}{(p+2)_{(t)}}.$$

The reliability function of the size-biased of order α version of a discrete distribution is given by,

$$\overline{F}^*_X(x)=\frac{1}{E(X)}\overline{F}_X(x)E[X/X>t]$$

Dimaki and Xekalaki (1996) showed that for every strictly monotonic function $w(\cdot)$, $E[w(X)/X>t]$ can be expressed in terms of $w(\cdot)$ and $\overline{F}_X(\cdot)$ only, namely:

$$E[w(X)/X>t]=w(t+1)+[\overline{F}_X(t)]^{-1}\cdot\sum_{x=t+1}^{\infty}[w(x+1)-w(x)]\overline{F}_X(x). \tag{4.47}$$

Applying the above result for w(x) = x we have that,

$$E(X/X>t)=(t+1)E(X) \tag{4.48}$$

Consequently, by substitution back to (4.14) we obtain that,

$$\bar{F}_X(t) = \frac{t!}{(p+2)_{(t)}}(t+1) = \frac{t+1}{p}\frac{p_{(2)}}{(p+1)_{(2)}}\frac{1_{(t-1)}2_{(t-1)}}{(p+3)_{(t-1)}}\frac{1}{(t-1)!} =$$

$$= \frac{t+1}{p}P(X^* = t)$$

from which follows that X* is distributed according to the shifted UGWD(1,2;p) or equivalently to the zero truncated Yule(p).

$$\bar{F}_{X^*}(x) = \frac{t+1}{p}P(X^* = t)$$

**Theorem 41.** *Let $X^*$ be the size-biased version of order α=1 of a random variable X taking values on {1,2,3...}. The hazard rate function is* $h_{X^*}(x) = \frac{p}{p+t+1}$, *if and only if X follows a shifted Yule(p+1) distribution.*

*Symbolically, X~shifted Yule(p+1)* $\Leftrightarrow h_{X^*}(x) = \frac{p}{p+t+1}$

**Proof.**

Let X be a non-negative integer valued random variable distributed according to the shifted Yule(p+1), p > 0.

It have been prooved that,

If X~Yule(p)$\Leftrightarrow h_X(t) = \frac{p}{p+t+1}$ , t =0,1,2,...

Therefore, in the present case

$$X\sim\text{shifted Yule}(p+1) \Leftrightarrow h_X(t) = \frac{p+1}{p+t+1}, \ t = 1,2,... \tag{4.49}$$

The hazard rate function of a weighted distribution is given by,

$$h_{X^w}(t) = \frac{h_X(t)}{\frac{E[w(X)/X>t]}{w(t)} + \left[1 - \frac{E[w(X)/X>t}{w(t)}\right]h_X(t)} \tag{4.50}$$

Applying equation (4.15) for w(x)=x we have that for the shifted Yule (p+1)

$$E[X / X > t] = \frac{(t+1)(p+1)}{p} \tag{4.51}$$

Substituting (4.49) and (4.51) in (4.50) it follows that,

$$h_{X^*}(x) = \frac{\dfrac{p+1}{p+t+1}}{\dfrac{(t+1)(p+1)}{tp} + \left[1 - \dfrac{(t+1)(p+1)}{tp}\right]\dfrac{p+1}{t+p+1}} =$$

$$= \frac{\dfrac{p+1}{p+t+1}}{\dfrac{(t+1)(p+1)}{tp} - \dfrac{t+p+1}{tp}\dfrac{p+1}{p+t+1}} =$$

$$= \frac{\dfrac{p+1}{p+t+1}}{\dfrac{p+1}{tp}t} = \frac{p}{p+t+1}$$

Consequently,

$$h_{X^*}(x) = \frac{p}{p+t+1}$$

**Theorem 42.** *Let $X^*$ be the size-biased version of order $a=1$ of a random variable X taking values on {1,2,3...}. The mean residual life function is $\mu^{X^*}(x) = \dfrac{tp+p+1}{p-1}$, $t=0,1,2,...$ if and only if X follows a shifted Yule (p+1) distribution.*

*Symbolically, X~shifted Yule(p+1)* $\Leftrightarrow \mu^{X^*}(x) = \dfrac{tp+p+1}{p-1}$, $t=0,1,2,...$

**Proof.** Let X be a non-negative integer valued random variable distributed according to the shifted Yule(p+1), $p > 0$.

It have been prooved that,

If $X \sim Yule(p) \Leftrightarrow \mu^{X}(t)=\frac{p+t+1}{p-1}$ , t=0,1,2,...

Therefore, in the present case

$$X \sim \text{shifted Yule}(p+1) \Leftrightarrow \mu^{X}(t)=\frac{p+t+1}{p}, \; t=1,2,... \tag{4.52}$$

The mean residual life function of a weighted distribution is given by,

$$\mu^{X^w}(t)=\frac{1}{\bar{F}_X(t)E[w(X)/X>t]}\sum_{x=t+1}^{\infty}\bar{F}_X(x)E[w(X)/X>x] \tag{4.53}$$

Applying equation (4.15) for w(x)=x we have that for the shifted Yule (p+1)

$$E[X/X>t]=\frac{(t+1)(p+1)}{p} \tag{4.54}$$

The reliability function of a discrete random variable X which is distributed according to the shifted Yule(p+1) is,

$$\bar{F}_X(t)=\bar{F}_X(1)\prod_{i=1}^{t-1}\frac{\frac{p+i+1}{p}-1}{\frac{p+i+2}{p}}=$$

$$=\bar{F}_X(1)\prod_{i=1}^{t-1}\frac{1+i}{p+2+i}=$$

$$=\bar{F}_X(1)\frac{1_{(t-1)}}{(p+2)_{(t-1)}}=$$

$$=\frac{1}{p+2}\frac{t!}{(p+2)_{(t-1)}} \tag{4.55}$$

Substituting (4.53) and (4.55) in (4.54) it follows that,

$$\mu^{X^*}(t)=\frac{1}{\frac{1}{p+2}\frac{t!}{(p+2)_{(t-1)}}\frac{(t+1)(p+1)}{p}}\sum_{x=t+1}^{\infty}\frac{1}{p+2}\frac{x!}{(p+2)_{(x-1)}}\frac{(x+1)(p+1)}{p}$$

Consequently,

$$\mu^{X^*}(x) = \frac{tp+p+1}{p-1}$$

**Theorem 43.** *Let a random variable taking values in {1,2,...} with hazard rate function $h_X(t)$. The hazard rate function of the corresponding size-biased of order α=1 distribution is $h_{X^*}(x)$ the ratio is $\frac{h_X(x)}{h_{X^*}(x)}$ is equal to $\frac{p+1}{p}$ if and only if the original random variable X follows shifted Yule(p+1) distribution.*

*Symbolically, X~shifted Yule(p+1))* $\Leftrightarrow \frac{h_X(x)}{h_{X^*}(x)} = \frac{p+1}{p}$.

**Proof.**

**Necessity:** Let X be a random variable distributed according to a shifted Yule (p+1).

Obviously,

$$P(X=t) = \frac{(p+1)(t-1)!}{(p+2)_{(t)}}, t=1,2,...$$

and

$$E(X) = \frac{p+1}{p}.$$

Also,

$$\begin{aligned} P(X \geq t) &= P(X=t) + P(X>t) \\ &= P(X=t) + \frac{t}{p+1}P(X=t) \\ &= \frac{p+1+t}{p+1}P(X=t). \end{aligned}$$

Therefore,

$$h_X(t) = \frac{p+1}{p+1+t}.$$

Moreover,

$$P(X^*=t)=\frac{tP(X=t)}{E(X)}=\frac{p(p+1)t!}{(P+2)_{(t)}}=\frac{p_{(2)}}{(p+1)_{(2)}}\frac{2_{(t-1)}}{(p+3)_{(t-1)}}.$$

i.e. X*~ shifted UGWD(1,2;p).

Therefore,

$$P(X^*\geq t)=P(X^*=t)+\frac{t+1}{p}P(X^*=t)=\frac{p+t+1}{p}P(X^*=t).$$

That implies that

$$h_X(t)=\frac{p}{p+t+1}$$

which leads to the result.

**Sufficiency:** From the definition of the hazard rate function it follows that:

$$\frac{h_X(t)}{h_{X^*}(t)}=\frac{\dfrac{P(X=t)}{P(X\geq t)}}{\dfrac{P(X^*=t)}{P(X^*\geq t)}}=\frac{\dfrac{P(X=t)}{\sum_{x=t}^{\infty}P(X=x)}}{\dfrac{\dfrac{tP(X=t)}{E(X)}}{\dfrac{\sum_{x=t}^{\infty}xP(X=x)}{E(X)}}}\Rightarrow\frac{\sum_{x=t}^{\infty}xP(X=x)}{t\sum_{x=t}^{\infty}P(X=x)}=\frac{p+1}{p}\Rightarrow$$

i.e.

$$p\sum_{x=t}^{\infty}xP(X=x)=(p+1)t\sum_{x=t}^{\infty}P(X=x). \tag{4.57}$$

Specializing (4.57) for t = r and t = r+1 we get

$$p\sum_{x=t}^{\infty}xP(X=x)=(p+1)t\sum_{x=t}^{\infty}P(X=x)$$

$$p\sum_{x=t+1}^{\infty}xP(X=x)=(p+1)(t+1)\sum_{x=t+1}^{\infty}P(X=x).$$

By subtracting the resulting equations we obtain

$$tP(X=t)=(p+1)\sum_{x=t+1}^{\infty}P(x=x), t=r,r+1,.. \tag{4.58}$$

Applying the same procedure in relation (4.58), yields

$(p+t+2)P(X=t+1)-tP(X=t)=0$

or, equivalently

$$P(X=t+1)-\frac{t}{p+t+2}P(X=t)=0$$

The unique solution of this difference equation is given by,

$$P(X=t)=P(X=1)\prod_{i=1}^{t-1}\frac{i}{p+i+2}, i=1,2,..$$

$$=P(X=1)\frac{(t-1)!}{(p+3)_{(t-1)}}$$

or equivalently,

$$P(x=t)=P(X=1)\frac{(t-1)!}{(p+3)_{(t-1)}}$$

But $\sum_{t=1}^{\infty}P(X=t)=1.$ Therefore,

$$\sum_{t=1}^{\infty}P(X=1)\frac{(t+1)!}{(p+3)_{(t-1)}}=1$$

leading to

$$P(X=1)=\frac{p+1}{p+2}.$$

Consequently,

$$P(X=t)=\frac{p+1}{p+2}\frac{(t-1)!}{(p+3)_{(t-1)}}=\frac{(p+1)(t-1)!}{(p+2)_{(t)}},$$

i.e. X~ shifted Yule(p+1).

## 4.4 Characterizations of the Weibull Distribution

Roy and Mukherjee (1986) proposed four different characterizations of the Weibull distribution the first two in terms of two reliability measures, the reliability function and the hazard rate function respectively. The third one have to do with the Fisher information minimization in a restricted class of distributions and the fourth states that the Weibull law has the maximum entropy within the class of distributions

If $R(x) = -\log(1-F(x)) = -\log \overline{F}_X(x)$, where F is the distribution function of a non-negative random variable X, then for the Weibull distribution where $F(x) = 1-\exp(\lambda x^{\alpha})$, it is clear that $R(x) = \lambda x^{\alpha}$ so that, for all $x > 0$, $y > 0$,

$$R(xy) R(1) = R(x)R(y), \qquad (R(1) > 0) \tag{4.59}$$

If, conversely (4.59) holds for all $x > 0$ and all $y > 0$, then $R(x) = \lambda x^{\alpha}$ for some $\alpha > 0$, $\lambda > 0$, as guaranteed by the multiplicative version of the classical Cauchy functional equation. Thus we have for $R(1) > 0$,

**Theorem 44.** *$R(xy)$ $R(1) = R(x)R(y)$ for all $x > 0$, $y > 0$ if and only if $X \sim$ Weibull$(\alpha, \lambda)$.*

It is easy to verify that if (4.59) holds for all $x > 0$ and for two values $y_1$ and $y_2$ of $y > 0$ such that $\log y_2/\log y_1$ is irrational, then again the same conclusions holds.

**Theorem 45.** *F, with support $\subset [0, \infty)$, is a weibull distribution function if and only if R satisfies the conditions:*

*(a) $R(xy)R(1) = R(x)R(y)$, with $R(1) > 0$, for at least one value of $y > 0$, $y \neq 1$, for all $x > 0$, and*

*(b) $xR'(x)/R(x)$ is non-decreasing for $x > 0$.*

**Proof:**

If F is Weibull, $R(x) = \lambda x^{\alpha}$ and assertions (a) and (b) are immediate.

Conversely suppose (b) holds and (a) holds for $y = y_0 \neq 1$ and for all $x > 0$.

If $v_0 = \log y_0$ $(\neq 0)$,

Then defining $\varphi(u) = R(e^u)/R(1)$, we have $\varphi(u+v_0) = \varphi(u)\varphi(v_0)$.

If then we define

$\psi(u) = \log\varphi(u)-(u/v_0)\log\varphi(v_0)$,

then $\psi$ has period $v_0$ and assumption (b) implies that $\psi'$is non-decreasing. These two facts together imply that $\psi$ is constant $= \psi(v_0) = 0$, so that

$\log\varphi(u)/u =$ constant $= \log\varphi(1)$, or $R(x) = R(1) x^{\alpha}$, as desired to prove.

If X > 0 is a random variable, and its distribution function F has a continuous probability density function f, then the hazard rate of X at x is defined as

$h_x(x) = \frac{f_x(x)}{\bar{F}_x(x)}$, and $R(x) = \int_0^x h_x(t)dt$. Considering $h_X(x)$ and $R(x)$ as functions of the random variable x it may be of interest to examine their distributions from the characterization point of view. In fact it can be observed that R(x) is an increasing and continuous function in x and whatever be the functional form of $f_X(x)$

$\Pr[R(x) \le u] = 1 - e^{-u}$.

Hence the distribution of R(x) cannot characterize that of X. however, it may be observed that if X follows an exponential distribution, $h_X(x)$ has a degenerate distribution and conversely, if $h_X(x)$ is degenerate, x follows an exponential distribution. So the possibility of characterizing the Weibull distribution through the distribution of $h_X(x)$ have to be examined.

**Theorem 46.** *(i) h is strictly increasing with $h_X(0) = 0$, and (ii) $h_X(X) \sim$ Weibull($\alpha'$, $\lambda'$) with $\alpha' > 1$ if and only if $X \sim$ Weibull($\alpha$, $\lambda$) where $\alpha > 1$,*

$\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$, *and* $\lambda' = \lambda(\alpha\lambda')^{\alpha}$.

**Proof:**

Let $X \sim$ Weibull($\alpha$, $\lambda$) with $\alpha > 1$, so that $h_X(x) = \lambda\alpha x^{\alpha-1}$. Since $\alpha > 1$, (i) is immediate. Also hX(x) has its probability density function given by f(x(h)) $x'(h)$ where f is the probability density function of X and $x(h) = (h/\lambda\alpha)^{1/(\alpha-1)}$, whence (ii) readily follows.

Conversely, let (i) and (ii) hold. Let then f be the inverse function of h on $[0, \infty)$, so that, in particular, $f(0) = 0$. Then, by virtue of our assumption,

$F_x(f(u)) = P[X \le f(u)] = P[h(x) \le u] = 1 - \exp(-\lambda' u^{\alpha'})$. (4.60)

From the definition of R(x) as $-\log[1-F(x)]$, we have therefore that $R(f(u)) = \lambda' u^{\alpha'}$.

Since $R(x) = \int_0^x h_x(t)dt$, we have $\int_0^{f(u)} h_x(t)dt = \lambda' u^{\alpha'}$ so that

$\int_0^u vf'(v)dv = \lambda' u^{\alpha'}$ whence $f'(u) = \alpha'\alpha\lambda'\lambda' u^{\alpha'-2}$.

This implies that $f(u) = \alpha'\lambda' u^{\alpha-1}/\alpha\text{-}1$, using $f(0) = 0$ and $\alpha' > 1$. Substituting this back in (4.60), we see that $X \sim$ Weibull($\alpha$, $\lambda$), where $\alpha$ and $\lambda$ are related to $\alpha'$,$\lambda'$ as stated.

A distribution function with probability density function f(x) given by

$$\frac{\alpha\lambda^{\beta}}{\Gamma(\beta)} x^{\alpha\beta-1} \exp(-\lambda x^{\alpha}), x > 0,$$

may be called a generalized gamma distribution and denoted by GG($\alpha,\beta,\lambda$). It is clear that $X \sim GG(\alpha,\beta,\lambda)$ if and only if $X' \sim G(\beta,\lambda)$. Also $\beta = 1$ corresponds to the Weibull distribution, $\alpha = 1$ to a Gamma distribution, and $\alpha = \beta = 1$ to an exponential distribution.

Let $\mathcal{F}_\alpha$ be the class of all distribution function's with support $\subset [0, \infty)$ and having a density function f with the properties

(i) f is continuously differentiable on $(0, \infty)$,

(ii) $xf(x) \to 0$ as $x \to 0^+$, $x^{1+\alpha} f(x) \to 0$ as $x \to \infty$.

(iii) (a) $\int x^{\alpha} f(x)dx = 1$, (b) $\int x^{2\alpha} f(x)dx = 2$.

**Theorem 47.** *Among all members of $\mathcal{F}_\alpha$, Fisher-information is minimized by GG($\alpha$,1,1).*

The maximum entropy distribution is the Weibull distribution

**Theorem 48.** *If $X \geq 0$, then among all distributions for X with ElogX= $g_1$ and $EX^{\alpha} = g_2$ (for an arbitrary, fixed $\alpha > 0$), where $g_1$,$g_2$ are (admissible) constants, the entropy of the distribution is maximum for an X whch follows a GG($\alpha,\beta,\lambda$) distribution (with $\beta$ and $\lambda$ determined by $g_1$ and $g_2$).*

Corollary: In particular, the maximum entropy distribution is the Weibull distribution if $g_2^{1/\alpha} = ((EX^{\alpha})^{1/\alpha} = )\ \exp(g_1 + v/\alpha)$, where v is the Euler's constant.

Scholz (1990) presented a characterization of the three parameter Weibull distribution given in terms of relationships between particular triads of quantiles as delineated by the set C below. These relationships stipulate that a certain function of the quantile triad is always proportional to a second function of the same triad, the proportionality factor remaining constant over all such triads. Of course, this characterization of the three parameter Weibull distribution is easily specialized to the case of a two parameter Weibull distribution.

In order to state the characterization theorem the following notation is introduced. Let

$C = \{(u, v, w): 0 < u < v < w < 1, \log(1-u)\log(1-w) = (\log(1-v))^2\}$

And let F be a cumulative distribution function with quantiles

$x_F(u) = x(u) = F^{-1}(u) = \text{Inf}\{x: F(x) \geq u\}$ for $0 < u < 1$. For the three parameter Weibull distibution function, defined for $\alpha > 0$, $\beta > 0$and $\lambda \in R$ by

$$G(x) = 1 - \exp\left(-\left(\frac{x-\lambda}{\alpha}\right)^{\beta}\right) \text{ for } x \geq \lambda$$

and $G(x) = 0$ for $x < \lambda$ , it is known that for same fixed t, namely $t = \lambda$, the following relationship holds between its quantiles, $x(u) = x_G(u)$:

$x(u)x(w)-x^2(v) = t(x(u)+x(w)-2x(v))$ for all $(u, v, w) \in C$

(4.61)

The following theorem states that this relationship actually characterizes the three-parameter Weibull distribution.

**Theorem 49.** *Any random variable X with cumulative distribution function F(x) and quantiles* $x_F(u) = x(u)$ *satisfying the relationships (4.61) is either degenerate or X has a three parameter Weibull distribution with* $\lambda = t$.

*Of course the degenerate case could be subsumed in the Weibull model with* $\alpha \geq 0$.

**Proof:**

The proof consists of the following steps.

1. The support of F cannot be $(-\infty, \infty)$.
2. The support is finite only in the degenerate case.

3. Assuming that the support is $[\alpha, \infty)$ or $(-\infty, \alpha]$ it follows that $\alpha = t$.
4. Finally, it is shown that the quantile relationship (4.61) translates into a linearity relation from which the Weibull characterization follows.

**Proof of 1:** This follows by contradiction upon dividing the relation (4.61) by $x(u)\, x(w)$ and letting $u \to 0$ and $w \to 1$ while holding v fixed.

**Proof of 2:** Suppose F has finite support $[\alpha, b]$. Let $Y = X - \alpha$ with corresponding quantiles $y(u)$. The quantile relation (4.61) translates to

$y(u)y(w) - y^2(v) = (t - \alpha)(y(u)+y(w) - 2y(v))$ for all $(u, v, w) \in C$.

Writing $s = t - \alpha$ and letting $u \to 0$ and $w \to 1$ while holding v fixed, with $y(v) = y$, leads to the following equation

$-y^2 = s(b-\alpha-2y)$ with solutions $y = s \pm \sqrt{s^2 - (b-\alpha)s}$

For any s this equation yields at most one solution $y \in [0, b - \alpha]$. This implies the degenerate case of the characterization.

**Proof of 3:** Dividing the relationship (4.61)by $x(w)$ (or $x(u)$, whichever becomes unbounded) and letting $u \to 0$ and $w \to 1$ while v is fixed one obtains $t = \alpha$.

**Proof of 4:** Proceeding as in step 2, the quantile relation becomes

$y(u)y(w)-y^2(v) = 0$ for all $(u, v, w) \in C$.

Let $h(z) = \log(y(\rho^{-1}(z)))$ for all $z \in R$, where $\rho(p) = \log(-\log(1 - p))$. For all $(u, v, w) \in C$ one now has

$h(\rho(u)) + h(\rho(w)) = 2h(\rho(v))$ and $\rho(u)+\rho(w) = 2\rho(v)$.

this implies the following functional equation

$$h\left(\frac{z_1 + z_2}{2}\right) = \frac{h(z_1) + h(z_2)}{2} \quad \text{for all } z_1, z_2 \in R.$$

Since $h(z)$ is bounded on any finite interval it follows that h is convex, concave and continuous, thus linear, i.e., $h(z) = A + Bz$ with $B > 0$ since $h(z)$ is strictly increasing. Hence

$y(p) = \exp(h(\rho(p))) = \exp(A + B\rho(p)) = \exp(A)(-\log(1-p))^B$ ,

which is the p-quantile of a two parameter Weibull distribution with $\alpha = \exp(A)$, and $\beta = 1/B$. Hence $x(p) = y(p) + \lambda$ is the p-quantile of $G(x)$.

**Theorem 49: In**

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# Chapter 5

# Applications of the Weibull Distribution in lab experiments

## 5.1 Introduction

Pike (1966) suggested a model to describe the process underlying the phenomena of carcinogenesis, that is the time of occurrence of cancer in a tissue follow a weibull distribution. And Berry (1975) examined Pike's model from the viewpoint of experimental design. Two design problems are discussed in his paper, the first concerns whether a time has been reached when it is optimum to terminate an experiment by sacrificing all surviving animals, and the second is the design of an experiment to test if the carcinogenic effect is related to the age of injection. This chapter includes a brief review of these papers illustrating only a sort part of the Weibull's distribution application field.

## 5.2 Weibull Distributions for carcinogenesis experiments

Pike (1966) suggested a model to describe the process underlying the phenomena of carcinogenesis. Pike's model will be described next.

Suppose that for any given normal cell (and its linear descendants in the tissue under consideration) the comulative distribution function (cdf) of its time to occurrence is $F(x)$, and that the tissue is originally composed of n independent cells. The cdf of the time, $X_n$, to occurrence of cancer in the tissue is then

$$\text{prob}(X_n < x) = G_n(x) = 1 - [1 - F(x)]^n$$

with the associated probability density function (pdf)

$$g_n(x) = nf(x)[1 - F(x)]^{n-1}$$

In this case n is very large and this suggests that although both $F(x)$ and n are unknown the asymptotic theory of extreme values which deals with the form of $G_n(x)$ as $n \to \infty$ may provide some guidance in a search for plausible distributions of time to occurrence of cancer in a tissue.

There are three possible asymptotic distributions of $G_n(x)$ :

$$G(x) = 1 - \exp[- \exp[a(x - u)]] \quad a > 0, -\infty < x < \infty$$

$$G(x) = 1 - \exp\{- [(s - v)/(s - x)]^k\} \quad k > 0, \ x \le s, v < s.$$

$$G(x) = 1 - \exp [-b(x - w)^k ] \ k > 0, \ x \ge w, b > 0. \text{ (weibull distribution).}$$

The second asymptotic distribution is not applicable to this situation. For the conditions required for either the first or third asymptotic distributions to hold $F(x)$ must be extremely smooth over the range of interest and $F(\infty)$ must equal 1.

One may hope to have most success in applying these distributions to experiments in which the carcinogenic insult to the experimental animal was applied continuously (or regularly) until the appearance of a tumor or death, and to have some success in experiments in which the insult was delivered for only a short time stopping before the first possible time of tumour appearance. Experiments in which the insult is delivered over a limited period would, however, lead to a sharp change in $F(x)$ at (or shortly after) the time the insult was stopped, and the asymptotic distributions would, therefore, not be expected to apply. The theoretical derivation of these two distributions also shows that one may more legitimately hope for success when the experiments are conducted with inbred strain of animals of a single sex and where the experimental conditions of insult are finely controlled. The author has chosen the third asymptotic distribution to the analysis of certain experiments. He chose this distribution because human mortality data on certain cancers show just the age specific mortality pattern that the third asymptotic distribution has, that is mortality rates rising as a power of age, and because certain animal experiments have been shown to behave in this way too. The parameters have a relatively clear cut interpretation with k and w being independent of the carcinogenic insult which is measured by the parameter b

Consider experiments of the type in which a comparison is made between a number of groups of mice, where the groups are distinguished though being subject to different carcinogenic insults (e.g. different intensities of carcinogen or different sized areas of tissue exposed).

The individual mice fall into two categories depending on their manner of 'death'.

Category 1: those mice showing the accepted criteria for diagnosis of carcinoma (in the relevant tissue) before they die (or removed from the experiment for any other reason).

Category 2: those mice not in Category 1

The time (after the start of the experiment) x at which the carcinoma was diagnosed is recorded for each mouse in category 1 and time y of removal from the experiment (by death or otherwise) is recorded for each mouse in category 2.

So that writing the general cdf as

$G(x \mid k, w, b) = 1 - \exp[- b(x - w)^k]$

with associated pdf

$g(x \mid k, w, b) = bk(x - w)^{k-1} \exp[- b(x - w)^k]$

The contribution to the likelihood function from a mouse in category 1 diagnosed at time x is given by $g(x \mid k, w, b)$, while the contribution of a mouse in category 2 'dying' at time y is

$1 - G(x \mid k, w, b)$ .

The general experiment may be described as one in which r groups of mice are insulted with different carcinogenic stimuli. In general the cdf for the i-th group may be written G(x | k, w, $b_i$) so that the contribution to the likelihood function from this group is

$$\prod_{j=1}^{s_i}\left\{b_i k\left(x_{ij}-w\right)^{k-1}\exp\left[-b_i\left(x_{ij}-w\right)^k\right]\right\}\cdot\prod_{j=1}^{t_i}\left\{\exp\left[-b_i\left(y_{ij}-w\right)^k\right]\right\} \qquad (5.1)$$

where in this group $s_i$ is the number of mice in category 1 and $t_i$ the number in category 2, $x_{ij}$ is the time of diagnosis of carcinoma for the j-th mouse in category 1 and $y_{ij}$ the time of 'death' of the j-th mouse in category 2. Let $L_i$ be the natural logarithm of 95.1) so that the complete likelihood function L(k, w, $b_1$, $b_2$, ..., $b_i$) = L(k, w, **b**) may be written

L(k, w, **b**) =

$$\sum_{i=1}^{r} L_i = \sum_{i=1}^{r} s_i \ln b_i + (\ln k)\sum_{i=1}^{r}\sum_{j=1}^{s_i}\ln(x_{ij}-w) - \sum_{i=1}^{r} b_i\left[\sum_{j=1}^{s_i}\left(x_{ij}-w\right)^k + \sum_{j=1}^{t_i}\left(y_{ij}-w\right)^k\right]$$

We will deal only with the case where k, w are known. Then,

$$\frac{\partial L(k,w,b)}{\partial b_i} = \frac{s_i}{b_i} - \sum_{j=1}^{s_i}(x_{ij}w)^k - \sum_{j=1}^{t_i}(y_{ij}-w)^k$$

so that,

$$\hat{b}_i = \frac{s_i}{\sum_{j=1}^{s_i}(x_{ij}-w)^k - \sum_{j=1}^{t_i}(y_{ij}-w)^k}$$

$\hat{b}_i$ has the approximate variance $\frac{b_i^{\,2}}{s_i}$ and is uncorrelated with $\hat{b}_j$ (i≠j)..

The most common problem faced in carcinogenecis experiments for which the above model might be appropriate involves testing the 'goodness of fit' and then testing whether two or more $b_i$'s are equal, that is whether their treatments have equal effect.

'Goodness of fit' may be tested graphically by plotting the observed proportion survivors against their predicted values. A chi-squared 'goodness of fit' test may also be made. If, however, the number of groups of animals is large, this test will be difficult to interpret since the number of degrees of freedom will be determined only within a wide range of values, unless the parameters themselves are calculated using the data grouped as for the chi-squared test itself.

Tests of special assumptions about the parameters can, of course, be made by standard likelihood ratio method.

This model (Pike's model) describes adequately the observed age distribution of many human cancers (Doll(1971)), of infiltrating skin cancers in experimental mice (Peto et.al(1972)) and of all skin tumors in experimental mice (Lee et.al (1971)). Moreover, the model has been successfully applied to experiments with rats in which an intrapleural inoculation of asbestos resulted in tumors, mesothliomas, of the pleura (Berry,et.al(1969) and Wagner et.al (1973)).

In Pike's model when the values of k and w are known, maximum likelihood (ML) estimation of, comparison of, and significance levels for the various constants of proportionality are straightforward. Peto (1973) suggested that multiple regression models by ML can be fitted, in which the log of the constant of proportionality is a linear combination (with unknown coefficients) of one or more explanatory variables.

In standard multiple regression techniques to compare two hypothesis in one of which the regressor variables from a subset of the regressor variables in the other, one tests differences in sum of squares for the two ML solutions. To compare two such hypotheses with Weibull methods, one compares the two maximum likelihood's, taking twice the difference as being distributed approximately as chi-squared with degrees of freedom equal to the difference in the number of regressor variables.

*A numeric example*

Pike's model will be fitted to a two group experiment of Glüsksmann and Cherry (of Stangeways Laboratories Cambridge) on vaginal cancer in female rats insulted with the carcinogen DMBA.

The two groups were distinguished by pretreatment regime. The data are as follows (x's and y's given in days after start of treatment with DMBA):

$s_1 = 17$, $t_1 = 2$

$x_{1i} =$ 143, 164, 188, 188, 190, 192, 206, 209, 213, 216, 220, 227, 230, 234, 246, 265, 304

$y_{1i} =$ 216, 244

$s_2 = 19$, $t_2 = 2$

$x_{2i} =$ 142, 156, 163, 198, 205, 232, 232, 233 ,233, 233, 233, 240, 261, 280, 296, 296, 323

$y_{2i} =$ 204, 344.

With w = 100 and k = 3 assumed known we find

$\hat{b}_1 = 4.51\times10^{-7}$ and $\hat{b}_2 = 2.38\times10^{-7}$

with respective approximate variances $1.19\times10^{-14}$ $2.99\times10^{-15}$

The log-likelihood function equals to −191.96.

With w = 100 and k = 3 assumed known and $b_1 = b_2$ we find

$\hat{b} = 3.07\times10^{-7}$

with log-likelihood function equal to −218.63

The test for $b_1 = b_2$ is thus $2\times(-191.96+218.63)\sim X_1^{\,2} = 53.3$

Thus the two treatments have different effect due to the different pretreatment regimes.

## 5.3 Design of carcinogenesis experiments using the Weibull Distribution

Berry (1975) examined Pike's model from the viewpoint of experimental design. Two design problems are discussed in this paper, the first concerns whether a time has been reached when it is optimum to terminate an experiment by sacrificing all surviving animals. This question is concerned in terms of return per unit cost, and it is shown that the optimum strategy is to allow all animals to live out their lives. The second example is the design of an experiment to test if the carcinogenic effect of asbestos is related to the age of injection, by allocating animals to two groups, one to be injected immediately and the other when older. Under certain assumptions it is possible to compute the optimum delay period before the second group is injected and the optimum distributions of animals between the two groups.

The approach based on the Weibull distribution was introduced for the case where the dose is applied continuously and Peto & Lee (1973) considered it unlikely to be relevant where a single application is given, and observations are continued for a long time thereafter. Superficially, experiments involving a single injection of asbestos into the pleural cavity belong to the later category, but nevertheless the author used an analysis based on the Weibull distribution for two reasons. First, asbestos is not easily destructible and remains in the animal for some considerable time after injection, so that the animal is exposed for more than a sort period of time. Secondly, as already have been mentioned the Weibull distribution fits the data (Berry and Wagner, 1969) and this overweighs any theoretical considerations. This approach is only applicable to tumours, which either result in death, e.g. a mesothelioma, or are observable in life, e.g. a skin tumour, and not to tumours which are only observed incidentally after death from some other cause.

Suppose that animals are continuously exposed to a carcinogen and as a result are at risk of developing a specific type of tumour. This tumour may occur during an animal's life or the animal may die, or be killed, before the tumour has occurred. For the sake of brevity the specific type of tumour will be referred to as simply a tumour, and the occurrence of death without the specific tumour as due to natural causes. The risks of a tumour developing or death occurring due to natural causes are assumed to be independent, and at time t days after first exposure the tumour incidence rate in surviving animals is $ck(t-w)^{k-1}$ for $t \geq w$, and zero for $t < w$.

This formulation leads to the time of tumour occurrence having a Weibull distribution but, as demonstrated by Peto and Lee (1973), the model is overparametrized since the estimates of k and w are highly interdependent. Fortunately k and w may be dependent only on the general type of carcinogen and tumour, and be independent of dose, and if previous data are available may be assumed known. Attention will be restricted to this case and the tumour incidence rate rewritten as $c\gamma(t)$. Let f(t) be the probability density of the times of tumour occurrence in the absence of mortality from other causes, and let F(t) be the corresponding comulative probability distribution. Then

$c\gamma(t) = f(t) / \{1-F(t)\}$,

and by integration

$1 - F(t) = \exp\{-cC(t)\}$,

where

$$C(t) = \int_0^t \gamma(u)du.$$

Suppose that in a group of n animals, m developed the tumour and the remainder died of natural causes, thus providing right-censored information on the time of tumour occurrence. Then the log likelihood L is given by

$L = m \log c + \Sigma' \log\{\gamma(t)\} - c\Sigma\, C(t)$,

Where $\Sigma'$ denotes summation over animals with the tumour and $\Sigma$ over all animals. Hence the maximum likelihood estimate of c is $\hat{c} = m / \sum C(t)$ with asymptotic variance $c^2 / E(m)$, where $E(.)$ denotes expected value. Therefore mortality from natural causes reduces the accuracy of the experiment by decreasing $E(m)$. For the purpose of experimental design it is necessary to know something about the natural death rate so that experiments may be designed to give a required accuracy. Suppose that the natural death rate is $\gamma'(t)$ which is such as to lead a Gomprtz distribution for the times of deaths in the absence of exposure to carcinogen. That is, $\gamma'(t) = \exp(\alpha + bt)$, where $\alpha$ and b are parameters. Then the total death rate is $c\gamma(t)+\gamma'(t)$. We then have that the expected proportion of animals surviving to t, S(t), is given by

S(t) = exp{-cC(t) – C′(t)},

Where C′(t) is defined similarly to C(t). Thus there is an explicit expression for the expected number of animals surviving at any time, but the number expected to develop the tumour by t, m(t), may only be obtained by numerical integration,

$$E\{m(t)\} = n\int_0^t S(u)c\gamma(u)du. \tag{5.2}$$

## 5.3.1 Choice of Duration

In an experiment of the type we are considering suppose that, instead of allowing each animal to live out its life, the experiment is terminated at a predetermined time T by killing all survivors (this is a common practice). A tumour in an early stage may be discovered in an animal so sacrificed, but it is not clear how to take account of this in the analysis, and Lee & Peto (1970), who terminated an experiment with mice after 18 months because of an epidemic, decided to ignore it. Therefore sacrificed animals provide right-censored information as if they died of natural causes at the time.

We consider whether a time is reached when it is optimum in terms of return per unit cost to terminate the experiment. The cost of carrying out an experiment will be taken as having two components, a fixed overhead per animal and running expenses. The former consists of the cost of obtaining the animal, applying the treatment, carrying out the post mortem, etc. the running expenses consist of the cost of food, space in the animal house, cleaning out, etc., and are taken as a daily rate per animal. The cost function is probably a reasonable assumption for an animal house in which there is a continual input of new experiments, as space becomes available, although one detail makes it not quite appropriate to our situation. That is, that our rats are caged in fours and therefore the average time a cage is in use is not proportional to the mean survival time, but to the mean of the maximum of four survival times. However, the effect of this may be shown to be negligible.

The return will be defined in terms of the accuracy with which the tumour rate is estimated, i.e. as Fisher's information or the reciprocal of the variance of $\hat{c}$, which is proportional to E{m(T)}. This may be criticized since it is assumed that information is of equal value independently of when it becomes available. It might be considered more realistic to place more value on information, which is available early.

Without loss of generality let the running expenses be one unit per animal per day and the overhead cost be X units per animal. Then the total expected cost is n{X+s(T)}, where s(T) is the expected survival time in days when the experiment is terminated at time T. the cost per unit of information, K(T), is given by,

$$K(T) = n\{X + s(T)\}/E\{m(T)\} = \tag{5.3}$$

$$= \left\{ X + \int_0^T S(u)du \right\} \Bigg/ \int_0^T c\gamma(u)S(u)du \tag{5.4}$$

If $\gamma$ (u) is monotonically increasing in u then it may be seen from the above, or alternatively may be provided by differentiation, that K(T) has no true minimum but declines monotonically in T. the condition that $\gamma$ (u) increases with u is certainly true for the Weibull case and is likely to be so in all practical situations with a homogeneous group of animals. Therefore K(T) is least when all animals are allowed to live out their life.

If the efficiency of termination at T, relative to allowing all animals to live out their life, is denoted by R(T) then

$$R(T) = K(\infty)/K(T) \tag{5.5}$$

It has been shown above that R(T) approaches one asymptotically with increasing T.

In some circumstances it would be useful to know how quickly the asymptote is approached. This can be obtained by evaluating (5.4) for a known natural death rate and a specified tumour rate. It is also necessary to specify X but limits can be given for R(T) without doing so since from (5.3) and (5.5)

$$\frac{1}{R(T)} = \frac{X}{X+s(\infty)}\left[\frac{E\{m(\infty)\}}{E\{m(T)\}}\right] + \frac{s(\infty)}{X+s(\infty)}\left[\frac{E\{m(\infty)\}}{E\{m(T)\}}\frac{s(T)}{s(\infty)}\right]$$
$$= \alpha/R(T \mid X=\infty)+(1-\alpha)/R(T \mid X=0),$$

where $\alpha = X|\{X+s(\infty)\}$ and therefore $0 \le \alpha \le 1$. Thus

$R(T|X = 0) \ge R(T) \ge R(T|X = \infty)$.

If we suppose that X is known and the other parameters have been given values estimated from experimental results. Then we can plot the relative efficiency against time and against the proportion of animals expected to have died.

The author made these plots for 4 different values of c chosen to give expected percentages of 10, 30, 50 and 70 developing the tumour, if the experiment was terminated prematurely. He observed that the variation is wide at the first plot but it is much reduced at the second. And also that if an experiment is terminated prematurely then the loss in efficiency is expected to be a little less than the proportion of animals alive at that time.

## 5.3.2 Age at Exposure

If individuals are exposed to a carcinogen then an unresolved question is whether the age at first exposure has any effect on the subsequent tumour incidence rate. To investigate this using asbestos injection in rats, the experimental strategy is to divide a group of young rats into two groups. Group 1 is injected with asbestos immediately and group 2 after a delay. There are three decisions to be made in specifying the design. First, what size of experiment is necessary? Secondly, what should be the relative size of the two groups? Thirdly, what should be the length of the delay period? The longer the delay period the longer the effect to be measured, but the greater the dilution of group2 though natural mortality.

It is reasonable to plan on the basis that any effect of the delay in injection is on the parameter c of the Weibull distribution, i.e. at any given time after injection of the respective groups, the tumour incident rate in group 2 is λ that of group 1. It is required to make inferences on λ and in particular to test whether $\lambda = 1$.

Then the log likelihood is given by

$L = (m_1 + m_2)\log c + m_2 \log\lambda - c\Sigma_1 C(t) - \lambda c \Sigma_2 C(t)$

where Σi indicates summation over group 1 and t is measured from the time of injection of the group concerned. The maximum likelihood estimates are,

$$\hat{c} = \frac{m_1 + m_2}{\Sigma_1 C(t) + \hat{\lambda}\Sigma_2 C(t)}$$

$$\hat{\lambda} = \frac{m_2 \Sigma_1 C(t)}{m_1 \Sigma_2 C(t)}$$

Evaluation of the log likelihood with both c and λ estimated by maximum likelihood and also with λ fixed as 1, and taking twice the difference, gives a test statistic, $z^2$, for testing the hypothesis that $\lambda = 1$,

$$z^2 = 2m2\log\hat{\lambda} - 2(m_1 + m_2)\log\left\{\frac{\hat{\lambda}(m_1 + m_2)}{\hat{\lambda} m_1 + m_2}\right\} \qquad (5.6)$$

which asymptotically should be distributed as chi – squared with one degree of freedom.

To proceed further it is necessary to specify values for α, b, c, k and w. Reasonable values $\alpha = -11.9633$, $b = 0.008627$, $c = 1.37 \times 10^{-3}$, $k = 3$ and $w = 350$, are estimated from

an earlier experiment injecting the same material into young rats. For any λ the expected values of $m_1$ and $m_2$ may be computed from (5.2) and insertion of these for $m_1$ and $m_2$ and λ for $\hat{\lambda}$ in (5.6) yields an expected value of $z^2$.

Now if $\lambda \neq 1$, then λ would be dependent on the delay, d. in the absence of any information on the form of relationship the author postulate $\lambda = 1 + \beta d$ and consider tow values of β, 1/350 and 2/350, that is λ approximately equal to 2 or 3 with a delay of one year. Then z was evaluated for d = 150(50)350 and for an experiment of 100 animals with $n_1$ = 20(2)60 in group 1.

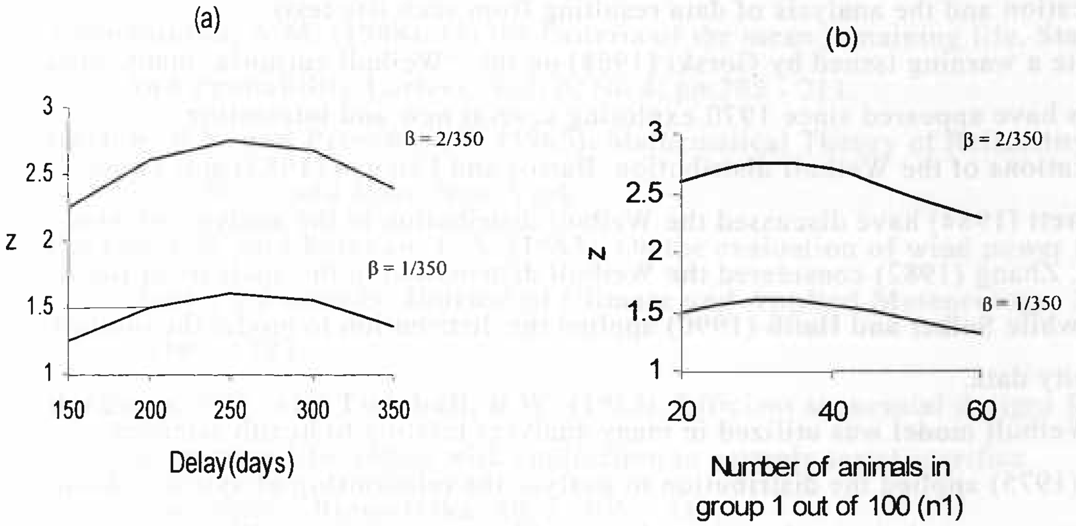


**Figure 5.1** 'Expected' value of test statistic z: (a) maximized with respect to $n_1$, plotted against delay, (b) for a delay of 250 days, plotted against $n_1$.

The highest values of z with respect to $n_1$ are plotted against d in figure 5.1 (a). for both values of β the optimum delay is between 250 and 300 days but, as the exact choice is not very critical, is taken as 250 days. In figure 5.1 (b), for a delay of 250 days, z is plotted against $n_1$, and the optimum value of $n_1$ is 34 for both values of β. The final decision is to choose the size of the experiment to give a 90% chance of detecting an age effect by a two- sided significance test for the case β = 2/350, that is λ = 2.43. This requires z = 3.24 compared with 2.73 in figure 5.1. Therefore a total of 141 animals are required.

## 5.4 Other Applications of the Weibull Distribution

As mentioned earlier in Chapter 3 the fact that the hazard rate of the Weibull distribution is a decreasing function when the shape parameter β is less than 1, a constant when β equals 1 (the exponential case), and an increasing function when β is greater than 1, has made this distribution highly useful as a lifetime model. Naturally numerous papers appeared dealing with this particular type of application and the analysis of data resulting from such life tests.

Despite a warning issued by Gorski (1968) on the 'Weibull eurhoria' many more papers have appeared since 1970 exploring several new and interesting applications of the Weibull distribution. Barros and Estevan (1983) and Tuller and Brett (1984) have discussed the Weibull distribution in the analysis of wind speed. Zhang (1982) considered the Weibull distribution in the analysis of flood data, while Selker and Haith (1990) applied the distribution to model the rainfall intensity data.

The Weibull model was utilized in many analyses relating to health sciences. Dyer (1975) applied the distribution to analyze the relationship of systolic blood pressure, serum cholesterol, and smoking to 14-year mortality in Chicago Gass Company, coronary and cardiovascular-renal mortality were also compared in two competing risk models in this study. Potier and Dinse (1987) made use of the Weibull distribution in their semiparametric analysis of tumor incidence rates in survival/ sacrifice experiments.

In addition to the above-mentioned applications, the Weibull distribution also found important uses in a variety of other problems. For example, Wong (1977) illustrated the uses of the distribution in analyzing hydrometeorogical data. The application of the Weibull distribution to the analysis of the reaction time data has been introduced by Ida (1980). Barry (1981) used the distribution as a human performance descriptor, Kanaroglou, Liaw and Papageorgiou (1986) applied it in the analysis of migratory systems.

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