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Portfolio optimization under uncertainty utilizing Stochastic Dominance

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of the necessary conditions
for obtaining the
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Abstract

We examine a different approach to the portfolio optimization problem. We employ stochastic dominance as our tool of optimization under uncertainty. Before proceeding to describe the model, its application and our results we include some chapters guiding the reader through concepts that should be known and understood to comprehend the work we are attempting and the reasons behind it.

We employ a first and second degree of stochastic dominance efficiency test on our data, to optimize our portfolio consisted of thirty shares listed on S&P500 and we use the index as our benchmark. We provide proof of concept for our portfolio outperforming the index.

We will apply five portfolio performance measures to appraise our results and reach a conclusion regarding the viability and profitability of a stochastically optimized portfolio.

As we will demonstrate in the closing chapter of this thesis, such a portfolio exists and truly achieves, as an out-of-sample test, showed higher than market returns most of the time.

Keywords: Stochastic dominance, portfolio optimization, portfolio management, risk management

Introduction

Investors and financial analysts are troubled by the market instability and the extreme fluctuations that we observed in the latter years. As the financial instruments and their management get more complex day by day, it is essential to adjust the models that we use in financial decision making.

Selecting the proper portfolio is a problem that tantalizes the economic science. The rapid advancements in information sciences allow us to use multi-criteria decision analysis models and specifically stochastic dominance of first and second order in our efforts to create the optimum portfolio.

In the process of selecting the optimal portfolio, the investor has to choose the assets with the highest yield and distribute his disposable income in a way to achieve the best result.

In this thesis, we have attempted to break down this problem in smaller, easier to digest parts and provide the reader with all the basic knowledge that will be needed to understand the work that we have attempted. We sought to present the various concepts, the model that we have applied and our conclusions in a language that would allow this document to act as material for the whole scientific community, including students who may want to access material on related topics.

On the first two chapters of the thesis we present the basic notion that should be known before any attempt of proceeding and reading the actual research is made, we will simplify and present the concepts of uncertainty, risk, investing and portfolio management.

The third chapter is dedicated to utility theory and modern portfolio theory, two concepts that we will require to grasp the meaning of the tool that we will be using to optimize our portfolio and understand the perspective from which we approach the portfolio selection problem.

Chapter four presents the theoretical background behind our tool in both degrees that we will be using and includes a presentation of the theory behind the third order of stochastic dominance, which we will not employ in our thesis.

Finally, chapter five and six contain information on the dataset that we used for our research, the model that we employed as well results and our comments on the said outcomes.

At the end of the thesis, the reader will find a table of references where all the scientific papers that have been used or mentioned in this text are displayed and a table of bibliography where all the textbooks that have been used are outlined.

This text concludes with an appendix that presents tables and diagrams of results and data that are mentioned throughout this thesis and are referenced to it.

Chapter 1

Understanding and measuring risk

Through this chapter, we will try to provide the reader with some insight on the basic concepts that govern the financial science. We will define risk, explain the notions of uncertainty, investments under uncertainty and finally provide some metrics of risk.

1.1 Definition of risk

The concept of risk is intrinsically linked with the financial theory. In our efforts to define risk and as we go through the economic and general literature the reader will soon come to the conclusion that there is not a universal definition of risk. While there are many definitions some simple and some more descriptive the most common are:

"possibility of loss or injury."

"someone or something that creates or suggests a hazard."

"the chance of loss or the perils to the subject matter of an insurance contract."

"a person or thing that is a specified hazard to an insurer."

"an insurance hazard from a specified cause or source."

"the chance that an investment (as a stock or commodity) will lose value."

Taking all these into consideration we will try to define risk, as this concept is essential for this thesis, through the definition of riskless and risky position on an asset.

We hold a riskless position on an asset when the economic outcome of the investment is known and will be realized with certainty. A typical example in the bibliography is the Treasury bill notes issued by the US Treasury Department to finance the US Federal Government debt.

An investor willing to pay today 987.65$ to acquire a 3-month Treasury bill with a face value of 1000$ holds a riskless position where he *knows* that in 3 months' time his invested income of 987.65$ will grow with *certainty* by 5% annual rate to 1000$. The certainty derives from the fact that the US Federal Government cannot go bankrupt as it can always raise more taxes or even print new money to pay its obligations.

It is easy to define formally the riskless position as the situation where the investor expects his investment of *I* currency units to grow to *X* currency *units* with a *probability* of X being realized p(x) = 1.

Having set the baseline for the riskless position we can now define the risky position as the situation where there are numerous economic outcomes for investing *I* currency units to asset A:

$X = \{X_1, X_2, X_3, \ldots\ldots, X_n\}$

with probability of X being realized $P(X_i) = \{ P(X_1), P(X_2), P(X_3), \ldots\ldots, P(X_n)\}$ and $\sum P(Xi) = 1$

A risky position is a situation where the economic outcome is not known with *absolute* certainty but only an estimation about *expected* outcome can be derived: $E_{\Pi(X)} = P(X_1) * X_1 + P(X_2) * X_2 + \ldots\ldots + P(X_n) * X_n.$

The value of $E_{\Pi(X)}$ relative to invested income and the *expected* rate of return is simply an estimation based on the available data and cannot exclude extreme fluctuations to investor's payoff.

A significant distinguishment was made by Frank Knight between *risk* and *uncertainty*; this will be thoroughly described in the following section.

Having described the basic differences between the riskless and risky position; with both being derived from the fact that the investor may hold a riskless or risky asset we can assume that risk in financial analysis that we are going to attempt through this thesis is defined as a situation where our future payoffs are determined through a random known process of which we are aware with certainty the probability under which each event is realized.

1.2 Uncertainty

In the previous section we tried to define the concept of risk in finance; now we will introduce the notion of uncertainty as it was first described by Frank Knight in 1921. The differentiation between risk and uncertainty is quite essential as it will lay the path for us to work towards our goal of portfolio optimization through stochastic dominance.

Decision making in finance is decision making under uncertainty: the outcome of today's decision depends on quantities (like future asset prices, interest rates or exchange rates), which are not known yet. The usual approach to deal with this uncertainty is to represent these quantities by a stochastic model. As a consequence, the outcome of the decision (e.g. the future wealth) is a random variable.

Stochasticity of the objective adds a new dimension to the decision making process: Whereas deterministic problems are characterized by costs, returns or wealth as real numbers, these quantities are random distributions in stochastic problems. It is the possible random variability which adds the risk dimension to the problem.

We've previously mentioned that future payoffs of a risky investment are defined through a random known process. Trying to apply this model in real world we come to realize that when talking, for instance about the performance of Apple's share in tomorrow's stock market trade we cannot state that there will be

an increase of 2% with a probability of 25%

an increase of 5% with a probability of 30%

a decrease of 3% with a probability of 30%

no change in stock's price with a probability of 15%

with *absolute certainty*. The fact that we do not know these probabilities of each event being realized but we can only estimate them through various econometric models is what introduces the concept of uncertainty in our analysis about risk. The probabilities under which each economic result may be realized depend on the conditions prevailing economy, political environment, the phase of the business cycle, monetary policy and many other factors.

Although, the main point where all definitions agree is that risk is defined in terms of changes in values between two dates, some papers argue that because risk is related to variability of the future value of a position, due to uncertain events, it is better to consider future values only. The principle "bygones are bygones" leads us to this future wealth approach.

Generalizing this, we can say that the definition of risk presented in section 1.1, once we include the notion of uncertainty will now state that risk under uncertainty is a situation where our future expected payoffs are defined through a random known process for which we can only estimate through observable historical data the probability under which each event is realized.

Those only lead us to the conclusion that:

1: Decision makers crudely operate in a world of random uncertainty

2: Risk is a condition in which the decision makers assign formal estimated mathematical probabilities to specify the uncertainty.

Understanding this two phrases is the key to explaining the need for a model that fits the conditions a decision maker operates in and if possible reduces uncertainty to a minimum; that is a role that stochastic dominance covers the best possible way.

1.3 Investing under uncertainty

Contemplating on the concepts outlined in the two previous sections we understand the need for some tools that would help an investor compare different investment opportunities presented with various characteristics and select the one which will provide him with the greatest certainty and expected payoffs in return for taking over the risk of the investment.

Many criteria have been proposed in this effort to assist investors; through this section, we will mention some of them and describe the weaknesses that created the need for a better decision analysis model.

The simplest criterion the reader could use is the *state-by-state dominance*, according to which a preferable investment is the one that can guarantee a better outcome in every possible situation than any other investment proposed. Although it is a logical metric, it is inefficient in the modern world where there are many investment opportunities, complex financial instruments, and the expected payoffs depend on many different variables many of which are unrelated to the investment plan itself.

Another criterion that has been suggested in literature is the *mean-variance* criterion under which a rational investor should choose the plan with the greatest mean and smallest variance. At this point, it is essential to note that in finance we consider the mean to be representative of the plan's expected payoff while the variance suggests the risk that is included in the investment under uncertainty. It would be rational to choose to invest our disposable income where we are promised greater payoffs with less risk; but what happens if the expected payoffs and the risk of a plan are higher than the fundamentals of another? That is a case where the mean-variance metric remains silent and can't guide the investor to a proper choice.

Finally, the last proposed criterion that tried to cope with all deficiencies mentioned earlier is the *Sharpe ratio*; it is an evolution of the mean-variance criterion, and it is simply defined as the ratio of the mean to the standard deviation for the proposed investment opportunity. Quite elegant in its simplicity, Sharpe's ratio provides the investor with the average of expected payoff per unit of standard deviation; providing the decision maker with the information of what is his average expected payoff for each unit of risk he is taking over by selecting to invest in that plan. Unfortunately, it still is not able to classify two investments if the one has greater standard deviation and mean than the other. An investment like the latter may be better than the former if for example the payoff distribution is asymmetrical towards the right tail of the normal distribution.

The other disadvantage that all above metrics share derives from the fact that, economic theory wise, they do not take into account neither investor's preferences concerning the trade between risk and payoff nor the asymmetry of the payoff distribution. Investors preferences determine the trade above regarding a potential income loss.

Through this last paragraph we outlined the need to include *utility theory* in our investment analysis, and as we will try to demonstrate in the following chapters, stochastic dominance will analyze investor preferences as will be described through utility theory and help us select under those restrictions the optimal portfolio.

1.4 Measures of risk – weaknesses

The driving force behind the effort to measure and quantify risk is the need to contain the losses that our invested income may sustain; while at the same time trying to rate the investments from the one that incurs the highest risk to the "safe heaven" of the riskless investments. A number of different metrics have been proposed in literature with a sole target to quantify the incurred risk and distinguish between investments. In this section, only eight of those metrics will be presented accompanied by their deficiencies.

1.4.1 Domar and Musgrave risk indexes

The first metric that we will present is the Domar and Musgrave Risk Index. According to their paper of 1944, they defined risk as the additive inverse of the sum of all the negative or relatively low possible payoffs of investment. The mathematical representation of this is:

$$\text{RI} = -\sum_{x_i \leq 0} p_i x_i$$

Since only the negative possible outcomes are taken into account; the RI is expected to be a positive number; thus the higher the RI, the riskier the investment.

The mathematical description presented above can be used only in an investment where the possible payoffs, represented in this formula with x_i variable, are discrete numbers. As we already know, though, the possible outcomes of an investment tend to be infinite, and we usually use the normal distribution as a proxy in the analysis. In these cases, the formula is written as:

$$\text{RI} = -\int_{-\infty}^{0} f(x)x\,dx$$

Last, but not least Domar and Musgrave understood that the rational risk-averse decision maker will feel that he invested poorly if his choice is not providing, at least, the risk-free interest rate. With that in mind they amended their risk index suggesting the following mathematical version of the RI:

$$\text{RI} = -\sum_{x_i \le r} p_i x_i (x_i - r)$$

With the same modification taking force to the continuous variables as well:

$$\text{RI} = -\int_{-\infty}^{r} f(x)(x-r)\,dx$$

Domar and Musgrave, a measure of risk may be very appealing. However they do contain a certain amount of disadvantages; the main being that they do not take into account the differential damage of the various negative monetary returns.

1.4.2 Roy's Safety First Rule

In his paper of 1952, A. D. Roy suggests that the primary goal of all investors is to ensure that they will not find themselves in a nightmare or "disaster" scenario. Based on this, he suggested that the risk as perceived by a decision maker is the probability of his future income being lower than a “disaster” level of d. The formulation of that is:

$$\mathrm{RI} = p(x \le d)$$

Note that Roy’s risk index does not take into account the size of the actual loss an investor may sustain but rather the probability of such a loss. The main disadvantage of this metric is that since each investor determines the d, it lacks the credibility to rank investments objectively.

1.4.3 Variance & Standard Deviation

In the previous section, we’ve mentioned that in finance we consider the standard deviation and the variance to be representative indexes of the risk that are incurred in investment. We will now, provide more information on this notion. Since as we stated earlier risk occurs when there are more than one possible outcomes and in an investment the number of potential payoffs tends to be infinite, it would be a reasonable act to measure it using one of the common dispersion metrics. Risk measured with variance and standard deviation would be for a discrete distribution:

$$\sigma_x^2 = \sum P(x_i)(x_i - \mathrm{E}x)^2 \text{ , with } Ex = \mu$$

moreover, for a continuous distribution:

$$\sigma_x^2 = \int f(x)(x - \mathrm{E}x)^2 dx \text{, with } Ex = \mu$$

Having calculated the variance of a distribution one can easily extrapolate the standard deviation as it is known that the standard deviation is equal to the square root of the variance. Decision makers are interested in the investment’s profitability as best estimated by the expected value of the returns. The standard deviation indicates possible deviations of the realized returns from their expected value;

hence, a high standard deviation is intuitively identified with high risk. Because of its simplicity and intuitive grasp as a risk measure, this index of risk is widely accepted among professional investors as well as academics.

Despite it is a widely accepted and applied metric it has its drawbacks, the most import being that it takes deviations due to asymmetry towards both the right and left tail of the distribution into account. While a distribution with high left asymmetry means that extreme losses may be more frequently presented; a strong asymmetry to the right tail would be translate as a great chance to incur superprofits more frequently – a desirable trait in investments. To overcome this defect, the semi-variance index was introduced.

1.4.4 Semi-Variance

This index takes into account, only deviations to the left of certain critical values, meaning that it measures only the risk of extreme losses. The mean of the distribution tends to be selected as the critical value.

For discrete distributions, the semi-variance index is defined as:

$$\mathrm{SV} = \sum_{x_i \le A} P(x_i)(x_i - A)^2$$

while for continuous distributions:

$$\mathrm{SV} = \int_{-\infty}^{A} f(x)(x-A)^2 \, dx$$

where the A in both equations presented above is the critical value that has been selected. Again, as in the “Safety First” rule the main disadvantage of this method is that the selection of the critical value A is something left to each investor’s judgment; thus, making the ranking subjective according to each’s preferences.

1.4.5 Baumol's Risk Measure

William Baumol in 1963 published a paper in which he argues that risk be perceived and should be measured as the likelihood of earning less than some critical level or "floor." This notion reminds us of Roy's "safety first", as well as modified Domar and Musgrave risk indexes; all of the above agree that risk is interwoven with a minimum required return and as such it should be treated.

According to that, an investment with high standard deviation but sufficiently high expected value would be relatively safe. The mathematical representation of this concept is:

$$RI = E - k\sigma$$

Where k is some constant, selected by the investor, representing his safety requirement; that would mean that the higher the risk index as depicted above, the safer the investment. The main disadvantages of this metric are that is it subjective as each investor would select his own k and it does not take into account the probability that a return may fall below the "floor".

1.4.6 Value at Risk ($VaR_{(\alpha)}$)

The metric of Value at Risk is a popular measure of risk which has achieved the high status of being written into industry regulations; it indicates the maximum possible loss at α confidence level. The confidence level at VaR is used in the sense that the α left part of the distribution is ignored, and only the less frequent (but possibly more damaging) 1-α part is taken into account. In general, it is defined as:

$$VaR_{(\alpha)} = \mu - L$$

where μ is the mean of the distribution and L is the value such that $P(x \le L) = \alpha$. Here the risk is measured as the maximum deviation from the mean when the left tail of the distribution is ignored.

It suffers, however, from being unstable and difficult to work with numerically when losses are not "normally" distributed – which in fact is often the case, because loss distributions tend to exhibit "fat tails" or empirical discreteness. VaR in general turns out to be not even weakly coherent and in particular not subadditive.

Thus VaR, that was introduced in the attempt of measuring risk for weird distributions, can be used only when the computationally simpler variance can also be used. Indeed, VaR, if applied to most (not elliptical) return distributions is not an acceptable risk measure:

- it does not measure losses exceeding VaR;
- a reduction of VaR may lead to stretch the tail exceeding VaR;
- it may provide conflicting results at different confidence levels;
- non-sub-additivity implies that portfolio diversification may lead to an increase of risk and prevents to add up the VaR of different risk sources;
- non-convexity makes it impossible to use VaR in optimization problems;
- VaR has many local extremes leading to unstable VaR ranking.

Thus VaR is an inadequate risk measure.

1.4.7 Condition Value-at-Risk

The CVaR, which is coincides with the results of the expected shortfall in cases of continuous random variables, is an evolution of VaR as described above. An alternative measure that does quantify the losses that might be encountered in the tail. As a tool in optimization modeling, CVaR has superior properties in many respects. It maintains consistency with VaR by yielding the same results. The main difference is that CVaR focusses on the less frequently occurring but more damaging events of the left tail of the distribution, ignoring any possible positive outcomes. For continuous random variables, CVaR is the expected value of the losses exceeding VaR_k,

$$CVaR_k = VaR_k + E[f(x,y) - VaR_k \mid f(x,y) > VaR_k]$$

1.4.8 Minmax regret

This index was proposed by Leonard Savage in 1951. The main thrust of this rule is that investors should choose the investment that offers the minimum risk of possible losses due to a wrong choice; hence, the regret measures danger of making a wrong investment choice.

The minimax regret criterion is as follows: the investor calculates the maximum possible regret for each stock and the stock with the minimum of these maximum regrets should be chosen. The stock with the minimax regret is the one with the lowest risk.

Although the notion of alternative costs is intuitively very appealing, this measure of risk has two major drawbacks. First, adding one more stock may change the relative risk of the stock itself even if the additional stock is irrelevant because it is not chosen. The second major drawback of the minimax regret is that the regret function measures risk due to the wrong choice but it does not take into account the likelihood of the different states of nature and, therefore, it does not fully gauge the risk of each stock.

Chapter 2

Portfolio composition and management

Having understood the notions of risk and uncertainty, we made the first step in the long process of portfolio optimization. In this chapter we will continue to provide the reader with the basic notions that are necessary to know and understand before we progress any further. We will explain the meaning of portfolio, demonstrate the process a portfolio manager follows and depict the different strategies that he has to choose from.

2.1 The concept of portfolio

In this thesis, we will widely mention the term portfolio. Driven by the methodology of the research that we will present it is deemed necessary to inform the reader on the meaning and importance of the term above.

The concept of the portfolio is defined as the sum of an investor's assets. These assets may include tradable securities, commodities, cars, houses, art, antiques, furniture and more.

The main purpose of holding a portfolio of assets is to invest a person's disposable income aiming to maximize expected a return and minimize the risk that is intrinsically linked with this return. Each portfolio's composition is different and affected by a series of various factors such as investor's risk appetite, the time frame of the investment and individual objectives and constraints. Portfolios may be managed by amateur investors as well as professional financial analysts.

Every human being gets daily in the process of selecting a portfolio. We all acquire good and services, place a part of our income to current or savings accounts, we pay a pensions fund contribution and make other financial choices. In essence, we distribute our income in the best possible way managing our *portfolio* as efficiently as possible.

Of course, a portfolio does not remain static over time. An investment that today seems promising may be proved injurious over time for our collection of assets. Daily, numerous events take place that affects the global economy and causes it constantly to shift. These lead us to the conclusions that it is essential to monitor constantly every portfolio and rebalance it, to achieve client's investment objective.

In our analysis, we will focus on one of the riskier portfolios an investor may hold, the stock-based investment portfolio. Our effort in the following sections will be to provide the reader with the fundamentals of portfolio management before progressing to modern portfolio theory.

2.2 Portfolio Management

Having understood the principles that govern an investment portfolio it is conspicuous why there is the need for help from a professional financial analyst in its creation and management.

Through the eyes of the professional, portfolio management can be described as a process. It is an integrated set of activities that combine in logical, orderly manner to produce the desired product. Like any other business process, the three elements that rule portfolio management are:

- Planning phase

- Execution phase
- Feedback phase

Which may be as loose or as disciplined their operators desire. In this section we will visit and provide some basic insight, in each step of the portfolio management process.

2.2.1 Planning phase

This is the first step in investment planning. The goal here is to identify and specify accurately investment objectives and constraints based on which the analyst develops investment strategies. Objectives of investment are the desired investor outcomes; usually expressed as risk and returns. Constraints of the investment may be either internal or external, where internal have to do with the investor himself while external are tax, legal and regulatory requirements.

Using the information that portfolio manager has acquired regarding the investor, he proceeds with choosing an investment strategy that matches clients profile. There is a wide selection of possible investment strategies in the quivers of the manager which we will demonstrate thoroughly in the following section.

Once a strategy has been selected the manager stars forming capital market expectation in micro as well as in macro level. This is made possible through various econometric, industry and market analysis models that provide forecasts on the expected performance of an economy, a sector, industry or even a transboundary market.

Finally, at the fourth step of the planning phase, the investment manager is ready to construct client's portfolio structure. At this step, the disposable income of the investor is apportioned to various financial instruments that manager has access to. Depending on whether the investor has a single-period or multi-period perspective, this is known as tactical or strategic asset allocation. The main difference between the two is that while the multiperiod perspective is more costly to implement, it addresses liquidity issues raised by the constant rebalancing of the single period perspective.

2.2.2 Execution phase

Reaching to this node in portfolio management process, we have already accumulated information regarding the capital market expectations and the strategies that will be used. Combining those brings us to the actual portfolio selection where the manager selects specific securities for the portfolio. It is evident that this is not a static, but rather a dynamic process where constant revision is needed based on analysts input, market circumstances or investors objectives shift; thus, the execution phase is in constant interaction with the feedback phase.

A significant part of a manager’s efforts at this phase is the portfolio optimization, as it bonds strategies with expectations; stochastic dominance would be our tool for achieving that.

This phase incurs a large part of implicit and explicit cost such as transaction fees, missed trade opportunities and delay costs since it is the step where all elaborate plans of the planning phase turn to reality.

2.2.3 Feedback phase

In every process, the feedback phase plays a vital part as it allows us to identify the underperforming part of said process and apply correcting actions to improve both process and outcome. Those two parts are known in the investment management process as monitoring and rebalancing, and performance evaluation.

The main reason that urges investors to trust portfolio managers to handle their disposable income is the hassle-free process of keeping their goals satisfied. Monitoring and rebalancing a portfolio is something that guarantees the manager that his client's objectives and constrains will continue to be satisfied. To keep those two in line we have to monitor investor-related factors as well as economic and market-based factors as both of them may affect portfolio compliance with investment policy. In any occasion where a condition has not met the manager has to rebalance the portfolio, changing the asset allocation the most efficient way to ensure that investor's profile will still be honored.

Any rebalancing will drive us back to the execution phase demonstrating the tight relationship between the two phases that we expected.

The last step of the feedback phase is the one that completes the portfolio management cycle. As in any investment, asset performance is periodically evaluated by the investor to ensure progress towards the investment objectives as well as portfolio management skills.

The usual practice in evaluating portfolio performance is to compare ours, with a benchmark with similar targets and constraints and determine if we outperformed most of the benchmarks or not.

The scientific process of properly evaluating the portfolio, thought, would be to measure performance and then attribute it to various factors besides management that may have led to this outcome; thus leaving the heaping payoff to be attributed to management. Once we determine the part of the return that we attribute to the manager we can further progress our analysis by examining if this is an outcome generated from the strategic asset allocation made in the planning phase, tactical decision made by the manager during the monitoring and rebalancing step known as market timing or selecting rising star securities in first place. All these factors provide both manager and investor with feedback regarding portfolio management efficiency.

Following this analysis of investment management process and the different factors that affect every phase of it, the reader understands the complexity of decision analysis that we should implement to achieve the best possible outcome.

2.3 Portfolio management strategies

As discussed earlier a crucial part of the portfolio management process, determined at the planning phases is the strategy that the manager will follow. The extent to which the strategy affects the portfolio goes all the way to characterizing the portfolio and even the manager for supporting it.

The prevailing strategies in investment management can be placed into two broad categories, the:

- Passive portfolio management strategies
- Active portfolio management strategies

A middle ground between those two may be found, and a hybrid management strategy may arise. The hybrid strategy through does not constitute a category of its own as it basically resembles one of the two broad categories with some aspects of the other being used depending on the manager. The main difference between the two lies on the assumption investors make on market efficiency. The most common way to distinguish between these strategies is by decomposing the returns the managers are trying to provide.

2.3.1 Passive management strategies

Portfolio managers that choose a passive investment strategy generally believe in market efficiency and set at their target the construction of a portfolio that tries to capture the market's expected return for the risk level their undertaking. The manager of such portfolios has a lot less freedom of choice and their role is limited in complying with certain well-defined criteria. A passively managed portfolio tries to combine the long-term investment horizon with little or no change in portfolio composition, features that will ensure the containment of transaction costs. The full spectrum of passive investment strategies can be separated into two sets of strategies. We will proceed by providing insight into each set.

2.3.1.1 Buy and hold strategies

In this set, we classify the strategies that typically are laid out by non-professional individual investors. The concept governing a buy and hold strategy is the efficient diversification of an investor's portfolio, usually consisted of shares and bonds, in a way that it will ensure long-term positive rates of return for the investor. There is not a specific technique that an investor willing to apply this strategy will use to decide on the assets he will hold. The selection is rather arbitrary under the subjective judgment of the individual with a sole scope the selection of assets that will allow the investment to follow the basic characteristics of a passively managed portfolio.

2.3.1.2 Index tracking strategies

This is the most representative set of passive management strategies. The target of those strategies is to mirror the return and risk of the selected index. As they are costlier to implement both on the number of securities required to purchase and on the analysis they required in their design phase they are mostly implemented by financial institutions and professional financial managers. We will mention and shortly explain the mechanics of three main techniques used in portfolio construction.

Firstly, we will visit the *full replication* technique. As its name suggests, this is simply achieved by purchasing all the securities that make up the index in proportion to their weights. While this will ensure a complete mirroring of index performance, it comes at a very high cost since there is a need to buy a large number of different securities skyrocketing transaction fees. The second drawback of this technique is the transaction costs related to dividends reinvestment, required to keep the portfolio balanced on the index.

The second technique that we will mention, which is known as *sampling*, was developed in order to tackle the problem of having to buy numerous shares. To achieve that, a sample of index representative shares is acquired. Instead of purchasing all the shares that make up the index, the manager acquires heavily weighted shares in proportion to their weights and less significant, index wise, shares are purchased in a way that their proportion and characteristics like beta, industry and dividend yield represent aggregately a large number of smaller index stocks. That ensures that larger positions may be taken, thus minimizing transaction costs as well as that the dividends are reinvested easily during the portfolio rebalance process. The main disadvantage of this is that it most certainly won't track index performance as efficiently as the replication technique.

The most modern technique is referred to literature as *quadratic optimization* or *programming*. In an effort to mirror the index as effectively as possible, while at the same time avoiding purchase all the shares making up the index computer analyzed mathematical models are used to determining a portfolio composition minimizing return deviations from the benchmark. As this technique, though, relies almost entirely on historical price changes and correlations it cannot ensure that, over significant time, changes will not drive the portfolio away from the index.

2.3.2 Active management strategies

Conversely to previously presented strategies, this category signifies investment managers' attempt to outperform the market. Every manager that identifies to this category firmly believes that in the real world the markets are not efficient, and there are opportunities for returns exceeding the market projected. This may be achieved by either identifying and selecting underrated shares or by tactical adjustments to the portfolio. The main advantage of these strategies being that they allow returns unrelated to the market index; resulting in, if implemented correctly, higher positive and less negative returns than their passive counterparts. On the other hand, the disadvantage of said strategies is that they incur both systematic and non-systematic risk.

2.3.2.1 Fundamental analysis strategies

This is the most subjective active management strategies set. In an effort to identify undervalued securities, the manager utilizes financial statement analysis, and sector analysis to estimate the intrinsic value of stock. Some managers will go even to the extent to analyze an economy to try to identify the driving forces behind animal spirits that govern, from time to time, a capital market. Once a promising security has been identified the manager will open a position to aforementioned security and monitor closely. A popular approach based on fundamental analysis is the pivot strategy, which aims to exploit certain circumstances that may rise in the market.

2.3.2.2 Technical analysis strategies

Opposite to fundamental analysis, technical analysis strategies heavily rely on econometric models based on historical data to analyze and identify early market trends. The portfolio is generated on the basis that either past stock price trends will continue, or they will reverse; always in accordance with the model analysis. Technical analysis models will even try to identify possible investor overreactions based on the known high correlation between the present and past share prices.

2.3.2.3 Anomalies and Attributes strategies

These strategies examine sectors and markets on a whole in an effort to either locate sectors or securities that display certain attributes such as high P/E ratio, or company book value that may signal an imminent share price increase. They, furthermore, try to identify and exploit market anomalies which may even allow riskless profit; in an effort to achieve that managers use models such as APT.

Having gone to this extent to grasp the implications and complexity of portfolio management we understand the need for proper portfolio selection to ensure minimum transaction costs and high expected returns for the assumed risk level.

Chapter 3

Investor utility and optimal portfolios

Continuing our introduction to the fundamental concepts related to our thesis we will now focus on the mathematical representation of an individual's preferences, as well as describe the different type of investor according to their risk preferences. Finally, we will present Modern Portfolio Theory, the theory that shaped the portfolio management sector.

3.1 Utility theory

Up to this point in our analysis, we've made clear that there are two main factors an investor takes into account when he decides upon a proposed investment: risk and return.

We will now present the building block in which our research will be based. This notion is no other than the expected utility criterion, a theory that takes into account the whole distribution of returns making nonessential to define risk separately, thus operating better under uncertainty.

The expected utility theory is a decision analysis tool and as such it should be treated. To grasp the mechanics behind the need for such a tool let us construct a simple game. There are two players 1 & 2, and they are presented with the following situation: there is one dollar on a table, and player 1 may choose to take it or leave it. If he chooses to take it our game end with the first player having one dollar and the second zero. If he chooses not to take it, the money quadruples and the second player is presented with the choice to either take four dollars or share them with the first player. In case his choice is to take them he has four dollars and the first player zero; in the sharing scenario both of them have two dollars.

The rational thing to do for the second player, in case he is presented with the chance, is to take the four dollars. We may say that he *prefers* to maximize his returns and have four dollars than two. This is consistent with payoffs numerical value where 4$ > 2$. However, would that be the case if player 1 and player 2 were related? For our argument let's assume that player 2 is mother and player 1 son. In that case, we probably think that she would choose to share the money with her son. In other words, she *prefers* to have two dollars and her son happy than four and a disappointed son. We now see that the numerical ordering we used earlier is not consistent with player 2 *preferences*, to solve that we have to order differently player's 2 preferences, and there is the part where utility comes to the rescue.

Instead of monetary outcomes we assign *utility units* which rank *preferences*, to both events and say that 2$ have a *utility* of 6 while 4$ a *utility* of 3; thus 6 > 3 and making our analysis consistent again.

Our effort now will be to expand the analysis above and present the expected utility theory scientifically as it was developed by von-Neumann and Morgenstern and is used in investment decision analysis.

The Maximum Expected Utility Criterion is based on six axioms that we will demonstrate bellow. If all six axioms hold true, then it has been proved that the expected utility criterion should be used to choose among alternative investments. Before we start, allow us to define some basic symbolism that we will use. We will use an example where the investor would be presented with two investment opportunities from which he will have to choose one. These are

$$L_1 = \{p_1A_1, p_2A_2, \ldots, p_nA_n\}$$

$$\text{and } L_2 = \{q_1A_1, q_2A_2, \ldots, q_nA_n\}$$

With A_i being the payoff of the investment and p_i or q_i the probability under which this payoff is realized; it is easily conceived that $\sum p_i = \sum q_i = 1$. We will, also, use the symbol $\succ$ which means "prefers" and $\sim$ which translates to "indifferent." We are now ready to proceed:

3.1.1 Axiom 1: Comparability

By this axiom, when our investor is presented with the investment choice dilemma he has to state his preference. Acceptable answers are $A_1 \succ A_2$, $A_2 \succ A_1$, $A_1 \sim A_2$. The answer "I do not know" is not acceptable under this axiom.

3.1.2 Axiom 2: Continuity

If the investor $A_3 \succ A_2 \succ A_1$, then there must be a probability $U(A_2)$ with $U(A_2) \in [0,1]$ such that our investor will be indifferent between receiving A_2 with certainty or investing in plan $L = \{(1- U(A_2))\ A_1, U(A_2)\ A_3\}$.

It is called continuity axiom because if you increase continuously $U(A_2)$ from 0 to 1, you will eventually hit the value $U(A_2)$ where $L \sim A_2$.

3.1.3 Axiom 3: Interchangeability

We are presented with two investment opportunities:

$$L_1 = \{p_1A_1, p_2A_2, p_3A_3\}$$

$$\text{and } L_2 = \{p_1A_1, p_2B, p_3A_3\}$$

Payoff B is an investment combining A_1 and A_3: B = $\{qA_1, (1-q)A_3\}$. Under interchangeability axiom, if we are indifferent between A_2 and B then we will be indifferent between L_1 *and* L_2.

3.1.4 Axiom 4: Transitivity

If it holds true that our investor $L_1 \succ L_2$ and $L_2 \succ L_3$ then under this axiom, we derive that $L_1 \succ L_3$. This holds true for indifference between investments as well.

3.1.5 Axiom 5: Decomposability

We have three investment plans, two *simple* and a *complex*. An investment plan is called simple when the possible outcomes are payoffs while complex where they are other investments. To be specific:

$$L_1 = \{p_1A_1, (1-p_1)A_2\}$$

$$L_2 = \{ p_2A_1, (1-p_2)A_2\}$$

$$L^* = \{q\, L_1, (1-q)\, L_2\}$$

Under this axiom, we may convert the complex investment into a simple by decomposing it, specifically:

$L^* \sim L = \{p^*A_1, (1-p^*)A_2\}$ where $p^* = qp_1 + (1-q)p_2$

3.1.6 Axiom 6: Monotonicity

Under *certainty* this Axiom states that if $A_2 > A_1$ then $A_2 \succ A_1$. While under uncertainty is formulated in two ways as:

$$\text{Let } L_1 = \{pA_1, (1-p)A_2\}$$

$$\text{and } L_2 = \{ qA_1, (1-q)A_2\}$$

$$\text{If } A_3 > A_2, \text{ hence } A_3 \succ A_1 \text{ then } L_2 \succ L_1$$

Second:

$$\text{Let } L_1 = \{pA_1, (1-p)A_2\}$$

$$\text{and } L_2 = \{ qA_1, (1-q)A_2\}$$

$$\text{If } p < q \text{ (or if } [1-p] > [1-q]) \text{ then } L_1 \succ L_2$$

Earlier, we tried to provide a preview of the utility theory usage through a simple game, now that we articulated the axioms of the theory we can try to apply this to investments. Our analysis of the axioms demonstrated that preference is a fundamental part of utility theory as it is in investment decision making.

Applying continuity axiom, we understand that investor preferences may be represented by a function. This non-decreasing function is called *utility function* and is symbolized as $U_{(x)}$ and as every function, it may be represented by a curve in the Cartesian coordinate system. The curve that connects various utility functions that provide an investor with the same level of utility is known as *indifference curve* and is measured by a comparison between uncertain investment and a certain cash flow.

A utility function comes in any form and thus there is no limitation in the shape of the utility curve, it may be either convex, curve, linear or a mix of the three. Although a utility function may be in any mathematical formulation and there is no limitation on the type of the curve, we will soon find out that each curve type signifies an investor stance towards risk; we will elaborate on that later in this chapter.

Up to this point, we showed that $U_{(x)}$ is just another way of writing probabilities, and a logical assumption would be that utility values should be restricted to probability values in the closed set of [0,1]. In our first example, though, as the reader may recall we used a utility unit greater than 1. That is allowed since one of the most important theorems states that a utility function is determined up to a positive linear transformation, where "determined" means that the ranking of the options by expected utility criterion does not change. This allows to unbound $U_{(x)}$ from [0,1] and let it take any real value, meaning that $U_{(x)} \in \mathbb{R}$.

Another important fact that we should keep in mind is that utility units, which are called *utiles*, have no meaning. They depict numerically the ranking a person sets to his available choices. Since that ranking is arbitrary, even when it is the result of a function, we cannot use it to say that L_2 is better than L_1 but merely say that L_2 is preferable according to this person to L_1.

Finally, we should be aware that investors tend to focus on the change of wealth by x monetary units and not to the total wealth they accumulate (w+x monetary units). Although this contradicts the expected utility paradigm, it does not contradict Stochastic Dominance, which states that if L_2 dominates L_1 for all $U_{(w+x)}$, the same holds true for all $U_{(x)}$ in a given set of preferences.

3.2 Investor types and utility functions

Each individual when it comes to decision making uses his set of criteria to rank the possible outcomes and make a decision. The same applies to investors, who are usually using the return and the risk related to investment as criteria in the decision-making process. Closely observing their behavior, we can easily separate them into three broad categories:

- The risk averse investor
- The risk neutral investor

- The risk lover investor

To clarify the differences between each investor category, we will use through this section an example where a representative individual of each class will have to choose between two similar investment plans with the same expected return but different risk levels.

This concept should remind us the expected utility criterion that we developed earlier and as we stated there is a connection between the convexity of the utility curve and investor's stance towards risk.

We will now visit each investor category, provide its basic characteristics and made the final connection between risk appetite and utility curve convexity.

3.2.1 Utility function of the Risk Averse investor

We will first examine the risk averse or rational investor. In this category, we classify all investors who when presented with the investment selection problem we've mentioned earlier will choose to undertake the plan that has the smallest amount of risk incurred. They require an increasing return to match the growing risk they may undertake. Most people will identify with this category, and this is why it is considered that it represents the rational investor. As it would be expected the risk averse investor's utility function in a Cartesian system with horizontal axis calibrated as the potential income level Y and the vertical axis as the utility of that income level $U_{(Y)}$ would be concave. This is reasonable because this type of investor will suffer a greater impact from an income change when he is close to axis start point but, he will feel a difference of decreasing importance as it moves up the line.

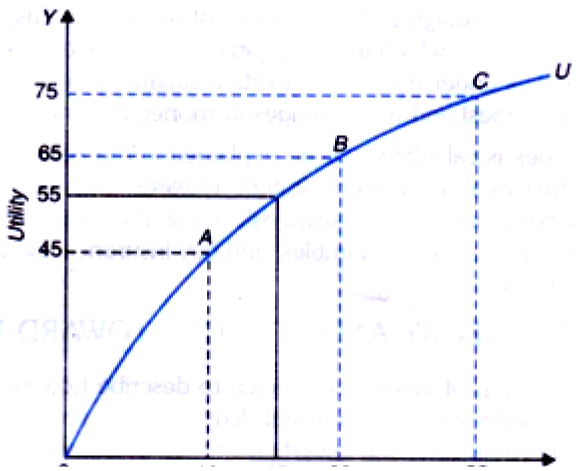

Diagram 1 – The risk averse investor's utility function

The form of the utility diagram provides us with some information regarding the utility function. It could have any form as long as the first derivative U'(Y) = $\frac{dU(Y)}{dY}$ is negatively correlated with the income Y and the second derivate takes only negative values, meaning U''(Y) = $\frac{dU^2(Y)}{dY^2} \in (-\infty, 0)$. A typical example may be the natural logarithmic function.

3.2.2 Utility function of the Risk Neutral investor

Continuing our analysis, we will examine the type of investor that is the most representative of portfolio and fund managers. A representative investor of this category, when presented with our decision dilemma will be indifferent between the two proposed plans. In other words, he will not draw extra utility from the certainty the reduced risk may provide since both plans are promising the same return.

The utility function of an investor of this category is expected to be linear, and thus, its diagram will simply be a diagonal line starting from axis start point.

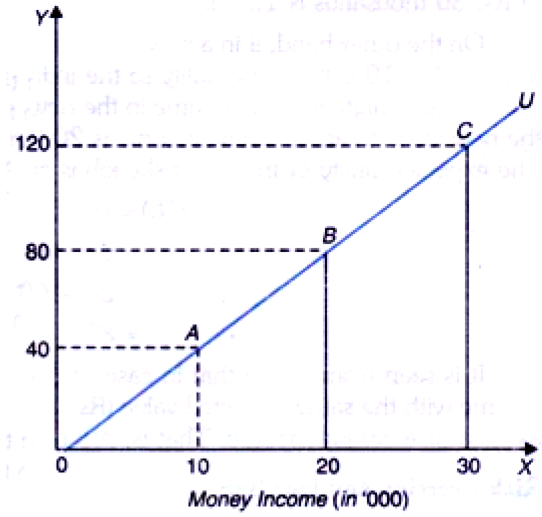

Diagram 2 – The risk neutral investor's utility function

Examining this depiction, we can extrapolate that the utility function would be of any form as long as its first derivative is any positive number, meaning $U'_{(Y)} = \frac{dU(Y)}{dY} \in (0,+\infty)$ and its second derivative is zero. A typical example of this may be the linear function $U_{(Y)} = ax + b$, where $a \in (0,+\infty)$ and $b \in [0,+\infty)$.

3.2.3 Utility function of the Risk Lover investor

Our final classification category, as irrational as it may seem, is the one that includes all decision makers that are drawn to risk and not returns. A representative decision maker of this category would be an addicted casino player.

This type of investor, when presented with our decision dilemma will select the plan that incurs the highest amount of risk, although there is no difference in said plans returns. His utility is greater at a higher risk, not with greater expected income. The utility function of this decision maker is supposed to be convex as utility increases with risk.

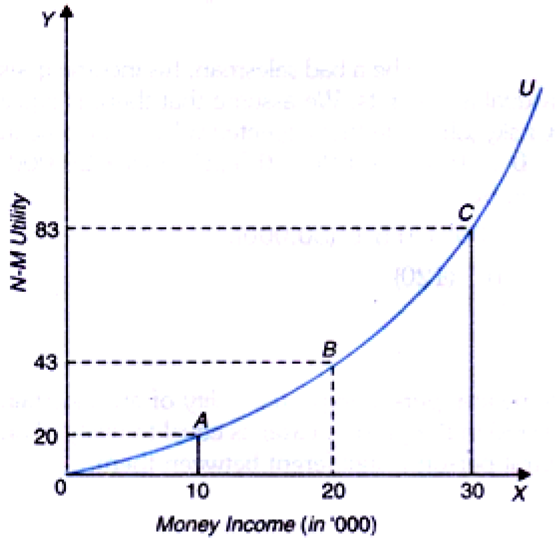

Diagram 3 – The risk lover investor's utility function

Examining the diagram of the utility function, we understand that it may have any form, as long as its first derivative $U'_{(Y)} = \frac{dU(Y)}{dY}$ is positively correlated with the income Y and its second derivative $U''_{(Y)} = \frac{dU^2(Y)}{dY^2} \in (0, +\infty)$. A typical example may be the natural exponential function.

As we conclude this section, we come to realize the importance of different investor types in portfolio management and the reason the expected utility theory is widely used in classifying investments. All this knowledge will be essential in our effort to optimize a portfolio through stochastic dominance.

3.3 Modern Portfolio Theory

Everything that we presented up to this point was a prelude to Modern Portfolio Theory as it was developed by Professor Harry Markowitz in 1952 in his article "Portfolio Selection" published in

Journal of Finance. An article that leads to him being awarded the Nobel Prize in Economic Sciences in 1990. Our final effort in this section will be to present Markowitz's theory; something that will allow the reader to connect the dots between the concepts that have been submitted and prepare for the analysis that will follow.

The main innovation introduced by Markowitz is to measure the risk of portfolio as the joint (multivariate) distribution of returns of all assets. He showed that the variance of return was a meaningful measure of portfolio risk under a reasonable set of assumptions. More important, he derived the formula for computing the variance of a portfolio which not only showed the importance of *diversification* but also demonstrated how to diversify efficiently. The assumptions that were formulated by Markowitz regarding investor behavior were:

1. Investors consider each investment alternative as being represented by a probability distribution of expected returns over some holding period.
2. Investors maximize one-period expected utility, and their utility curves demonstrate diminishing marginal utility of wealth.
3. Investors estimate the risk of the portfolio on the basis of the variability of expected returns.
4. Investors base decision solely on expected return and risk, so their utility curves are a function of expected return and the expected variance (or standard deviation) of returns only.
5. For a given risk level, investors prefer higher returns to lower returns. Similarly, for a given level of expected return, investors prefer less risk to more risk.

Under these assumptions, a portfolio is efficient when no other portfolio offers higher expected return with the same or lower risk.

In his analysis Markowitz used the metrics of dispersion as a measure of risk incurred at a portfolio, a logical choice considering the first assumption under which we are working. To include uncertainty in his analysis, he used the expected rate of return as the metric of portfolio performance. The portfolio expected the rate of return is simply calculated as the weighted average of the expected rates of return of the individual investments. The weights are the proportion of total value for each investment.

$$E(R_p) = \sum W_i R_i$$

Where

R_p: Portfolio return

W_i: Weight of $_i$ investment

R_i: Expected rate of return on investment

To calculate, though, the portfolio Standard Deviation, we will first have to calculate the covariance and correlation of individual assets making up the portfolio. The covariance is a statistical metric that shows us the degree to which two variable, in our case the expected rates of return, "move" together. It is easily calculated, through the following mathematical formula:

$$\mathrm{Cov}_{ij} = E\left\{\left[\bar{R}_i - E(R_i)\right]\left[\bar{R}_j - E(R_j)\right]\right\}$$

The magnitude of the covariance depends on the variances of the individual return series, as well as on the relationship between the series. Using the covariance, we've managed to quantify the relationship between the expected returns of two assets. As the function result may be any real number it is hard to interpret the outcome between many assets and it does not allow comparability.

Since the covariance is affected by the individual series variability we "standardize" it, and create the *correlation coefficient*, a metric that can vary only in range -1 to +1; thus allowing us to draw conclusions. The mathematical formulation of this is:

$$r_{ij} = \frac{Cov_{i,j}}{\sigma_i \sigma_j}$$

where

r_{ij}: Correlation coefficient

σ_i: Standard deviation of R_i

σ_j: Standard deviation of R_j

Using these tools Harry Markowitz, managed to calculate the portfolio standard deviation as:

$$\sigma_p = \sqrt{\sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} w_i w_j Cov_{i,j}}$$

Thus, this formula includes besides the weighted average of individual variances, the weighted covariances between all the assets in the portfolio.

This notion brings us to the Efficient Frontier, the line that represents every set of portfolios, for all possible combinations of a given asset or assets, which have the maximum rate of return for every given risk level. Meaning that it is the line that represents every portfolio for a minimum σ_p for a certain level of R_p and vice versa. As an investor, everyone would target a point along the efficient frontier based on his utility function, which as the reader may recall, reflects investor's attitude towards risk.

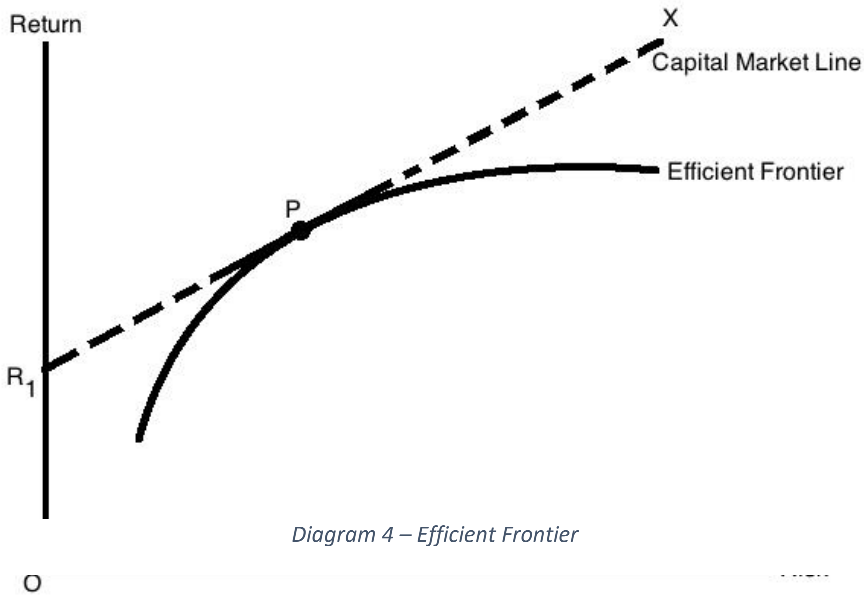

Diagram 4 – Efficient Frontier

As someone may notice, the slope of the efficient frontier curve decreases steadily as we move upward. This implies that adding equal increments of risk as we move up the efficient frontier gives diminishing increments of expected return. Markowitz uses a person's utility curve to determine an efficient portfolio that best suits investor's needs. Two investors will choose the same portfolio only if their utility curves are identical.

Contemplating on those, we understand that there is no absolute optimal portfolio that everyone should choose but rather an optimal portfolio is the efficient portfolio that has the highest utility for a given investor. It lies at the point of tangency between the efficient frontier and the curve with the highest possible utility. This revolutionary analysis by H. Markowitz, and many more that followed shaped a significant part of financial science as we know it today.

Concluding this chapter, the reader has accumulated the knowledge of the basic principles that will be required and widely mentioned in our analysis as well as informed on the forces that power the need for better, more analytical models of portfolio optimization. We are now ready to proceed on the main theory behind the thesis and our research on portfolio optimization with stochastic dominance.

Chapter 4

Concepts of Stochastic Dominance

The driving force behind investment selection is individual's preferences. Previously, we demonstrated utility theory and how the maximum expected utility criterion in synergy with the Modern Portfolio Theory could lead us to the optimal portfolio.

Although, this concept is promising and easy to implement it lacks the key characteristic of real world application. It most certainly may help us analyze theoretical models and partially explain the diverse phenomena that we observe, but it cannot be actively used by an investment analyst to propose a portfolio. The reason behind this is that we cannot know with certainty the exact form of an investor's utility function. This partial information leads us to inconsistencies that we will try to address.

In a case where we know exactly the form of the investors utility function we achieve complete preference ordering, and we can definitively state the optimal portfolio for the specific investor. As we turn to real world and the information becomes scarce and difficult to accumulate we start to order investor preferences partially into two broad sets:

The efficient set (ES)

and

The inefficient set (IS)

Both sets contain investments that are available to an individual, and together they create the feasible set (FS). The foremost difference between the two, being that all the investments in the inefficient set, with decision maker preferences in mind, underperform when compared to efficient set.

To separate the possible investments in those two sets, we will need a tool, which is *Stochastic Dominance (SD)*; the theory that we will present through this chapter. We will progressively increase our degree of information on investor preferences and thus gradually explore the orders of stochastic dominance.

On a last note before we proceed we should keep in mind that stochastic dominance is the *optimal* tool that an investment analyst would use to objectively separate the available investments into those that he will present to an investor and those that he won't. The choice of the actual investment plan is a subjective choice made by the investor himself as only he knows the true form of his utility function; thus, stochastic dominance applies only to the investment analyst's decision and as such a tool it will be examined.

4.1 First Order of Stochastic Dominance (FSD)

The first degree of stochastic dominance assumes the least amount of information on investor preference being available. The analyst has to identify the investment that classifies as efficient knowing only that all individuals prefer *more money than less.*

At this point and before we go any further let us explain the meaning of the term stochastic dominance as it is in the center of our thesis. To fully comprehend the term, we will account for the significance of the two components; stochastic process and dominance. A *stochastic process* is a random process evolving with time. More precisely, it is a collection of random variables X_i indexed by time. The study of processes changing with time leads to the study of either differential equations, for continuous variables, or difference equations for discrete variables. In our case and as the returns of a portfolio or asset can be represented by distributions, we will study the density function for continuous variables or the cumulative probability function for discrete variables. We will revert on that as our analysis progresses. The second term that we have to understand is the concept of *dominance*. We will present dominance in its weak sense, which simply states that a security dominates another if it promises the same payoff for every occasion and at least better on one occasion to the decision maker. Combining the two terms that were described, we get stochastic dominance; the concept that in its degrees examine a random process, which in finance are asset returns, to identify occasions where either an asset or a portfolio *dominates* another by offering the investor at least the same returns and in times better that its counterparts.

Understanding the meaning of the theory that we will present and its use we turn to the first order of stochastic dominance, denoted as FSD. The available level of information here should be translated into information on investor's utility function as this is our model for numerically ranking an individual's preferences. The knowledge of preferring more money over less translates into the first derivative of the utility function being either positive or zero. Thus, we created a set which includes all utility function with $U' \geq 0$; we will call that set U_1 where the number one signifies that we are working under FSD.

For our paradigm we will work with two investment plans F & G. In case the returns of the investments are discrete we will be working with the cumulative probability function:

$$F(x) = P(X \leq x) = \sum_{X \leq x} P(x)$$

While for investment whose returns are better depicted by continuous variables we will use the distribution density function:

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

Both of them will provide us with the probability the return of the investment being below the critical value x. Under the First Degree of Stochastic Dominance rule, a preferable investment between the two is the one that dominates the other and as such is situated in the efficient set.

This is easily identified if we examine the probability function. The investment that dominates would be the one for which:

$$F(x) \leq G(x)$$

holds true for all x, with a strong inequality for at least one x_0. Considering that the investor prefers more money than less, the investment with the smaller exposition to probabilities for low or negative returns will be the one ensuring a greater utility level for our investor, thus ensuring $E_F U_{(x)} \geq E_G U_{(x)}$ for all $U \in U_1$ with a strong inequality for at least one $U_0 \in U_1$. In cases where those hold true, we state that investment plan F dominates G under FSD, which is written as FD_1G. To formulate officially, we define:

$$G(x) - F(x) = I_1(x)$$

<u>Then</u>

FD_1G if and only if $I_1(x) \geq 0$ for all x and $I_1(x_0) > 0$ for some x_0.

To provide an intuition behind the FSD rule, we will work with the cumulative probability function for discrete variables and the FSD rule:

If FD_1G holds true, then $F(x) \leq G(x) \Rightarrow 1\text{-}F(x) \geq 1\text{-}G(x) \;\forall\; x$. At this point recall that $F(x) = p(X \leq x)$, if we substitute the latter equation into the former we will get $1\text{-}F(x) \geq 1\text{-}G(x) \Rightarrow p_F(X \geq x) \geq p_G(X \geq x)$. Thus, if FD_1G holds true, the probability of obtaining higher payoffs than the critical value x is larger under distribution F than under distribution G.

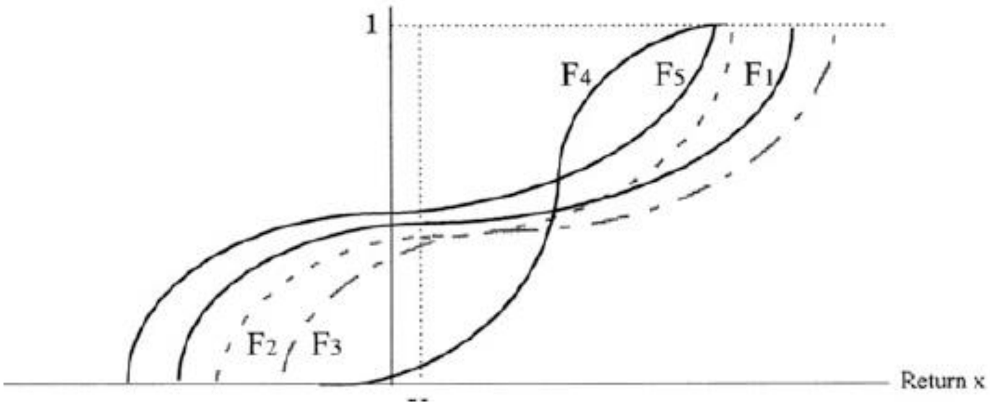

Diagram 5 – First Degree of Stochastic Dominance

Examining further the FSD rule as described we can break it down to five points:

1) FSD requires the two compared distributions not crossing but allows tangency
2) An investment is characterized as inefficient if there is at least one other investment dominating the former plan
3) An inefficient investment may dominate another inefficient investment
4) An inefficient investment cannot dominate an efficient investment
5) All investments within FSD efficient set must intercept.

The FSD rule is the best available rule for our set of information, as it provides us with the smaller efficient set for the given knowledge on preference. That falls in line with the typical description of an *optimal* rule, which is a *necessary and sufficient condition* for dominance without contradicting the maximum expected utility criterion. The mathematical representation of this is:

$$E_F U_{(x)} \geq E_G U_{(x)} \ \ \forall \ \ U \in U_1 \Leftrightarrow FD_1 G$$

There are many different rules which may be either sufficient but not necessary or necessary but not sufficient that may help us screen the feasible set. A sufficient rule alone may be used, but as it may not be powerful enough, it will lead us to Type I errors. A Type I error is induced when we accept to the efficient set a dominated investment and thus, a sufficient rule alone yields a broad efficient set.

On the other hand, the necessary rules are condition implied to hold true if we have dominance. The three most basic are:

1) If FD_1G then $E_F\left(x\right) \ > E_G\left(x\right)$

2) If FD_1G then $\overline{x}_{geo}(F) > \overline{x}_{geo}(G)$

3) If FD_1G then $Min_F(x) \geq Min_G(x)$

All three conditions may independently hold, but they do not necessarily suggest that there is first degree of stochastic dominance. On the contrary, FSD cannot exist if any of the necessary rules outlaid does not hold. Using a necessary condition alone to identify dominance may lead us to Type II errors. A Type II error is induced when we reject from the efficient set a non-dominated investment and that may mean poor investment decision. Only the First Degree of Stochastic Dominance rule as presented is a sufficient and necessary condition for $U \in U_1$ and thus is the optimal rule.

As the information on investor preference at this level is quite limited and consequently we are faced with a rather large efficient set, there is the need to include another layer of information on presumed preferences creating the second order of stochastic dominance.

4.2 Second Order of Stochastic Dominance (SSD)

As we concluded the presentation of the first order of stochastic dominance we have developed a necessary tool that helps us discriminate between efficient and inefficient investments. The investment analyst should present an individual with plans that are in the efficient set and then, the investor chooses from them. We may recall, though, that FSD operates under limited assumptions on investor preferences, and consequently, it yields a broad efficient set. We will now try to limit the efficient investments by adding another assumption on investor preference.

The new layer of information is based rather on the observation than the assumption, that all investors are risk averters. Most people prefer a higher return for the same amount of risk and less risk for the same level of returns. That said, we add the risk aversion assumption to the non-decreasing utility functions of investors examined previously.

The assumption that we have made up to this point was formulated as $U' \geq 0$ generating the equation set $U \in U_1$, which meets that criterion. By adding risk aversion, we define a new smaller equation set $U \in U_2$ with $U_2 \subseteq U_1$, which contains all the utility functions that have non-negative first derivative and non-positive second derivative with at least one point where they are strictly positive and negative respectively. Extrapolating from the fact that the sign of a function's derivative suggests its graph, we expect a risk averters utility function graph to be concave. From that simple statement on function convexity, we derive Jensen's Inequality, which states that the expected utility is smaller or equal to the utility of the expected return with the mathematical representation of this being: $U(E_x) \geq EU_{(x)}$. In addition to the mathematical formulation of the utility function, there are two facts intrinsically linked with the nature of a risk averter. The first is that a person who averts risk will never play a fair game; a game in which the price of the ticket to play the game is equal to the expected prize. The second is that risk averters will be willing to pay a positive premium to insure their wealth against negative outcomes. All these five mathematical and non-mathematical notions are linked with risk aversion and we may use them interchangeably in our analysis.

Before we present the SSD rule, we would like to remind the reader that under this model we are working on the *assumption* of risk aversion. This means that although any investor would agree on $U \in U_1$, there may be someone who disagrees on $U \in U_2$. The reason that makes worthwhile to extend our model to include that assumption is the fact that most investors would agree on $U \in U_2$, something that is indicated by the cost of capital, as investors require higher premiums for higher risk concentrations; a behavior that signals risk aversion.

We are now ready to present the SSD decision rule; we will use the same paradigm of two investment plans F & G with density functions $f_{(x)}$ and $g_{(x)}$. We state that F dominates G by second degree of stochastic dominance denoted by FD_2G for all risk averters if and only if:

$$I_2(x) = \int_a^x [G(t) - F(t)dt \geq 0$$

For all x $\in$[a,b] and with at least one x_o for which there is a strict inequality. As the investor would prefer a plan that would be more profitable with less risk incurred, we derive that the above definition, when FD_2G, may be used interchangeably with

$$E_F U(x) - E_G U(x) \geq 0$$

For all U $\in$ U_2, with at least one U_o $\in$ U_2 for which there is a strict inequality.

The rule presented previously for the second degree of stochastic dominance is a sufficient and necessary condition for dominance, thus, making the SSD rule the optimal decision rule when we are in U $\in U_2$ information set.

To help us grasp this concept better and the reason ensuring that any investment dominating another, while not dominated itself is an efficient investment under SSD we will use the graphical exposition of SSD.

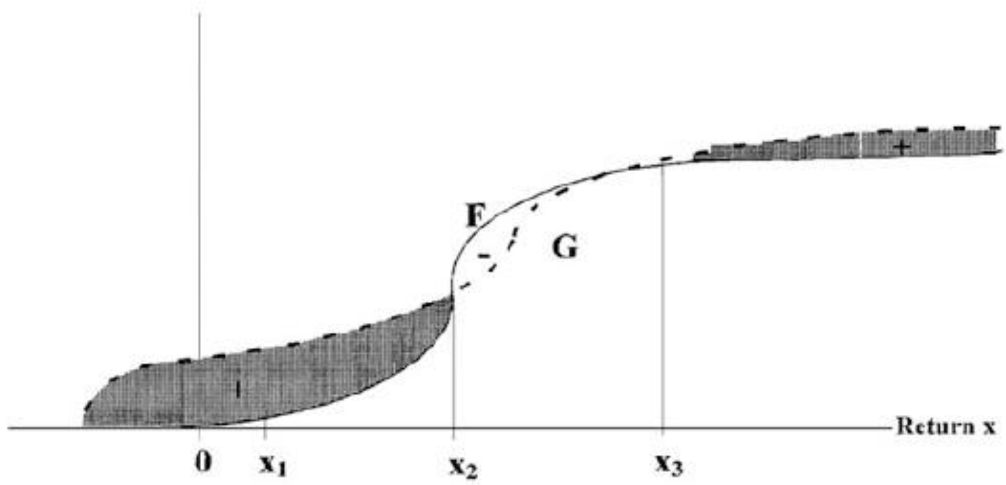

Diagram 6 – Second Degree of Stochastic Dominance

To slowly build our argument we will start from the SSD integral condition. $I_2(x) \geq 0$, this implies that the area between the two distributions that we are studying should be non-negative up to every x. When F dominates G we mark the area where F is below G as "+"

and the area where G is below F as "-". When applying the SSD rule, we are looking for all the x up to which, the one distribution dominates the other. This can happen irrespectively:

May F dominate G for $x \leq x_1$ and $x \geq x_2$

and

May G dominate F for $x_1 < x < x_2$

However, to state definitively, we should know that the positive area is greater in total than the negative area, that is implied when the integral rule

$$\int_a^x [G(t) - F(t)]dt \geq 0$$

holds true. To achieve that there must be a positive space preceding every negative area such that the sum of the positive areas is larger than the sum of the negative areas accumulated up to x. The following inequality represents this:

$$S_i^- \leq \sum_{j=1}^{i-1} S_j^+$$

Where $S_1, S_2, S_3, \ldots, S_n$ is the absolute value of all areas ordered from lowest to highest. In case, the distributions cross more than once, hence, we have more negative areas, the inequality is written as

$$\sum_{i=1}^{k} S_i^- \leq \sum_{j=1}^{m} S_j^+$$

Where m is the positive areas *before* the k^{th} negative area. Namely, the condition states that the sum of all positive areas m preceding the k negative area should be greater or at least equal to the sum of all negative areas up to k. Finally, note that to check if $I_2(x) \geq 0 \ \forall x$ holds true it is sufficient to test the SSD integral rule for the intersection points of F and G.

To understand the reason why when SSD integral rule holds true we are expecting the distribution that dominates the other to have a higher utility value for the investor we should analyze the equation

$$E_F U(x) - E_G U(x) = \int_a^b [G(x) - F(x)] U'(x) dx$$

Examining the equation, we notice that, by the assumption of risk aversion (U" ≤ 0), the utility is a declining function of x; a fact that means that the positive area is multiplied by a larger number U'(x) than the negative area and therefore the total integral is non-negative. This implies that

$$E_F U(x) \geq E_G U(x) \forall U \in U_2$$

The intuition prevailing SSD is that if FD_2G, as F has a positive area for a lower wealth and U' is declining the monetary value of the positive area in *utility terms* is larger than the value in utility terms of the negative area that F loses in comparison to G. Hence, F will be preferred over G by all risk averters.

To complete our presentation of SSD, it is crucial to mention the sufficient and necessary conditions that are needed to support dominance.

Let us start with the three most important sufficient rules for dominance. As in FSD, if one of this conditions alone is met it does not necessarily suggest SSD and if we implement any of these conditions alone we risk a Type I error. The first rule suggests that if the FSD rule holds true, then that is a sufficient condition for SSD. Although any investment relegated to the inefficient set with FSD will also be relegated to the inefficient set with SSD, this rule alone will result in a rather large efficient set since it cannot recognize dominance when the two distributions cross. The second rule suggests that if the $Min_F(x) > Max_G(x)$ that is a sufficient rule for SSD; this implies that $F(x) \leq G(x) \ \forall$ x. That condition, and any other that is a sufficient rule for FSD, hold true for SSD as well since the existence of FSD dominance implies SSD dominance. The last sufficient rule that we will mention is the "k rule" which states that F dominates G if:

$$\int_a^x [G(t)-F(t)]dt \geq k \quad \forall \text{ x where k} > 0$$

It is a sufficient rule since it holds for all x then $I_2(x) \geq 0$ for some x_o.

To complete this section, we will shortly mention the three necessary rules that are implied by SSD, remember that while SSD cannot exist unless all three condition are met, SSD is not signified by the fulfillment of one rule alone; if we ignore that we are risking a Type II error.

The first condition is known as the means, which states:

$$E_F(x) \geq E_G(x)$$

As a prerequisite for dominance of F over G in U_2. Namely that if FD_2G then the expected returns of F must be greater or equal to the expected returns of G. The second rule is the geometric means, which suggests that if FD_2G then $\bar{x}_{geo}(F) \geq \bar{x}_{geo}(G)$. The last condition if the "left tail" rule which states that $Min_F(x) \geq Min_G(x)$, namely the left tail of G must be "thicker." Although there are more sufficient and necessary rules for SSD, the ones described above are the most important and we will not include others in our analysis.

Concluding our SSD presentation, we note that by adding the risk aversion assumption we've managed to yield a significantly smaller efficient set, but as we are only conducting our work on two assumptions we still have a relative large efficient set. In the following section, we will try to use to our advantage one of the distributions' properties to limit even further the efficient set while providing the investor with the best of his procurable investment opportunities.

4.3 Third Order of Stochastic Dominance (TSD)

As our analysis progressed, we made various assumptions on investor preference driven by reasoning and observation. The next degree of stochastic dominance that we will attempt to present is more challenging than the previous and for that reason, we need to start by addressing the assumption that will increase our level of information.

The assumptions that we have formulated were that all investor prefer more money than less money (U' ≥ 0) and that most investors are risk averters (U" ≤ 0). We are adding now the assumption that most investors prefer a return distribution to be positively skewed (U"' ≥ 0) than negatively skewed. To facilitate our transition to the TSD decision rule and understand its importance, it is first deemed necessary to provide a definition for skewness and explain the economic rationale and importance of skewness in decision making.

In statistics skewness is a metric that help us quantify the extent and magnitude of a distribution's asymmetry. Namely, it helps us understand and measure the degree to which the two tails of the distribution are of an even size, or one tends to be "fatter" than the other. It is calculated as the third central moment denoted by μ_3, as follows:

$$\mu_3 = \sum_{i=1}^{n} p_i (x_i - E(x))^3$$

For discrete distributions, where with n is denoted the number of observations and with (p_i, x_i) the probability function and for continuous distributions:

$$\mu_3 = \int_{-\infty}^{\infty} f(x)(x - E(x))^3 dx$$

To help us understand even better the importance of skewness note that the prizes of a lottery are positively skewed due to the small probability of winning an enormous prize while the value of an uninsured house is negatively skewed because of the small likelihood of a heavy loss due to fire. Distributions that are symmetric have zero skewness.

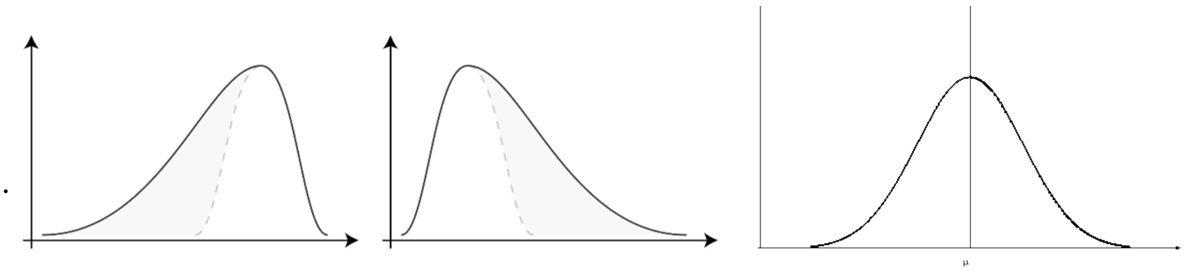

Diagram 7 - Distributions with negative, positive and zero skewness

Understanding now the meaning and mathematical calculation of the term skewness we turn to the interrelationship between skewness and U''' to start connecting the dots that will lead us to TSD.

It was Friedman and Savage, and Kahneman and Tversky that suggested a positive approach toward understanding investor preference. Their suggestion consists of the notion that by observing investor *behavior* we can draw conclusions regarding their *preferences*. Following their doctrine and observing behavior, we will soon notice that most people tend to insure their homes and are willing to buy lottery tickets. Through home insurance individuals reduce the variance of future value and annihilate the skewness, generalizing that we may say that people tend to insure against negative skewness and are willing to pay a premium for positive skewness

Using a Taylor series expansion about the point (w+E(x)) we can prove that ceteris paribus, the higher the σ_x^2 the lower will be the expected utility of a risk averter due to the fact that U" ≤ 0 and the higher the positive skewness, the higher the expected utility, provided that U"' ≥ 0. Summarizing, the investor utility will increase as the variance tends to zero (U"≤ 0) or as the skewness tends to be zero or positive (U"' ≥ 0).

To expand this analysis to investor preference in the stock market, we will first have to mention that stock rates of return are positively skewed as at most a stock price can drop to zero resulting in a -100% rate of return.

In the case of lotteries and home insurance it is hard to separate the effect of changes in the variance and changes in the skewness; hence, we could not definitively conclude that U"' >0 from the fact that individuals buy insurance and lottery tickets. Stock market rates of return can be used to ascertain whether U"' is indeed positive: The effect of the variance can be separated from the effect of the skewness by conducting multiple regression analysis. To be more specific, the following cross-section regression can be performed:

$$\overline{R_i} = a_1 + a_2\sigma_i^2 + \alpha_3\mu_{i3} + \alpha_4\mu_{i4} + \ldots\ldots + \alpha_k\mu_{ik}$$

where μ_{ik} is the k^th^ central moment of the i^th^ mutual fund (the first k moments are included in the regression), $\sigma_i^2 = \mu_{i2}$ is the variance, and R_i is the i^th^ stock average rate of return. The regression coefficients (if significant) determine how the various moments of the distribution affect the expected rate of return R_i.

The market dynamic for price determination of risky assets is as follows: Suppose that a firm takes an action such that the skewness of the returns on the stock increases. Then, if investors like positive skewness, the demand for the stock will increase the stock's price and, therefore, for a given future profitability, the average rate of return with the new high price will be lower.

Two last notes before presenting the TSD decision rule would be that relying on the observation that the higher the investor's wealth, the smaller the premium he is willing to pay to insure against a given loss we support even further the assumption of $U''' \geq 0$. At this point, we should stress enough that our hypothesis is supported by empirical data for *most* but not *all* investors.

As in previous degrees of dominance, the optimal investment rule for $U \in U_3$ information set would be given in the form of the rule that follows:

Let F(x) and G(x) be the cumulative distributions of two investments under consideration whose density functions are f(x) and g(x), respectively. Then F dominates G by *Third Degree Stochastic Dominance (TSD)* if and only if the following two conditions hold:

$$\text{a)} \quad I_3(x) = \int_a^x \int_a^z [G(t) - F(t)]dtdz \geq 0 \text{ for all x}$$

$$\text{b)} \quad E_F(x) \geq E_G(x) \text{ (or } I_2(b) \geq 0)$$

with at least one strict inequality, namely: $I_3(x) \geq 0$ and $I_2(b) > 0 \Leftrightarrow E_F U(x) \geq E_G U(x)$ for all $U \in U_3$, the same holds true if $I_3(x) > 0$ and $I_2(b) = 0 \Leftrightarrow E_F U(x) \geq E_G U(x)$ for all $U \in U_3$.

To have dominance, we require that either:

$$I_3(x_0) > 0$$

OR

$I_2(b) > 0$

Which guarantees a strong inequality holds for some $U \in U_3$.

Although it is tempting to believe that for TSD dominance, the dominating investment is sufficient to have larger skewness alone, this is not true. It is possible for F to dominate G even if they have the same or zero asymmetries. The conclusion draw from that is that it is possible to have TSD, without having FSD or SSD but the result may not be necessarily linked with skewness.

To understand how TSD is possible without its existence being related directly with skewness, we will try to provide an intuitive explanation of the rule. The most notable impact of assuming $U_{(x)}''' \geq 0$ is that it implies that $U_{(x)}''$ is a non-decreasing function, although $U_{(x)}'''$ may not be a declining function itself. Besides that, the assumption that we've formulated on the third derivative of the utility function also implies that $U_{(x)}'$ is a declining convex function. By that we allow U' to decline at the same pace at some range and because of the required condition $U_{(x)}''' \geq 0$, for some x, it must be strictly convex at some range. Based on that we will explain how it is possible to have TSD without SSD even for symmetric distributions. We have:

$$E_F U(x) - E_G U(x) = \int_a^b [G(x) - F(x) U'_{(x)} dx$$

Because U'(x) is a declining convex function, we allow a first positive area to be followed by a second larger negative area such that SSD does not hold, but TSD may hold. If U' is strictly declining, that makes the following area to worth *less* in utility terms; thus in utility terms in $U \in U_3$ the positive area worth more hence FD_3G is possible. Of course, the larger negative area that is allowed is a function of the positive area that precedes it as well as the relative location of these two areas on the horizontal axis. Finally, we should note that a convex U'(x) is required to have TSD but U'(x) = 0 is possible in some ranges.

Concluding our presentation of the third order of stochastic dominance it is essential to mention that TSD is a necessary and sufficient decision rule for all $U \in U_3 \subseteq U_2 \subseteq U_1$ and as such it is the optimal rule for that information set. However, we can here establish various sufficient and necessary rules for $U \in U_3$ dominance.

We will start by mentioning two basic sufficient rules for dominance. As always the use of those standards should be avoided as they may lead us to Type I errors and yield a larger than the optimum efficient set.

1)FSD is a sufficient rule for TSD

To understand this, suppose that FD_1G, then $F(x) \leq G(x)$ for all x with a least one strict inequality. This implies that: $E_F(x) > E_G(x)$ and $l_3(x) > 0$ because FSD implies that the superior investment has a higher mean and that $I_1(x) = [G(x) - F(x)]$ is nonnegative. However, because the integral of $I_1(x)$ is l2(x); l2(x) > 0 and $l_3(x)$, which is the integral of $l_2(x)$ is also non-negative.

2)SSD is a sufficient rule for TSD

If FD_2G then that means that the fundamental rule for SSD holds true, namely $I_2(x) \geq 0$; but because the $I_3(x)$ may be written as $I_3(x) = \int_a^x I_2(t)dt \geq 0$, that would inevitably mean that $I_3(x) \geq 0$. I2(x) > 0 for all x implies that it holds also for x = b; hence, $E_F(x) > E_G(x)$. Thus, FD_2G implies that the two conditions required for TSD dominance hold; hence, FD_3G.

Finally, we will describe three necessary rules for TSD. Again, the use of those standards should be avoided as they may lead us to

Type II errors and assign an efficient investment to the inefficient set. We should, though, keep in mind that third order stochastic dominance may not exist if any of those necessary rules are broken.

1)The means

Unlike FSD and SSD, TSD explicitly requires that $E_F(x) > E_G(x)$ to have FD_3G. This condition on the expected values is a necessary condition for dominance in U_3. Note that for FSD and SSD we had to prove that this condition was necessary for dominance but TSD, there is nothing to show because it is explicitly required by the dominance condition.

2)The geometric means if FD_3G then $\bar{x}_{geo}(F) > \bar{x}_{geo}(G)$

3)The “left tail” condition

Like FSD and SSD, for FD_3G, the left tail of the cumulative distribution of G must be "thicker" than the left tail of F. In other words, $Min_F(x) > Min_G(x)$ is a necessary condition for FD_3G.

At this point, we have provided an outline of the properties, mechanics and formulation of all three degrees of stochastic dominance as they will be used in the following chapter to optimize our portfolio. The reason why this model presents a tempting alternative to investment analysts should be clear. It allows them to identify prosperous investments, consistent with the objectives of investor majority, with limited need for information, something that allows opportunities for cost-effective solutions in investments management.

On a last note, it is essential to mention that stochastic dominance may be extended even further to include risk-seeking investors, n^{th} order stochastic dominance, as well as it may extend to include the decreasing absolute risk aversion model. In this thesis, we will limit ourselves to the three degrees presented in this chapter.

Chapter 5

Portfolio optimization under Stochastic Dominance

For us to proceed to the essence of this thesis and demonstrate the use of stochastic dominance as an optimization instrument, we will have to construct a portfolio that will be well diversified while limiting the securities selection to a number that would be rational and achievable for an investor.

5.1 Data Selection Process

Taking into consideration the constant flow of information and the ease of access that modern world allows to an individual, the global securities universe should be considered when dealing with an investment selection problem. Through this section, we will try to unveil and guide the reader through our reasoning for selecting the stocks that make up our portfolio.

Before deciding on the individual shares, we dealt with a series of dilemmas, the first of which was if the trading venue should be in an emerging or a developed market. The answer to that question arises from the fact that we are working on a proof of concept and working on an emerging market would entrust our portfolio with numerous problems and market inefficiencies that will make harder to identify the root of an exceeding return. Having decided on the venue category, our following criterion should be market depth, data availability, ability to verify information and previous researches on the specific market.

Keeping those in mind, we will soon come the conclusion that the Standard & Poor's 500 Composite Index fits best all the criteria. S&P500 created in 1957 as the first market capitalization weighted index in the United States. It includes the 500 largest U.S. corporations having their stock traded on either NYSE or NASDAQ. The index constituents are selected by a committee assessing a broad range of variables such as market capitalization, liquidity, domicile, public float, sector classification, financial viability, length of time publicly traded and listed exchange.

The vast range of criteria that are used to decide on introducing or substituting a listed company on the S&P500 index, its active maintenance, and its widespread use led the National Bureau of Economic Research listing common stocks traded on S&P500 as a leading indicator of U.S. business cycles. It is easily understood the reason that drove us to select S&P500 to serve as a proxy for our portfolio.

The next step in this process was to acquire historical data on stock and index prices, extending to reasonable time into the past that would allow us not alone to construct our model based on them but to test it as well. Data on the closing price, total returns index (RI) and market capitalization were extracted from Reuters' DataStream for the period from December 31st, 1999 until Friday, June 3rd, 2016, both on daily and monthly intervals. In addition to the above information were acquired on the sector and sub-sector that under which each share was classified.

Combining the average market capitalization values on monthly intervals and sector classification on shares listed continuously on the S&P500 index during the whole period, we have managed to arrange the stocks per market capitalization and per sector. We have limited the maximum number of stocks that an investor would like to keep in his portfolio at a given time at thirty. This restriction when applied to the stocks arrangement that we have achieved, allowed us to select the top three shares, in average market capitalization terms, from each sector.

Table 1 – Data Descriptive Statistics

Sector	Stock	Mean	Std. Dev	Variance	Kurtosis	Skewness	J-B
Index	S&P500	0,00301	0,04372	0,00191	0,89694	-0,49643	44,39561
Consumer Discr.	Home Depot	0,00640	0,07578	0,00574	0,24418	-0,19245	63,55438
	The Walt Disney Company	0,00861	0,07395	0,00547	1,50566	-0,25099	20,39798
Consumer Staples	Wal-Mart Stores	0,00176	0,05609	0,00315	1,25228	-0,24394	27,02654
	Procter & Gamble	0,00431	0,05314	0,00282	10,89232	-1,82913	621,13747
	The Coca Cola Company	0,00424	0,05154	0,00266	1,12645	-0,40933	34,31398
	PepsiCo Inc.	0,00692	0,04534	0,00206	2,90578	-0,37426	4,67182
Energy	Exxon Mobil Corp.	0,00593	0,04973	0,00247	1,65243	0,33431	18,57561
	Chevron Corp.	0,00764	0,06065	0,00368	0,80543	0,16356	40,41085
	Schlumberger Ltd.	0,00928	0,09310	0,00867	1,20861	-0,08916	26,60233
Financials	JPMorgan Chase & Co.	0,00654	0,09158	0,00839	0,99984	-0,18672	33,98321
	Bank of America Corp	0,00566	0,12030	0,01447	9,07154	0,49227	310,54523
	Citigroup Inc.	-0,00224	0,12600	0,01588	8,80171	0,36610	280,69183
Health Care	Johnson & Johnson	0,00636	0,04745	0,00225	1,68151	-0,27687	16,78628
	Pfizer Inc.	0,00341	0,05796	0,00336	-0,04522	-0,11255	76,53472
	Merck & Co.	0,00367	0,07302	0,00533	1,16066	-0,22220	29,39136
Industrials	General Electric	0,00144	0,07700	0,00593	1,59984	-0,15057	16,83632
	3M Company	0,00860	0,05791	0,00335	0,83304	0,10193	38,88519
	United Technologies	0,00822	0,06508	0,00424	3,45162	-0,64050	15,14365
Inform. Tech.	Apple Inc.	0,02419	0,12360	0,01528	3,28243	-0,64943	14,50230
	Microsoft Corp.	0,00400	0,08970	0,00805	3,00356	0,32042	3,37110
	International Bus. Machines	0,00436	0,07349	0,00540	4,10527	0,60198	21,92557
Materials	Dow Chemical	0,00815	0,11328	0,01283	19,43227	2,35658	2398,74776
	Du Pont (E.I.)	0,00472	0,07807	0,00609	1,57330	0,29857	19,63466
	Praxair Inc.	0,00958	0,06195	0,00384	2,43827	0,04053	2,64402
Telecom. Services	AT&T Inc	0,00375	0,06658	0,00443	2,07812	0,15379	7,75240
	Verizon Communications	0,00427	0,06864	0,00471	5,31860	0,92408	72,16425
	CenturyLink Inc	0,00255	0,08117	0,00659	5,25392	-0,05593	41,80238
Utilities	Duke Energy	0,00753	0,06258	0,00392	3,41374	-0,71021	17,96622
	Southern Co.	0,01004	0,04688	0,00220	3,28570	0,23941	2,55191
	Exelon Corp.	0,00715	0,06219	0,00387	1,77096	-0,11982	12,87037

Thus, resulting in a well-diversified portfolio that included stocks from all industry sectors as demonstrated in the previous table of descriptive statistics.

Finally, during this phase, we calculated the descriptive statistics values that would be needed in our analysis (see appendix for the full table). To ensure that our computations are not affected by events such stock splits or reverse splits, dividend payments et cetera we used Return Index to calculate monthly stock yields. Return Index, is a calculated index for each stock from the date that it was issued until today, providing a representation of the share return adjusted towards external factors to demonstrate the actual performance. Its price on the stock issue date is 100 and then it is calculated with the following mathematical formula:

$$RI_t = RI_{t-1} * \frac{PI_t}{PI_{t-1}} * (1 + \frac{DY_t}{100} * \frac{1}{N})$$

The index assumes 260 weekdays in a year, and that dividends are reinvested to purchase additional units of equity or unit trust at the closing price application on the ex-dividend date.

Observing **Table 1**, a reader with a keen eye will notice that most stock excess returns present distribution signs of abnormality, a characteristic that it is reinforced by the values of the Jarque-Bera test performed for each stock and the index. This is a goodness of fit test of whether sample data have the skewness and kurtosis matching a normal distribution. If the data comes from a normal distribution, the J-B statistic asymptotically has a chi-squared distribution with two degrees of freedom, so the statistic can be used to test the hypothesis that the data are from a normal distribution. The hypothesis of the test where:

H_o: $J - B = \frac{n}{6}(S^2 + \frac{1}{4}(K-3)^2) \sim X_2^2$ w*hich signifies that the excess returns distribution is normal and*

H_1: $J - B = \frac{n}{6}(S^2 + \frac{1}{4}(K-3)^2) \nsim X_2^2$ *which signifies that the excess returns follow a non-normal distribution*

Where n is the number of observations, S^2 the sample variance, K the sample kurtosis and X_2^2 the chi-squared distribution with two degrees of freedom.

The null hypothesis is rejected for all distributions tested at 95% confidence level indicating that we could not work under the normality assumption for our observations. Stochastic dominance proves to be the tool of choice to work towards portfolio optimization under those condition as it does not require a normal distribution to operate appropriately and thus we do not have to try to normalize the returns a process that may result in loss of information.

The mean and variance of the excess returns are relatively small, indicating that the selected stocks should be classified as "blue chips" belonging to the most reliable and stable stocks representing each category. It should, as well, be noted that the are evidence of strong leptokurtic distributions while only a handful presents mesokurtic characteristics and in addition to the negative skew that is presented allows us to understand the reason for relative small variances and means.

5.2 Description of the Stochastic Dominance test

We employ the Scaillet and Topaloglou (2010) test for Stochastic Dominance Efficiency. Let the asset returns be described by a strictly stationary process {Yt} taking values in Rn. The observations consist of a realization of {Yt ; t = 1,...,T}. We denote by F(y), the continuous cumulative distribution function of Y=(Y1,…Yn)' at point y=(y1,…yn)'. Let a portfolio consisting of n assets and the vector λ of portfolio weights in L, where L = {λ ∈ Rn: e' λ=1} with e being a vector of units. Let G(z,λ;F) denote the cumulative density function of the portfolio return λ'Y at portfolio return point z given by

$$G(z,\lambda;F) := \int_{R^n} I\{\lambda' u \le z\} dF(u)$$

where *I*( ) denotes the indicator function taking the value of 1 if $\lambda' u \le z$ and 0 otherwise and z is a given risk level. Define for z ∈R:

$$J_1(z,\lambda;F) := G(z,\lambda;F)$$

$$J_2(z,\lambda;F) := \int_{-\infty}^{z} G(u,\lambda;F)du = \int_{-\infty}^{z} J_1(u,\lambda;F)du$$

Following Scaillet and Topaloglou (2010) work the hypothesis for testing SDE of order j (j=1 for first SDE and j=2 for second SDE) can be written compactly as:

$$H_0^j : J_j(z,\tau;F) \le J_j(z,\lambda;F) \quad \forall z \in \mathbb{R} \text{ and } \forall \lambda \in L,$$

$$H_1^j : J_j(z,\tau;F) > J_j(z,\lambda;F) \text{ for some } z \in \mathbb{R} \text{ and some } \lambda \in L$$

Under the null hypothesis, H_0^j there is no portfolio λ formed from the set of assets that dominates the benchmark τ at any order j, i.e. the reference portfolio τ is stochastic dominance efficient. In this case, the function $J_j(z,\tau;F)$ is always lower than the function $J_j(z,\lambda;F)$ for any possible portfolio λ constructed from the set of alternative assets for any point z. Under the alternative hypothesis H_1^j, we can build a portfolio λ that for some points z, the function $J_j(z,\tau;F)$ is greater than the function $J_j(z,\lambda;F)$, i.e. the benchmark portfolio τ is not SDE.

We test the null hypothesis by employing the Scaillet and Topaloglou (2010) test which uses a $\hat{S}_j$ Kolmogorov-Smirnov type test statistic of order j:

$$\hat{S}_j := \sqrt{T}\frac{1}{T} sup_{z,\lambda}[J_j(z,\tau;\hat{F}) - J_j(z,\lambda;\hat{F})]$$

where $\hat{F}$ is the empirical distribution of *F*. We reject if where H_0^j if $\hat{S}_j$ $>c_j$ is some critical value (for the test properties, see Scaillet and

Topaloglou, 2010). Given that the distribution of $\hat{S}_j$ is not known, we calculate the *p*-value corresponding to c_j by bootstrap. We use Abadie's (2002) block bootstrap method.

In the case where the null hypothesis is rejected we trace the stochastic dominance efficient portfolio. We will now describe the method that we use to identify said portfolio by Daskalaki, Skiadopoulos, and Topaloglou (2016). Portfolio λ is termed to dominate a benchmark portfolio τ under the first order and second order stochastic dominance efficiency criteria (FSDE, SDEE) respectively if it satisfies the following respective equations

$$Max_{z,\lambda}[G(z,\tau;F)-G(z,\lambda;F)]$$

$$Max_{z,\lambda}[\int_{-\infty}^{z} G(u,\tau;F)du-\int_{-\infty}^{z} G(u,\lambda;F)du]$$

The resulting portfolio is also termed efficient. Therefore, as we have already outlined in previous sections, a portfolio is defined to be effective when it stochastically dominates all other portfolios constructed from a given asset universe for any given stochastic dominance efficiency criterion under consideration. Notice that the construction of optimal portfolios under the FSDE and SSDE criteria does not require an assumption of the specific form of a utility function. This is because both SDE criteria are consistent with a broad class of utility functions. FSDE is appropriate for both risk lovers and risk averters and permits a preliminary screening of investment alternatives eliminating those which no rational investor will ever choose. The SSDE criterion adds the assumption of global risk aversion.

5.3 Portfolio optimization under FSDE

To identify the portfolios that are stochastic dominance efficient under the first degree of stochastic dominance we employee the Scaillet and Topaloglou (2010) test as it has been presented in the previous section. We will guide the reader through the mechanics of the test that we employed to identify the efficient portfolios and construct the optimized under FSDE portfolio.

5.3.1 Mathematical formulation of the test

To test for first order stochastic dominance efficiency (FSDE), we optimize the test statistic

$$\hat{S}_1 := \sqrt{T}\,\tfrac{1}{T}\,sup_{z,\lambda}[J_1(z,\tau;\widehat{F}) - J_1(z,\lambda;\widehat{F})]$$

The above formulation permits testing the dominance of a given portfolio strategy τ over any potential linear combination λ of the set of the available assets. Hence, we implement a test of stochastic dominance efficiency and not a standard stochastic dominance. The mathematical formulation of the problem is the following:

$$Max_{z,\lambda} S_1 = \sqrt{T}\,\frac{1}{T}\sum_{t=1}^{T}[L_t - Q_t]$$

such that

$$M(L_t - 1) \le z - \tau' Y_t \le ML_t, \forall t$$

$$M(Q_t - 1) \le z - \lambda' Y_t \le MQ_t, \forall t$$

$$e'\lambda = 1$$

$$Q_t \in \{0,1\}, L_t \in \{0,1\}, \forall t$$

Where M is the greatest portfolio return. The model is a mixed integer program maximizing the distance between two binary variables

$\frac{1}{T}\sum_{t=1}^{T} L_t$, $\frac{1}{T}\sum_{t=1}^{T} Q_t$, which represent $J_1(z,\tau;\hat{F})$ and $J_1(z,\lambda;\hat{F})$ respectively; sums are taken over all possible values of portfolio returns. According to the first inequality, Lt equals 1 for each return t for which $z \geq \tau' Y_t$, and 0 otherwise. Similarly, the second inequality ensures that Q_t equals 1 for each scenario for which $z \geq \lambda' Yt$. These two inequalities ensure that the two binary variables are cumulative distribution functions. The third equation defines the sum of all weights to be unity.

To solve the problem, we discretize the variable z, and we solve smaller problems P(r) in which z is fixed to a given return r. Then, we take the value for z that yields the maximum distance

It takes six hours to solve the model and yield the efficient portfolios for all 29 iterations that were employed. The problem is solved with Gurobi solver on a dual core HP Pavilion with 2.20 GHz Intel i5 processor and4 GB of RAM. The Gurobi solver uses the branch and bound technique. We model the optimization problems by using GAMS (General Algebraic Modeling System).

5.3.2 Model implementation and results

Following the description of the Scaillet and Topaloglou (2010) test, we implement it in the mathematical form that we presented previously as it has been formulated by Daskalaki, Skiadopoulos, and Topaloglou (2016). We simulated the application of the model in real market conditions through in sample and out of sample analysis. This was achieved by separating our dataset in two periods, before and after January 1st, 2014, the model was executed for the first time containing historical monthly data from all the years preceding that date up to January 1st, 2000. That would result in 168 observations for 30 shares plus the index for its first execution. For each following month and up to May 31st, 2016, the model was executed again including the observation of the actual realized excessive returns that the individual shares and the market yielded the previous month, increasing its level of information. That resulted in 29 iterations in total, increasing the number of observation gradually from 168 initially to 196 on its last execution that would be on May 1st, 2016.

In every execution, the model provided a number of results equal to the number of observations that were provided for the specific iteration. Even following the rejection of the portfolios that presented a negative test statistic the number of optimal portfolios for each iteration under FSDE was vast, as it was expected. To tackle this, we choose to calculate the mean optimal portfolio for each iteration, under the restriction of a possitive test statistic; that resulted in one portfolio optimized under FSDE per period representing the average outcome of optimization.

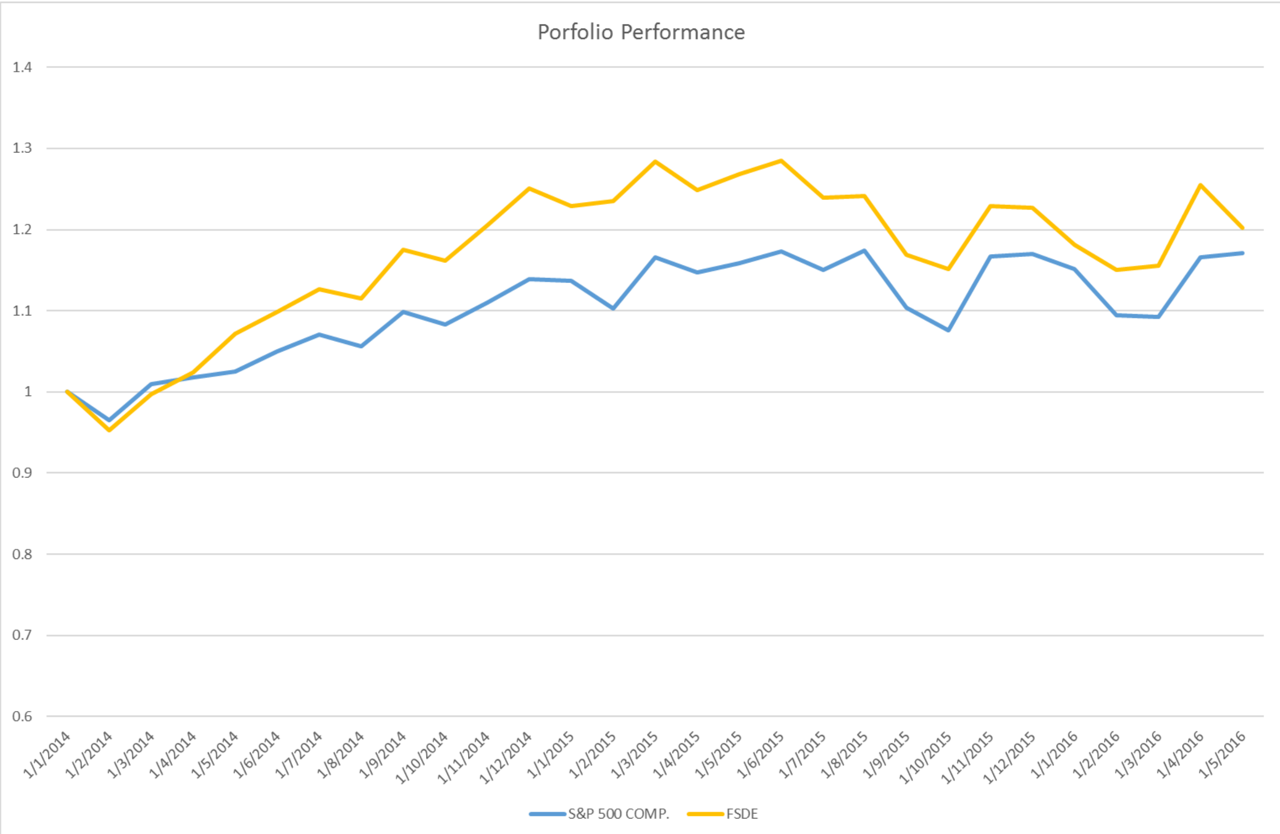

Diagram 8 - FSDE vs Index Portfolio performance

The excessive returns of the portfolio optimized under FSDE compared to the S&P500 index for the same period are being displayed on **Diagram 8**. A table presenting index and FSDE excessive returns per month is available on appendix for further consideration. The chart shows the expected value that a dollar invested in either the index or the FSDE portfolio will have in each period taking into account the excessive returns that each investment option offers. Analyzing the graph, we will notice that the optimized portfolio offers at most periods higher value, as a result of higher excessive returns, to the investor when compared to the market portfolio. This comes at a cost as a reader with a keen eye will notice that is more volatile since the line that represents the FSDE portfolio exhibits far more extreme fluctuations when compared to the smoother S&P500 representation. To that effect we may attribute the underperformance of the FSDE portfolio during the first four months of its life. As our portfolio has included more volatile stocks, it tends to magnify the market performance, as a results our first results are to magnify the negative market performance and from that point on, use that information on our model as well as monthly rebalancing to achieve the desired result.

A set of portfolio performance measures has been calculated and is available for the reader in appendix; we will expand further on those and discuss our findings in the following chapter.

5.4 Portfolio optimization under SSDE

Once again we will use the Scaillet and Topaloglou (2010) test as it has been presented in a previous section to identify the SSDE optimal portfolios. We will guide the reader through the mathematical formulation of the test that we employed to determine the efficient portfolios and construct the optimized under SSDE portfolio.

5.4.1 Mathematical formulation of the test

The model for second order stochastic dominance efficiency is formulated regarding standard linear programming. Numerical implementation of first-order stochastic dominance efficiency is much more computationally demanding because we need to develop mixed integer programming formulations. To test for second order stochastic dominance efficiency (SSDE) of portfolio τ over any potential linear combination λ, we optimize the test statistic

$$\hat{S}_2 := \sqrt{T}\,\frac{1}{T}\,sup_{z,\lambda}[J_2(z,\tau;\widehat{F}) - J_2(z,\lambda;\widehat{F})]$$

The mathematical formulation of the problem is the following:

$$Max_{z,\lambda} S_2 = \sqrt{T}\,\frac{1}{T}\sum_{t=1}^{T}[L_t - W_t]$$

such that

$$M(F_t - 1) \le z - \tau' Y_t \le MF_t$$

$$-M(1-F_t) \le L_t - (z - \tau' Y_t) \le M(1-F_t)$$

$$-MF_t \le L_t \le MF_t$$

$$W_t \ge z - \lambda' Y_t$$

$$e'\lambda = 1$$

$$W_t \ge 0, F_t \in [0,1] \forall t$$

The model is a linear programming maximizing the distance between the sum of all scenarios of two variables, $\frac{1}{T}\sum_{t=1}^{T} L_t - \frac{1}{T}\sum_{t=1}^{T} W_t$ for each given value of z, which represent $J_2(z,\tau;\hat{F})$ and $J_2(z,\lambda;\hat{F})$ respectively. According to the first inequality, F_t equals 1 for each return t for which $z \ge \tau' Y$, and 0 otherwise. Analogously, the second and third inequalities ensure that the variable L_t equals $z - \tau' Y_t$ for the scenarios for which the difference is positive, and 0 otherwise. The fourth and last inequalities ensure that W_t equals $z - \lambda' Y_t$ for the scenarios for which the difference is positive, and 0 otherwise. The fifth equation defines the sum of all weights to be unity.

It takes only 10 minutes to solve the model and yield the efficient portfolios for all 29 iterations that were employed. The problem is solved with Gurobi solver on a dual core HP Pavilion with 2.20 GHz Intel i5 processor and 4 GB of RAM. The Gurobi solver uses the branch and bound technique. We model the optimization problems by using GAMS (General Algebraic Modeling System).

5.4.2 Model implementation and results

We followed the mathematical form that we presented previously as it has been formulated by Daskalaki, Skiadopoulos, and Topaloglou (2016). We used the same dataset segmentation as stated for the FSDE and we run the model for 29 iterations yielding one efficient portfolio per period. This significant reduction in the number of efficient solutions that were determined by the first and second order of stochastic dominance efficiency indicates the important difference in results that may be achieved simply by increasing our level of information by one level.

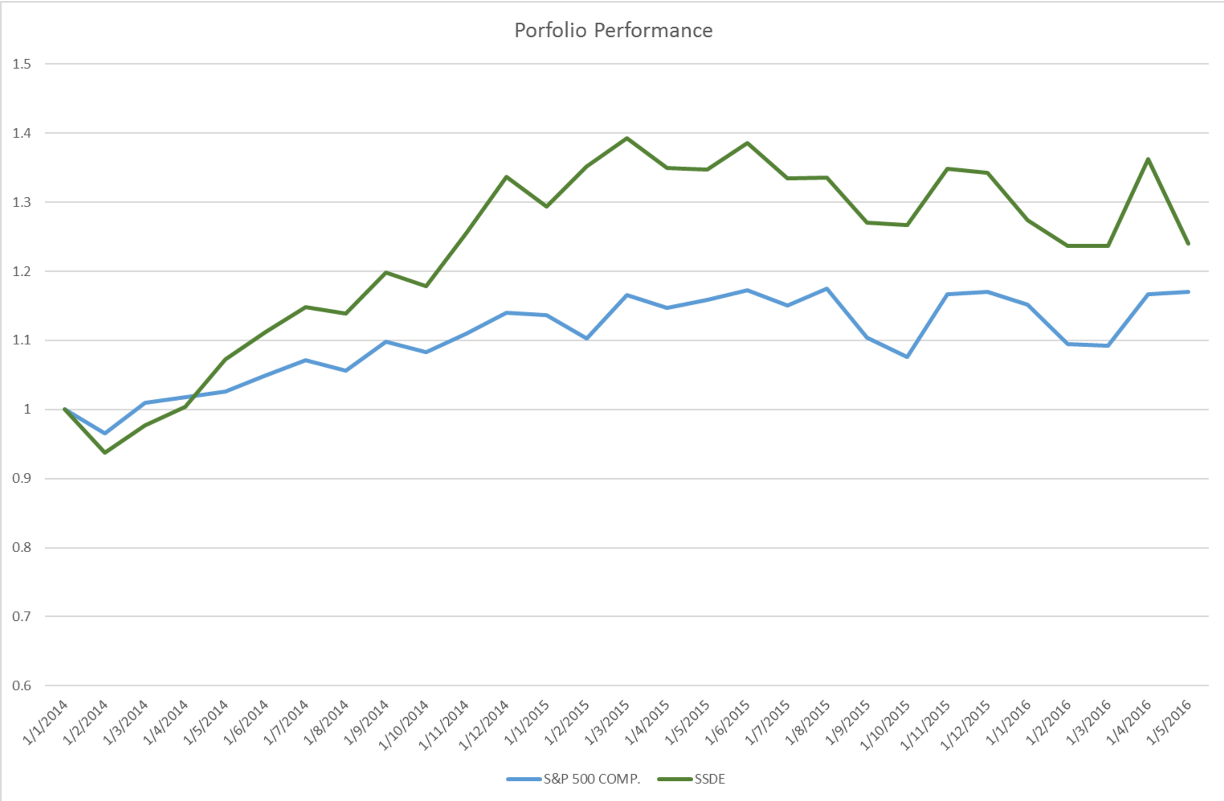

Diagram 9 - SSDE vs. Index portfolio performance

Diagram 9 plots the monthly performance of the SSDE optimized portfolio when compared to the index performance for the same period. The performance is measured by displaying the returns that an investor would accumulate for every one dollar he chooses to invest in either our optimized portfolio or the index taking into consideration the excessive returns both investment options provide.

It should be noted that our expectations are verified and the SSDE optimized portfolio not only outperforms the index most of the time, but there are also strong indications of outperforming the FSDE optimized portfolio as it would be expected. Once more, and this time more easily, we notice that the optimized portfolio exhibits higher volatility than the market, this trait allows us to capitalize higher gains on bull market conditions, but it will result in stronger than market losses in bear market. This is the cause of the underperformance of our portfolio during the first four months of its life. Once the model increases its level of information and there is a tendency for higher returns our SSDE optimized portfolio will result in a higher than market profit level. Furthermore, the constant rebalancing will ensure that our investor's income is protected against negative market movements, and the premium they luxuriate compensates the risk they are undertaking.

A more thorough analysis on the performance of each portfolio accompanied by its relative measures will follow in the next chapter.

5.5 Performance evaluation measures

Following work by DeMiguel et al. (2009), Kostakis et al. (2010) and Daskalaki and Skiadopoulos (2011) we employ five commonly used parametric performance measures: the Sharpe ratio (SR), opportunity cost, portfolio turnover, a measure of the portfolio risk-adjusted returns net of transaction costs and upside potential and downside risk ratio (UPratio) proposed by Sortino and van der Meer (1991).

To fix ideas, let a specific strategy denoted by c. The estimate of the strategy's SR_c is defined as the fraction of the sample mean of out-of-sample excess returns $\widehat{\mu_c}$ divided by their sample standard deviation $\widehat{\sigma_c}$, i.e.

$$\widehat{SR}_c = \frac{\widehat{\mu_c}}{\hat{\sigma}_c}$$

To test whether the SRs of the two portfolio strategies based on the index and stochastically optimized portfolio are statistically different, we use the statistic proposed by Jobson and Korkie (1981) and corrected by Memmel (2003). The use of SR is in line with the finance industry practice however it is suitable to assess the performance of a strategy only in the case where the strategy's returns are normally distributed.

Next, we use the concept of opportunity cost (Simaan, 1993) to assess the economic significance of the difference in performance of the two optimal portfolios, respectively. Denote by r_{sd},r_m ,the optimal portfolio realized returns obtained by an investor with the optimized portfolio under stochastic dominance and the investment opportunity set restricted to tracking the index, respectively. The opportunity cost θ is defined to be the return that needs to be added (or subtracted) to the portfolio return r_m so that the investor becomes indifferent (in utility terms) between the two strategies imposed by the different investment opportunity sets, i.e.

$$\mathrm{E}\left[U\left(1+r_m+\theta\right)\right]=\mathrm{E}\left[U\left(1+r_{sd}\right)\right]$$

Therefore, a positive (negative) opportunity cost implies that the investor is perceived to be better (worse) off in case of investing in a stochastic dominance optimized the portfolio. Notice that the opportunity cost takes into account all the characteristics of the utility function and hence it is suitable to evaluate strategies even when the assets return distribution is not normal. To calculate the opportunity cost, we use an exponential and a power utility function alternatively.

The portfolio turnover (PT) is computed so as to get a feel of the degree of rebalancing required to implement each one of the two strategies. For any portfolio strategy c, the portfolio turnover, PT_c is defined as the average absolute change in the weights over the T-K (T-168) rebalancing periods in time and across the N (30) available assets, i.e.

$$PT_c=\frac{1}{T-K}\sum_{t=1}^{T-K}\sum_{j=1}^{N}(|\, w_{c,j,t+1}-w_{c,j,t+}\,|)$$

where $w_{c,j,t}$, $w_{c,j,t+1}$ are the derived optimal weights of asset j under strategy c at time t and t+1, respectively; $w_{c,j,t+}$ is the portfolio weight before the rebalancing at time t+1; the quantity $|w_{c,j,t+1} - w_{c,j,t+}|$ shows the magnitude of trade needed for asset j at the rebalancing point t+1. The PT quantity can be interpreted as the average fraction (in percentage terms) of the portfolio value that has to be reallocated over the whole period.

We also evaluate the two investment strategies under the risk-adjusted, net of transaction costs; returns measure proposed by DeMiguel et al. (2009). This metric provides an economic interpretation of the PT; it shows how the proportional transaction costs generated by the portfolio turnover affect the returns from any given strategy. To fix ideas, let pc be the proportional transaction cost and $r_{c,p,r+1}$ the realized portfolio return at t+1 (before rebalancing). The evolution of the net of transaction costs wealth NW_c for strategy c, is given by:

$$NW_{c,t+1} = NW_{c,t}(1 + r_{c,p,t+1})\left[1 - pc \times \sum_{j=1}^{N}(|w_{c,j,t+1} - w_{c,j,t+}|)\right]$$

Therefore, the return net of transaction costs is defined as:

$$RNTC_{c,t+1} = \frac{NW_{c,t+1}}{NW_{c,t}} - 1$$

The return-loss measure is calculated as the additional return needed for the index tracking investor to perform as well as the optimized portfolios. Let μ_m, μ_{sd} the monthly out-of-sample mean of RNTC from the strategy with the market and the optimized portfolios, respectively, and σ_m, σ_{sd}, be the corresponding standard deviations. Then, the return-loss measure is given by:

$$return - loss = \frac{\mu_{sd}}{\sigma_{sd}} \times \sigma_m - \mu_m$$

To calculate $NW_{c,t+1}$, we set the proportional transaction cost pc equal to 50 basis points per transaction for stocks and the index (for a similar choice, see DeMiguel et al., 2009).

Finally, we calculated the upside potential and downside risk ratio (UPratio) proposed by Sortino and van der Meer (1991). This ratio contrasts the average excess return on some target with a measure of the shortfall from the same benchmark, as suggested by Sortino et al. (1999). We use the S&P500 index as the benchmark. Let r_{sd} be the realized return of a portfolio in month t = 1,…,n of the model; n = 29 is the number of months in the simulation period 01/2014-05/2016. Let r_m be the return of the benchmark (S&P500 index) at the same period. Then the UPratio is

$$UP_{ratio} = \frac{\frac{1}{n}\sum_{t=1}^{n} \max\left[0, r_{sd,t} - r_{m,t}\right]}{\sqrt{\frac{1}{n}\sum_{t=1}^{n} (\max\left[0, r_{m,t} - r_{sd,t}\right])^2}}$$

5.6 Discussion on Results

In this section, we will proceed in describing and commenting on our results. All tables and charts that are mentioned throughout this and following sections are provided for the reader in the appendix of the thesis.

We used the Scailet and Topaloglou (2010) test for stochastic dominance to test if null hypothesis that the constructed portfolio stochastically dominates the benchmark set holds true. The test statistics returned for each iteration of the trial for the both degrees of stochastic dominance are available in appendix **Table 5**. We determine that we accept that the null hypothesis holds true meaning that the optimized under FSDE and SSDE portfolios dominate the benchmark market portfolio represented here by the S&P 500 index. This holds true both for the first and second degree of stochastic dominance.

Once, we have constructed the optimal portfolios and verified through the test above that they stochastically dominate the benchmark portfolio we note, the excessive returns that are achieved during the out-of-sample period for our portfolios and the reference index. These data are available in appendix **Table 6**, and their graphical plot has been included in **Diagram 10**. As it may be noted in most periods, the optimized portfolio achieves returns significantly higher than the index. Even in occasions where they underperform the index the cumulative gain of previous periods in synergy with the higher returns of following periods, are high enough to ensure a constant growth in investors income greater than the one provided by the index.

To verify said deduction and present it in a more comprehensive manner we calculated the monetary value of an investment in the index and both our portfolios in dollars. These data are available in **Table 7**, and their graphical plot in **Diagram 11** both included in the following pages. Once we deconstruct the difficulty of comparing percentile returns and convert them to actual dollar values, we start to mold a picture regarding the performance of the three proposed investment options and is easier to compare the outcome that an individual will enjoy. The optimized portfolios provide our investor with a higher income when compared to the market portfolio thus constituting a better investment option. Unfortunately, the investor to achieve higher returns would have to accept a higher level of volatility in his future income, which is compensated by the higher level of returns he enjoys in every possible scenario.

Following this, we compare the out-of-sample performance of the two optimal portfolios and the benchmark using the five standard measures of performance. **Table 8** reports results for each one of the five performance measures. In the case of opportunity cost, we assume various levels of (absolute/relative) risk aversion (ARA, RRA=2,4,6) for the individual investor. To assess the statistical significance of the superiority in SRs, we also report the p-values of Memmel's (2003) noted as JKM test p-value. The null hypothesis is that the SRs obtained from the benchmark portfolio, and the optimized portfolios are equal.

Considering the Sharpe ratio of all three investment opportunities, results determine that a stock portfolio optimized with either first or second order of stochastic dominance be a preferable investment relative to the market index. When comparing the ratios between the two optimized portfolios, it becomes evident that the SSDE investment provides a higher return for every unit of volatility that the investor undertakes as a risk. JKM test p-values only strengthen our results and verify that indeed there is a statistically significant difference between the Sharpe ratio of the benchmark, FSDE and SSDE portfolios; a fact that justifies investor preference of a stochastically optimized portfolio over the benchmark when only the Sharpe ratio is being used as a decision tool.

Shifting our attention to the opportunity cost, we notice that it is positive on most occasions. This indicates that an investor would require a premium to substitute an investment in a stochastically optimized portfolio with an index tracking investment. It worth to notice thought that the opportunity cost follows two different paths in the portfolio that have been optimized with the first and a portfolio that has been optimized with the second degree of stochastic dominance. Since the opportunity cost increases as the degree of risk aversion increases for the FSDE portfolio, it is easy to assume that an investor would require a higher premium and would be less willing to trade a portfolio that has been optimized using FSDE with an index replicating portfolio. In this occasion, the volatility of the expected returns is not high enough to avert an individual from our portfolio; instead, the skewness of the expected returns is such that compensate the investor for the extra risk he is undertaking by investing in our portfolio.

On the other hand, the opportunity cost for the SSDE portfolio retreats as we increase the level of risk aversion, in these case the investor would require a decreasing premium as his loathe of risk increases to select our portfolio over the markets and he would even require a premium for ARA of 6 in order to invest in our portfolio than the index. This should be attributed to the higher volatility of expected returns presented by the portfolio that has been optimized with the second order of stochastic dominance. Although higher returns are promised to an investor and up to a point those returns would be high enough to make him rather select our portfolio than markets, as its risk aversion increases the skewness of the expected returns is not sufficient to compensate for the perceived risk and the investor would feel more comfortable with the less volatile market portfolio.

Following, we examine the return-loss measure that takes into account transaction costs. The sign of this measure is rather important as it indicates the actual performance of our portfolio net of transaction costs; which in our occasion are expected to have a significant impact on the investor's future income as a result of the monthly rebalancing of the portfolio. The positive sign here confirms that both our optimized portfolios are superior to the index, even when we deduct the cash outflow generated by the transactions needed to maintain our positions. It is worth noticing that although the FSDE portfolio has a lower turnover when compared to the SSDE portfolio, the returns net of transaction costs indicates that a good proportion of the SSDE portfolio superiority is lost in the more aggressive transaction while the FSDE portfolio achieves a better economic overall result.

Finally, we compare the two portfolios using the UP_{ratio} proposed by Sortino et al. which shows a strong preference towards the SSDE optimized portfolio when compared to the FSDE portfolio $UP_{ratio.}$ This may be expected as the SSDE portfolio yields significantly higher returns when compared to the benchmark index portfolio.

Chapter 6

Conclusions

On this final section, we will comment on the impact of our findings in the portfolio construction process as has been presented in previous chapters and mention a few other paths that may be pursued in the future in the direction of portfolio optimization through stochastic dominance.

6.1 Impact on portfolio management

As we have seen previously portfolio management is a process that consists of many stages and follows one of two main strategies, active or passive management.

Our findings have a significant impact on this process. As we have demonstrated that a portfolio optimized with stochastic dominance, either of first or second degree will always result in a portfolio that outperforms the index and will yield returns that will be preferable to most investors. This result allows the fund managers to replicate our process and try to optimize their portfolios while rebalancing the weights of the selected shares monthly always to include the latest market information to the investment set. Examining the outcome from the investor point of view, we can now understand why somebody who does not possess the skills, the equipment or the capital to perform such an analysis would be willing to invest in shares of a mutual fund that would be managed by a professional and ensure profits such as those achieved by our portfolios.

Throughout this thesis, we outlaid some basic concepts that are needed to understand the variables that affect an investor motives in a choice of investment strategy, motives that in turn guide portfolio managers to choices as their primary goal is to provide value to their customer through acceptable risk undertaking. This signifies that although a fund manager may be willing to accept an additional level of risk as his expertise, experience and knowledge suggests ensuring a higher level of return, he will be restrained up to a point by his goal of maintaining a happy clientele.

Considering that our optimized portfolios outperform the market suggests that active portfolio management and optimization through SD degrees will ensure the manager a satisfied clientele, higher than market returns and a prosperous fund. Introducing this tool in investment management, the fund management achieves all of his primary and secondary goals, while it is easy to amend the model and the yielded portfolio to meet separate investor needs

Those above suggest that there is a strong call for active rather than passive portfolio management as it allows the model to adjust better to new information while as we demonstrated the transaction costs remain at acceptable levels and does not significantly affect the net returns.

6.2 What lies ahead

On our final section, which may be viewed as an epilog, we will suggest some different approaches to the same optimization problem we may explore in the future to achieve even the same or higher returns while decreasing the incurred investment risk.

Before we move any further, it is imperative to mention that judging our final results it is evident that we have achieved the main goal of this thesis. We have managed to create and optimize a portfolio that was based on only 30 out of 500 shares listed on S&P500, which we then optimized using SD and achieve an investment proposal that it is at all times preferable to the index; in essence, we have beaten the market. Our portfolio besides the theoretical outperformance of the market was a viable and profitable investment choice even considering the transaction costs that we had to undertake to maintain our optimized positions. This, though, didn't come at no cost as it became apparent that to achieve those higher returns we had to accept with them a higher volatility level of our future income.

Higher volatility for higher returns is something that should be expected during the portfolio construction phase since returns are in essence our premium for undertaking the increased risk. However, there is room for further improvement in our model which may allow us to reduce even further our volatility levels. Firstly, we should recall that the choice of the stocks to compose our portfolio was arbitrary. We have simply selected the three stocks per sector with the higher market capitalization, but there is no scientific ground for not selecting stocks with mid or low capitalization. If in the future, we choose to enforce our model to all 500 shares of the S&P500 we most likely will achieve very different results. We know now, as our proof of concept demonstrated, that we should expect a portfolio yielding higher returns than the index at all times but we do not have a clear indication of the course that the volatility will follow. This exercise may achieve a portfolio with even lower volatility while still outperforming the market or a portfolio as volatile as our but with significantly higher returns.

An analyst may even choose to reduce portfolio volatility by selecting to include in the portfolio some fixed income securities or indices and then optimize said portfolio using stochastic dominance.

The returns still are expected to be higher than the market returns while the investor will enjoy a more stable future income level. The actual results may be the subject of future research.

On a final note, we should stress out that stochastic dominance has been proven to be a powerful tool in finance which allows optimization on different levels, with minimum information and should be used even from the security selection phase.

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Appendix: Charts & Tables

In the following pages, the reader may find the complete tables and charts that are mentioned in chapters 5 and 6 of this thesis.

Information on the methodology and applications used to extrapolate the data are provided in the form of a short description in a section preceding each page. A complete list of tables and a list of diagrams is available in the pages following the contents at the start of the thesis.

Table 2 - Detail Data Descriptive Statistics

This table provides the full data descriptive statistics analysis data performed on the initial observations consisting of the monthly excess returns on the market and shares. The data were calculated using Data Analysis tool of Microsoft Excel.

Sector	Stock	Mean	Std. Error	Median	Std. Dev	Variance	Kurtosis	Skewness	Range
Index	S&P500	0.00301	0.00311	0.00876	0.04372	0.00191	0.89694	-0.49643	0.27760
Consumer Discr.	Home Depot	0.00640	0.00540	0.00902	0.07578	0.00574	0.24418	-0.19245	0.42765
	The Walt Disney Company	0.00861	0.00527	0.01235	0.07395	0.00547	1.50566	-0.25099	0.50669
Consumer Staples	Wal-Mart Stores	0.00176	0.00400	0.00356	0.05609	0.00315	1.25228	-0.24394	0.36517
	Procter & Gamble	0.00431	0.00379	0.00726	0.05314	0.00282	10.89232	-1.82913	0.47813
	The Coca Cola Company	0.00424	0.00367	0.00502	0.05154	0.00266	1.12645	-0.40933	0.30897
	PepsiCo Inc.	0.00692	0.00323	0.00673	0.04534	0.00206	2.90578	-0.37426	0.39278
Energy	Exxon Mobil Corp.	0.00593	0.00354	0.00260	0.04973	0.00247	1.65243	0.33431	0.35029
	Chevron Corp.	0.00764	0.00432	0.01139	0.06065	0.00368	0.80543	0.16356	0.38694
	Schlumberger Ltd.	0.00928	0.00663	0.00791	0.09310	0.00867	1.20861	-0.08916	0.64727
Financials	JPMorgan Chase & Co.	0.00654	0.00652	0.00839	0.09158	0.00839	0.99984	-0.18672	0.53566
	Bank of America Corp	0.00566	0.00857	0.00302	0.12030	0.01447	9.07154	0.49227	1.26410
	Citigroup Inc.	-0.00224	0.00898	0.00371	0.12600	0.01588	8.80171	0.36610	1.26418
Health Care	Johnson & Johnson	0.00636	0.00338	0.00775	0.04745	0.00225	1.68151	-0.27687	0.33472
	Pfizer Inc.	0.00341	0.00413	0.00165	0.05796	0.00336	-0.04522	-0.11255	0.32687
	Merck & Co.	0.00367	0.00520	0.00570	0.07302	0.00533	1.16066	-0.22220	0.46452
Industrials	General Electric	0.00144	0.00549	-0.00036	0.07700	0.00593	1.59984	-0.15057	0.52815
	3M Company	0.00860	0.00413	0.01013	0.05791	0.00335	0.83304	0.10193	0.34779
	United Technologies	0.00822	0.00464	0.00849	0.06508	0.00424	3.45162	-0.64050	0.55796
Inform. Tech.	Apple Inc.	0.02419	0.00881	0.02871	0.12360	0.01528	3.28243	-0.64943	1.03219
	Microsoft Corp.	0.00400	0.00639	0.01143	0.08970	0.00805	3.00356	0.32042	0.75195
	International Bus. Machines	0.00436	0.00524	0.00572	0.07349	0.00540	4.10527	0.60198	0.58037
Materials	Dow Chemical	0.00815	0.00807	-0.00355	0.11328	0.01283	19.43227	2.35658	1.28033
	Du Pont (E.I.)	0.00472	0.00556	0.00949	0.07807	0.00609	1.57330	0.29857	0.52336
	Praxair Inc.	0.00958	0.00441	0.01187	0.06195	0.00384	2.43827	0.04053	0.43395
Telecom. Services	AT&T Inc	0.00375	0.00474	0.00279	0.06658	0.00443	2.07812	0.15379	0.48110
	Verizon Communications	0.00427	0.00489	0.00309	0.06864	0.00471	5.31860	0.92408	0.60573
	CenturyLink Inc	0.00255	0.00578	0.00998	0.08117	0.00659	5.25392	-0.05593	0.75249
Utilities	Duke Energy	0.00753	0.00446	0.01469	0.06258	0.00392	3.41374	-0.71021	0.49026
	Southern Co.	0.01004	0.00334	0.01342	0.04688	0.00220	3.28570	0.23941	0.36481
	Exelon Corp.	0.00715	0.00443	0.00553	0.06219	0.00387	1.77096	-0.11982	0.43699

Table 3 - Detail Data Descriptive Statistics Cont'd

Sector	Stock	Min.	Max.	Sum	Obs.	Largest(4)	Smallest(1)	Confidence Level(95,0%)	J-B
Index	S&P500	-0.16832	0.10928	0.59258	197	0.08911	-0.16832	0.00614	44.39561
Consumer Discr.	Home Depot	-0.20744	0.22021	1.26089	197	0.16044	-0.20744	0.01065	63.55438
	The Walt Disney Company	-0.26973	0.23696	1.69547	197	0.18583	-0.26973	0.01039	20.39798
Consumer Staples	Wal-Mart Stores	-0.21245	0.15271	0.34683	197	0.14265	-0.21245	0.00788	27.02654
	Procter & Gamble	-0.36169	0.11644	0.84896	197	0.09741	-0.36169	0.00747	621.13747
	The Coca Cola Company	-0.16716	0.14181	0.83474	197	0.10973	-0.16716	0.00724	34.31398
	PepsiCo Inc.	-0.20045	0.19233	1.36324	197	0.09101	-0.20045	0.00637	4.67182
Energy	Exxon Mobil Corp.	-0.11965	0.23064	1.16729	197	0.11278	-0.11965	0.00699	18.57561
	Chevron Corp.	-0.15393	0.23301	1.50454	197	0.14679	-0.15393	0.00852	40.41085
	Schlumberger Ltd.	-0.33895	0.30832	1.82909	197	0.20848	-0.33895	0.01308	26.60233
Financials	JPMorgan Chase & Co.	-0.28195	0.25371	1.28761	197	0.21734	-0.28195	0.01287	33.98321
	Bank of America Corp	-0.53287	0.73123	1.11576	197	0.28232	-0.53287	0.01690	310.54523
	Citigroup Inc.	-0.57768	0.68649	-0.44147	197	0.23324	-0.57768	0.01770	280.69183
Health Care	Johnson & Johnson	-0.16494	0.16978	1.25315	197	0.10241	-0.16494	0.00667	16.78628
	Pfizer Inc.	-0.17693	0.14994	0.67234	197	0.11111	-0.17693	0.00814	76.53472
	Merck & Co.	-0.26131	0.20321	0.72360	197	0.16853	-0.26131	0.01026	29.39136
Industrials	General Electric	-0.27703	0.25112	0.28296	197	0.17450	-0.27703	0.01082	16.83632
	3M Company	-0.14594	0.20186	1.69433	197	0.15314	-0.14594	0.00814	38.88519
	United Technologies	-0.32211	0.23584	1.61957	197	0.13586	-0.32211	0.00914	15.14365
Inform. Tech.	Apple Inc.	-0.58235	0.44984	4.76600	197	0.26140	-0.58235	0.01737	14.50230
	Microsoft Corp.	-0.34813	0.40382	0.78858	197	0.23570	-0.34813	0.01260	3.37110
	International Bus. Machines	-0.22773	0.35264	0.85817	197	0.17659	-0.22773	0.01033	21.92557
Materials	Dow Chemical	-0.38245	0.89788	1.60480	197	0.23136	-0.38245	0.01592	2398.74776
	Du Pont (E.I.)	-0.20807	0.31528	0.92929	197	0.20264	-0.20807	0.01097	19.63466
	Praxair Inc.	-0.19880	0.23515	1.88663	197	0.14239	-0.19880	0.00870	2.64402
Telecom. Services	AT&T Inc	-0.18883	0.29227	0.73938	197	0.15420	-0.18883	0.00936	7.75240
	Verizon Communications	-0.21446	0.39128	0.84023	197	0.13638	-0.21446	0.00964	72.16425
	CenturyLink Inc	-0.34467	0.40783	0.50182	197	0.14239	-0.34467	0.01140	41.80238
Utilities	Duke Energy	-0.27261	0.21765	1.48372	197	0.14129	-0.27261	0.00879	17.96622
	Southern Co.	-0.12919	0.23562	1.97806	197	0.11287	-0.12919	0.00659	2.55191
	Exelon Corp.	-0.18509	0.25190	1.40878	197	0.12968	-0.18509	0.00874	12.87037

Table 4 - Variance-Covariance Matrix

This table provides the variance-covariance matrix of the initial observations consisting of the monthly excess returns on the market and shares. The data were calculated using Data Analysis tool of Microsoft Excel.

	Apple Inc.	Microsoft Corp.	Exxon Mobil Corp.	Johnson & Johnson	General Electric	AT&T Inc	JPMorgan Chase & Co.	Wal-Mart Stores	Procter & Gamble	Pfizer Inc.	Verizon Communications	The Coca Cola Company	Chevron Corp.	Home Depot	The Walt Disney Company
Apple Inc.	0.01520	0.00507	0.00108	-0.00001	0.00251	0.00024	0.00330	0.00019	-0.00081	0.00007	0.00126	0.00012	0.00221	0.00266	0.00315
Microsoft Corp.	0.00507	0.00800	0.00097	0.00053	0.00289	0.00173	0.00338	0.00111	-0.00027	0.00096	0.00280	0.00098	0.00173	0.00219	0.00205
Exxon Mobil Corp.	0.00108	0.00097	0.00246	0.00060	0.00131	0.00108	0.00107	0.00018	0.00059	0.00096	0.00126	0.00083	0.00226	0.00056	0.00115
Johnson & Johnson	-0.00001	0.00053	0.00060	0.00224	0.00138	0.00076	0.00051	0.00077	0.00113	0.00124	0.00076	0.00107	0.00060	0.00052	0.00082
General Electric	0.00251	0.00289	0.00131	0.00138	0.00590	0.00177	0.00350	0.00091	0.00096	0.00159	0.00157	0.00116	0.00184	0.00251	0.00307
AT&T Inc	0.00024	0.00173	0.00108	0.00076	0.00177	0.00441	0.00117	0.00104	0.00067	0.00114	0.00337	0.00107	0.00116	0.00123	0.00112
JPMorgan Chase & Co.	0.00330	0.00338	0.00107	0.00051	0.00350	0.00117	0.00834	0.00112	0.00065	0.00182	0.00133	0.00103	0.00171	0.00334	0.00325
Wal-Mart Stores	0.00019	0.00111	0.00018	0.00077	0.00091	0.00104	0.00112	0.00313	0.00021	0.00073	0.00124	0.00065	0.00040	0.00174	0.00039
Procter & Gamble	-0.00081	-0.00027	0.00059	0.00113	0.00096	0.00067	0.00065	0.00021	0.00281	0.00068	0.00006	0.00107	0.00033	0.00027	0.00024
Pfizer Inc.	0.00007	0.00096	0.00096	0.00124	0.00159	0.00114	0.00182	0.00073	0.00068	0.00334	0.00138	0.00097	0.00110	0.00106	0.00159
Verizon Communications	0.00126	0.00280	0.00126	0.00076	0.00157	0.00337	0.00133	0.00124	0.00006	0.00138	0.00469	0.00097	0.00138	0.00119	0.00159
The Coca Cola Company	0.00012	0.00098	0.00083	0.00107	0.00116	0.00107	0.00103	0.00065	0.00107	0.00097	0.00097	0.00264	0.00097	0.00069	0.00099
Chevron Corp.	0.00221	0.00173	0.00226	0.00060	0.00184	0.00116	0.00171	0.00040	0.00033	0.00110	0.00138	0.00097	0.00366	0.00127	0.00172
Home Depot	0.00266	0.00219	0.00056	0.00052	0.00251	0.00123	0.00334	0.00174	0.00027	0.00106	0.00119	0.00069	0.00127	0.00571	0.00212
The Walt Disney Company	0.00315	0.00205	0.00115	0.00082	0.00307	0.00112	0.00325	0.00039	0.00024	0.00159	0.00159	0.00099	0.00172	0.00212	0.00544
Merck & Co.	-0.00044	0.00105	0.00102	0.00149	0.00115	0.00180	0.00093	0.00049	0.00088	0.00225	0.00201	0.00130	0.00069	-0.00006	0.00139
Bank of America Corp	0.00271	0.00310	0.00086	0.00129	0.00515	0.00094	0.00752	0.00113	0.00117	0.00257	0.00090	0.00099	0.00184	0.00327	0.00361
International Bus. Machines	0.00429	0.00335	0.00095	0.00060	0.00237	0.00114	0.00321	0.00093	-0.00019	0.00083	0.00177	0.00039	0.00141	0.00202	0.00227
PepsiCo Inc.	0.00001	0.00073	0.00045	0.00117	0.00122	0.00130	0.00110	0.00077	0.00078	0.00105	0.00123	0.00124	0.00061	0.00083	0.00113
Citigroup Inc.	0.00371	0.00409	0.00167	0.00177	0.00581	0.00162	0.00783	0.00146	0.00117	0.00301	0.00171	0.00154	0.00263	0.00360	0.00454
Schlumberger Ltd.	0.00371	0.00175	0.00221	0.00048	0.00274	0.00093	0.00305	0.00000	0.00031	0.00088	0.00083	0.00069	0.00291	0.00171	0.00277
3M Company	0.00139	0.00089	0.00087	0.00100	0.00221	0.00077	0.00195	0.00055	0.00096	0.00086	0.00083	0.00093	0.00136	0.00182	0.00188
United Technologies	0.00178	0.00164	0.00113	0.00090	0.00275	0.00109	0.00247	0.00074	0.00067	0.00127	0.00105	0.00061	0.00175	0.00205	0.00255
Dow Chemical	0.00274	0.00280	0.00135	0.00119	0.00515	0.00180	0.00537	0.00108	0.00152	0.00149	0.00159	0.00135	0.00219	0.00296	0.00429
Du Pont (E.I.)	0.00236	0.00248	0.00147	0.00128	0.00376	0.00135	0.00377	0.00086	0.00093	0.00151	0.00157	0.00123	0.00208	0.00241	0.00323
Duke Energy	-0.00001	0.00026	0.00105	0.00082	0.00113	0.00151	0.00113	0.00035	0.00070	0.00111	0.00132	0.00092	0.00130	0.00079	0.00096
Southern Co.	-0.00027	-0.00073	0.00051	0.00072	0.00041	0.00041	-0.00021	0.00007	0.00087	0.00049	0.00030	0.00042	0.00058	0.00019	0.00012
Exelon Corp.	0.00004	-0.00014	0.00103	0.00066	0.00069	0.00092	0.00000	-0.00001	0.00076	0.00060	0.00090	0.00087	0.00099	0.00012	0.00108
Praxair Inc.	0.00298	0.00157	0.00099	0.00077	0.00228	0.00094	0.00241	0.00074	0.00041	0.00085	0.00121	0.00069	0.00195	0.00171	0.00201
CenturyLink Inc	0.00073	0.00239	0.00119	0.00072	0.00160	0.00231	0.00152	0.00073	0.00074	0.00071	0.00254	0.00155	0.00164	0.00126	0.00136
S&PCOMP(RI)	0.00275	0.00232	0.00103	0.00081	0.00246	0.00125	0.00271	0.00076	0.00051	0.00122	0.00138	0.00087	0.00150	0.00197	0.00223

	Merck & Co.	Bank of America Corp	International Bus. Machines	PepsiCo Inc.	Citigroup Inc.	Schlumberger Ltd.	3M Company	United Technologies	Dow Chemical	Du Pont (E.I.)	Duke Energy	Southern Co.	Exelon Corp.	Praxair Inc.	CenturyLink Inc	S&PCOMP(RI)
Apple Inc.	-0.00044	0.00271	0.00429	0.00001	0.00371	0.00371	0.00139	0.00178	0.00274	0.00236	-0.00001	-0.00027	0.00004	0.00298	0.00073	0.00275
Microsoft Corp.	0.00105	0.00310	0.00335	0.00073	0.00409	0.00175	0.00089	0.00164	0.00280	0.00248	0.00026	-0.00073	-0.00014	0.00157	0.00239	0.00232
Exxon Mobil Corp.	0.00102	0.00086	0.00095	0.00045	0.00167	0.00221	0.00087	0.00113	0.00135	0.00147	0.00105	0.00051	0.00103	0.00099	0.00119	0.00103
Johnson & Johnson	0.00149	0.00129	0.00060	0.00117	0.00177	0.00048	0.00100	0.00090	0.00119	0.00128	0.00082	0.00072	0.00066	0.00077	0.00072	0.00081
General Electric	0.00115	0.00515	0.00237	0.00122	0.00581	0.00274	0.00221	0.00275	0.00515	0.00376	0.00113	0.00041	0.00069	0.00228	0.00160	0.00246
AT&T Inc	0.00180	0.00094	0.00114	0.00130	0.00162	0.00093	0.00077	0.00109	0.00180	0.00135	0.00151	0.00041	0.00092	0.00094	0.00231	0.00125
JPMorgan Chase & Co.	0.00093	0.00752	0.00321	0.00110	0.00783	0.00305	0.00195	0.00247	0.00537	0.00377	0.00113	-0.00021	0.00000	0.00241	0.00152	0.00271
Wal-Mart Stores	0.00049	0.00113	0.00093	0.00077	0.00146	0.00000	0.00055	0.00074	0.00108	0.00086	0.00035	0.00007	-0.00001	0.00074	0.00073	0.00076
Procter & Gamble	0.00088	0.00117	-0.00019	0.00078	0.00117	0.00031	0.00096	0.00067	0.00152	0.00093	0.00070	0.00087	0.00076	0.00041	0.00074	0.00051
Pfizer Inc.	0.00225	0.00257	0.00083	0.00105	0.00301	0.00088	0.00086	0.00127	0.00149	0.00151	0.00111	0.00049	0.00060	0.00085	0.00071	0.00122
Verizon Communications	0.00201	0.00090	0.00177	0.00123	0.00171	0.00083	0.00083	0.00105	0.00159	0.00157	0.00132	0.00030	0.00090	0.00121	0.00254	0.00138
The Coca Cola Company	0.00130	0.00099	0.00039	0.00124	0.00154	0.00069	0.00093	0.00061	0.00135	0.00123	0.00092	0.00042	0.00087	0.00069	0.00155	0.00087
Chevron Corp.	0.00069	0.00184	0.00141	0.00061	0.00263	0.00291	0.00136	0.00175	0.00219	0.00208	0.00130	0.00058	0.00099	0.00195	0.00164	0.00150
Home Depot	-0.00006	0.00327	0.00202	0.00083	0.00360	0.00171	0.00182	0.00205	0.00296	0.00241	0.00079	0.00019	0.00012	0.00171	0.00126	0.00197
The Walt Disney Company	0.00139	0.00361	0.00227	0.00113	0.00454	0.00277	0.00188	0.00255	0.00429	0.00323	0.00096	0.00012	0.00108	0.00201	0.00136	0.00223
Merck & Co.	0.00530	0.00140	0.00067	0.00114	0.00230	0.00083	0.00073	0.00107	0.00145	0.00166	0.00133	0.00052	0.00135	0.00071	0.00163	0.00108
Bank of America Corp	0.00140	0.01440	0.00224	0.00163	0.01207	0.00258	0.00265	0.00318	0.00691	0.00464	0.00093	0.00059	0.00006	0.00254	0.00158	0.00286
International Bus. Machines	0.00067	0.00224	0.00537	0.00037	0.00348	0.00215	0.00118	0.00151	0.00190	0.00222	0.00090	0.00000	0.00016	0.00177	0.00143	0.00197
PepsiCo Inc.	0.00114	0.00163	0.00037	0.00204	0.00184	0.00064	0.00096	0.00088	0.00100	0.00123	0.00090	0.00048	0.00063	0.00078	0.00151	0.00084
Citigroup Inc.	0.00230	0.01207	0.00348	0.00184	0.01579	0.00376	0.00298	0.00391	0.00735	0.00554	0.00155	0.00050	0.00069	0.00306	0.00229	0.00364
Schlumberger Ltd.	0.00083	0.00258	0.00215	0.00064	0.00376	0.00862	0.00200	0.00243	0.00396	0.00275	0.00098	0.00020	0.00138	0.00228	0.00147	0.00233
3M Company	0.00073	0.00265	0.00118	0.00096	0.00298	0.00200	0.00334	0.00202	0.00355	0.00283	0.00054	0.00039	0.00055	0.00187	0.00131	0.00146
United Technologies	0.00107	0.00318	0.00151	0.00088	0.00391	0.00243	0.00202	0.00421	0.00364	0.00304	0.00079	0.00026	0.00096	0.00235	0.00147	0.00188
Dow Chemical	0.00145	0.00691	0.00190	0.00100	0.00735	0.00396	0.00355	0.00364	0.01277	0.00635	0.00064	-0.00044	0.00070	0.00355	0.00240	0.00302
Du Pont (E.I.)	0.00166	0.00464	0.00222	0.00123	0.00554	0.00275	0.00283	0.00304	0.00635	0.00606	0.00060	-0.00007	0.00061	0.00288	0.00282	0.00242
Duke Energy	0.00133	0.00093	0.00090	0.00090	0.00155	0.00098	0.00054	0.00079	0.00064	0.00060	0.00390	0.00177	0.00203	0.00082	0.00074	0.00079
Southern Co.	0.00052	0.00059	0.00000	0.00048	0.00050	0.00020	0.00039	0.00026	-0.00044	-0.00007	0.00177	0.00219	0.00163	0.00029	-0.00027	0.00014
Exelon Corp.	0.00135	0.00006	0.00016	0.00063	0.00069	0.00138	0.00055	0.00096	0.00070	0.00061	0.00203	0.00163	0.00385	0.00070	0.00039	0.00059
Praxair Inc.	0.00071	0.00254	0.00177	0.00078	0.00306	0.00228	0.00187	0.00235	0.00355	0.00288	0.00082	0.00029	0.00070	0.00382	0.00157	0.00169
CenturyLink Inc	0.00163	0.00158	0.00143	0.00151	0.00229	0.00147	0.00131	0.00147	0.00240	0.00282	0.00074	-0.00027	0.00039	0.00157	0.00655	0.00157
S&PCOMP(RI)	0.00108	0.00286	0.00197	0.00084	0.00364	0.00233	0.00146	0.00188	0.00302	0.00242	0.00079	0.00014	0.00059	0.00169	0.00157	0.00190

Table 5 - FSDE and SSDE Portfolio test statistics

In the following table we display the Scailet and Topaloglou (2010) test statistics for the first and second order of stochastic dominance for the null hypothesis that the stochastically optimized portfolio generated by each iteration dominates on the specific period the benchmark portfolio (S&P500 index)

Date	*FSDE Test Statistc*	*SSDE Test Statistic*
31/1/2014	*0.01049*	*0.01567*
28/2/2014	*0.01042*	*0.01552*
31/3/2014	*0.01031*	*0.01541*
30/4/2014	*0.01044*	*0.01538*
30/5/2014	*0.01072*	*0.01609*
30/6/2014	*0.01065*	*0.01586*
31/7/2014	*0.01063*	*0.01647*
29/8/2014	*0.01064*	*0.01648*
30/9/2014	*0.01057*	*0.01583*
31/10/2014	*0.01061*	*0.01573*
28/11/2014	*0.01059*	*0.01587*
31/12/2014	*0.01068*	*0.01645*
30/1/2015	*0.01058*	*0.0165*
27/2/2015	*0.01103*	*0.01682*
31/3/2015	*0.01089*	*0.01632*
30/4/2015	*0.01066*	*0.01615*
29/5/2015	*0.01051*	*0.016*
30/6/2015	*0.01053*	*0.016*
31/7/2015	*0.01040*	*0.01582*
31/8/2015	*0.01033*	*0.01563*
30/9/2015	*0.01023*	*0.0156*
30/10/2015	*0.01022*	*0.01564*
30/11/2015	*0.01005*	*0.01545*
31/12/2015	*0.01009*	*0.01533*
29/1/2016	*0.00975*	*0.01501*
29/2/2016	*0.00984*	*0.01503*
31/3/2016	*0.00959*	*0.01512*
29/4/2016	*0.00978*	*0.01573*
31/5/2016	*0.00945*	*0.01583*

Table 6 - Portfolio Monthly Excessive Returns

This table provides the excessive returns in percentage that each portfolio (S&P500, FSDE Optimized, SSDE Optimized) yielded in the period 31/1/2014 to 31/5/2016

Date	S&P 500 COMP.	FSDE Optimized	SSDE Optimized
31/1/2014	-3.4593%	-4.7509%	-6.2972%
28/2/2014	4.5704%	4.6628%	4.2770%
31/3/2014	0.8362%	2.7413%	2.7469%
30/4/2014	0.7366%	4.6473%	6.7914%
30/5/2014	2.3443%	2.4972%	3.6876%
30/6/2014	2.0623%	2.5927%	3.2459%
31/7/2014	-1.3817%	-1.0237%	-0.7935%
29/8/2014	3.9981%	5.3670%	5.2489%
30/9/2014	-1.4040%	-1.1485%	-1.6244%
31/10/2014	2.4416%	3.7421%	6.4947%
28/11/2014	2.6877%	3.7706%	6.4268%
31/12/2014	-0.2553%	-1.7852%	-3.1922%
30/1/2015	-3.0037%	0.5449%	4.5098%
27/2/2015	5.7457%	3.9540%	2.9949%
31/3/2015	-1.5841%	-2.7210%	-3.0764%
30/4/2015	0.9586%	1.5114%	-0.1362%
29/5/2015	1.2850%	1.3256%	2.8045%
30/6/2015	-1.9367%	-3.5417%	-3.6784%
31/7/2015	2.0885%	0.2125%	0.1215%
31/8/2015	-6.0401%	-5.8819%	-4.9487%
30/9/2015	-2.4735%	-1.5182%	-0.2439%
30/10/2015	8.4288%	6.8091%	6.4624%
30/11/2015	0.2790%	-0.1912%	-0.4769%
31/12/2015	-1.5905%	-3.7032%	-5.0666%
29/1/2016	-4.9889%	-2.6369%	-2.9852%
29/2/2016	-0.1624%	0.4689%	0.0417%
31/3/2016	6.7662%	8.5676%	10.1673%
29/4/2016	0.3695%	-4.2396%	-8.9733%
31/5/2016	1.7674%	1.9222%	4.6254%

Diagram 10 - Portfolio Monthly Excess Returns

The following chart plots a graphical representation of the yielded excess returns that were presented in **Table 6**.

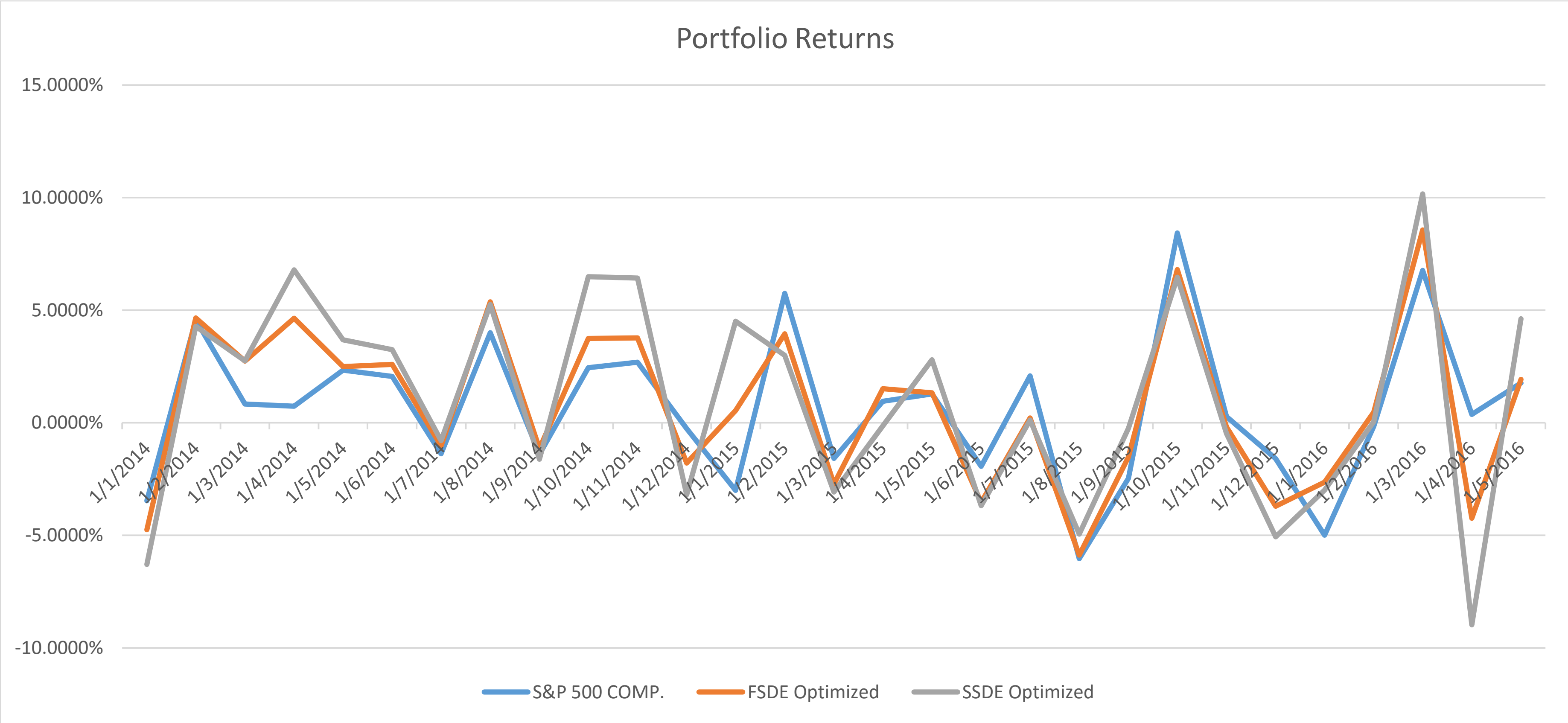

Table 7 - Portfolio dollar performance

In this section, we express the monetary value, in dollars, of the excess returns that an investor will accumulate through our separate portfolios. It should be noted that although the stochastically optimized portfolios underperform when compared to the market in the first four months, the money outcome from the fifth period and onwards is always positive for the investor.

Date	S&P 500 COMP.	FSDE Optimized	SSDE Optimized
31/1/2014	$ 0.97	$ 0.95	$ 0.94
28/2/2014	$ 1.01	$ 1.00	$ 0.98
31/3/2014	$ 1.02	$ 1.02	$ 1.00
30/4/2014	$ 1.03	$ 1.07	$ 1.07
30/5/2014	$ 1.05	$ 1.10	$ 1.11
30/6/2014	$ 1.07	$ 1.13	$ 1.15
31/7/2014	$ 1.06	$ 1.12	$ 1.14
29/8/2014	$ 1.10	$ 1.18	$ 1.20
30/9/2014	$ 1.08	$ 1.16	$ 1.18
31/10/2014	$ 1.11	$ 1.21	$ 1.26
28/11/2014	$ 1.14	$ 1.25	$ 1.34
31/12/2014	$ 1.14	$ 1.23	$ 1.29
30/1/2015	$ 1.10	$ 1.24	$ 1.35
27/2/2015	$ 1.17	$ 1.28	$ 1.39
31/3/2015	$ 1.15	$ 1.25	$ 1.35
30/4/2015	$ 1.16	$ 1.27	$ 1.35
29/5/2015	$ 1.17	$ 1.28	$ 1.39
30/6/2015	$ 1.15	$ 1.24	$ 1.33
31/7/2015	$ 1.17	$ 1.24	$ 1.34
31/8/2015	$ 1.10	$ 1.17	$ 1.27
30/9/2015	$ 1.08	$ 1.15	$ 1.27
30/10/2015	$ 1.17	$ 1.23	$ 1.35
30/11/2015	$ 1.17	$ 1.23	$ 1.34
31/12/2015	$ 1.15	$ 1.18	$ 1.27
29/1/2016	$ 1.09	$ 1.15	$ 1.24
29/2/2016	$ 1.09	$ 1.16	$ 1.24
31/3/2016	$ 1.17	$ 1.25	$ 1.36
29/4/2016	$ 1.17	$ 1.20	$ 1.24
31/5/2016	$ 1.19	$ 1.22	$ 1.30

Diagram 11 - Portfolio Dollar Performance

The following chart plots a graphical representation of the yielded dollar returns that were presented in **Table 7** for easier comparison between the fulfilment of each portfolio. It is becoming obvious that a stochastically optimized portfolio will most of the time ensure higher monetary returns for an investor.

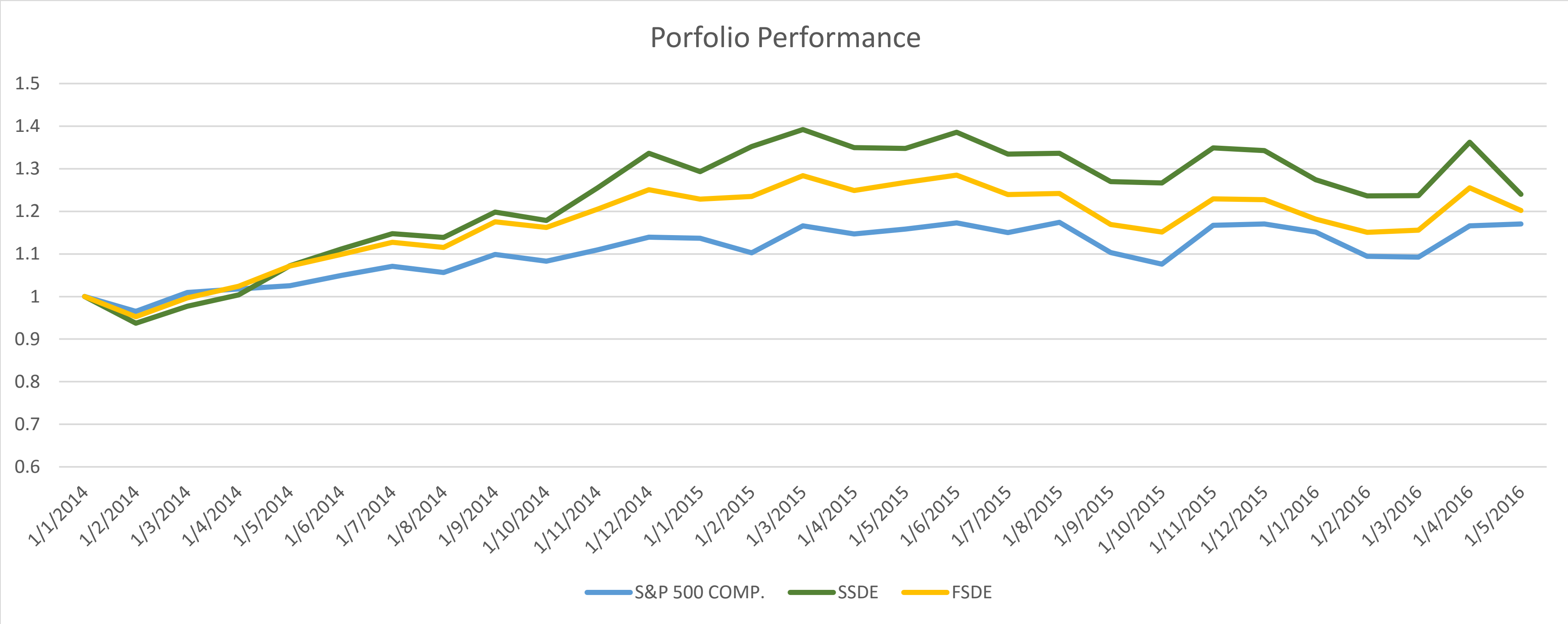

Table 8 - Portfolio Performance Measures

Entries report the performance measures (annualized Sharpe Ratio, annualized Opportunity Cost, Portfolio Turnover, annualized Return-Loss, U-P Ratio) when the benchmark set includes the S&P 500 Equity Index. The p-values of Memmel's (2003) test are also reported; the null hypothesis is that the SR obtained from the index is equal to that derived from the optimized portfolios. The results for the opportunity cost are reported for different degrees of absolute risk aversion (ARA=2,4,6) and varying degrees of relative risk aversion (RRA=2,4,6). The dataset spans the period from Jan. 2000-May. 2016. The out-of-sample analysis is conducted over the period from Jan. 2014-May. 2016.

Metric		S&P500	FSDE Optimized	SSDE Optimized
Sharpe Ratio		0.33541	0.3692	0.4243
JKM Test P-value			0.0362	0.0017
Portfolio Turnover			6.87%	9.33%
Return-Loss			13.24%	10.32%
UPratio			59.79%	239.86%
Opportunity Cost				
Exponential Utility				
	ARA=2		10.72%	6.68%
	ARA=4		18.55%	1.55%
	ARA=6		26.37%	-3.59%
Power Utility				
	RRA=2		10.72%	6.68%
	RRA=4		18.55%	1.55%
	RRA=6		26.37%	-3.59%