BINOMIAL TESTS FOR SEQUENCES OF POINTS FALLING BETWEEN SYMMETRICAL QUALITY CONTROL LIMITS

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The active presence of assignable causes affecting the population mean of variables, fraction defective and number of defects can be signalled by binomial tests conducted on sequences of points falling in the band between symmetrical control limits of the statistical quality control charts Np, p, C and $\bar{X}$. Three FORTRAN IV computer programs produce the critical values needed for the application of such tests which assume that constant size samples are used and certain population parameters are known.

1. *INTRODUCTION*

During the last 50 years various statistical methods have been employed in the service of quality control[1]. The well-known control charts, proposed originally by Shewhart (1931), are still used extensively having proved very effective and simple in their construction and interpretation.

A typical control chart is formed by three parallel lines of which the central one is drawn at a level corresponding to the population mean; the upper and lower lines are the so called upper control limit (UCL) and lower control limit (LCL) respectively.

A production process is tested for the active presence of assignable causes by drawing random samples and determining whether the observed critical quality characteristics appear in the area demarcated by the control limits; if they do, the statistical decision is made that assignable causes are not present. In the case of sample observations falling outside the limits, the presence of assignable causes is statistically accepted.

The two control limit lines are either symmetrically located at S standard deviation units on each side of the central line or spaced in such a way that —under conditions of statistical control— the probability of an observation falling outside the upper limit equals the probability

1. See, for instance, Burr (1976).

of falling outside the lower limit. The limits of the first case are called symmetrical while those of the second are termed probability limits.

The occurence of a sample point between the limits is never a guarantee that no assignable causes are in action at the time of drawing the sample. The same is true in the case where a sequence of such points is found between the limits. This means that the active presence of assignable causes might be signalled not only by points falling outside the limits but also by conducting suitable statistical tests on the distribution of those found between the limits. As a matter of fact the well-known theory of runs[2] has been used for such investigations[3].

The present paper refers to the same problem; it concerns the distribution above and below the central line of the control chart of a sequence of points, all lying between the control limits, the purpose being to identify extremities that indicate the presence of assignable causes. The relevant statistical analysis is based on the binomial distribution; it makes use of the three control lines mentioned above —which are common in control charting— and works explicitly with the probability of any point falling between the central line and the upper control limit when all the assignable causes remain inactive.

This paper does not take into account the number or length of runs and it does not require that the central line be necessarily at median values.

The whole investigation refers to the control charts Np, p, C and $\overline{X}$, their control limits being at S standard deviation units on each side of the central line — with the constraint that the lower control line of the first three charts cannot be set at negative levels.

2. *THE Np AND p CONTROL CHARTS FOR ATTRIBUTES*

The designed quality characteristics of manufactured products are observed as variables or attributes. Variables are those characteristics which in practice can be measured and expressed in numbers by appropriate measuring means. Attributes are not measurable; in this case each inspected item is simply classified into two classes, conforming and non-conforming to the attribute specifications. However, all the quality characteristics belonging to the category of variables may be techni-

2. Mood (1940).
3. Mosteller (1941).

cally changed to attributes; this is done by «go-not go» gages or other similar classifying mechanisms.

In the case where the inspected item is classified to the non-conforming class, with respect to one or more attributes, the item is termed defective. Considering the universe of product units that under given conditions can be produced, a certain number of them will inevitably be defective because of the uncontrolled role of a stable system of numerous, independent and unidentified chance causes associated with the production process; the corresponding fraction of such defective units has the standard value π. In the short run —when no technological, managerial or other changes take place— π deviates from its standard level only as a result of systematic effects exercised by assignable causes. Thus, the product quality is tested by investigating the active presence of assignable causes. To this end, random samples of size n are drawn from the production line, the defective units contained are identified, counted and the corresponding fraction of defectives is calculated. Statistical theory is employed to test the significance of the sample findings. The basic tools, in this respect, are the statistical distribution of number or fraction of defective units and the corresponding control charts.

The distribution of number and fraction of defectives

The number W of defective units in each sample, in case all kinds of assignable causes are absent, is a binomial random variable, $b(n, \pi)$, having the following discrete density function:

$$\begin{aligned} f_W(w) &= f_W(w; n, \pi) \\ &= \binom{n}{w} \cdot \pi^w \cdot (1-\pi)^{n-w} \qquad \text{for } w = 0, 1, \ldots, n \qquad (1) \end{aligned}$$

where π satisfies $0 \leqslant \pi \leqslant 1$ and n ranges over the positive integers. The mean and variance of W are respectively $n\pi$ and $n\pi(1-\pi)$.

The sample fraction defective $P = W/n$ is distributed as follows:

$$\begin{aligned} f_P(p) &= f_P(p; n, \pi) \\ &= \binom{n}{w} \cdot (\pi)^w \cdot (1-\pi)^{n-w} \qquad \text{for } w = 0, 1, \ldots, n \qquad (2) \end{aligned}$$

where $p = w/n = 0/n, \ldots, n/n$. The mean and variance of P are respectively π and $\pi(1-\pi)/n$.

Symmetrical control limits

The symmetrical control limits of the Np chart —the control chart for the number of defective units— are placed about the central line at equal distances of S standard deviation units:

$$
\begin{aligned}
UCL_w &= n\pi + S\sqrt{n\pi(1-\pi)} \\
\text{central value} &= n\pi \qquad\qquad (3)\\
LCL_w &= n\pi - S\sqrt{n\pi(1-\pi)}
\end{aligned}
$$

The corresponding control limits of the p chart —control chart for the fraction of defectives— are drawn at the following levels:

$$
\begin{aligned}
UCL_p &= \pi + S\sqrt{\pi(1-\pi)/n} \\
\text{central value} &= \pi \qquad\qquad (4)\\
LCL_p &= \pi - S\sqrt{\pi(1-\pi)/n}
\end{aligned}
$$

The symmetrical control limits of both charts are not probability limits (except in the rare case of $\pi = 1/2$). In practice S is usually set equal to 3.

Points between control limits and their distribution

It has already been pointed out that a succession of random samples producing a series of points, all falling in the area between control limits of one and the same chart, may be used to diagnose the presence of active assignable causes. The distribution of such points in the two successive areas demarcated by the three control lines is used for this purpose. The theoretical foundation of the relevant analysis is summarized by the theorem and corollary that follow.

THEOREM I. If the random variable W is binomial, $b(n, \pi)$, then every increase (decrease) in π (ceteris paribus) leads to an increase (decrease) in the ratio

$$\pi_3 = \frac{\pi_2}{\pi_1} \qquad (5)$$

where:

$$P_r(b_1 \leqslant w \leqslant b_2) = \pi_1 > 0$$

$$P_r(b_2 < w \leqslant b_3) = \pi_2 > 0$$

$$0 < b_1 < b_2 < b_3 \leqslant n$$

A proof of this theorem was given by Tzorztopoulos (1980). Putting $b_1 = LCL_W$, $b_2 = n\pi$ and $b_3 = UCL_W$, the following specific definitions are obtained:

$$\pi_1 = \Pr(LCL_W \leqslant w \leqslant n\pi) = \Pr(LCL_p \leqslant p \leqslant \pi)$$

$$\pi_2 = \Pr(n\pi < w \leqslant UCL_W) = \Pr(\pi < p \leqslant UCL_p)$$

By means of theorem I it is easy to show that the following corollary will always hold.

COROLLARY. If the random variable W is binomial, $b(n, \pi)$, then every increase (decrease) in π (ceteris paribus) leads to an increase (decrease) in of the ratio

$$\pi_4 = \frac{\pi_2}{\pi_1 + \pi_2} \tag{6}$$

This corollary may also be proved independently of theorem I by use of the first derivative of (1) and (2):

$$\frac{\partial f_W(w)}{\partial \pi} = \binom{n}{w} \cdot \pi^w \cdot (1 - \pi)^{n-w} \cdot \left(\frac{w - n\pi}{\pi(1 - \pi)} \right) \tag{7}$$

$$\frac{\partial f_p(p)}{\partial \pi} = \binom{n}{np} \cdot \pi^{np} \cdot (1 - \pi)^{n-np} \cdot \frac{n(p - \pi)}{\pi(1 - \pi)} \tag{8}$$

Apparently, (7) and (8) are positive for $w > n\pi$ and $p > \pi$; they are negative for $w < n\pi$ and $p < \pi$.

Now, assuming that N independent samples of size n are drawn consecutively and that their corresponding values of W fall in the area $LCL_W \leqslant w \leqslant UCL_W$, for each sample the binary variable Z is defined so that

$$\begin{aligned} Z_i &= 1 \quad \text{if } n\pi < w \leqslant UCL_W \quad \text{or} \quad \pi < p \leqslant UCL_p \\ Z_i &= 0 \quad \text{if } LCL_W \leqslant w \leqslant n\pi \quad \text{or} \quad LCL_p \leqslant p \leqslant \pi \end{aligned} \tag{9}$$

$$i = 1, 2, \ldots, N$$

Z follows the point binomial or Bernoulli distribution:

$$f_Z(z) = (\pi_4)^z \cdot (1 - \pi_4)^{1-z} \qquad \text{for } z = 0,1 \qquad (10)$$

where π_4 is defined by (6) and varies in the range between 0 and 1.

For the N samples taken together, the sum

$$Y = \sum_{i=1}^{i=N} Z_i \qquad (11)$$

is binomially distributed, $b(N, \pi_4)$,

$$f_Y(y) = f_Y(y; N, \pi_4)$$

$$= \binom{N}{y} \cdot \pi_4{}^y \cdot (1 - \pi_4)^{N-y} \qquad \text{for } y = 0, 1, \ldots\ldots, N \qquad (12)$$

As it will be shown in the sequel, Y is the basic random variable of the present investigation.

Testing for the presence of assignable causes

When searching for the presence of assignable causes using N_p or p control charts, the following hypotheses are usually tested:

$$H_0 : E(W) = n\pi \text{ against } H_1 : E(W) \neq n\pi \qquad (13)$$

or

$$H_0 : E(P) = \pi \text{ against } H_1 : E(P) \neq \pi \qquad (14)$$

The alternative hypotheses may well be $H_1 : E(W) > n\pi$, $H_1 : E(W) < n\pi$, $H_1 : E(P) > \pi$ and $H_1 : E(P) < \pi$. The relevant statistical tests may be called single-point tests, because they are carried out each time a single point of the Np or p charts is produced; H_0 is accepted if the point in question falls in the area between the control limits —otherwise it is rejected. Equivalent testing work can be conducted by the same charts using the following hypotheses:

$$H_0 : E(Y) = N\pi_4 \qquad \text{against } H_1 : E(Y) \neq N\pi_4 \qquad (15)$$

or

$$H_0 : E(Y/N) = \pi_4 \qquad \text{against } H_1 : E(Y/N) \neq \pi_4 \qquad (16)$$

The testing of (15) and (16) requires N chart points, all forming a sequence located between the control limits. Therefore, if each of N successive points fails to reject H_0 of (13) or (14), the corresponding hypotheses (15) or (16) are put to the test; now H_0 is rejected if, given the available number of successive points, the value of Y is extremely small or large for predetermined probability levels α– otherwise H_0 is accepted.

The two-sided rejection regions of H_0— for both (15) and (16)— are:

$$Y \leqslant -1 + Y_{\alpha/2} \tag{17}$$

$$Y \geqslant 1 + Y_{1-\alpha/2} \tag{18}$$

where, $Pr(Y < Y_{\alpha/2}) \leqslant \alpha/2$, $Pr(Y > Y_{1-\alpha/2}) \leqslant a/2$ and α expresses the size of type I error associated with H_0 of (15) or (16).

If $1 + Y_{1-\alpha/2} > N$ or $-1 + Y_{\alpha/2} < 0$, the corresponding rejection regions do not exist.

One-sided tests may also be applied by means of the rejection regions (17) or (18) with half the value of α– the alternative hypotheses must be redefined accordingly ($H_1 : E(Y) > N\pi_4$, $H_1 : E(Y) < N\pi_4$, $H_1 : E(Y/N) > \pi_4$, $H_1 : E(Y/N) < \pi_4$).

In practice, the application of multi-point Y tests requires a proper control chart, a series of N consecutive points— all falling between the control limits— and the critical values $1 + Y_{1-\alpha/2}$ and $-1 + Y_{\alpha/2}$. Such values depend on π, n, N, π_4, S and α; their computation requires extensive work involving the families of distributions (1) or (2) and (12) and can be carried out accurately and quickly by an appropriate computer program[4]. Such a program, called here BINOM I, has been developped by the author and is shown in Appendix B.

The computer program BINOM I[5]

BINOM I is a program written in FORTRAN IV and concerns the Np and p charts which have S-sigma symmetrical control limits; it consists of the main program and the subroutines AREA and LIMITS.

4. Published statistical tables of the binomial distribution, if available, can be used to calculate certain of the critical values in question. See Harvard University Press (1955), Robertson (1960), Weintraub (1963). However, general use of such tables or application of the well-known normal approximations should be avoided as laborious and inaccurate.

5. BINOM I is an improved version of the original program introduced and used by Tzortzopoulos (1980).

The main program introduces the required values of N, n, π, S and α (written as A1 and A2) and prints the results produced by the two subroutines.

The output of the program is a set of two pairs of critical values K1-K4, K2-K3 and a corresponding set of probabilities B1-B4, B2-B3, where

$$K1 = -1 + Y_{A1} \qquad K4 = 1 + Y_{1-A1} \tag{19}$$

$$K2 = -1 + Y_{A2} \qquad K3 = 1 + Y_{1-A2} \tag{20}$$

$$Bi = \sum_{y=0}^{y=Ki} \binom{N}{y} \cdot (\pi 4)^{y} \cdot (1 - \pi 4)^{N-y}$$

$$i = 1, 2 \tag{21}$$

$$Bi = \sum_{y=Ki}^{y=N} \binom{N}{y} \cdot (\pi 4)^{y} \cdot (1 - \pi 4)^{N-y}$$

$$i = 3, 4$$

The probabilities B express the actual size of type I error corresponding to the adopted values of A1 and A2 ($B1 \leqslant A1$, $B4 \leqslant A1$, $B2 \leqslant A2$, $B3 \leqslant A2$).

Example. Tables 1-4 contain part of the output of BINOM I for n = 5(1)20 (5)50(10)100(100)500, N = 10(5)50(10)100, $\pi = 0.05$, A1 = 0.01, A2 = 0.05 and S = 3. Now, assume that a population containing 5% defectives is sampled with n = 15 units and that the observed number of defectives in each sample is depicted as a point in a 3-sigma Np control chart; if N = 25 consecutive samples produce points between the control limits and up to 7 or at least 20 of them fall above the central line of the chart, assignable causes must be present ($\alpha = 0.01 + 0.01 = 0.02$). As the tables show, though $\alpha = 0.02$, the actual probability of type I error, in this case, is B1 + B4 = 0.0091 + 0.0056 = 0.0147. The same example shows that if more than K3 = 18 points fall above the central line, assignable causes increasing the value of $\pi = 0.05$ are present ($\alpha = 0.05$, actual probability B4 = 0.0464). Similar two-tail and one-tail tests may be applied easily using tables 1-4.

It must be observed that as n increases, other things being equal, the value of K's changes irregularly; this means that interpolation is not always a safe way to calculate K values missing from the tables.

3. *THE C CONTROL CHART FOR NUMBER OF DEFECTS*

In certain cases the main quality control interest is concentrated on the number of defects rather than the number of defectives; the defects are observed in constant size random samples of n product units ($n = 1, 2, 3, \ldots$) and their number c may be zero or any non-negative integer ($c = 0, 1, 2, \ldots$). The basic problem of quality control is tackled again by the usual three-line control chart which distinguishes samples with extreme number of defects —indicative of active assignable causes— from the rest of samples— indicative of normal quality conditions.

The distribution of C

If all kinds of assignable causes are inactive and the chances of a defect occurring in any one sample are constant and small, while the opportunities for defects per sample are numerous, the number c is a Poisson random variable with probability density function:

$$P_C = f_C(c) = f_C(c;\lambda)$$

$$= \frac{e^{-\lambda}\lambda^c}{c\,!} \qquad \text{for } c = 0, 1, 2, \ldots, \text{ ad inf.} \qquad (22)$$

where $\lambda > 0$; λ is the mean and variance of C. Under given general production conditions with absent assignable causes, the parameter λ alone expresses the composite effect of all the random causes. Thus, tests operated to ascertain displacements of λ from its standard value are also tests for the active presence of assignable causes.

Symmetrical control limits

The three control lines of the symmetrical control chart for C are drawn at the following levels :

$$UCL_c = \lambda + S\sqrt{\lambda}$$

$$\text{central value} = \lambda \qquad (23)$$

$$LCL_c = \lambda - S\sqrt{\lambda}$$

Such S-sigma control limits are not probability limits because (22) is a discrete and skew distribution. Again, S is usually set equal to 3.

Points between control limits and their distribution

Any sequence of points, all falling between the control limits of a given C control chart, may be used to test for the existence of assignable causes according to what has already been mentioned about it in connection with the Np and p charts. The theoretical framework of the relevant analysis is based on the following theorem and corollary.

THEOREM II. If C is a Poisson random variable with mean and variance equal to λ, then every increase (decrease) in λ (ceteris paribus) leads to an increase (decrease) in the ratio

$$\pi_3' = \frac{\pi_2'}{\pi_1'}$$

where:

$$\Pr(b_1 \leqslant C \leqslant b_2) = \pi_1' > 0$$

$$\Pr(b_2 < C \leqslant b_3) = \pi_2' > 0$$

$$0 \leqslant b_1 < b_2 < b_3$$

Proof

From (22) the following are derived:

$$\frac{P_{C+1}}{P_C} = \frac{\lambda}{C+1}$$

$$\frac{P_Q}{P_N} = \frac{\lambda^{Q-N}}{(N+1)\,(N+2)\ldots\,Q}$$

(Q and N are non-negative integers, $Q > N$)

$$A = \frac{P_{K+1} + P_{K+2} + \ldots + P_{K+L}}{P_K}$$

$$= \frac{\lambda}{K+1} + \frac{\lambda^2}{(K+1)(K+2)} + \ldots + \frac{\lambda^L}{(K+1(K+2)\ \ldots\ (K+L)}$$

$$B = \frac{P_{K+L+1} + P_{K+L+2} + \ldots + P_{K+L+M}}{P_K}$$

$$= \frac{\lambda^{L+1}}{(K+1)(K+2)\ldots(K+L+1)} + \ldots + \frac{\lambda^{L+M}}{(K+1)(K+2)\ldots(K+L+M)}$$

$$\Lambda = \frac{B}{A} = \left[\frac{\lambda^{L+1}}{(K+1)(K+2)\ldots(K+L+1)} + \ldots\right.$$

$$\left.\ldots + \frac{\lambda^{L+M}}{(K+1)(K+2)\ldots(K+L+M)}\right] \div \left[\frac{\lambda}{K+1} + \ldots\right.$$

$$\left.\ldots + \frac{\lambda^{L}}{(K+1)(K+2)\ldots(K+L)}\right]$$

The first derivative of Λ with respect to λ takes the form

$$\frac{d\Lambda}{d\lambda} = \frac{1}{A^2}\cdot\left[\left(\frac{(L+1)\,\lambda^{L}}{K+1)\ldots(K+L+1)} + \ldots + \frac{(L+M)\,\lambda^{L+M-1}}{(K+1)\ldots(K+L+M)}\right)\cdot\right.$$

$$\cdot\left(\frac{\lambda}{(K+1)} + \ldots + \frac{\lambda^{L}}{(K+1)\ldots(K+L)}\right) - \left(\frac{1}{K+1} + \frac{2\lambda}{(K+1)(K+2)} + \ldots +\right.$$

$$\left.\ldots + \frac{L\,\lambda^{L-1}}{(K+1)\ldots(K+2)}\right)\cdot\left(\frac{\lambda^{L+1}}{(K+1)\ldots(K+L+1)} + \ldots\right.$$

$$\left.\left.\ldots + \frac{\lambda^{L+M}}{(K+1)\ldots(K+L+M)}\right)\right]$$

Now, for the first T powers of λ

$$\lambda,\ \lambda^2,\ \lambda^3,\ \ldots,\ \lambda^T$$

the following relationship will always hold:

$$(\lambda^U)'\cdot[\lambda+\lambda^2 + \ldots + \lambda^L] - \lambda^U\,[\lambda' + (\lambda^2)' + \ldots + (\lambda^L)'] > 0 \qquad (24)$$

where L and U are integers such that $1 \leqslant L < T$, $L < U \leqslant T$; primes signify first derivatives with respect to λ.

Indeed,

$$(\lambda^U)'\sum_{i=1}^{i=L}\lambda^i - \lambda^U\sum_{j=1}^{j=L}(\lambda^j)' =$$

$$= U\sum_{i=1}^{i=L}(\lambda^{U+i-1}) - \sum_{j=1}^{j=L}(j)\,(\lambda^{j+U-1})$$

$$= \sum_{i=1}^{i=L}(U-i)\,\lambda^{i+U-1} \qquad (25)$$

Since $U > i$ and $\lambda > 0$, the sum (25) is always positive; therefore, (24) is true.

By putting $U = L+1, L+2, \ldots, L+M$ and using (24) repeatedly it is easy to show that $d\Lambda/d\lambda > 0$; this completes the proof of theorem II.

At this stage the constants b1, b2 and b3 are replaced by LCL_c, λ and UCL_c respectively. Then,

$$\pi_1' = \Pr(LCL_c \leqslant c \leqslant \lambda)$$

$$\pi_2' = \Pr(\lambda < c \leqslant UCL_c)$$

On the basis of theorem II and the above definitions the following corollary can be proved easily:

COROLLARY. If the number of defects C is a Poisson random variable with mean and variance equal to λ, then every increase (decrease) in λ (ceteris paribus) leads to an increase (decrease) in the ratio

$$\pi_4' = \frac{\pi_2'}{\pi_1' + \pi_2'} \tag{26}$$

The proof of this corollary may also be derived directly from the first derivative of (22):

$$\frac{\partial f_C(c)}{\partial \lambda} = \frac{\partial (e^{-\lambda}\lambda^C/c!)}{\partial \lambda} = \frac{e^{-\lambda}\lambda^C}{c!} \cdot \left[\frac{c - \lambda}{\lambda}\right] \tag{27}$$

Apparently, this derivative takes up positive values for $c > \lambda$ and negative values for $c < \lambda$.

Now assume that N consecutive measurements of C have been obtained, all falling between the control limits:

$$LCL_C \leqslant c \leqslant UCL_C$$

For each measurement the binary variable Z is defined as before, i.e.

$$Z_i = 1 \qquad \text{if } \lambda < c \leqslant UCL_C$$

$$Z_i = 0 \qquad \text{if } LCL_C \leqslant c \leqslant \lambda$$

$$i = 1, 2, \ldots, N$$

The sum varible Y is also defined in the manner expressed by (11); Z and Y have densities (10) and (12) respectively with π_4' instead of π_4.

Testing for the presence of assignable causes

The usual application of the C chart technique concerns the statistical testing of:

$$H_0 : E(C) = \lambda \text{ against } H_1 : E(C) \neq \lambda \tag{28}$$

By introducing Y, the equivalent of the above statistical operation is testing the hypotheses

$$\begin{aligned} & H_0 : E(Y) = N\pi_4' \quad \text{against} \quad H_1 : E(Y) \neq N\pi_4' \\ \text{or} \quad & H_0 : E(Y/N) = \pi_4' \quad \text{against} \quad H_1 : E(Y/N) \neq \pi_4' \end{aligned} \tag{29}$$

The rejection of H_0 in the last case leads directly to the conclusion that π_4' has changed from its original level; this means that λ has changed as well. Thus, the rejection of H_0 in (29) signals the presence of assignable causes.

The rejection regions are those defined by (17) and (18); the reader is referred to the relevant part of previous analysis for more details.

Certainly, to test the hypotheses (29) one needs the necessary data and the appropriate values of K and B which have already been discussed; the calculation of such measures can easily be effected by a suitable computer program [6].

The computer program BINOM II

The FORTRAN IV computer program BINOM II has been developed to produce the required values of K's and B's; it is a program similar to BINOM I —it consists of the main program and two subroutines called again AREA and LIMITS. The main program introduces 30 values of λ, one value of S, two levels of α (written as A1 and A2) and 14 values of N. The subroutines compute the critical values K1, K2, K3 and K4 and the actual probabilities B1, B2, B3 and B4.

It goes without saying that by minor alterations the program can accept more values of λ, S, α and N; however, instead of such altera-

6. See footnote (4).

tions, it is suggested to repeat the execution of the program for the required values of the parameters.

Example. Tables 5-8 show part of the results produced by BINOM II for $\lambda = 0.05(0.05)0.50(0.1)1.0(0.5)8.5$, $A1 = 0.01$, $A2 = 0.05$, $S = 3$ and $N = 10(5)50(10)100$. From the tables in question one can see, for instance, that a 3-sigma C control chart with $\lambda = 0.6$ signals the presence of assignable causes if, among 25 consecutive points located between the limits, up to 4 or at least 18 fall above the central line ($\alpha = 0.02$). It is also observed that though in this example the size of type I error was predetermined at $\alpha = 2(0.01) = 0.02$, the nature of the binomial distribution has actually reduced it to $B1 + B4 = 0.0033 + 0.0041 = 0.0074$. The same tables can be used for one-tail tests.

It must be pointed out again that for given N, α and S the critical values of K vary irregularly with λ; therefore, the usual interpolations applied in practice for the omitted values of λ should be avoided here.

4. THE $\bar{X}$ CONTROL CHART FOR VARIABLES

Very often the quality characteristic under investigation is a variable, say X; in this case random samples of constant size n are drawn from the production process and the average $\bar{X}$ of each sample is calculated and presented in the $\bar{X}$ control chart.

The distribution of $\bar{X}$

According to the central limit theorem, the sample average $\bar{X}$ is normal, $N(\mu, \sigma^2/n)$, independent of the parent population distribution where μ and σ^2 are respectively the population mean and variance; if X is non-normal, $\bar{X}$ becomes normal when σ^2 is finite and n increases without bound. In practice, however, $\bar{X}$ approximates normality even at very small values of n, particularly if X does not deviate from normality a great deal .

Symmetrical and probability control limits

When μ and σ^2 are known, the three lines of the $\bar{X}$ conrol chart are drawn at the following levels:

$$UCL_{\bar{X}} = \mu + S \cdot \frac{\sigma}{\sqrt{n}}$$

$$\text{central value} = \mu \tag{30}$$

$$LCL_{\bar{X}} = \mu - S \cdot \frac{\sigma}{\sqrt{n}}$$

Since $\bar{X}$ is normal, the symmetrical limits (30) are also probability limits. Thus, if no assignable causes are active, the ratio

$$\frac{Pr(\mu < \bar{x} \leqslant UCL\bar{x})}{Pr(LCL\bar{x} \leqslant \bar{x} \leqslant UCL\bar{x})} = \frac{1}{2} \tag{31}$$

holds for any values of n, σ^2, S and n — the only minor reservation being that of non-normal X combined with small n.

Points between control limits and their distribution

Assume now that N independent random samples of size n are drawn consecutively and that all their $\bar{X}$'s fall in the region between the limits:

$$LCL\bar{x} \leqslant \bar{x} \leqslant UCL\bar{x}$$

For each sample the binary variable Z is defined so that

$$Zi = 1 \quad \text{if} \quad \mu < \bar{x} \leqslant UCL\bar{x}$$

$$Zi = 0 \quad \text{if} \quad LCL\bar{x} \leqslant \bar{x} \leqslant \mu$$

$$i = 1, 2, \ldots, N$$

The sum variable Y is defined again as (11). Z and Y have the den sities (10) and (12) respectively with $\pi_4 = 0.5$.

Testing for the presence of assignable causes

The active role of assignable causes that affect μ can be sensed statistically by testing the hypotheses:

$$H_0 : E(\bar{X}) = \mu \quad \text{versus } H_1 : E(\bar{X}) \neq \mu \tag{32}$$

This single-point testing work is carried out directly by the $\bar{X}$ chart and is based on each $\bar{x}$ point falling inside or outside the area between the control limits (30). Equivalent testing work is done by sequences of N points all falling between the control limits and thus producing values of the variable Y; the hypotheses to be tested in this case are:

$$H_0 : E(Y) = N/2 \qquad \text{versus} \quad H_1 : E(Y) \neq N/2$$

$$\text{or} \quad H_0 : E(Y/N) = 1/2 \qquad \text{versus} \quad H_1 : E(Y/N) \neq 1/2 \tag{33}$$

Indeed, if $\bar{X}$ is $N(\mu, \sigma^2/n)$, any assignable causes that increase (decrease) the value of μ will also increase (decrease) the value 1/2 of the ratio (31) on condition that the original $\bar{X}$ control chart remains unchanged (this can be shown by derivation of the normal density function).

Therefore, extreme values of Y can be used as indications of quality deterioration (or improvement) caused by active assignable causes that affect μ.

Such values form the rejection regions (17) and (18), their boundaries being the critical values K —as defined by (19) and (20)— obtainable from the symmetrical binomial distribution for given N and α. An appropriate computer program has been prepared to serve the purposes of calculating any critical value of this type together with the probabilities B, that may be of interest both on the practical and on the theoretical level [7].

The computer program BINOM III

The program BINOM III, written in FORTRAN IV, produces four critical values presented in two pairs as L1 - L2 with probability levels $\alpha = A1$ and $\alpha = A2$ for the first and second pair respectively.

Since (12), with $\pi_4 = 1/2$ is a discrete probability density function, BINOM III computes the actual probability levels P1 and P2, where:

$$P1 = Pr(Y \leqslant L1) = Pr(Y \geqslant L2) \leqslant A1 \qquad \text{first pair L1-L2}$$

$$P2 = Pr(Y \leqslant L1) = Pr(Y \geqslant L2) \leqslant A2 \qquad \text{second pair L1-L2}$$

Example. As an example, part of the results obtained by BINOM

7. If extensive statistical tables of the binomial distribution for $\pi = 1/2$ are available, any K's and P's can be easily obtained from them. The well-known normal approximations may also be used; Raff (1956) has shown that the error of such approximations is never greater than 0.05 if $n(\pi)^{3/2} > 1.07$.

III are presented in Table 9 with A1 = 0.0005, A2 = 0.0025 and selected values of N. The results show, for instance, that if 15 successive samples of constant size produce values of $\bar{X}$ between the control limits and up to 1 or at least 14 of them fall above the central line, the hypothesis H_0 of (33) is rejected —certain assignable causes are in action (α = 2A1 = = 0.0010). It is noted that, though in this example, α was set equal to 0.0010, the actual size of type I error is 2P1 = 2(0.0004) = 0.0008.

The critical values produced by BINOM III apply to control limits located at any symmetrical positions about the central line of the $\bar{X}$ chart.

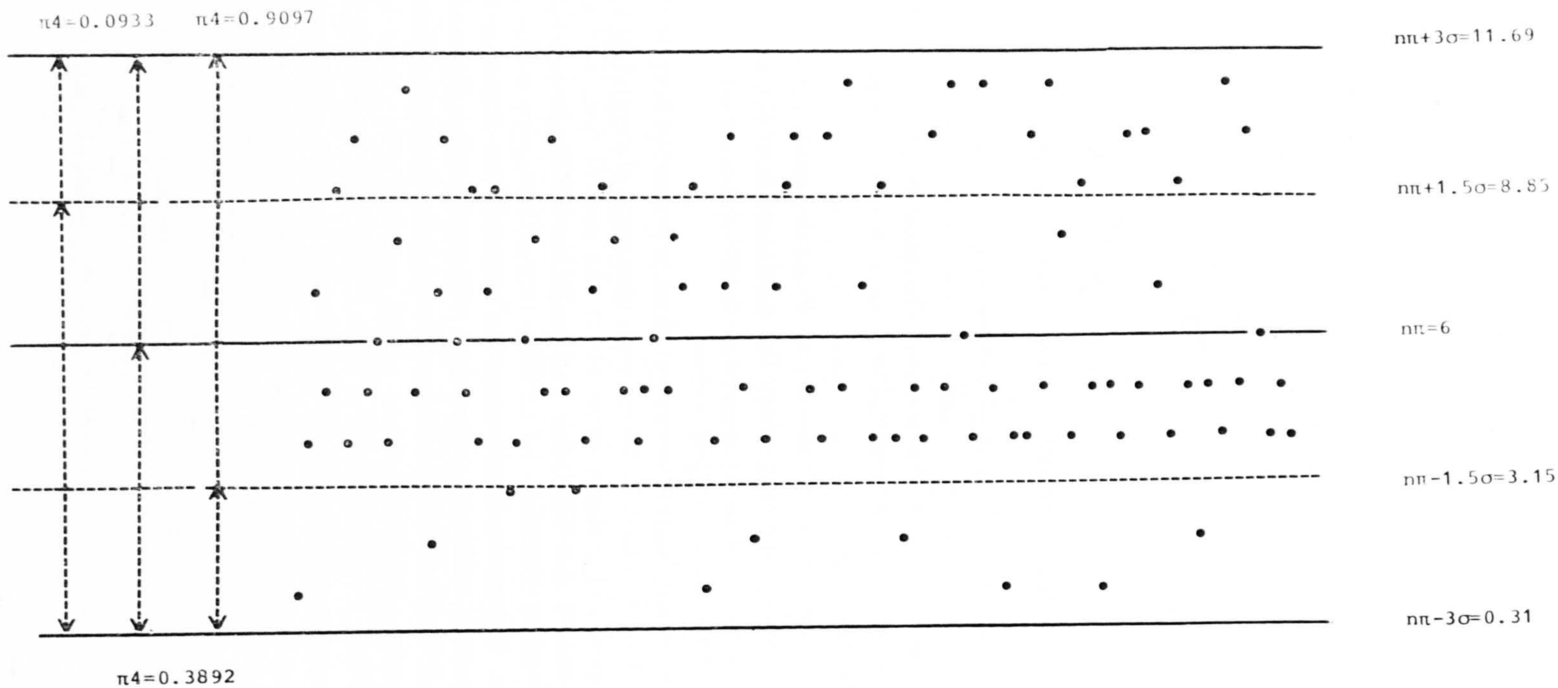
5. *CHARTS WITH MORE THAN TWO CONTROL TIMITS*

Instead of a single pair of symmetrical control limits two or more might be considered. Sometimes in practice an inner pair of warning limits is combined with another one, the latter consisting of the so called action limits which are identical to those that have already been examined in this paper.

When using more than one pair of limits, multi-point Y tests may be conducted by applying either the multinomial or the binomial distribution. In both cases appropriate computer programs are needed to produce the required critical values.

As an example, the Np control chart is presented here with $\pi = 0.4$ and n = 15; the chart is drawn with two pairs of control limits located at 1.5-sigma and 3-sigma units about the central line. The population mean equals 6 and the outer and inner control limits are respectively 6 ± 5.69 and 6 ± 2.85.

Now, assume that 100 consecutive points are found between the outer limits of the chart in question dispersed in the depicted way; 45 points fall above the central line, this being an indication that the process is under statistical control as the proposed two - sided test shows ($\pi_4 = 0.3892$, $\alpha = 0.05$, $Pr(Y \leqslant 28) = 0.015$, $Pr(Y \geqslant 50) = 0.016$). The upper three bands contain 90 points wich are not an extreme enough case to reject the hypothesis that no assignable causes are present (two-sided test, $\pi_4 = 0.9097$, $\alpha = 0.05$, $Pr(Y \leqslant 84) = 0.017$, $Pr(Y \geqslant 97) = = 0.017$). In the upper band 25 points are found; this number is extreme in the sense that —if no assignable causes are present— the chances of finding 25 or more points in the upper band are 4 in one million. Therefore, the allocation of 25 points in the upper band strongly indicates the


π4=0.0933
π4=0.9097
nπ+3σ=11.69
nπ+1.5σ=8.85
nπ=6
nπ-1.5σ=3.15
nπ-3σ=0.31
π4=0.3892

presence of assignable causes ($\pi_4 = 0.0933$, $\alpha = 0.05$, $Pr(Y \leqslant 3) = 0.013$, $Pr(Y \geqslant 16) = 0.023$).

The example shows that of the three tests only one refutes the hypothesis that no assignable causes are in action. More binomial tests can be applied, if required, on the same sequence of points[8].

8. As a matter of interest, 16 such tests are possible with charts having two pairs of control limits. However, if enough warning criteria are used to evaluate a given sequence of points practically all data will be «out of control». See Burr (1976 p. 86).

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TABLE 1

N_p and p control chart. Fraction defective $\pi = 0.050$ — Critical values K1 and K4

	N=10		N=15		N=20		N=25		N=30		N=35		N=40		N=45		N=50		N=60		N=70		N=80		N=90		N=100	
n	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4
5		6		8	0	10	0	11	1	13	1	14	2	16	3	17	3	18	5	21	6	24	8	26	9	29	11	32
6		7		9	0	11	1	12	1	14	2	16	3	17	4	19	4	20	6	23	8	26	10	29	12	32	14	35
7		8	0	10	1	12	2	14	3	16	4	18	5	20	6	22	7	24	9	27	11	31	14	35	16	38	19	42
8		8	0	10	1	13	2	15	3	17	4	19	6	21	7	24	8	26	11	30	13	34	16	38	19	42	22	45
9		8	0	11	2	13	3	16	4	18	5	21	7	23	8	25	10	27	12	32	15	36	18	40	21	45	24	49
10	0	9	1	11	2	14	3	17	5	19	6	22	8	24	9	26	11	29	14	34	17	38	21	43	24	47	27	52
11	0	9	1	12	3	15	4	17	6	20	7	23	9	25	10	28	12	30	16	35	19	40	23	45	26	50	30	55
12	0	9	1	12	3	15	5	18	6	21	8	24	10	26	12	29	13	32	17	37	21	42	25	47	29	52	32	57
13	0	9	2	13	4	16	5	19	7	22	9	25	11	28	13	31	15	33	19	39	23	45	27	50	32	56	36	61
14	1	10	2	13	4	16	6	19	8	23	10	26	12	29	14	32	16	35	21	41	25	46	29	52	34	58	38	64
15	1	10	3	13	5	17	7	20	9	23	11	26	13	30	15	33	18	36	22	42	27	48	31	54	36	60	41	66

TABLE 2

N_p and p control chart. Fraction defective $\pi = 0.05$ — Critical values K2 and K3

	N=10		N=15		N=20		N=25		N=30		N=35		N=40		N=45		N=50		N=60		N=70		N=80		N=90		N=100	
n	K2	K3	K2	K3	K2	K3	K2	K3	K2	K3	K2	K3	K2	K3	K2	K3	K2	K3	K2	K3	K2	K3	K2	K3	K2	K3	K2	K3
5		5	0	7	0	8	1	10	2	11	3	12	3	14	4	15	5	16	7	19	8	21	10	24	12	26	13	29
6		6	0	7	1	9	2	11	3	12	3	14	4	15	5	17	6	18	8	21	10	24	12	27	14	29	16	32
7	0	6	1	8	2	10	3	12	4	14	5	16	6	18	8	20	9	21	11	25	14	28	16	32	19	35	22	39
8	0	7	1	9	2	11	4	13	5	15	6	17	8	19	9	21	10	23	13	27	16	31	19	35	22	33	25	42
9	0	7	2	10	3	12	4	14	6	16	7	18	9	21	10	23	12	25	15	29	18	33	21	37	24	41	28	45
10	1	8	2	10	3	13	5	15	7	17	8	20	10	22	11	24	13	26	17	31	20	35	23	40	27	44	30	49
11	1	8	2	10	4	13	6	16	7	18	9	21	11	23	13	25	14	28	18	33	22	37	26	42	29	47	33	51
12	1	8	3	11	4	14	6	16	8	19	10	22	12	24	14	27	16	29	20	34	24	39	28	44	32	49	36	54
13	1	8	3	11	5	14	7	17	9	20	11	23	13	26	15	28	17	31	22	36	26	42	30	47	35	52	39	58
14	2	9	3	12	6	15	8	18	10	21	12	24	14	27	16	29	19	32	23	38	28	44	32	49	37	55	42	60
15	2	9	4	12	6	15	8	18	11	21	13	25	15	28	18	31	20	33	25	39	30	45	34	51	39	57	44	63

TABLE 3

N_p and p control chart. Fraction defective $\pi = 0.05$ — Actual probability levels B1 and B4

	N=10		N=15		N=20		N=25		N=30		N=35		N=40	
n	B1	B4	B1	B4	B1	B4	B1	B4	B1	B4	B1	B4	B1	B4
5		0.0079		0.0055	0.0094	0.0036	0.0029	0.0077	0.0080	0.0045	0.0029	0.0077	0.0057	0.0045
6		0.0027		0.0031	0.0041	0.0028	0.0093	0.0076	0.0028	0.0056	0.0048	0.0040	0.0069	0.0076
7		0.0016	0.0048	0.0036	0.0078	0.0050	0.0092	0.0058	0.0096	0.0062	0.0094	0.0062	0.0089	0 0060
8		0.0034	0.0023	0.0084	0.0034	0.0037	0.0036	0.0055	0.0034	0.0071	0.0030	0.0084	0.0081	0.0093
9		0.0064	0.0011	0.0041	0.0087	0.0089	0.0066	0.0047	0.0048	0.0073	0.0035	0.0039	0.0076	0.0054
10	0.0066	0.0015	0.0058	0.0082	0.0042	0.0056	0.0028	0.0036	0.0067	0.0069	0.0041	0.0043	0.0074	0.0068
11	0.0041	0.0027	0.0032	0.0034	0.0096	0.0031	0.0052	0.0083	0.0092	0.0060	0.0049	0.0043	0.0074	0.0078
12	0.0026	0.0044	0.0017	0.0062	0.0051	0.0062	0.0089	0.0056	0.0041	0.0048	0.0060	0.0040	0.0077	0.0083
13	0.0013	0.0084	0.0051	0.0026	0.0085	0.0040	0.0032	0.0048	0.0042	0.0051	0.0050	0.0051	0.0056	0.0049
14	0.0091	0.0012	0.0029	0.0046	0.0046	0.0076	0.0055	0.0097	0.0059	0.0037	0.0060	0.0043	0.0058	0.0047
15	0.0060	0.0019	0.0089	0.0077	0.0095	0.0033	0.0091	0.0056	0.0082	0.0077	0.0072	0.0094	0.0062	0.0042

	N=45		N=50		N=60		N=70		N=80		N=90		N=100	
n	B1	B4	B1	B4	B1	B4	B1	B4	B1	B4	B1	B4	B1	B4
5	0.0092	0.0068	0.0039	0.0097	0.0082	0.0078	0.0051	0.0063	0.0083	0.0099	0.0052	0.0077	0.0075	0.0060
6	0.0088	0.0053	0.0033	0.0088	0.0051	0.0094	0.0068	0.0096	0.0081	0.0095	0.0092	0.0092	0.0100	0.0087
7	0.0083	0.0057	0.0076	0.0053	0.0061	0.0095	0.0049	0.0076	0.0083	0.0060	0.0063	0.0087	0.0094	0.0067
8	0.0065	0.0043	0.0052	0.0048	0.0079	0.0054	0.0049	0.0057	0.0065	0.0058	0.0080	0.0057	0.0094	0.0097
9	0.0053	0.0069	0.0094	0.0084	0.0046	0.0056	0.0051	0.0073	0.0054	0.0090	0.0055	0.0058	0.0054	0.0068
10	0.0046	0.0097	0.0071	0.0060	0.0064	0.0051	0.0056	0.0083	0.0095	0.0066	0.0078	0.0094	0.0064	0.0073
11	0.0041	0.0054	0.0057	0.0086	0.0092	0.0087	0.0065	0.0085	0.0089	0.0080	0.0062	0.0075	0.0079	0.0068
12	0.0094	0.0065	0.0047	0.0051	0.0064	0.0066	0.0078	0.0078	0.0089	0.0088	0.0098	0.0095	0.0059	0.0099
13	0.0059	0.0046	0.0061	0.0095	0.0061	0.0074	0.0058	0.0057	0.0054	0.0082	0.0089	0.0061	0.0078	0.0081
14	0.0055	0.0050	0.0051	0.0051	0.0090	0.0050	0.0071	0.0093	0.0055	0.0081	0.0078	0.0070	0.0060	0.0060
15	0.0053	0.0050	0.0099	0.0056	0.0067	0.0066	0.0089	0.0071	0.0059	0.0074	0.0072	0.0074	0.0085	0.0073

TABLE 4

N_p and p control chart. Fraction defective $\pi = 0.05$ — Actual probability levels B2 and B3

	N=10		N=15		N=20		N=25		N=30		N=35		N=40	
n	B2	B3	B2	B3	B2	B3	B2	B3	B2	B3	B2	B3	B2	B3
5		0.0386	0.0301	0.0224	0.0094	0.0402	0.0220	0.0228	0.0353	0.0338	0.0480	0.0456	0.0215	0.0271
6		0.0161	0.0163	0.0463	0.0302	0.0320	0.0407	0.0222	0.0483	0.0381	0.0187	0.0261	0.0224	0.0394
7	0.0286	0.0466	0.0359	0.0491	0.0362	0.0470	0.0340	0.0432	0.0309	0.0390	0.0276	0.0349	0.0245	0.0310
8	0.0175	0.0195	0.0196	0.0305	0.0178	0.0371	0.0468	0.0409	0.0360	0.0428	0.0278	0.0435	0.0491	0.0433
9	0.0108	0.0327	0.0492	0.0170	0.0335	0.0273	0.0230	0.0362	0.0421	0.0437	0.0286	0.0498	0.0441	0.0276
10	0.0499	0.0111	0.0299	0.0305	0.0181	0.0185	0.0334	0.0301	0.0496	0.0419	0.0303	0.0256	0.0412	0.0332
11	0.0344	0.0179	0.0181	0.0499	0.0333	0.0340	0.0471	0.0234	0.0253	0.0380	0.0328	0.0259	0.0398	0.0372
12	0.0236	0.0269	0.0432	0.0249	0.0193	0.0209	0.0264	0.0429	0.0321	0.0325	0.0365	0.0247	0.0398	0.0393
13	0.0137	0.0449	0.0232	0.0467	0.0283	0.0437	0.0306	0.0394	0.0312	0.0348	0.0309	0.0304	0.0300	0.0264
14	0.0478	0.0127	0.0144	0.0213	0.0478	0.0257	0.0435	0.0275	0.0389	0.0278	0.0344	0.0273	0.0303	0.0262
15	0.0345	0.0184	0.0338	0.0325	0.0299	0.0413	0.0255	0.0464	0.0488	0.0491	0.0391	0.0231	0.0314	0.0246

	N=45		N=50		N=60		N=70		N=80		N=90		N=100	
n	B2	B3	B2	B3	B2	B3	B2	B3	B2	B3	B2	B3	B2	B3
5	0.0287	0.0350	0.0358	0.0433	0.0494	0.0327	0.0305	0.0451	0.0391	0.0339	0.0471	0.0437	0.0305	0.0331
6	0.0255	0.0273	0.0280	0.0384	0.0316	0.0364	0.0337	0.0339	0.0349	0.0313	0.0354	0.0475	0.0353	0.0427
7	0.0485	0.0274	0.0415	0.0462	0.0306	0.0349	0.0428	0.0458	0.0315	0.0345	0.0407	0.0427	0.0500	0.0324
8	0.0377	0.0426	0.0290	0.0415	0.0349	0.0388	0.0394	0.0357	0.0428	0.0326	0.0453	0.0473	0.0471	0.0422
9	0.0306	0.0312	0.0433	0.0343	0.0411	0.0392	0.0383	0.0429	0.0354	0.0455	0.0324	0.0473	0.0474	0.0484
10	0.0259	0.0408	0.0336	0.0483	0.0498	0.0365	0.0392	0.0469	0.0311	0.0354	0.0408	0.0430	0.0322	0.0327
11	0.0462	0.0495	0.0271	0.0316	0.0351	0.0314	0.0421	0.0475	0.0480	0.0412	0.0333	0.0358	0.0372	0.0479
12	0.0423	0.0297	0.0441	0.0429	0.0463	0.0445	0.0471	0.0448	0.0469	0.0443	0.0462	0.0433	0.0450	0.0420
13	0.0287	0.0450	0.0272	0.0383	0.0434	0.0491	0.0367	0.0354	0.0311	0.0425	0 0422	0 0489	0.0353	0.0359
14	0.0265	0.0486	0.0441	0.0445	0.0327	0.0371	0.0419	0.0309	0.0311	0.0425	0.0377	0.0349	0.0439	0.0447
15	0.0491	0.0255	0.0393	0.0492	0.0451	0.0466	0.0495	0.0434	0.0326	0.0401	0.0351	0.0368	0.0371	0.0336

TABLE 5

C control chart. Critical values: K1 and K4

	N=10		N=15		N=20		N=25		N=30		N=35		N=40		N=45		N=50		N=60		N=70		N=80		N=90		N=100	
λ	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4	K1	K4
0.050		1		1		1		1		1		1		1		1		1		1		1		1		1		1
0.100		4		5		6		7		8		9		9		10	0	11	0	12	0	13	1	15	2	16	2	17
0.150		5		6		8		9		10	0	11	0	12	0	13	1	14	1	15	2	17	3	19	4	21	5	22
0.200		6		7		9		10	0	11	0	12	1	14	1	15	2	16	3	18	4	20	5	23	6	25	8	27
0.250		6		8		9	0	11	0	12	1	14	2	15	2	17	3	18	4	21	6	23	7	26	9	28	10	31
0.300		7		9	0	10	0	12	1	14	2	15	3	17	3	18	4	20	6	23	7	26	9	29	11	31	13	34
0.350		7	0	10	1	12	1	14	2	16	3	18	4	20	5	21	6	23	9	27	11	30	13	34	16	38	18	41
0.400		8	0	10	1	13	2	15	3	17	4	19	5	21	7	23	8	25	10	29	13	33	16	37	18	41	21	45
0.450		8	0	11	1	13	3	16	4	18	5	20	6	22	8	25	9	27	12	31	15	35	18	40	21	44	24	48
0.500	0	8	1	11	2	14	3	16	5	19	6	21	7	24	9	26	10	28	14	33	17	38	20	42	23	46	26	51
0.600	0	9	1	12	3	15	4	18	6	20	8	23	9	26	11	28	13	31	16	36	20	41	24	46	28	51	31	56

TABLE 6

C control chart. Critical values: K2 and K3

	N=10		N=15		N=20		N=25		N=30		N=35		N=40	
λ	K2	K3	K2	K3	K2	K3	K2	K3	K2	K3	K2	K3	K2	K3
0.050		1		1		1		1		1		1		1
0.100		4		4		5		6		7	0	7	0	8
0.150		4		5		6	0	7	0	8	1	9	1	10
0.200		5		6	0	7	0	8	1	10	1	11	2	12
0.250		5	0	7	0	8	1	9	2	11	2	12	3	13
0.300		6	0	7	1	9	1	10	2	12	3	13	4	15
0.350	0	6	1	8	2	10	3	12	4	14	5	16	6	17
0.400	0	7	1	9	2	11	3	13	5	15	6	17	7	19
0.450	0	7	1	9	3	12	4	14	5	16	7	18	8	20
0.500	0	7	2	10	3	12	5	15	6	17	8	19	9	21
0.600	1	8	2	11	4	13	6	16	8	19	10	21	11	24

	N=45		N=50		N=60		N=70		N=80		N=90		N=100	
λ	K2	K3	K2	K3	K2	K3	K2	K3	K2	K3	K2	K3	K2	K3
0.050		1		1		1		1		1		1		1
0.100	0	8	0	9	1	10	2	12	2	13	3	14	4	15
0.150	1	11	2	12	3	13	4	15	5	17	6	18	7	20
0.200	3	13	3	14	4	16	6	18	7	20	8	22	10	24
0.250	4	15	5	16	6	18	8	21	9	23	11	25	13	28
0.300	5	16	6	18	8	20	10	23	11	26	13	28	15	31
0.350	7	19	8	21	11	24	13	28	16	31	18	34	21	38
0.400	9	21	10	23	13	27	15	30	18	34	21	38	24	41
0.450	10	22	11	24	14	28	17	33	20	37	24	41	27	44
0.500	11	24	13	26	16	30	19	35	23	39	26	43	30	48
0.600	13	26	15	29	19	34	23	39	27	43	31	48	35	53

TABLE 7

C control ehart. Actual probability levels: B1 and B4

λ	N=10		N=15		N=20		N=25		N=30		N=35		N=40	
	B1	B4	B1	B4	B1	B4	B1	B4	B1	B4	B1	B4	B1	B4
0.050		0.0000		0.0000		0.0000		0.0000		0.0000		0.0000		0.0000
0.100		0.0092		0.0086		0.0071		0.0057		0.0044		0.0034		0.0086
0.150		0.0054		0.0086		0.0024		0.0031		0.0035	0.0075	0.0038	0.0037	0.0039
0.200		0.0024		0.0066		0.0028		0.0047	0.0042	0.0067	0.0017	0.0087	0.0061	0.0039
0.250		0.0064		0.0042		0.0100	0.0038	0.0056	0.0012	0.0095	0.0040	0.0053	0.0079	0.0079
0.300		0.0021		0.0023	0.0053	0.0077	0.0014	0.0055	0.0038	0.0038	0.0067	0.0074	0.0097	0.0050
0.350		0.0089	0.0057	0.0029	0.0094	0.0039	0.0021	0.0045	0.0028	0.0047	0.0033	0.0046	0.0036	0.0044
0.400		0.0028	0.0028	0.0068	0.0042	0.0028	0.0045	0.0042	0.0044	0.0053	0.0040	0.0061	0.0035	0.0068
0.450		0.0054	0.0014	0.0033	0.0018	0.0070	0.0084	0.0035	0.0064	0.0054	0.0048	0.0073	0.0036	0.0092
0.500	0.0078	0.0094	0.0071	0.0066	0.0053	0.0043	0.0037	0.0086	0.0090	0.0051	0.0057	0.0080	0.0037	0.0047
0.600	0.0031	0.0036	0.0022	0.0049	0.0066	0.0047	0.0033	0.0041	0.0058	0.0098	0.0084	0.0075	0.0042	0.0056

λ	N=45		N=50		N=60		N=70		N=80		N=90		N=100	
	B1	B4	B1	B4	B1	B4	B1	B4	B1	B4	B1	B4	B1	B4
0.050		0.0000		0.0000		0.0000		0.0000		0.0000		0.0000		0.0000
0.100		0.0063	0.0085	0.0046	0.0033	0.0071	0.0013	0.0098	0.0044	0.0053	0.0094	0.0069	0.0044	0.0086
0.150	0.0019	0.0039	0.0078	0.0039	0.0023	0.0089	0.0037	0.0076	0.0050	0.0064	0.0062	0.0053	0.0071	0.0091
0.200	0.0027	0.0048	0.0066	0.0057	0.0063	0.0074	0.0058	0.0089	0.0051	0.0049	0.0044	0.0056	0.0095	0.0062
0.250	0.0032	0.0044	0.0057	0.0063	0.0039	0.0048	0.0080	0.0079	0.0053	0.0058	0.0086	0.0083	0.0057	0.0061
0.300	0.0036	0.0083	0.0051	0.0056	0.0080	0.0057	0.0042	0.0056	0.0057	0.0054	0.0072	0.0095	0.0085	0.0085
0.350	0.0037	0.0096	0.0036	0.0085	0.0086	0.0066	0.0071	0.0100	0.0058	0.0074	0.0096	0.0055	0.0076	0.0074
0.400	0.0089	0.0072	0.0072	0.0074	0.0048	0.0075	0.0073	0.0073	0.0099	0.0069	0.0064	0.0064	0.0080	0.0058
0.450	0.0075	0.0048	0.0054	0.0058	0.0068	0.0076	0.0079	0.0091	0.0085	0.0055	0.0089	0.0063	0.0091	0.0070
0.500	0.0066	0.0067	0.0042	0.0089	0.0097	0.0068	0.0088	0.0052	0.0078	0.0074	0.0069	0.0099	0.0060	0.0072
0.600	0.0057	0.0099	0.0072	0.0074	0.0047	0.0085	0.0063	0.0092	0.0079	0.0096	0.0093	0.0096	0.0059	0.0095

TABLE 8

C control chart. Actual probability levels: B2 and B3

λ	N=10 B2	N=10 B3	N=15 B2	N=15 B3	N=20 B2	N=20 B3	N=25 B2	N=25 B3	N=30 B2	N=30 B3	N=35 B2	N=35 B3	N=40 B2	N=40 B3
0.050		0.0000		0.0000		0.0000		0.0000		0.0000		0.0000		0.0000
0.100		0.0092		0.0412		0.0302		0.0220		0.0161	0.0356	0.0357	0.0221	0.0256
0.150		0.0317		0.0366		0.0375	0.0304	0.0365	0.0151	0.0345	0.0469	0.0321	0.0261	0.0296
0.200		0.0155		0.0274	0.0261	0.0371	0.0105	0.0447	0.0295	0.0197	0.0135	0.0232	0.0274	0.0261
0.250		0.0328	0.0352	0.0181	0.0115	0,0321	0.0274	0.0468	0.0442	0.0256	0.0190	0.0344	0.0285	0.0432
0.300		0.0132	0.0195	0.0381	0.0368	0.0251	0.0120	0.0440	0.0188	0.0286	0.0249	0.0430	0.0302	0.0285
0.350	0.0319	0.0413	0.0409	0.0425	0.0421	0.0397	0.0404	0.0357	0.0375	0.0316	0.0342	0.0276	0.0309	0.0493
0.400	0.0198	0.0168	0.0229	0.0258	0.0214	0.0308	0.0186	0.0334	0.0441	0.0343	0.0349	0.0343	0.0277	0.0336
0.450	0.0124	0.0284	0.0128	0.0464	0.0400	0.0222	0.0284	0.0291	0.0202	0.0347	0.0365	0.0391	0.0258	0.0425
0.500	0.0078	0.0442	0.0353	0.0254	0.0222	0.0419	0.0412	0.0238	0.0257	0.0329	0.0390	0.0416	0.0249	0.0498
0.600	0.0276	0.0229	0.0134	0.0204	0.0242	0.0465	0.0336	0.0339	0.0414	0.0248	0.0477	0.0398	0.0255	0.0290

λ	N=45 B2	N=45 B3	N=50 B2	N=50 B3	N=60 B2	N=60 B3	N=70 B2	N=70 B3	N=80 B2	N=80 B3	N=90 B2	N=90 B3	N=100 B2	N=100 B3
0.050		0.0000		0.0000		0.0000		0.0000		0.0000		0.0000		0.0000
0.100	0.0137	0.0480	0.0085	0.0347	0.0230	0.0431	0.0407	0.0233	0.0198	0.0281	0.0315	0.0326	0.0446	0.0368
0.150	0.0144	0.0271	0.0333	0.0248	0.0377	0.0434	0.0403	0.0346	0.0416	0.0276	0.0421	0.0414	0.0420	0.0329
0.200	0.0446	0.0286	0.0238	0.0307	0.0202	0.0338	0.0409	0.0359	0.0328	0.0371	0.0264	0.0377	0.0427	0.0379
0.250	0.0382	0.0250	0.0480	0.0308	0.0308	0.0427	0.0437	0.0303	0.0287	0.0388	0.0378	0.0474	0.0469	0.0342
0.300	0.0348	0.0397	0.0387	0.0267	0.0447	0.0457	0.0490	0.0397	0.0274	0.0344	0.0298	0.0495	0.0317	0.0423
0.350	0.0277	0.0418	0.0248	0.0355	0.0402	0.0466	0.0309	0.0335	0.0432	0.0411	0.0332	0.0482	0.0432	0.0352
0.400	0.0482	0.0325	0.0381	0.0312	0.0466	0.0283	0.0298	0.0435	0.0347	0.0377	0.0389	0.0326	0.0425	0.0443
0.450	0.0401	0.0451	0.0288	0.0471	0.0300	0.0496	0.0301	0.0301	0.0296	0.0312	0.0474	0.0318	0.0444	0.0495
0.500	0.0347	0.0303	0.0453	0.0356	0.0381	0.0455	0.0320	0.0326	0.0455	0.0390	0.0375	0.0450	0.0492	0.0327
0.600	0.0291	0.0417	0.0322	0.0306	0.0372	0.0308	0.0407	0.0302	0.0431	0.0472	0.0448	0.0439	0.0458	0.0406

TABLE 9

Control chart for the mean of variable X

	A1=0.0005			A2 = 0.0025		
N	L1	L2	P1	L1	L2	P2
5	—2	—2	—2	—2	—2	—2
6	—2	—2	—2	—2	—2	—2
7	—2	—2	—2	—2	—2	—2
8	—2	—2	—2	—2	—2	—2
9	—2	—2	—2	0	9	19
10	—2	—2	—2	0	10	9
11	0	11	4	0	11	4
12	0	12	2	0	12	2
13	0	13	1	1	12	17
14	0	14	0	1	13	9
15	1	14	4	1	14	4
16	1	15	2	2	14	20
17	1	16	1	2	15	11
18	1	17	0	2	16	6
19	2	17	3	3	16	22
20	2	18	2	3	17	12
21	2	19	1	3	18	7
22	3	19	4	4	18	21
23	3	20	2	4	19	12
24	3	21	1	4	20	7
25	4	21	4	5	20	20
26	4	22	2	5	21	12
27	4	23	1	5	22	7
28	5	23	4	6	22	18
29	5	24	2	6	23	11
30	5	25	1	6	24	7

Comments

N = Number of consecutive sample points all located
between the control limits; of these points Y(=0, 1, 2, ..., N) fall above the line of the mean of X
L1, L2: Number of points above the mean of X satisfying:
PR(Y. LE. L1) = PR(Y. GE.L2). LE. A1 (or A2)
P1, P2: Actual probability levels:
PR(Y. LE. L1) = PR(Y. GE. L2) = P1 (for A1), = P2(for A2)
P1 and P2 are multiplied by 10000
Assignable causes assumed to be absent
Negative signs mean that the corresponding values do not exist.