The Two-tier Stochastic Frontier (2TSF) framework: Theory and Applications, Models and Tools.

Alecos Papadopoulos

PhD Thesis.

November 2018.

vol.2: TECHNICAL APPENDIX

The Two-tier Stochastic Frontier (2TSF) framework: Theory and Applications, Models and Tools.

PhD Thesis, vol.2: Technical Appendix.

Alecos Papadopoulos
papadopalex@aueb.gr

Athens University of Economics and Business (AUEB), Greece.

TECHNICAL APPENDIX - Contents

Chapter 3
Independence and Exogeneity.

TECHNICAL APPENDIX 3.I.

The 2TSF Exponential specification.

We present here the calculations only for our contributions to this specification.

A. eq. [3.3]: The distribution function $F_{\varepsilon}(\varepsilon)$ of the composite error term.

The three-component error density is

$$f_{\varepsilon}(\varepsilon) \;=\; \frac{\exp\{a_1\}\Phi(b_1)\; + \exp\{a_2\}\Phi(b_2)}{\sigma_w + \sigma_u}$$

$$a_1 = \frac{\sigma_v^2}{2\sigma_u^2} + \frac{\varepsilon}{\sigma_u}\,, \quad b_1 = -\left(\frac{\varepsilon}{\sigma_v} + \frac{\sigma_v}{\sigma_u}\right), \quad a_2 = \frac{\sigma_v^2}{2\sigma_w^2} - \frac{\varepsilon}{\sigma_w}, \quad b_2 = \frac{\varepsilon}{\sigma_v} - \frac{\sigma_v}{\sigma_w}$$

So

$$f_{\varepsilon}(\varepsilon) = \psi_1 I_1 \;+\; \psi_2 I_2 \quad \Rightarrow \quad F_{\varepsilon}(\varepsilon) = \psi_1 \int_{-\infty}^{\varepsilon} I_1 ds \;+ \psi_2 \int_{-\infty}^{\varepsilon} I_2 ds$$

$$\psi_1 = \frac{1}{\sigma_w + \sigma_u} \exp\left\{\frac{\sigma_v^2}{2\sigma_u^2}\right\}, \;\; I_1 = \exp\{\varepsilon/\sigma_u\}\,\Phi\left(-\frac{1}{\sigma_v}\left(\varepsilon + \sigma_v^2/\sigma_u\right)\right)$$

$$\psi_2 = \frac{1}{\sigma_w + \sigma_u} \exp\left\{\frac{\sigma_v^2}{2\sigma_w^2}\right\}, \; I_2 = \exp\{-\varepsilon/\sigma_w\}\,\Phi\left(\frac{1}{\sigma_v}\left(\varepsilon - \sigma_v^2/\sigma_w\right)\right)$$

In Owen (1980), p. 409, (eq. 101,000), we find the indefinite integral

$$\int \exp\{\gamma x\}\Phi(\delta x)dx = \frac{1}{\gamma}\exp\{\gamma x\}\Phi(\delta x) - \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}\Phi\left(\delta x - \frac{\gamma}{\delta}\right)$$

To calculate $F_\varepsilon(\varepsilon)$ we want the definite integral

$$\int_{-\infty}^{x} \exp\{\gamma s\}\Phi(\delta s)ds = \frac{1}{\gamma}\exp\{\gamma x\}\Phi(\delta x) - \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}\Phi\left(\delta x - \frac{\gamma}{\delta}\right)$$

$$-\frac{1}{\gamma}\lim_{s\to-\infty}\exp\{\gamma s\}\Phi(\delta s) + \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}\lim_{s\to-\infty}\Phi\left(\delta s - \frac{\gamma}{\delta}\right)$$

Looking at our integrands I_1, I_2 , in order to determine the limits, we have to consider alternating pairs of signs for γ, δ .

A) For integrand I_1 the corresponding signs are $\gamma = 1/\sigma_u > 0,\ \delta = -\frac{1}{\sigma_v} < 0$

In this case, $\lim_{s\to-\infty}\exp\{\gamma s\}\Phi(\delta s) = 0\cdot 1 = 0,\quad \lim_{s\to-\infty}\Phi\left(\delta s - \frac{\gamma}{\delta}\right) = \Phi(\infty) = 1$. So the general formula to be used, after eliminating the zero-terms, is

$$\int_{-\infty}^{x} \exp\{\gamma s\}\Phi(\delta s)ds = \frac{1}{\gamma}\exp\{\gamma x\}\Phi(\delta x) - \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}\Phi\left(\delta x - \frac{\gamma}{\delta}\right) + \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}$$

$$= \frac{1}{\gamma}\exp\{\gamma x\}\Phi(\delta x) + \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}\Phi\left(-\delta x + \frac{\gamma}{\delta}\right)$$

where we have used the reflective symmetry of $\Phi(\cdot)$.

We have

$$\int_{-\infty}^{\varepsilon} I_1 ds = \int_{-\infty}^{\varepsilon} \exp\{s/\sigma_u\}\,\Phi\left(-\frac{1}{\sigma_v}\left(\varepsilon+\sigma_v^2/\sigma_u\right)\right)ds = \int_{-\infty}^{\varepsilon+\sigma_v^2/\sigma_u} \exp\{s/\sigma_u-\sigma_v^2/\sigma_u^2\}\,\Phi\left(-\frac{s}{\sigma_v}\right)ds$$

$$=\exp\{-\sigma_v^2/\sigma_u^2\}\int_{-\infty}^{\varepsilon+\sigma_v^2/\sigma_u} \exp\{s/\sigma_u\}\,\Phi\left(-\frac{s}{\sigma_v}\right)ds$$

which is now in the appropriate form to use Owen's formula. The correspondence of coefficients is $\gamma=1/\sigma_u>0,\ \delta=-\dfrac{1}{\sigma_v}<0$. We get

$$\int_{-\infty}^{\varepsilon} I_1 ds = \exp\{-\sigma_v^2/\sigma_u^2\}\Bigg[\sigma_u \exp\{\varepsilon/\sigma_u+\sigma_v^2/\sigma_u^2\}\,\Phi\left(-\frac{\varepsilon}{\sigma_v}-\frac{\sigma_v}{\sigma_u}\right)$$

$$+\,\sigma_u \exp\{\tfrac{1}{2}\sigma_v^2/\sigma_u^2\}\,\Phi\left(\frac{\varepsilon+\sigma_v^2/\sigma_u}{\sigma_v}-\frac{\sigma_v}{\sigma_u}\right)\Bigg]$$

$$\Rightarrow \int_{-\infty}^{\varepsilon} I_1 ds = \sigma_u\left[\exp\{\varepsilon/\sigma_u\}\,\Phi\left(-\frac{\varepsilon}{\sigma_v}-\frac{\sigma_v}{\sigma_u}\right)+\exp\{-\tfrac{1}{2}\sigma_v^2/\sigma_u^2\}\,\Phi\left(\frac{\varepsilon}{\sigma_v}\right)\right]$$

and

$$\psi_1\int_{-\infty}^{\varepsilon} I_1 ds = \frac{1}{\sigma_w+\sigma_u}\exp\left\{\frac{\sigma_v^2}{2\sigma_u^2}\right\}\left[\exp\{\varepsilon/\sigma_u\}\,\Phi\left(-\frac{\varepsilon}{\sigma_v}-\frac{\sigma_v}{\sigma_u}\right)+\exp\{-\tfrac{1}{2}\sigma_v^2/\sigma_u^2\}\,\Phi\left(\frac{\varepsilon}{\sigma_v}\right)\right]$$

$$=\frac{\sigma_u}{\sigma_w+\sigma_u}\left[\exp\left\{\frac{\sigma_v^2}{2\sigma_u^2}\right\}\exp\{\varepsilon/\sigma_u\}\,\Phi\left(-\frac{\varepsilon}{\sigma_v}-\frac{\sigma_v}{\sigma_u}\right)+\Phi\left(\frac{\varepsilon}{\sigma_v}\right)\right]$$

We turn now to the second integral.

B) For the integrand I_2 the corresponding signs are $\gamma=-1/\sigma_w<0,\ \delta=\dfrac{1}{\sigma_v}>0$

In this case, $\lim_{s\to-\infty} \exp\{\gamma s\}\Phi(\delta s) = 0$, $\lim_{s\to-\infty} \Phi\left(\delta s - \frac{\gamma}{\delta}\right) = \Phi(-\infty) = 0$. So the general formula to be used for I_2 is

$$\int_{-\infty}^{x} \exp\{\gamma s\}\Phi(\delta s)\,ds = \frac{1}{\gamma}\exp\{\gamma x\}\Phi(\delta x) - \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}\Phi\left(\delta x - \frac{\gamma}{\delta}\right)$$

We have

$$\int_{-\infty}^{\varepsilon} I_2 ds = \int_{-\infty}^{\varepsilon} \exp\{-s/\sigma_w\}\Phi\left(\frac{1}{\sigma_v}\left(s - \sigma_v^2/\sigma_w\right)\right)ds = \int_{-\infty}^{\varepsilon-\sigma_v^2/\sigma_w} \exp\{-s/\sigma_w - \sigma_v^2/\sigma_w^2\}\Phi\left(\frac{s}{\sigma_v}\right)ds$$

$$= \exp\{-\sigma_v^2/\sigma_w^2\}\int_{-\infty}^{\varepsilon-\sigma_v^2/\sigma_w} \exp\{-s/\sigma_w\}\Phi\left(\frac{s}{\sigma_v}\right)ds$$

which is now in the appropriate form. Matching coefficients, we have

$$\int_{-\infty}^{\varepsilon} I_2 ds = \exp\{-\sigma_v^2/\sigma_w^2\}\left[-\sigma_w \exp\{-\varepsilon/\sigma_w + \sigma_v^2/\sigma_w^2\}\Phi\left(\frac{\varepsilon - \sigma_v^2/\sigma_w}{\sigma_v}\right)\right.$$
$$\left. + \ \sigma_w \exp\{\tfrac{1}{2}\sigma_v^2/\sigma_w^2\}\Phi\left(\frac{\varepsilon - \sigma_v^2/\sigma_w}{\sigma_v} + \frac{\sigma_v}{\sigma_w}\right)\right]$$

$$= \sigma_w\left[-\exp\{-\varepsilon/\sigma_w\}\Phi\left(\frac{\varepsilon - \sigma_v^2/\sigma_w}{\sigma_v}\right) + \exp\{-\tfrac{1}{2}\sigma_v^2/\sigma_w^2\}\Phi\left(\frac{\varepsilon}{\sigma_v}\right)\right]$$

and

$$\psi_2\int_{-\infty}^{\varepsilon} I_2 ds = \frac{1}{\sigma_w + \sigma_u}\exp\left\{\frac{\sigma_v^2}{2\sigma_w^2}\right\}\sigma_w\left[-\exp\{-\varepsilon/\sigma_w\}\Phi\left(\frac{\varepsilon - \sigma_v^2/\sigma_w}{\sigma_v}\right) + \exp\{-\tfrac{1}{2}\sigma_v^2/\sigma_w^2\}\Phi\left(\frac{\varepsilon}{\sigma_v}\right)\right]$$

$$= \frac{\sigma_w}{\sigma_w + \sigma_u}\left[\Phi\left(\frac{\varepsilon}{\sigma_v}\right) - \exp\left\{\frac{\sigma_v^2}{2\sigma_w^2}\right\}\exp\{-\varepsilon/\sigma_w\}\Phi\left(\frac{\varepsilon}{\sigma_v} - \frac{\sigma_v}{\sigma_w}\right)\right]$$

So the cumulative distribution function of the composite error term is

$$F_\varepsilon(\varepsilon) = \int_{-\infty}^{\varepsilon} f_\varepsilon(s)\,ds = \psi_1 \int_{-\infty}^{\varepsilon} I_1 ds + \psi_2 \int_{-\infty}^{\varepsilon} I_2 ds$$

$$F_\varepsilon(\varepsilon) = \frac{\sigma_u}{\sigma_w + \sigma_u}\left[\exp\left\{\frac{\sigma_v^2}{2\sigma_u^2}\right\}\exp\{\varepsilon/\sigma_u\}\Phi\left(-\frac{\varepsilon}{\sigma_v} - \frac{\sigma_v}{\sigma_u}\right) + \Phi\left(\frac{\varepsilon}{\sigma_v}\right)\right]$$

$$+ \frac{\sigma_w}{\sigma_w + \sigma_u}\left[\Phi\left(\frac{\varepsilon}{\sigma_v}\right) - \exp\left\{\frac{\sigma_v^2}{2\sigma_w^2}\right\}\exp\{-\varepsilon/\sigma_w\}\Phi\left(\frac{\varepsilon}{\sigma_v} - \frac{\sigma_v}{\sigma_w}\right)\right]$$

and finally,

$$F_\varepsilon(\varepsilon) = \Phi\left(\frac{\varepsilon}{\sigma_v}\right) + \frac{\sigma_u}{\sigma_w + \sigma_u}\exp\left\{\frac{\sigma_v^2}{2\sigma_u^2}\right\}\exp\{\varepsilon/\sigma_u\}\Phi\left(-\frac{\varepsilon}{\sigma_v} - \frac{\sigma_v}{\sigma_u}\right)$$

$$- \frac{\sigma_w}{\sigma_w + \sigma_u}\exp\left\{\frac{\sigma_v^2}{2\sigma_w^2}\right\}\exp\{-\varepsilon/\sigma_w\}\Phi\left(\frac{\varepsilon}{\sigma_v} - \frac{\sigma_v}{\sigma_w}\right)$$

Using the shorthands used also for the density,

$$a_1 = \frac{\sigma_v^2}{2\sigma_u^2} + \frac{\varepsilon}{\sigma_u}, \quad b_1 = -\left(\frac{\varepsilon}{\sigma_v} + \frac{\sigma_v}{\sigma_u}\right), \quad a_2 = \frac{\sigma_v^2}{2\sigma_w^2} - \frac{\varepsilon}{\sigma_w}, \quad b_2 = \frac{\varepsilon}{\sigma_v} - \frac{\sigma_v}{\sigma_w}$$

we arrive at

$$[3.3]: F_\varepsilon(\varepsilon) = \Phi\left(\frac{\varepsilon}{\sigma_v}\right) + \frac{\sigma_u}{\sigma_w + \sigma_u}\exp\{a_1\}\Phi(b_1) - \frac{\sigma_w}{\sigma_w + \sigma_u}\exp\{a_2\}\Phi(b_2)$$

B. The composite error density and firm-specific measures.

Equation [3.9]: $E\left(e^w \middle| \varepsilon\right)$

The conditional density of the positive error term is given in the Appendix of Kumbhakar and Parmeter (2009) as

$$f\left(w|\varepsilon\right) = \frac{\lambda \exp\{-\lambda w\}\Phi\left(w/\sigma_v+\beta\right)}{\chi_2}$$

So the expected value we want is

$$E\left(e^w|\varepsilon\right) = \int_0^\infty e^w \frac{\lambda \exp\{-\lambda w\}\Phi\left(w/\sigma_v+\beta\right)}{\chi_2}dw$$

$$=\left(\lambda/\chi_2\right)\int_0^\infty \exp\{(1-\lambda)w\}\Phi\left(w/\sigma_v+\beta\right)dw$$

We can proceed assuming $\lambda>1$, which is the most likely case. Since $d\left(\exp\{(1-\lambda)w\}\right) = (1-\lambda)\exp\{(1-\lambda)w\}dw$ we can write

$$E\left(e^w|\varepsilon\right) = \frac{\lambda}{(1-\lambda)\chi_2}\int_0^\infty \Phi\left(w/\sigma_v+\beta\right)d\left(\exp\{(1-\lambda)w\}\right)$$

Integrating by parts,

$$E\left(e^w|\varepsilon\right) = \frac{\lambda}{(1-\lambda)\chi_2}\Big[\exp\{(1-\lambda)w\}\Phi\left(w/\sigma_v+\beta\right)\Big|_0^\infty$$

$$- \int_0^\infty \exp\{(1-\lambda)w\}d\left[\Phi\left(w/\sigma_v+\beta\right)\right]\Big]$$

$$= \frac{\lambda}{(1-\lambda)\chi_2}\left[-\Phi(\beta) - \int_0^\infty \exp\{(1-\lambda)w\}\phi\left(w/\sigma_v+\beta\right)d\left(w/\sigma_v\right)\right]$$

$$= \frac{\lambda}{(\lambda-1)\chi_2}\left[\Phi(\beta) + \int_0^\infty \exp\{(1-\lambda)w\}\phi\left(w/\sigma_v+\beta\right)d\left(w/\sigma_v\right)\right]$$

Define $z\equiv \tilde{w}\equiv w/\sigma_v$ and subtract β from the variable of integration to obtain

$$E\left(e^{w}|\varepsilon\right) = \frac{\lambda}{(\lambda-1)\chi_2}\left[\Phi(\beta) + \int_{\beta}^{\infty}\exp\left\{(1-\lambda)\sigma_v(\tilde{w}-\beta)\right\}\phi(\tilde{w})d\tilde{w}\right]$$

Calculating the integrand we have

$$\exp\left\{(1-\lambda)\sigma_v(\tilde{w}-\beta)\right\}\phi(\tilde{w}) = \frac{1}{\sqrt{2\pi}}\exp\left\{(\lambda-1)\sigma_v\beta\right\}\exp\left\{-\frac{1}{2}\tilde{w}^2-(\lambda-1)\sigma_v\tilde{w}\right\}$$

So

$$E\left(e^{w}|\varepsilon\right) = \frac{\lambda}{(\lambda-1)\chi_2}\left[\Phi(\beta) + \frac{1}{\sqrt{2\pi}}e^{(\lambda-1)\sigma_v\beta}\int_{\beta}^{\infty}\exp\left\{-\frac{1}{2}\tilde{w}^2-(\lambda-1)\sigma_v\tilde{w}\right\}d\tilde{w}\right]$$

Using the relevant formula from Gradshteyn and Ryzhik (2007) (p.336), we obtain

$$\int_{\beta}^{\infty}\exp\left\{-\frac{1}{2}\tilde{w}^2-(\lambda-1)\sigma_v\tilde{w}\right\}d\tilde{w} = \sqrt{\frac{\pi}{2}}\exp\left\{\frac{1}{2}(\lambda-1)^2\sigma_v^2\right\}\left[1-\operatorname{erf}\left(\frac{(\lambda-1)\sigma_v}{\sqrt{2}}+\frac{\beta}{\sqrt{2}}\right)\right]$$

$$= 2\sqrt{\frac{\pi}{2}}\exp\left\{\frac{1}{2}(\lambda-1)^2\sigma_v^2\right\}\left[1-\Phi\left((\lambda-1)\sigma_v + \beta\right)\right]$$

Inserting into the full expression and simplifying constants we obtain

$$E\left(e^{w}|\varepsilon\right) = \frac{\lambda}{(\lambda-1)\chi_2}\left[\Phi(\beta) + \exp\left\{(\lambda-1)\sigma_v\beta + \frac{1}{2}(\lambda-1)^2\sigma_v^2\right\}\left[1-\Phi\left((\lambda-1)\sigma_v + \beta\right)\right]\right]$$

Here $\beta = -\left(\frac{\varepsilon}{\sigma_v}+\frac{\sigma_v}{\sigma_u}\right)$. So $(\lambda-1)\sigma_v + \beta = \frac{\sigma_v}{\sigma_w} + \frac{\sigma_v}{\sigma_u} - \sigma_v - \frac{\varepsilon}{\sigma_v} - \frac{\sigma_v}{\sigma_u} = -b-\sigma_v$

Inserting into the expression we obtain

$$E\left(e^{w}|\varepsilon\right) = \frac{\lambda}{(\lambda-1)\chi_2}\left[\Phi(\beta) + \exp\left\{(\lambda-1)\sigma_v\beta + \frac{1}{2}(\lambda-1)^2\sigma_v^2\right\}\Phi(b+\sigma_v)\right]$$

Manipulating further the exponent expression we have

$$(\lambda-1)\sigma_v\beta + \frac{1}{2}(\lambda-1)^2\sigma_v^2 = (\lambda-1)\sigma_v\left[\beta + \frac{1}{2}(\lambda-1)\sigma_v\right]$$

$$=(\lambda-1)\sigma_v\left[\beta + (\lambda-1)\sigma_v - \frac{1}{2}(\lambda-1)\sigma_v\right] = (\lambda-1)\sigma_v\left[-b-\sigma_v - \frac{1}{2}(\lambda-1)\sigma_v\right]$$

Now given how composite coefficients are defined here we have

$b+\beta = -\lambda\sigma_v$. Using this relation we get

$$(\lambda-1)\sigma_v\beta + \frac{1}{2}(\lambda-1)^2\sigma_v^2 = (-b-\beta-\sigma_v)\left[-b - \sigma_v - \frac{1}{2}(-b-\beta-\sigma_v)\right]$$

$$=(-b-\beta-\sigma_v)\left[-\frac{1}{2}b - \frac{1}{2}\sigma_v + \frac{1}{2}\beta\right] = \frac{1}{2}(b+\sigma_v+\beta)(b+\sigma_v-\beta)$$

$$=\frac{1}{2}\left[(b+\sigma_v)^2 - \beta^2\right]$$

So finally we obtain

$$E\left(e^{w}|\varepsilon\right) = \frac{\lambda}{(\lambda-1)\chi_2}\left[\Phi(\beta) + \exp\left\{\frac{1}{2}\left[(b+\sigma_v)^2 - \beta^2\right]\right\}\Phi(b+\sigma_v)\right]$$

Equation [3.12]: $E\left(e^{w_i}e^{-u_i}|\varepsilon_i\right)$.

Since the variables are conditionally dependent, we set $z = w - u$ and we want to obtain

$$E\left(e^{w}e^{-u}|\varepsilon\right) = E\left(e^{z}|\varepsilon\right) = \int_{-\infty}^{\infty} e^{z} f_{z|\varepsilon}(z|\varepsilon)\,\mathrm{d}z$$

We note that z is independent of v, while we have $\varepsilon - z = v$. Then the joint density function of z and ε can be obtained as

$f_{z,\varepsilon}(z,\varepsilon) = f_{z,v}(z,\varepsilon - z) = f_z(z)\, f_v(\varepsilon - z)$. So the conditional density function we need is

$$f_{z|\varepsilon}(z \mid \varepsilon) \;=\; \frac{f_z(z)\, f_v(\varepsilon - z)}{f_\varepsilon(\varepsilon)}$$

It is a known result that the density of z, the difference of two independent Exponential random variables, has two branches,

$$f_z(z) = \frac{1}{\sigma_w + \sigma_u}\begin{cases} \exp\{z/\sigma_u\} & z \le 0 \\[2ex] \exp\{-z/\sigma_w\} & z > 0 \end{cases}$$

We also have that $f_v(\varepsilon - z)$ is a normal density function, while $f_\varepsilon(\varepsilon)$ is known. So

$$\int_{-\infty}^{\infty} e^z f_{z|\varepsilon}(z \mid \varepsilon)\mathrm{dz} = \int_{-\infty}^{0} e^z f_{z|\varepsilon}(z \mid \varepsilon)\mathrm{dz} \;+\; \int_{0}^{\infty} e^z f_{z|\varepsilon}(z \mid \varepsilon)\mathrm{dz}$$

$$= \frac{1}{f_\varepsilon(\varepsilon)}\frac{1}{\sigma_w + \sigma_u}\frac{1}{\sigma_v\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^z e^{z/\sigma_u}\exp\left\{-\frac{1}{2\sigma_v^2}(\varepsilon - z)^2\right\}\mathrm{dz}\right.$$

$$\left. + \int_{0}^{\infty} e^z e^{-z/\sigma_w}\exp\left\{-\frac{1}{2\sigma_v^2}(\varepsilon - z)^2\right\}\mathrm{dz}\right]$$

We calculate each integral in turn.

$$I_1 = \int_{-\infty}^{0} e^z e^{z/\sigma_u}\exp\left\{-\frac{1}{2\sigma_v^2}(\varepsilon - z)^2\right\}\mathrm{dz} = \int_{-\infty}^{0}\exp\left\{\frac{1+\sigma_u}{\sigma_u}z\right\}\exp\left\{-\frac{1}{2\sigma_v^2}(\varepsilon - z)^2\right\}\mathrm{dz}$$

$$= \exp\left\{-\frac{1}{2\sigma_v^2}\varepsilon^2\right\}\int_{-\infty}^{0}\exp\left\{\frac{1+\sigma_u}{\sigma_u}z\right\}\exp\left\{-\frac{1}{2\sigma_v^2}z^2+\frac{\varepsilon}{\sigma_v^2}z\right\}\mathrm{d}z$$

$$= \exp\left\{-\frac{1}{2\sigma_v^2}\varepsilon^2\right\}\int_{-\infty}^{0}\exp\left\{-\frac{1}{2\sigma_v^2}z^2\right\}\exp\left\{\left(\frac{1+\sigma_u}{\sigma_u}+\frac{\varepsilon}{\sigma_v^2}\right)z\right\}\mathrm{d}z$$

$$= \exp\left\{-\frac{1}{2\sigma_v^2}\varepsilon^2\right\}\int_{0}^{\infty}\exp\left\{-\frac{1}{2\sigma_v^2}z^2\right\}\exp\left\{-\left(\frac{1+\sigma_u}{\sigma_u}+\frac{\varepsilon}{\sigma_v^2}\right)z\right\}\mathrm{d}z$$

Combining formula 6.3(13) p. 313 from Erdelyi et al. (1954) with expression 9.254(1) p. 1030 from Gradshteyn & Ryzhik (2007), we have that

$$\int_0^{\infty}\exp\left\{-az^2\right\}\exp\left\{-bz\right\}\mathrm{d}z = \frac{\sqrt{\pi}}{\sqrt{a}}\exp\left\{\frac{b^2}{4a}\right\}\Phi\left(-\frac{b}{\sqrt{2a}}\right)$$

In our case,

$$a=\frac{1}{2\sigma_v^2},\quad b=\left(\frac{1+\sigma_u}{\sigma_u}+\frac{\varepsilon}{\sigma_v^2}\right),\quad \sqrt{a}=\frac{1}{\sqrt{2}\sigma_v},\quad \frac{b}{\sqrt{2a}}=\frac{\left(\frac{1+\sigma_u}{\sigma_u}+\frac{\varepsilon}{\sigma_v^2}\right)}{1/\sigma_v}=\frac{(1+\sigma_u)\sigma_v}{\sigma_u}+\frac{\varepsilon}{\sigma_v}$$

$$\frac{b^2}{4a}=\frac{\left(\frac{1+\sigma_u}{\sigma_u}+\frac{\varepsilon}{\sigma_v^2}\right)^2}{2/\sigma_v^2}=\frac{1}{2}\left(\frac{(1+\sigma_u)\sigma_v}{\sigma_u}+\frac{\varepsilon}{\sigma_v}\right)^2=\frac{1}{2}\frac{\varepsilon^2}{\sigma_v^2}+\frac{(1+\sigma_u)}{\sigma_u}\varepsilon+\frac{1}{2}\left(\frac{(1+\sigma_u)\sigma_v}{\sigma_u}\right)^2$$

Inserting these we obtain

$$\exp\left\{-\frac{1}{2\sigma_v^2}\varepsilon^2\right\}\int_{0}^{\infty}\exp\left\{-\frac{1}{2\sigma_v^2}z^2\right\}\exp\left\{-\left(\frac{1+\sigma_u}{\sigma_u}+\frac{\varepsilon}{\sigma_v^2}\right)z\right\}\mathrm{d}z$$

$$=\exp\left\{-\frac{1}{2\sigma_v^2}\varepsilon^2\right\}\sigma_v\sqrt{2\pi}\exp\left\{\frac{1}{2}\frac{\varepsilon^2}{\sigma_v^2}+\frac{(1+\sigma_u)}{\sigma_u}\varepsilon+\frac{1}{2}\left(\frac{(1+\sigma_u)\sigma_v}{\sigma_u}\right)^2\right\}\Phi\left(-\frac{(1+\sigma_u)\sigma_v}{\sigma_u}-\frac{\varepsilon}{\sigma_v}\right)$$

$$\Rightarrow I_1 = \sigma_v\sqrt{2\pi}\exp\left\{\frac{(1+\sigma_u)}{\sigma_u}\varepsilon \;+\; \frac{(1+\sigma_u)^2\sigma_v^2}{2\sigma_u^2}\right\}\Phi\left(-\frac{(1+\sigma_u)\sigma_v}{\sigma_u}-\frac{\varepsilon}{\sigma_v}\right)$$

For the 2nd integral we have

$$\int_0^\infty e^z e^{-z/\sigma_w}\exp\left\{-\frac{1}{2\sigma_v^2}(\varepsilon-z)^2\right\}\mathrm{dz} = \int_0^\infty \exp\left\{-\frac{1-\sigma_w}{\sigma_w}z\right\}\exp\left\{-\frac{1}{2\sigma_v^2}(\varepsilon-z)^2\right\}\mathrm{dz}$$

$$= \exp\left\{-\frac{1}{2\sigma_v^2}\varepsilon^2\right\}\int_0^\infty \exp\left\{-\frac{1}{2\sigma_v^2}z^2\right\}\exp\left\{-\left(\frac{1-\sigma_w}{\sigma_w}-\frac{\varepsilon}{\sigma_v^2}\right)z\right\}\mathrm{dz}$$

Matching coefficients here,

$$a=\frac{1}{2\sigma_v^2},\quad b=\left(\frac{1-\sigma_w}{\sigma_w}-\frac{\varepsilon}{\sigma_v^2}\right),\quad \sqrt{a}=\frac{1}{\sqrt{2}\sigma_v},\quad \frac{b}{\sqrt{2a}}=\frac{\left(\frac{1-\sigma_w}{\sigma_w}-\frac{\varepsilon}{\sigma_v^2}\right)}{1/\sigma_v}=\frac{(1+\sigma_w)\sigma_v}{\sigma_w}-\frac{\varepsilon}{\sigma_v}$$

$$\frac{b^2}{4a}=\frac{\left(\frac{1-\sigma_w}{\sigma_w}-\frac{\varepsilon}{\sigma_v^2}\right)^2}{2/\sigma_v^2}=\frac{1}{2}\left(\frac{(1-\sigma_w)\sigma_v}{\sigma_w}-\frac{\varepsilon}{\sigma_v}\right)^2=\frac{1}{2}\frac{\varepsilon^2}{\sigma_v^2}\;-\;\frac{(1-\sigma_w)}{\sigma_w}\varepsilon\;+\;\frac{1}{2}\left(\frac{(1-\sigma_w)\sigma_v}{\sigma_w}\right)^2$$

and so

$$\exp\left\{-\frac{1}{2\sigma_v^2}\varepsilon^2\right\}\int_0^\infty \exp\left\{-\frac{1}{2\sigma_v^2}z^2\right\}\exp\left\{-\left(\frac{1-\sigma_w}{\sigma_w}-\frac{\varepsilon}{\sigma_v^2}\right)z\right\}\mathrm{dz} =$$

$$= \exp\left\{-\frac{1}{2\sigma_v^2}\varepsilon^2\right\}\sigma_v\sqrt{2\pi}\exp\left\{\frac{1}{2}\frac{\varepsilon^2}{\sigma_v^2}\;-\;\frac{(1-\sigma_w)}{\sigma_w}\varepsilon\;+\;\frac{1}{2}\left(\frac{(1-\sigma_w)\sigma_v}{\sigma_w}\right)^2\right\}\Phi\left(\frac{\varepsilon}{\sigma_v}-\frac{(1-\sigma_w)\sigma_v}{\sigma_w}\right)$$

$$\Rightarrow I_2 = \sigma_v\sqrt{2\pi}\exp\left\{\frac{(1-\sigma_w)^2\sigma_v^2}{2\sigma_w^2}\;-\;\frac{(1-\sigma_w)}{\sigma_w}\varepsilon\right\}\Phi\left(\frac{\varepsilon}{\sigma_v}-\frac{(1-\sigma_w)\sigma_v}{\sigma_w}\right)$$

Bringing it all together, while simplifying the constants $\sigma_v\sqrt{2\pi}$,

$$\int_{-\infty}^{\infty} e^z f_{z|\varepsilon}\left(z \mid \varepsilon\right)\mathrm{dz} = \frac{1}{f_\varepsilon\left(\varepsilon\right)}\frac{1}{\sigma_w+\sigma_u}\left[\exp\left\{\frac{\left(1+\sigma_u\right)}{\sigma_u}\varepsilon + \frac{\left(1+\sigma_u\right)^2\sigma_v^2}{2\sigma_u^2}\right\}\Phi\left(-\frac{\left(1+\sigma_u\right)\sigma_v}{\sigma_u}-\frac{\varepsilon}{\sigma_v}\right)\right.$$

$$\left. +\exp\left\{\frac{\left(1-\sigma_w\right)^2\sigma_v^2}{2\sigma_w^2} - \frac{\left(1-\sigma_w\right)}{\sigma_w}\varepsilon\right\}\Phi\left(\frac{\varepsilon}{\sigma_v}-\frac{\left(1-\sigma_w\right)\sigma_v}{\sigma_w}\right)\right]$$

Using the shorthands already in use, we can compact the above into

[3.12]: $\int_{-\infty}^{\infty} e^z f_{z|\varepsilon}\left(z \mid \varepsilon\right)\mathrm{dz} =$

$$E\left(e^{w_i}e^{-u_i}\left|\varepsilon_i\right.\right) = \frac{\exp\left\{\left(1+\sigma_u\right)\left(a_{1i}+\frac{\sigma_v^2}{2\sigma_u}\right)\right\}\Phi\left(b_{1i}-\sigma_v\right)+\exp\left\{\left(1-\sigma_w\right)\left(a_{2i}-\frac{\sigma_v^2}{2\sigma_w}\right)\right\}\Phi\left(b_{2i}+\sigma_v\right)}{\exp\left\{a_{1i}\right\}\Phi\left(b_{1i}\right)+\exp\left\{a_{2i}\right\}\Phi\left(b_{2i}\right)}$$

Equation [3.13]: $\Pr\left(w>u\right)$.

$$\Pr\left(w>u\right) = \int_0^{\infty}\left(1/\sigma_u\right)e^{-u/\sigma_u}\int_u^{\infty}\left(1/\sigma_w\right)e^{-v/\sigma_w}\mathrm{d}v\mathrm{d}u$$

$$= \int_0^{\infty}\left(1/\sigma_u\right)e^{-u/\sigma_u}\left[1-\left(1-e^{-u/\sigma_w}\right)\right]\mathrm{d}u = \frac{1}{\sigma_u}\int_0^{\infty}\exp\left\{-\left(\frac{1}{\sigma_u}+\frac{1}{\sigma_w}\right)u\right\}\mathrm{d}u$$

$$= \frac{1}{\sigma_u}\int_0^{\infty}\exp\left\{-\left(\frac{\sigma_w+\sigma_u}{\sigma_u\sigma_w}\right)u\right\}\mathrm{d}u = \frac{\sigma_u\sigma_w}{\sigma_u\left(\sigma_w+\sigma_u\right)}\cdot 1 = \frac{\sigma_w}{\sigma_w+\sigma_u}$$

$$\boxed{[3.13]:\ \Pr\left(w>u\right) = \frac{\sigma_w}{\sigma_w+\sigma_u}}$$

Equation [3.14]: $\Pr\left(w_i > u_i \left| \varepsilon_i \right.\right)$.

We want to calculate $\Pr\left(z_i > 0 \left| \varepsilon_i \right.\right) = \int_0^\infty f_{z|\varepsilon}\left(z \mid \varepsilon\right) \mathrm{d}z$

From before, we have that

$$f_{z|\varepsilon}\left(z \mid \varepsilon\right) = \frac{f_z\left(z\right) f_v\left(\varepsilon - z\right)}{f_\varepsilon\left(\varepsilon\right)} = \frac{\dfrac{\exp\left\{-z/\sigma_w\right\}}{\sigma_w + \sigma_u} \dfrac{1}{\sigma_v\sqrt{2\pi}} \exp\left\{-\dfrac{1}{2\sigma_v^2}\left(\varepsilon - z\right)^2\right\}}{\dfrac{\exp\left\{a_1\right\}\Phi\left(b_1\right) + \exp\left\{a_2\right\}\Phi\left(b_2\right)}{\sigma_w + \sigma_u}}$$

$$= \frac{\dfrac{1}{\sigma_v\sqrt{2\pi}} \exp\left\{-\dfrac{\varepsilon^2}{2\sigma_v^2}\right\} \exp\left\{-\dfrac{z^2}{2\sigma_v^2} - \left(\dfrac{1}{\sigma_w} - \dfrac{\varepsilon}{\sigma_v^2}\right) z\right\}}{\exp\left\{a_1\right\}\Phi\left(b_1\right) + \exp\left\{a_2\right\}\Phi\left(b_2\right)}$$

The only term that contains z is the rightmost exponential expression in the numerator so we integrate using the relevant formula from Gradshteyn and Ryzhik (2007) (p.336),

$$\int_0^\infty \exp\left\{-\frac{z^2}{2\sigma_v^2} - \left(\frac{1}{\sigma_w} - \frac{\varepsilon}{\sigma_v^2}\right) z\right\} \mathrm{d}z =$$

$$= \sqrt{\pi}\,\frac{\sigma_v}{\sqrt{2}} \exp\left\{\frac{\sigma_v^2}{2}\left(\frac{1}{\sigma_w} - \frac{\varepsilon}{\sigma_v^2}\right)^2\right\}\left[1 - \mathrm{erf}\left(\left(\frac{1}{\sigma_w} - \frac{\varepsilon}{\sigma_v^2}\right)\frac{\sigma_v}{\sqrt{2}}\right)\right]$$

$$= \sqrt{\pi}\,\frac{\sigma_v}{\sqrt{2}} \exp\left\{\frac{\sigma_v^2}{2}\left(\frac{1}{\sigma_w^2} - 2\frac{\varepsilon}{\sigma_w\sigma_v^2} + \frac{\varepsilon^2}{\sigma_v^4}\right)\right\} 2\Phi\left(\frac{\varepsilon}{\sigma_v} - \frac{\sigma_v}{\sigma_w}\right)$$

$$= \sigma_v\sqrt{2\pi} \cdot \exp\left\{\frac{\sigma_v^2}{2\sigma_w^2} - \frac{\varepsilon}{\sigma_w} + \frac{\varepsilon^2}{2\sigma_v^2}\right\}\Phi\left(\frac{\varepsilon}{\sigma_v} - \frac{\sigma_v}{\sigma_w}\right)$$

$$= \sigma_v\sqrt{2\pi} \cdot \exp\left\{a_2 + \frac{\varepsilon^2}{2\sigma_v^2}\right\}\Phi\left(b_2\right)$$

Substituting in the integral,

$$\int_0^{\infty} f_{z|\varepsilon}(z|\varepsilon)\mathrm{dz} = \frac{\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\frac{\varepsilon^2}{2\sigma_v^2}\right\}\sigma_v\sqrt{2\pi}\cdot\exp\left\{a+\frac{\varepsilon^2}{2\sigma_v^2}\right\}\Phi(b)}{\exp\{a_1\}\Phi(b_1)\ +\exp\{a_2\}\Phi(b_2)}$$

$$\boxed{\Rightarrow [3.14]:\ \Pr\left(w_i > u_i \,|\varepsilon_i\right) = \frac{\exp\{a_2\}\Phi(b_2)}{\exp\{a_1\}\Phi(b_1)\ +\exp\{a_2\}\Phi(b_2)}}$$

C. Skewness and Excess Kurtosis of the composite error term in the 2TSF Exponential specification.

C.1 Skewness.

For the composite error term $\varepsilon = v + w - u$, the skewness coefficient γ_1 is

$$\gamma_1(\varepsilon) = \frac{E(\varepsilon^3) - 3E(\varepsilon)\mathrm{Var}(\varepsilon) - \left[E(\varepsilon)\right]^3}{\left[\mathrm{Var}(\varepsilon)\right]^{3/2}}$$

$$\Rightarrow \mathrm{sign}\{\gamma_1(\varepsilon)\} = \mathrm{sign}\left\{E(\varepsilon^3) - 3E(\varepsilon)\mathrm{Var}(\varepsilon) - \left[E(\varepsilon)\right]^3\right\}$$

Set $z = w - u$ and use where appropriate the symbols μ, σ for the mean and standard deviation respectively. Use also the assumption that the three components are jointly independent, and that the odd moments of v are zero, $E(v) = E(v^3) = 0$. We have

$$E(\varepsilon^3) - 3E(\varepsilon)\mathrm{Var}(\varepsilon) - \left[E(\varepsilon)\right]^3 = E(v+z)^3 - 3\mu_z\left(\sigma_v^2 + \sigma_z^2\right) - \mu_z^3$$

$$= E\left[(v+z)\left(v^2 + 2vz + z^2\right)\right] - 3\mu_z\left(\sigma_v^2 + \sigma_z^2\right) - \mu_z^3$$

$$= E\left[v^3 + 2v^2z + vz^2 + zv^2 + 2vz^2 + z^3\right] - 3\mu_z\left(\sigma_v^2 + \sigma_z^2\right) - \mu_z^3$$

$$= 3\sigma_v^2\mu_z + E\left(z^3\right) - 3\mu_z\left(\sigma_v^2 + \sigma_z^2\right) - \mu_z^3$$

$$= E\left(z^3\right) - 3\mu_z\sigma_z^2 - \mu_z^3$$

$$= E\left[(w-u)\left(w^2 - 2uw + u^2\right)\right] - 3(\mu_w - \mu_u)\left(\sigma_w^2 + \sigma_u^2\right) - (\mu_w - \mu_u)^3$$

$$= E\left(w^3 - 2uw^2 + wu^2 - uw^2 + 2u^2w - u^3\right) - 3(\mu_w - \mu_u)\left(\sigma_w^2 + \sigma_u^2\right) - (\mu_w - \mu_u)^3$$

$$= E\left(w^3\right) - E\left(u^3\right) - 3\mu_u E\left(w^2\right) + 3\mu_w E\left(u^2\right) - 3(\mu_w - \mu_u)\left(\sigma_w^2 + \sigma_u^2\right) - (\mu_w - \mu_u)^3$$

$$= E\left(w^3\right) - E\left(u^3\right) - 3\mu_u\left(\sigma_w^2 + \mu_w^2\right) + 3\mu_w\left(\sigma_u^2 + \mu_u^2\right) - 3(\mu_w - \mu_u)\left(\sigma_w^2 + \sigma_u^2\right) - (\mu_w - \mu_u)^3$$

Decomposing, simplifying and taking common factors,

$$= E\left(w^3\right) - E\left(u^3\right) - 3\mu_u\sigma_w^2 - 3\mu_u\mu_w^2 + 3\mu_w\sigma_u^2 + 3\mu_w\mu_u^2$$

$$- 3\mu_w\left(\sigma_w^2 + \sigma_u^2\right) + 3\mu_u\left(\sigma_w^2 + \sigma_u^2\right) - (\mu_w - \mu_u)^3$$

$$\Rightarrow \text{sign}\{\gamma_1(\varepsilon)\} = \text{sign}\left\{E\left(w^3\right) - E\left(u^3\right) - 3\mu_w\mu_u(\mu_w - \mu_u) - 3\mu_w\sigma_w^2 + 3\mu_u\sigma_u^2 - (\mu_w - \mu_u)^3\right\}$$

For the 2TSF Exponential specification, $w \sim \text{Exp}\left(1/\sigma_w\right),\ u \sim \text{Exp}\left(1/\sigma_u\right)$ we have that $\mu = \sigma \Rightarrow \mu^2 = \sigma^2 = \text{Var}(x),\ E\left(x^3\right) = 6\sigma^3$. Using these we have

$$\text{sign}\{\gamma_1(\varepsilon)\} = \text{sign}\left\{6\mu_w^3 - 6\mu_u^3 - 3\mu_w\mu_u(\mu_w - \mu_u) - 3\mu_w^3 + 3\mu_u^3 - (\mu_w - \mu_u)^3\right\}$$

$$= \text{sign}\left\{3\mu_w^3 - 3\mu_u^3 - (\mu_w - \mu_u)\left(3\mu_w\mu_u + (\mu_w - \mu_u)^2\right)\right\}$$

$$= \text{sign}\left\{3(\mu_w - \mu_u)\left(\mu_w^2 + \mu_w\mu_u + \mu_u^2\right) - (\mu_w - \mu_u)\left(\mu_w^2 + \mu_w\mu_u + \mu_u^2\right)\right\}$$

$$= \text{sign}\{\mu_w - \mu_u\}$$

So the skewness of the composite error term will be positive iff $\mu_w > \mu_u \Rightarrow \sigma_w > \sigma_u$.

C.2 Excess Kurtosis.

The density of the composite error term in the 2TSF exponential specification is

$$f(\varepsilon)\big|_{ExpSpec} = \frac{1}{\sigma_u + \sigma_w}\left[\exp\left\{\frac{\varepsilon}{\sigma_u} + \frac{\sigma_v^2}{2\sigma_u^2}\right\}\Phi\left(-\frac{\varepsilon}{\sigma_v} - \frac{\sigma_v}{\sigma_u}\right) + \exp\left\{-\frac{\varepsilon}{\sigma_u} + \frac{\sigma_v^2}{2\sigma_w^2}\right\}\Phi\left(\frac{\varepsilon}{\sigma_v} - \frac{\sigma_v}{\sigma_w}\right)\right]$$

The density becomes an even function in ε when $\sigma_u = \sigma_w = \sigma$,

$$f(\varepsilon)\big|_{ExpSpec,Sym} = \frac{1}{2\sigma}\exp\left\{\frac{\sigma_v^2}{2\sigma^2}\right\}\left[\exp\left\{\frac{\varepsilon}{\sigma}\right\}\Phi\left(-\frac{\varepsilon}{\sigma_v} - \frac{\sigma_v}{\sigma}\right) + \exp\left\{-\frac{\varepsilon}{\sigma}\right\}\Phi\left(\frac{\varepsilon}{\sigma_v} - \frac{\sigma_v}{\sigma}\right)\right]$$

and so symmetric around $\varepsilon = 0$. At $\varepsilon = 0$ we have

$$f(\varepsilon = 0)\big|_{ExpSpec,Sym} = \frac{1}{2\sigma}\exp\left\{\frac{\sigma_v^2}{2\sigma^2}\right\}\left[\Phi\left(-\frac{\sigma_v}{\sigma}\right) + \Phi\left(-\frac{\sigma_v}{\sigma}\right)\right] = \frac{1}{\sigma}\exp\left\{\frac{\sigma_v^2}{2\sigma^2}\right\}\Phi\left(-\frac{\sigma_v}{\sigma}\right)$$

$$= \frac{\frac{1}{\sqrt{2\pi}}\Phi\left(-\frac{\sigma_v}{\sigma}\right)}{\sigma\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{\sigma_v}{\sigma}\right)^2\right\}} = \frac{\frac{1}{\sqrt{2\pi}}\Phi\left(-\frac{\sigma_v}{\sigma}\right)}{\sigma\phi\left(\frac{\sigma_v}{\sigma}\right)} = \frac{1}{\sigma\sqrt{2\pi}}m(\sigma_v/\sigma)$$

where $m(\sigma_v/\sigma)$ is the Mill's ratio of the standard normal distribution.

The variance of the composite error under symmetry is $Var(\varepsilon) = \sigma_v^2 + 2\sigma^2$.

The value of the density of a zero-mean normal random variable y at zero having the same variance as ε is

$$\frac{1}{\sigma_\varepsilon}\phi\left(\frac{y}{\sigma_\varepsilon} = 0\right) = \frac{1}{\left(\sqrt{\sigma_v^2 + 2\sigma^2}\right)\sqrt{2\pi}} = \frac{1}{\sigma\sqrt{2\pi}\sqrt{\left(\sigma_v^2/\sigma^2\right) + 2}}$$

For the composite error term to exhibit positive excess kurtosis, the value of its density at zero should be larger than the corresponding value of a normal random variable having the same variance. Therefore, and setting $z \equiv \sigma_v / \sigma$ we must have

$$\Rightarrow f(\varepsilon = 0)\big|_{ExpSpec,Sym} > \frac{1}{\sigma_\varepsilon}\phi\left(\frac{y}{\sigma_\varepsilon} = 0\right) \Rightarrow \frac{1}{\sigma\sqrt{2\pi}} m(z) > \frac{1}{\sigma\sqrt{2\pi}\sqrt{z^2 + 2}}$$

$$\Rightarrow m(z) > \frac{1}{\sqrt{z^2 + 2}}$$

Birnbaum (1942), improving on the better known results of Gordon (1941), showed that a tighter than Gordon's lower bound for the Mill's ratio of the standard normal is

$$m(z) > \frac{\left(\sqrt{z^2 + 4}\right) - z}{2} = \frac{2}{\left(\sqrt{z^2 + 4}\right) + z}$$

Combining, a sufficient condition for the positive excess kurtosis result to hold is that

$$\frac{2}{\left(\sqrt{z^2 + 4}\right) + z} > \frac{1}{\sqrt{z^2 + 2}} \Rightarrow 2\sqrt{z^2 + 2} > \left(\sqrt{z^2 + 4}\right) + z$$

$$\Rightarrow 4\left(z^2 + 2\right) > \left(z^2 + 4\right) + 2z\sqrt{z^2 + 4} + z^2$$

$$\Rightarrow 2z^2 + 4 > 2z\sqrt{z^2 + 4} \Rightarrow z^2 + 2 > z\sqrt{z^2 + 4}$$

$$\Rightarrow \left(z^2 + 2\right)^2 > z^2\left(z^2 + 4\right) \Rightarrow z^4 + 4z^2 + 4 > z^4 + 4z^2$$

which holds. QED.

--

TECHNICAL APPENDIX 3.II.

The Half-normal 2TSF specification.

A. The composite error density, eq. [3.18].

We have the composite error term $\varepsilon = v + w - u$, where

$$v \sim N\left(0, \sigma_u^2\right),\ w \sim HN\left(\sigma_1\right),\ u \sim HN\left(\sigma_2\right)$$

jointly independent. For this Appendix we identify for added clarity $\sigma_1 = \sigma_w$ and $\sigma_2 = \sigma_u$.

We start with the density of $z = w - u$, the difference of two independent Half-normals. This distribution has been studied in a wider context in Papadopoulos (2015b). From there, we have the density

$$f_z(z) = \begin{cases} \dfrac{4}{s_h}\phi\left(z/s_h\right)\Phi\left(\dfrac{\sigma_1}{\sigma_2}\left(z/s_h\right)\right), & z \le 0 \\[2ex] \dfrac{4}{s_h}\phi\left(z/s_h\right)\Phi\left(-\dfrac{\sigma_2}{\sigma_1}\left(z/s_h\right)\right), & z \ge 0 \end{cases} \qquad s_h = \sqrt{\sigma_1^2 + \sigma_2^2} \qquad [1]$$

Both branches are two times a Skew-normal density, with zero location parameter, same scale parameter but different skew parameter.

Then since $\varepsilon = v + z \Rightarrow v = \varepsilon - z$ and $f_{v,z}(v,z) = f_v(v)\, f_z(z)$ we have that

$$f_\varepsilon(\varepsilon) = \int_{z=-\infty}^{0} f_v(\varepsilon - z)\, f_z(z)\, dz + \int_{z=0}^{\infty} f_v(\varepsilon - z)\, f_z(z)\, dz$$

$$\Rightarrow f_\varepsilon(\varepsilon) = \int_{z=-\infty}^{0} \frac{1}{\sigma_v}\phi\left(\frac{\varepsilon - z}{\sigma_v}\right)\frac{4}{s_h}\phi(z/s_h)\Phi\left(\lambda_h(z/s_h)\right)dz$$

$$+ \int_{z=0}^{\infty} \frac{1}{\sigma_v}\phi\left(\frac{\varepsilon - z}{\sigma_v}\right)\frac{4}{s_h}\phi(z/s_h)\Phi\left(-\lambda_h^{-1}(z/s_h)\right)dz \qquad [2]$$

where we have set $\lambda_h \equiv \dfrac{\sigma_1}{\sigma_2}$.

For a general skew parameter λ, the integrand in both integrals takes the form

$$A(\lambda) = \frac{4(2\pi)^{-1}}{\sigma_v s_h}\exp\left\{-\frac{1}{2}\left(\frac{\varepsilon - z}{\sigma_v}\right)^2\right\}\exp\left\{-\frac{1}{2}\left(\frac{z}{s_h}\right)^2\right\}\int_{-\infty}^{\lambda z/s_h}\phi(y)\,dy$$

Setting $s^2 \equiv \sigma_v^2 + \sigma_1^2 + \sigma_2^2 = \sigma_v^2 + s_h^2$, and $C \equiv \dfrac{4(2\pi)^{-1}}{\sigma_v s_h}\exp\left\{-\dfrac{1}{2}\left(\dfrac{\varepsilon}{\sigma_v}\right)^2\right\}$, i.e containing the terms that do not depend on the integrating variables, we have

$$A(\lambda) = C\exp\left\{-\frac{1}{2}\left(\frac{z}{s_h}\right)^2 - \frac{1}{2}\left(\frac{z}{\sigma_v}\right)^2 + \frac{\varepsilon z}{\sigma_v^2}\right\}\int_{-\infty}^{\lambda z/s_h}\phi(y)\,dy = C\exp\left\{-\frac{1}{2}\frac{s^2}{\sigma_v^2 s_h^2}z^2 + \frac{\varepsilon}{\sigma_v^2}z\right\}\int_{-\infty}^{\lambda z/s_h}\phi(y)\,dy$$

Next, we change the order of integration in the two integrals of the density.

For the integral $\int_{z=-\infty}^{0} C\exp\left\{-\frac{1}{2}\frac{s^2}{\sigma_v^2 s_h^2}z^2 + \frac{\varepsilon}{\sigma_v^2}z\right\}\int_{-\infty}^{\lambda_h z/s_h}\phi(y)\,dydz$ we have

$$\left\{\begin{matrix} -\infty \le z \le 0 \\ -\infty \le y \le \lambda_h z/s_h \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} -\infty \le y \le 0 \\ (s_h/\lambda_h)\,y \le z \le 0 \end{matrix}\right\}$$

while for the integral $\int_{z=0}^{\infty} C\exp\left\{-\frac{1}{2}\frac{s^2}{\sigma_v^2 s_h^2}z^2 + \frac{\varepsilon}{\sigma_v^2}z\right\}\int_{-\infty}^{-\lambda_h^{-1}z/s_h}\phi(y)\,dydz$ we have

$$\left\{\begin{matrix} 0\le z\le\infty \\ -\infty\le y\le -\lambda_h^{-1}z/s_h \end{matrix}\right\}\longrightarrow\left\{\begin{matrix} -\infty\le y\le 0 \\ 0\le z\le -\left(s_h\lambda_h\right)y \end{matrix}\right\}$$

Now the outer limits of integration are the same so we can write the two inner integrals under one outer integral,

$$f_\varepsilon(\varepsilon)=C\int_{y=-\infty}^{0}\phi(y)\left[\int_{z=(s_h/\lambda_h)y}^{0}\exp\left\{-\frac{1}{2}\frac{s^2}{\sigma_v^2 s_h^2}z^2+\frac{\varepsilon}{\sigma_v^2}z\right\}dz\right.$$

$$\left.+\int_{z=0}^{-(s_h\lambda_h)y}\exp\left\{-\frac{1}{2}\frac{s^2}{\sigma_v^2 s_h^2}z^2+\frac{\varepsilon}{\sigma_v^2}z\right\}dz\right]dy$$

Note that the two inner integrals have the same integrand as well as consecutive intervals of integration and so we can combine them in one,

$$f_\varepsilon(\varepsilon)=C\int_{y=-\infty}^{0}\phi(y)\int_{z=(s_h/\lambda_h)y}^{-(s_h\lambda_h)y}\exp\left\{-\frac{1}{2}\frac{s^2}{\sigma_v^2 s_h^2}z^2+\frac{\varepsilon}{\sigma_v^2}z\right\}dz\,dy$$

...and decompose them again suitably:

$$f_\varepsilon(\varepsilon)=C\int_{y=-\infty}^{0}\phi(y)\left[\int_{z=(s_h/\lambda_h)y}^{\infty}\exp\left\{-\frac{1}{2}\frac{s^2}{\sigma_v^2 s_h^2}z^2+\frac{\varepsilon}{\sigma_v^2}z\right\}dz\right.$$

$$\left.-\int_{z=-(s_h\lambda_h)y}^{\infty}\exp\left\{-\frac{1}{2}\frac{s^2}{\sigma_v^2 s_h^2}z^2+\frac{\varepsilon}{\sigma_v^2}z\right\}dz\right]dy \qquad [3]$$

The inner integrals in [3] are now solvable by using a formula provided in Gradshteyn and Ryzhik (2007) (p.336),

$$\int_m^\infty \exp\left\{-\frac{1}{4\delta}\omega^2-\gamma\omega\right\}d\omega=\sqrt{\pi\delta}e^{\delta\gamma^2}\left[1-\operatorname{erf}\left(\gamma\sqrt{\delta}+\frac{m}{2\sqrt{\delta}}\right)\right]$$

The solutions for the two integrals will differ only in their error-function argument due to different lower limits of integration.

We match common coefficients:

$$\frac{1}{4\delta}=\frac{1}{2}\frac{s^2}{\sigma_v^2 s_h^2} \Rightarrow \delta=\frac{\sigma_v^2 s_h^2}{2s^2} \quad , \quad \gamma=\frac{-\varepsilon}{\sigma_v^2}$$

$$\sqrt{\delta}=\frac{\sigma_v s_h}{s\sqrt{2}}, \quad \delta\gamma^2=\frac{\sigma_v^2 s_h^2}{2s^2}\left(\frac{-\varepsilon}{\sigma_v^2}\right)^2=\frac{s_h^2}{2s^2\sigma_v^2}\varepsilon^2, \quad \gamma\sqrt{\delta}=\frac{-\varepsilon}{\sigma_v^2}\frac{\sigma_v s_h}{s\sqrt{2}}=-\frac{s_h}{s\sigma_v\sqrt{2}}\varepsilon$$

We first calculate the error function arguments.

For the 1st inner integral of [3] we have

$$m=\left(s_h/\lambda_h\right)y \Rightarrow 1-\operatorname{erf}\left(\gamma\sqrt{\delta}+\frac{m}{2\sqrt{\delta}}\right)=1-\operatorname{erf}\left(-\frac{s_h}{s\sigma_v\sqrt{2}}\varepsilon+\frac{s\left(s_h/\lambda_h\right)}{\sqrt{2}\sigma_v s_h}y\right)$$

$$=1-\operatorname{erf}\left(\frac{1}{\sqrt{2}}\left[-\frac{s_h}{s\sigma_v}\varepsilon+\frac{\left(s/\lambda_h\right)}{\sigma_v}y\right]\right)$$

$$=1-2\Phi\left(-\frac{s_h}{s\sigma_v}\varepsilon+\frac{\left(s/\lambda_h\right)}{\sigma_v}y\right)+1=2\Phi\left(\frac{s_h}{s\sigma_v}\varepsilon-\frac{\left(s/\lambda_h\right)}{\sigma_v}y\right)$$

For the 2nd inner integral of [3] we have

$$m=-\left(s_h\lambda_h\right)y \Rightarrow \; 1-\operatorname{erf}\left(\gamma\sqrt{\delta}+\frac{m}{2\sqrt{\delta}}\right)=1-\operatorname{erf}\left(-\frac{s_h}{s\sigma_v\sqrt{2}}\varepsilon-\frac{s\left(s_h\lambda_h\right)}{\sqrt{2}\sigma_v s_h}y\right)$$

$$=1-\operatorname{erf}\left(-\frac{1}{\sqrt{2}}\left[\frac{s_h}{s\sigma_v}\varepsilon+\frac{s\lambda_h}{\sigma_v}y\right]\right)=1+\operatorname{erf}\left(\frac{1}{\sqrt{2}}\left[\frac{s_h}{s\sigma_v}\varepsilon+\frac{s\lambda_h}{\sigma_v}y\right]\right)$$

$$=1+2\Phi\left(\frac{s_h}{s\sigma_v}\varepsilon+\frac{\left(s/\lambda_h\right)}{\sigma_v}y\right)-1=2\Phi\left(\frac{s_h}{s\sigma_v}\varepsilon+\frac{s\lambda_h}{\sigma_v}y\right)$$

Then the solutions to the two inner integrals are

$$\int_m^{\infty} \exp\left\{-\frac{1}{4\delta}\omega^2 - \gamma\omega\right\} d\omega = \begin{cases} \sqrt{\pi}\,\dfrac{\sigma_v s_h}{s\sqrt{2}} \exp\left\{\dfrac{s_h^2}{2s^2\sigma_v^2}\varepsilon^2\right\} 2\Phi\left(\dfrac{s_h}{s\sigma_v}\varepsilon - \dfrac{(s/\lambda_h)}{\sigma_v} y\right), & m = (s_h/\lambda_h)\, y \\[2ex] \sqrt{\pi}\,\dfrac{\sigma_v s_h}{s\sqrt{2}} \exp\left\{\dfrac{s_h^2}{2s^2\sigma_v^2}\varepsilon^2\right\} 2\Phi\left(\dfrac{s_h}{s\sigma_v}\varepsilon + \dfrac{s\lambda_h}{\sigma_v} y\right), & m = -(s_h\lambda_h)\, y \end{cases}$$

We now bring together all the terms in [3] that do not depend on the remaining integrating variable, y:

$$C\sqrt{\pi}\,\frac{\sigma_v s_h}{s\sqrt{2}}\, 2\exp\left\{\frac{s_h^2}{2s^2\sigma_v^2}\varepsilon^2\right\} = \frac{4(2\pi)^{-1}}{\sigma_v s_h} \exp\left\{-\frac{1}{2}\left(\frac{\varepsilon}{\sigma_v}\right)^2\right\} \sqrt{\pi}\,\frac{\sigma_v s_h}{s\sqrt{2}}\, 2\exp\left\{\frac{s_h^2}{2s^2\sigma_v^2}\varepsilon^2\right\}$$

$$= \frac{4}{s\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_v^2} - \frac{s_h^2}{s^2\sigma_v^2}\right)\varepsilon^2\right\} = \frac{4}{s}\phi(\varepsilon/s)$$

For compactness we also define temporarily $a \equiv \dfrac{s_h}{s\,\sigma_v}$, $\lambda_1 \equiv \dfrac{s}{\lambda_h\sigma_v}$, $\lambda_2 \equiv \dfrac{s\,\lambda_h}{\sigma_v}$.

Inserting all results back into [3] we obtain

$$f_\varepsilon(\varepsilon) = \frac{4}{s}\phi(\varepsilon/s)\left[\int_{-\infty}^{0} \phi(y)\Phi(a\varepsilon - \lambda_1 y)\,dy - \int_{-\infty}^{0} \phi(y)\Phi(a\varepsilon + \lambda_2 y)\,dy\right] \qquad [4]$$

The solution to the two integrals of [4] is given in Owen (1980), p. 403. The general solution adapted to our notation is

$$\int_{-\infty}^{0} \phi(y)\Phi(a\varepsilon + \lambda y)\,dy = \frac{1}{2}\left[\Phi\left(\frac{a\varepsilon}{\sqrt{1+\lambda^2}}\right) - 2T\left(\frac{a\varepsilon}{\sqrt{1+\lambda^2}}\,,\,\lambda\right)\right]$$

where $T(\cdot)$ is Owen's T-function tabulated in Owen(1956).

So

$$\Rightarrow f_{\varepsilon}(\varepsilon)=\frac{2}{s}\phi(\varepsilon/s)\left\{\Phi\left(\frac{a\varepsilon}{\sqrt{1+\lambda_1^2}}\right)-2T\left(\frac{a\varepsilon}{\sqrt{1+\lambda_1^2}}\,,-\lambda_1\right)\right.$$

$$\left.-\left[\Phi\left(\frac{a\varepsilon}{\sqrt{1+\lambda_2^2}}\right)-2T\left(\frac{a\varepsilon}{\sqrt{1+\lambda_2^2}}\,,\lambda_2\right)\right]\right\} \qquad [5]$$

But now the two terms in the curly brackets represent each a Skew-normal distribution function in the variable ε. Both have zero location parameter.

The first one has scale parameter $\omega_1=\dfrac{\sqrt{1+\lambda_1^2}}{a}$ and skew parameter $-\lambda_1$.

The 2nd has scale parameter $\omega_2=\dfrac{\sqrt{1+\lambda_2^2}}{a}$ and skew parameter λ_2.

With this notation we can write the density of the variable ε as

$$\boxed{[3.18]:\ f_{\varepsilon}(\varepsilon)=\frac{2}{s}\phi(\varepsilon/s)\left[G_1(\varepsilon;0,\omega_1,-\lambda_1)-G_2(\varepsilon;0,\omega_2,\lambda_2)\right]}$$

which is eq. [3.18] of the main text, and where $G(\cdot)$ denotes a Skew-normal distribution function.

At this point we have four composite coefficients $(s,a,\lambda_1,\lambda_2)$ for the three variances involved. To obtain a one-to-one reparametrization we calculate as follows:

$$\omega_1=\frac{\sqrt{1+\lambda_1^2}}{a}=\frac{\sqrt{1+\left(\dfrac{s}{\lambda_h\sigma_v}\right)^2}}{\dfrac{s_h}{s\sigma_v}}=\frac{s\sigma_v\sqrt{\lambda_h^2\sigma_v^2+s^2}}{\lambda_h\sigma_v s_h}=\frac{s\sqrt{\dfrac{\sigma_1^2}{\sigma_2^2}\sigma_v^2+s_h^2+\sigma_v^2}}{\dfrac{\sigma_1}{\sigma_2}s_h}$$

$$=\frac{s\sqrt{\sigma_1^2\sigma_v^2+\sigma_2^2\left(s_h^2+\sigma_v^2\right)}}{\sigma_1 s_h}=\frac{s\sqrt{\sigma_v^2\left(\sigma_1^2+\sigma_2^2\right)+\sigma_2^2 s_h^2}}{\sigma_1 s_h}=\frac{s\sqrt{s_h^2\left(\sigma_v^2+\sigma_2^2\right)}}{\sigma_1 s_h}=\frac{s\sqrt{\left(\sigma_u^2+\sigma_2^2\right)}}{\sigma_1}$$

Defining $\theta_1 \equiv \dfrac{\sigma_1}{\sigma_u}$, $\theta_2 \equiv \dfrac{\sigma_2}{\sigma_u}$ (and so $s \equiv \sqrt{\sigma_u^2 + \sigma_1^2 + \sigma_2^2} = \sigma_v\sqrt{1+\theta_1^2+\theta_2^2}$)

we arrive at $\omega_1 = \dfrac{s\sqrt{1+\theta_2^2}}{\theta_1}$ as shown in the main text.

For the skew parameter we have $\lambda_1 = \dfrac{s}{\lambda_h \sigma_u} = \dfrac{\sigma_v\sqrt{1+\theta_1^2+\theta_2^2}}{\left(\sigma_1/\sigma_2\right)\sigma_v} = \dfrac{\theta_2}{\theta_1}\sqrt{1+\theta_1^2+\theta_2^2}$

For the 2nd Skew-normal distribution function, we have

$$\omega_2 = \frac{\sqrt{1+\lambda_2^2}}{a} = \frac{\sqrt{1+\left(\dfrac{s\lambda_h}{\sigma_v}\right)^2}}{\dfrac{s_h}{s\sigma_v}} = \frac{s\sigma_v\sqrt{\sigma_u^2+\lambda_h^2 s^2}}{\sigma_v s_h} = \frac{s\sqrt{\sigma_2^2\sigma_v^2+\sigma_1^2\left(s_h^2+\sigma_v^2\right)}}{\sigma_2 s_h}$$

$$= \frac{s\sqrt{\sigma_v^2\left(\sigma_1^2+\sigma_2^2\right)+\sigma_1^2 s_h^2}}{\sigma_2 s_h} = \frac{s\sqrt{s_h^2\left(\sigma_v^2+\sigma_1^2\right)}}{\sigma_2 s_h} = \frac{s\sqrt{\left(\sigma_v^2+\sigma_1^2\right)}}{\sigma_2}$$

$$= \frac{s\sqrt{1+\theta_1^2}}{\theta_2}$$

The skew parameter λ_2 can be written as

$$\lambda_2 = \frac{s\lambda_h}{\sigma_v} = \frac{\sigma_v\left(\sigma_1/\sigma_2\right)\sqrt{1+\theta_1^2+\theta_2^2}}{\sigma_v} = \frac{\theta_1}{\theta_2}\sqrt{1+\theta_1^2+\theta_2^2}\,.$$

So by using $\left(s, \theta_1, \theta_2\right)$, we can eliminate a and treat $\left(\omega_1, \omega_2, \lambda_1, \lambda_2\right)$ as shorthands. The main benefits of this reparametrization is that a) the skew parameters do not depend on s and b) the density of ε depends on θ_1, θ_2 only through the Skew-normal distribution functions. These simplify somewhat the calculations needed in the rest of the paper.

A.2. eq. [3.21] : Alternative expression for the Skew-normal distribution function.

Consider the distribution function of a bivariate standard Normal random variable,

$$\Phi_2\left(\frac{\varepsilon-\xi}{\sigma}, 0 ; \rho\right)=\int_{-\infty}^{(\varepsilon-\xi)/\sigma}\int_{-\infty}^{0}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left(v^2+w^2-2\rho vw\right)\right\}dwdv$$

where one of the two standard Normals is evaluated at zero and the other at $(\varepsilon-\xi)/\sigma$, where ε is a random variable and ξ , σ are constants.

Separating we have

$$\Phi_2\left(\frac{\varepsilon-\xi}{\sigma}, 0 ; \rho\right)=\int_{-\infty}^{(\varepsilon-\xi)/\sigma}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{v^2}{2\left(1-\rho^2\right)}\right\}\int_{-\infty}^{0}\exp\left\{-\frac{w^2}{2\left(1-\rho^2\right)}+\frac{\rho v}{1-\rho^2}w\right\}dwdv$$

We calculate the inner integral $I_{IN}=\int_{-\infty}^{0}\exp\left\{-\frac{w^2}{2\left(1-\rho^2\right)}+\frac{\rho v}{1-\rho^2}w\right\}dw$.

Swapping the limits of integration and multiplying by minus one we obtain

$$I_{IN}=\int_{0}^{\infty}\exp\left\{-\frac{w^2}{2\left(1-\rho^2\right)}-\frac{\rho v}{1-\rho^2}w\right\}dw$$

This has general solution $\int_{0}^{\infty}\exp\left\{-\frac{1}{4\delta}\omega^2-\gamma\omega\right\}d\omega=\sqrt{\pi\delta}e^{\delta\gamma^2}\left[1-\operatorname{erf}\left(\gamma\sqrt{\delta}\right)\right]$

Matching coefficients we have

$$\frac{1}{4\delta}=\frac{1}{2\left(1-\rho^2\right)}\Rightarrow\delta=\frac{1-\rho^2}{2}\quad,\ \sqrt{\delta}=\frac{\sqrt{1-\rho^2}}{\sqrt{2}}\quad,\ \gamma=\frac{\rho v}{1-\rho^2}\ \Rightarrow\gamma^2=\left(\frac{\rho}{1-\rho^2}\right)^2v^2$$

Substituting we have

$$I_{IN} = \sqrt{\pi}\,\frac{\sqrt{1-\rho^2}}{\sqrt{2}}\exp\left\{\frac{1-\rho^2}{2}\left(\frac{\rho}{1-\rho^2}\right)^2 v^2\right\}\left[1-\operatorname{erf}\left(\frac{\rho v}{1-\rho^2}\frac{\sqrt{1-\rho^2}}{\sqrt{2}}\right)\right].$$

Simplifying and using the relation $\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) = 2\Phi(x)-1$ we get

$$I_{IN} = \sqrt{\pi}\,\frac{\sqrt{1-\rho^2}}{\sqrt{2}}\exp\left\{\frac{1}{2}\frac{\rho^2}{1-\rho^2}v^2\right\}\left[1-2\Phi\left(\frac{\rho}{\sqrt{1-\rho^2}}v\right)+1\right]$$

$$= \sqrt{2\pi}\sqrt{1-\rho^2}\exp\left\{\frac{1}{2}\frac{\rho^2}{1-\rho^2}v^2\right\}\Phi\left(\frac{-\rho}{\sqrt{1-\rho^2}}v\right)$$

Inserting this into to bivariate double integral we obtain

$$\Phi_2\left(\frac{\varepsilon-\xi}{\sigma},\,0\,;\,\rho\right) = \int_{-\infty}^{(\varepsilon-\xi)/\sigma}\frac{\sqrt{2\pi}\sqrt{1-\rho^2}}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{v^2}{2\left(1-\rho^2\right)}\right\}\exp\left\{\frac{\rho^2v^2}{2\left(1-\rho^2\right)}\right\}\Phi\left(\frac{-\rho}{\sqrt{1-\rho^2}}v\right)dv$$

$$= \int_{-\infty}^{(\varepsilon-\xi)/\sigma}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}v^2\right\}\Phi\left(\frac{-\rho}{\sqrt{1-\rho^2}}v\right)dv \;=\frac{1}{2}\int_{-\infty}^{(\varepsilon-\xi)/\sigma}2\phi(v)\Phi\left(\frac{-\rho}{\sqrt{1-\rho^2}}v\right)dv$$

$$\Rightarrow 2\Phi_2\left(\frac{\varepsilon-\xi}{\sigma},\,0\,;\,\rho\right) = \int_{-\infty}^{\varepsilon}\frac{2}{\sigma}\phi\left(\frac{v-\xi}{\sigma}\right)\Phi\left(\frac{-\rho}{\sqrt{1-\rho^2}}\frac{v-\xi}{\sigma}\right)dv$$

The right-hand side now represents the distribution function of a Skew-normal random variable ε with location parameter ξ, scale parameter σ and skew parameter $\lambda = \frac{-\rho}{\sqrt{1-\rho^2}}$.

We note that if the correlation between the two standard Normals is positive then the resulting skew parameter in the Skew-normal will be negative, and vice-versa.

Solving for ρ we obtain $\rho = \frac{-\lambda}{\sqrt{1+\lambda^2}}$. Hence, we arrive at

$$\boxed{[3.21]:\ G(\varepsilon,\xi,\sigma,\lambda)=2\Phi_2\left(\frac{\varepsilon-\xi}{\sigma},0;\ \rho=\frac{-\lambda}{\sqrt{1+\lambda^2}}\right)} \qquad [6]$$

which is eq [3.21] of the main text, with $\xi=0$, that inserted in [3.18] **leads to [3.22].**

B. Skewness and positive excess kurtosis of the 2TSF Half-normal density.

B.1 Sign of Skewness.

Earlier, we have obtained that

$$\operatorname{sign}\{\gamma_1(\varepsilon)\}=\operatorname{sign}\left\{E\left(w^3\right)-E\left(u^3\right)\ -3\mu_w\mu_u\left(\mu_w-\mu_u\right)-3\mu_w\sigma_w^2\ +\ 3\mu_u\sigma_u^2-\left(\mu_w-\mu_u\right)^3\right\}$$

For the Half-normal specification, $\varepsilon=v+w-u,\quad w\sim\mathrm{HN}\left(s_w\right),\ u\sim\mathrm{HN}\left(s_u\right)$

we have $\mu=\sqrt{\frac{2}{\pi}}s,\ \mu^2=\frac{2}{\pi}s^2,\ \sigma^2=\left(1-\frac{2}{\pi}\right)s^2,\ E\left(x^3\right)=2\sqrt{\frac{2}{\pi}}s^3$

So here, the expression to the right becomes

$$2\sqrt{\frac{2}{\pi}}s_w^3-2\sqrt{\frac{2}{\pi}}s_u^3\ -3\mu_w\mu_u\left(\mu_w-\mu_u\right)-3\mu_w\left(1-\frac{2}{\pi}\right)s_w^2\ +\ 3\mu_u\left(1-\frac{2}{\pi}\right)s_u^2-\left(\mu_w-\mu_u\right)^3$$

Taking common factors and using $\mu=\sqrt{\frac{2}{\pi}}s$

$$\ldots=2\sqrt{\frac{2}{\pi}}\left(s_w^3-s_u^3\right)-\left(\mu_w-\mu_u\right)\left[3\mu_w\mu_u+\left(\mu_w-\mu_u\right)^2\right]-3\mu_w\left(1-\frac{2}{\pi}\right)s_w^2\ +\ 3\mu_u\left(1-\frac{2}{\pi}\right)s_u^2$$

$$=2\sqrt{\frac{2}{\pi}}\left(s_w^3-s_u^3\right)-\sqrt{\frac{2}{\pi}}\frac{2}{\pi}\left(s_w-s_u\right)\left(s_w^2+s_w s_u+s_u^2\right)$$

$$-3\sqrt{\frac{2}{\pi}}\left(1-\frac{2}{\pi}\right)s_w^3 \;+\; 3\sqrt{\frac{2}{\pi}}\left(1-\frac{2}{\pi}\right)s_u^3$$

$$=2\sqrt{\frac{2}{\pi}}\left(s_w^3-s_u^3\right)-\sqrt{\frac{2}{\pi}}\frac{2}{\pi}\left(s_w^3-s_u^3\right)-3\sqrt{\frac{2}{\pi}}\left(1-\frac{2}{\pi}\right)\left(s_w^3-s_u^3\right)$$

$$=\left[2\sqrt{\frac{2}{\pi}}-\sqrt{\frac{2}{\pi}}\frac{2}{\pi}-3\sqrt{\frac{2}{\pi}}+3\sqrt{\frac{2}{\pi}}\frac{2}{\pi}\right]\left(s_w^3-s_u^3\right)=\sqrt{\frac{2}{\pi}}\left(\frac{4}{\pi}-1\right)\left(s_w-s_u\right)\left(s_w^2+s_w s_u+s_u^2\right)$$

and so $\operatorname{sign}\left\{\gamma_1(\varepsilon)\right\}=\operatorname{sign}\left\{s_w-s_u\right\}$.

B.2. Positive excess Kurtosis.

In terms of Owen's *T*-function, the density in the 2TSF Half-normal specification is written

$$f_\varepsilon(\varepsilon_i)=\frac{2}{s}\phi(\varepsilon_i/s)\Big\{\left[\Phi(\varepsilon_i/\omega_1)+2T(\varepsilon_i/\omega_1\,;\lambda_1)\right]$$

$$-\left[\Phi(\varepsilon_i/\omega_2)-2T(\varepsilon_i/\omega_2\,;\lambda_2)\right]\Big\}$$

$$\theta_1\equiv\frac{\sigma_w}{\sigma_v}\;,\;\theta_2\equiv\frac{\sigma_u}{\sigma_v}\;,\;s\equiv\sqrt{\sigma_v^2+\sigma_w^2+\sigma_u^2}\;=\;\sigma_v\sqrt{1+\theta_1^2+\theta_2^2}$$

$$\omega_1\equiv\frac{s\sqrt{1+\theta_2^2}}{\theta_1}\;,\;\omega_2\equiv\frac{s\sqrt{1+\theta_1^2}}{\theta_2},\quad\lambda_1\equiv\frac{\theta_2}{\theta_1}\sqrt{1+\theta_1^2+\theta_2^2}\;,\;\lambda_2\equiv\frac{\theta_1}{\theta_2}\sqrt{1+\theta_1^2+\theta_2^2}$$

If $\sigma_w=\sigma_u=\sigma$ we obtain $\theta_1=\theta_2=\sigma/\sigma_v$, $\omega_1=\omega_2=\omega_s=s\sqrt{1+\sigma_v^2/\sigma^2}$

$\lambda_1=\lambda_2=\lambda_s=s/\sigma_v$. Note that $\lambda_s>1$. The density becomes

$$f_\varepsilon\left(\varepsilon_i\Big|_{\sigma_w=\sigma_u=\sigma}\right)=\frac{2}{s}\phi(\varepsilon_i/s)\Big\{\big[\Phi(\varepsilon_i/\omega_s)+2T(\varepsilon_i/\omega_s\ ;\lambda_s)\big]$$

$$-\big[\Phi(\varepsilon_i/\omega_s)-2T(\varepsilon_i/\omega_s\,;\lambda_s)\big]\Big\}$$

$$=\frac{8}{s}\phi(\varepsilon_i/s)T(\varepsilon_i/\omega_s\ ;\lambda_s)$$

The *T*-function is an even function with respect to its first argument. So the above density is a product of two even functions and so itself an even function in ε_i, and so symmetric around $\varepsilon=0$. But given that $\lambda_s>1$, by the tables of the T-function (see Owen 1956), we have that $8T(0\ ;\lambda_s>1)>1$, and so

$$f_\varepsilon\left(0\Big|_{\sigma_w=\sigma_u=\sigma}\right)=\frac{8}{s}\phi(0)T(0\ ;\lambda_s)>\frac{1}{s}\phi(0)$$

Namely, the value of the density at zero will be higher than the corresponding value of a zero-mean Normal density with the same variance. Therefore the symmetric $f_\varepsilon\left(\varepsilon_i\Big|_{\sigma_w=\sigma_u=\sigma}\right)$ will exhibit positive excess kurtosis.

C. Conditional densities and expected values in the Half-normal specification.

C.1. Eq. [3.27]: The conditional density $f\left(w|\varepsilon\right)$

To obtain the density of the Half-normal variable w conditional on the composite variable ε we note that w is independent of $\xi=v-u$. So the joint distribution of w and ξ is

$f_{w,\xi}\left(w,\xi\right)=f_w\left(w\right)f_\xi\left(\xi\right)$. Since $\varepsilon=\xi+w\Rightarrow\xi=\varepsilon-w$, we have that

$f_{w,\xi}\left(w,\, \varepsilon - w\right) = f_w\left(w\right) f_\xi\left(\varepsilon - w\right)$ is the joint density of ε and w. The density of w conditional on ε is therefore

$$f_{w|\varepsilon}\left(w|\varepsilon\right) = \frac{f_{w,\xi}\left(w,\, \varepsilon - w\right)}{f_\varepsilon\left(\varepsilon\right)} = \frac{f_w\left(w\right) f_\xi\left(\varepsilon - w\right)}{f_\varepsilon\left(\varepsilon\right)} \qquad [7]$$

$f_w\left(w\right)$ is a Half-normal density, $f_\xi\left(\varepsilon - w\right)$ is eq.[10] of the main text, while $f_\varepsilon\left(\varepsilon\right)$ is equation [18]. So we have

$$f_{w|\varepsilon}\left(w|\varepsilon\right) = \frac{\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_1}\exp\left\{-\frac{1}{2}\left(\frac{w}{\sigma_1}\right)^2\right\}\frac{2}{s_2}\phi\left(\left(\varepsilon - w\right)/s_2\right)\Phi\left(-\theta_2\frac{\left(\varepsilon - w\right)}{s_2}\right)}{\frac{2}{s}\phi\left(\varepsilon/s\right)\left[G_1\ -\ G_2\right]}$$

$$= \left(G_1\ -\ G_2\right)^{-1}\sqrt{\frac{2}{\pi}}\frac{s}{\sigma_1 s_2}\exp\left\{-\frac{1}{2}\left(\frac{w}{\sigma_1}\right)^2 - \frac{1}{2}\left(\frac{\varepsilon - w}{s_2}\right)^2 + \frac{1}{2}\left(\frac{\varepsilon}{s}\right)^2\right\}\Phi\left(\theta_2\frac{w - \varepsilon}{s_2}\right)$$

where $s_2^2 = \sigma_v^2 + \sigma_2^2$.

Calculating the argument of the exponential we have

$$\left\{-\frac{1}{2}\left(\frac{w}{\sigma_1}\right)^2 - \frac{1}{2}\left(\frac{\varepsilon - w}{s_2}\right)^2 + \frac{1}{2}\left(\frac{\varepsilon}{s}\right)^2\right\} = -\frac{1}{2}\left\{\frac{w^2}{\sigma_1^2} + \frac{\varepsilon^2}{s_2^2} - 2\frac{\varepsilon w}{s_2^2} + \frac{w^2}{s_2^2} - \frac{\varepsilon^2}{s^2}\right\}$$

$$= -\frac{1}{2}\left\{\frac{s^2}{\sigma_1^2 s_2^2}w^2 - 2\frac{\varepsilon w}{s_2^2} + \frac{\sigma_1^2}{s^2 s_2^2}\varepsilon^2\right\} = -\frac{1}{2}\left(\frac{s}{\sigma_1 s_2}w\ -\ \frac{\sigma_1}{s s_2}\varepsilon\right)^2$$

Note that $\dfrac{\sigma_1}{s s_2} = \dfrac{\sigma_1}{s\sqrt{\sigma_v^2 + \sigma_2^2}} = \dfrac{\sigma_1}{s\sigma_v\sqrt{1+\theta_2^2}} = \dfrac{\theta_1}{s\sqrt{1+\theta_2^2}} = \dfrac{1}{\omega_1}$.

Also , we can write

$$\frac{\theta_2}{s_2}=\frac{\theta_2}{s_2}\frac{\omega_1}{\omega_1}=\frac{\theta_2}{s_2}\frac{s\sqrt{1+\theta_2^2}}{\theta_1}\frac{1}{\omega_1}=\frac{\theta_2}{\sigma_v\sqrt{1+\theta_2^2}}\frac{s\sqrt{1+\theta_2^2}}{\theta_1}\frac{1}{\omega_1}=\frac{\theta_2}{\theta_1}\sqrt{1+\theta_1^2+\theta_2^2}\,\frac{1}{\omega_1}=\lambda_1\frac{1}{\omega_1}$$

Set also $\omega_w \equiv \dfrac{\sigma_1 s_2}{s}$ and insert all into the conditional density while multiplying and dividing by 2 to obtain

$$[3.27]:\quad f_{w|\varepsilon}\left(w|\varepsilon\right)=\left(G_1-G_2\right)^{-1}\frac{2}{\omega_w}\phi\left(\frac{w}{\omega_w}-\frac{\varepsilon}{\omega_1}\right)\Phi\left(\lambda_1\frac{\left(w-\varepsilon\right)}{\omega_1}\right) \qquad [8]$$

which is eq. [3.27] of the main text.

C.2. Eq. [3.29]: The conditional expected value $E\left(w|\varepsilon\right)$.

We want to calculate

$$E\left(w|\varepsilon\right)=\int_0^{\infty} w f_{w|\varepsilon}\left(w|\varepsilon\right)dw=\int_0^{\infty} w\left(G_1-G_2\right)^{-1}\frac{2}{\omega_w}\phi\left(\frac{w}{\omega_w}-\frac{\varepsilon}{\omega_1}\right)\Phi\left(\lambda_1\frac{\left(w-\varepsilon\right)}{\omega_1}\right)dw \qquad [9]$$

We make the following change of variables:

$$\text{Set } w^*\equiv\frac{w}{\omega_w}-\frac{\varepsilon}{\omega_1}\Rightarrow\begin{cases} w=\omega_w w^*+\left(\omega_w/\omega_1\right)\varepsilon \\ dw=\omega_w dw^* \\ w=0\Rightarrow w^*=-\left(\varepsilon/\omega_1\right)\end{cases}$$

First we substitute in [9] only for the w outside the density:

$$E\left(w|\varepsilon\right) = \int_0^\infty \left[\omega_w w^* + \left(\omega_w/\omega_1\right)\varepsilon\right] f_{w|\varepsilon}\left(w|\varepsilon\right)dw$$

$$= \int_0^\infty \omega_w w^* f_{w|\varepsilon}\left(w|\varepsilon\right)dw \;+\; \left(\omega_w/\omega_1\right)\varepsilon \int_0^\infty f_{w|\varepsilon}\left(w|\varepsilon\right)dw$$

The 2nd intergal equals unity, so we eliminate it and complete the change of variables:

$$E\left(w|\varepsilon\right) = \int_{-(\varepsilon/\omega_1)}^{\infty} \omega_w w^* \left(G_1 - G_2\right)^{-1} \frac{2}{\omega_w} \phi\left(w^*\right)\Phi\left(\lambda_1 \frac{\left(\omega_v w^* + \left(\omega_w/\omega_1\right)\varepsilon - \varepsilon\right)}{\omega_1}\right)\omega_w dw^* + \left(\omega_w/\omega_1\right)\varepsilon$$

$$\Rightarrow E\left(w|\varepsilon\right) = \left(\omega_w/\omega_1\right)\varepsilon$$

$$+ \left(G_1 - G_2\right)^{-1} 2\omega_w \int_{-(\varepsilon/\omega_1)}^{\infty} w^* \phi\left(w^*\right)\Phi\left(\frac{\lambda_1\left(\omega_w - \omega_1\right)}{\omega_1^2}\varepsilon + \frac{\lambda_1\omega_w}{\omega_1} w^*\right)dw^* \qquad [10]$$

The integral in [10] can be solved by using the relevant equation from Owen (1980) p. 404, which is

$$\int x\phi(x)\Phi(a+bx)dx = \frac{b}{\sqrt{1+b^2}}\phi\left(\frac{a}{\sqrt{1+b^2}}\right)\Phi\left(x\sqrt{1+b^2} + \frac{ab}{\sqrt{1+b^2}}\right) - \phi(x)\Phi(a+bx)$$

Owen provides the indefinite form. Evaluated at limits $(c,\, \infty)$ the above becomes

$$\int_c^\infty x\phi(x)\Phi(a+bx)dx = \frac{b}{\sqrt{1+b^2}}\phi\left(\frac{a}{\sqrt{1+b^2}}\right)\Phi(\infty) - \phi(\infty)\Phi(\infty)$$

$$-\frac{b}{\sqrt{1+b^2}}\phi\left(\frac{a}{\sqrt{1+b^2}}\right)\Phi\left(c\sqrt{1+b^2} + \frac{ab}{\sqrt{1+b^2}}\right) + \phi(c)\Phi(a+bc)$$

$$= \frac{b}{\sqrt{1+b^2}}\phi\left(\frac{a}{\sqrt{1+b^2}}\right)\left[1 - \Phi\left(\frac{c\left(1+b^2\right)+ab}{\sqrt{1+b^2}}\right)\right] + \phi(c)\Phi(a+bc)$$

$$= \frac{b}{\sqrt{1+b^2}}\phi\left(\frac{a}{\sqrt{1+b^2}}\right)\Phi\left(\frac{c+b\left(a+bc\right)}{-\sqrt{1+b^2}}\right) + \phi(c)\Phi(a+bc) \qquad [11]$$

We now match coefficients between [11] and [10] and calculate the various expressions.

$$a \equiv \frac{\lambda_1\left(\omega_w - \omega_1\right)}{\omega_1^2}\varepsilon \ , \ b = \frac{\lambda_1 \omega_w}{\omega_1} \ , \ c = -\left(\varepsilon / \omega_1\right)$$

$$a + bc = \frac{\lambda_1\left(\omega_w - \omega_1\right)}{\omega_1^2}\varepsilon - \frac{\lambda_1 \omega_w}{\omega_1}\left(\varepsilon/\omega_1\right) = \frac{\lambda_1}{\omega_1^2}\left(\omega_w - \omega_1 - \omega_1\right)\varepsilon = -\frac{\lambda_1}{\omega_1}\varepsilon$$

Then $c + b\left(a + bc\right) = -\left(\varepsilon/\omega_1\right) - \frac{\lambda_1 \omega_w}{\omega_1}\frac{\lambda_1}{\omega_1}\varepsilon = -\frac{1}{\omega_1}\left(1 + \frac{\lambda_1^2 \omega_w}{\omega_1}\right)\varepsilon = -\frac{1}{\omega_1}\left(1 + \theta_2^2\right)\varepsilon$

The last equality comes from

$$\frac{\theta_2}{s_2} = \frac{\lambda_1}{\omega_1} \Rightarrow \frac{\lambda_1^2}{\omega_1} = \frac{\theta_2^2 \omega_1}{s_2^2} \text{ and } \omega_w \equiv \frac{\sigma_1 s_2}{s} \ , \ \frac{\sigma_1}{s s_2} = \frac{1}{\omega_1} \Rightarrow \omega_w \omega_1 = s_2^2$$

We also have $\omega_1 = \frac{s\sqrt{1+\theta_2^2}}{\theta_1}$ and so $c + b\left(a + bc\right) = -\frac{\theta_1\sqrt{1+\theta_2^2}}{s}\varepsilon$.

We turn to

$$\sqrt{1+b^2} = \sqrt{1 + \left(\frac{\theta_2 \omega_w}{s_2}\right)^2} = \sqrt{1 + \theta_2^2\left(\frac{\sigma_1}{s}\right)^2} = \frac{1}{s}\sqrt{s^2 + \theta_2^2 \sigma_1^2} =$$

$$= \frac{1}{s}\sqrt{s_2^2 + \left(1+\theta_2^2\right)\sigma_1^2} = \frac{1}{s}\sqrt{s_2^2 + \left(\sigma_v^2 + \sigma_2^2\right)\frac{\sigma_1^2}{\sigma_v^2}} = \frac{s_2}{s}\sqrt{1+\theta_1^2}$$

So
$$\frac{c + b\left(a+bc\right)}{-\sqrt{1+b^2}} = \frac{-\dfrac{\theta_1\sqrt{1+\theta_2^2}}{s}\varepsilon}{-\dfrac{s_2}{s}\sqrt{1+\theta_1^2}} = \frac{\theta_1\sqrt{1+\theta_2^2}}{s_2\sqrt{1+\theta_1^2}}\varepsilon$$

We have that $\omega_2 = \frac{s\sqrt{1+\theta_1^2}}{\theta_2} \Rightarrow \sqrt{1+\theta_1^2} = \frac{\omega_2 \theta_2}{s}$ and that

$s = \sigma_v \sqrt{1+\theta_1^2+\theta_2^2}$. Inserting both into the expression and rearranging we obtain

$$\frac{c+b(bc+a)}{-\sqrt{1+b^2}} = \frac{\theta_1\sqrt{1+\theta_1^2+\theta_2^2}}{\theta_2\omega_2}\frac{\sigma_v\sqrt{1+\theta_2^2}}{s_2}\varepsilon$$

But $\frac{\theta_1\sqrt{1+\theta_1^2+\theta_2^2}}{\theta_2} = \lambda_2$ while $\frac{\sigma_v\sqrt{1+\theta_2^2}}{s_2} = \frac{\sqrt{\sigma_v^2+\sigma_2^2}}{s_2} = 1$

So in the end , $\frac{c+b(bc+a)}{-\sqrt{1+b^2}} = \frac{\lambda_2}{\omega_2}\varepsilon$.

We also need to calculate $\frac{a}{\sqrt{1+b^2}} = \frac{\frac{\lambda_1(\omega_w-\omega_1)}{\omega_1^2}\varepsilon}{\frac{1}{s}s_2\sqrt{1+\theta_1^2}}$

We have that $\omega_w - \omega_1 = \frac{\sigma_1 s_2}{s} - \frac{ss_2}{\sigma_1} = \frac{\sigma_1^2 s_2 - s^2 s_2}{s\sigma_1} = -\frac{s_2^3}{s\sigma_1}$. We also have $\frac{\lambda_1}{\omega_1} = \frac{\theta_2}{s_2}$. Inserting both and simplifying we obtain

$$\frac{a}{\sqrt{1+b^2}} = -\frac{\frac{\theta_2}{s_2\omega_1}\frac{s_2^3}{s\sigma_1}}{\frac{1}{s}s_2\sqrt{1+\theta_1^2}}\varepsilon = -\frac{s_2}{\omega_1\sigma_1}\frac{\theta_2}{\sqrt{1+\theta_1^2}}\varepsilon = -\frac{s_2}{\frac{ss_2}{\sigma_1}\sigma_1}\frac{\theta_2}{\sqrt{1+\theta_1^2}}\varepsilon = -\frac{\theta_2}{s\sqrt{1+\theta_1^2}}\varepsilon = \frac{1}{\omega_2}\varepsilon$$

Finally, $\frac{b}{\sqrt{1+b^2}} = \frac{\frac{\lambda_1\omega_w}{\omega_1}}{\frac{1}{s}s_2\sqrt{1+\theta_1^2}} = \frac{s}{s_2^2}\frac{\theta_2}{\sqrt{1+\theta_1^2}}\omega_v = \frac{s^2\omega_w}{s_2^2\omega_2}$

Inserting all the results in [11] and then in [10] (and using the fact that the Normal density is an even function), we obtain

$$
\begin{aligned}
E\left(w\middle|\varepsilon\right) &= \left(\omega_w/\omega_1\right)\varepsilon \\
&\quad + \left(G_1 \,-\, G_2\right)^{-1} 2\omega_w \left\{ \frac{s^2\omega_w}{s_2^2\omega_2}\phi\left(\varepsilon/\omega_2\right)\Phi\left(\frac{\lambda_2}{\omega_2}\varepsilon\right) + \phi\left(\varepsilon/\omega_1\right)\Phi\left(-\frac{\lambda_1}{\omega_1}\varepsilon\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\omega_w/\omega_1\right)\varepsilon \; + \left(G_1 \,-\, G_2\right)^{-1}\left\{\omega_w\omega_1\left[\frac{2}{\omega_1}\phi\left(\varepsilon/\omega_1\right)\Phi\left(-\frac{\lambda_1}{\omega_1}\varepsilon\right)\right]\right. \\
&\qquad\qquad\qquad\qquad \left. + \frac{s^2\omega_w^2}{s_2^2}\left[\frac{2}{\omega_2}\phi\left(\varepsilon/\omega_2\right)\Phi\left(\frac{\lambda_2}{\omega_2}\varepsilon\right)\right]\right\}
\end{aligned}
$$

Note that we have obtained the two Skew-normal densities g_1 and g_2 that appear in the density $f_\varepsilon\left(\varepsilon\right)$, but each multiplied by a different parameter. We can further simplify the expression:

$$
\left(\omega_w/\omega_1\right) = \frac{\sigma_1^2}{s^2} \;,\quad \omega_w\omega_1 = \frac{\sigma_1\, s_2}{s}\,\frac{ss_2}{\sigma_1} = s_2^2 = \sigma_v^2 + \sigma_2^2 \;,\quad \frac{s^2\omega_w^2}{s_2^2} = \frac{s^2\left(\dfrac{\sigma_1\, s_2}{s}\right)^2}{s_2^2} = \sigma_1^2
$$

So , reinstating also the subscript *i*,

$$
E\left(w_i\middle|\varepsilon_i\right) = \frac{\sigma_1^2}{s^2}\varepsilon_i \; + \left(G_{1i} \,-\, G_{2i}\right)^{-1}\left[\left(\sigma_v^2 + \sigma_2^2\right)g_{1i} + \sigma_1^2 g_{2i}\;\right]
$$

$$
= \frac{\sigma_1^2}{s^2}\varepsilon_i \; + \left(G_{1i} - G_{2i}\right)^{-1}\left[\left(s^2 - \sigma_1^2\right)g_{1i} + \sigma_1^2 g_{2i}\;\right]
$$

$$
= \frac{\sigma_1^2}{s^2}\varepsilon_i \; - \sigma_1^2\,\frac{g_{1i} - g_{2i}}{G_{1i} - G_{2i}} + s^2 g_{1i}\left(G_{1i} - G_{2i}\right)^{-1}
$$

and with $\psi_{1i} \equiv \dfrac{g_{1i}}{G_{1i} - G_{2i}}$, $\psi_{2i} \equiv \dfrac{g_{2i}}{G_{1i} - G_{2i}}$, $\psi_i = \psi_{1i} - \psi_{2i} = \dfrac{g_{1i} - g_{2i}}{G_{1i} - G_{2i}}$, we arrive at

$$
\boxed{[3.29]\!:\; E\left(w_i\middle|\varepsilon_i\right) \;=\; s^2\psi_{1i} + \sigma_1^2\left(\frac{1}{s^2}\varepsilon_i - \psi_i\right)} \qquad \text{` [12]}
$$

which is eq. [3.29] in the main text.

C.3. Unconditional expected values. eq. [3.51], [3.52].

Equation [12] is useful for another reason also: It provides a shortcut way to show that $E\left(\frac{1}{s^2}\varepsilon_i - \psi_i\right) = 0$, a result that will be needed later on. To prove this, note that $E\left(E\left(w_i|\varepsilon_i\right)\right) = E\left(w_i\right) = \sqrt{\frac{2}{\pi}}\sigma_1$ by our initial assumptions. We now show that

$$\boxed{[3.51]:\ E\left(s^2\psi_{1i}\right) = \sqrt{\frac{2}{\pi}}\sigma_1 = E\left(w_i\right)} \qquad [13]$$

and therefore, from [12], that

$$E\left(\frac{1}{s^2}\varepsilon_i - \psi_i\right) = 0 \Rightarrow E\left(\varepsilon_i\right) = E\left(s^2\psi_i\right). \qquad [14]$$

We have

$$E\left[s^2 g_{1i}\left(G_{1i} - G_{2i}\right)^{-1}\right] = s^2\int_{-\infty}^{\infty}\frac{2}{s}\phi\left(\varepsilon_i/s\right)\left(G_{1i} - G_{2i}\right)g_{1i}\left(G_{1i} - G_{2i}\right)^{-1}d\varepsilon_i$$

$$= s^2\int_{-\infty}^{\infty}\frac{2}{s}\phi\left(\varepsilon_i/s\right)\frac{2}{\omega_1}\phi\left(\varepsilon_i/\omega_1\right)\Phi\left(-\frac{\lambda_1}{\omega_1}\varepsilon_i\right)d\varepsilon_i$$

Unifying the two Normal densities we have

$$E\left[s^2 g_{1i}\left(G_{1i} - G_{2i}\right)^{-1}\right] = \frac{2}{\sqrt{2\pi}}s^2\int_{-\infty}^{\infty}\frac{2}{s\omega_1}\phi\left(\frac{\sqrt{\omega_1^2 + s^2}}{s\omega_1}\varepsilon_i\right)\Phi\left(-\frac{\lambda_1}{\omega_1}\varepsilon_i\right)d\varepsilon_i$$

$$= \sqrt{\frac{2}{\pi}}\frac{s^2}{\sqrt{\omega_1^2 + s^2}}\int_{-\infty}^{\infty}2\frac{\sqrt{\omega_1^2 + s^2}}{s\omega_1}\phi\left(\frac{\sqrt{\omega_1^2 + s^2}}{s\omega_1}\varepsilon_i\right)\Phi\left(-\frac{\lambda_1}{\omega_1}\varepsilon_i\right)d\varepsilon_i$$

The argument of the standard Normal distribution function can be manipulated so that it includes the required scale parameter, with some resulting skew parameter other than λ_1. But whatever this skew parameter will be, the integral has become a proper Skew-normal density integrated over the whole of its domain, and so it equals unity. So we are left with

$$E\left[s^2 g_{1i}\left(G_{1i}-G_{2i}\right)^{-1}\right]= \sqrt{\frac{2}{\pi}}\frac{s^2}{\sqrt{\omega_1^2+s^2}}=\sqrt{\frac{2}{\pi}}\frac{s^2}{\sqrt{s^2\frac{\left(1+\theta_2^2\right)}{\theta_1^2}+s^2}}=\sqrt{\frac{2}{\pi}}\frac{s\,\theta_1}{\sqrt{1+\theta_2^2+\theta_1^2}}$$

$$=\sqrt{\frac{2}{\pi}}\frac{\theta_1\,\sigma_v\sqrt{1+\theta_2^2+\theta_1^2}}{\sqrt{1+\theta_2^2+\theta_1^2}}=\sqrt{\frac{2}{\pi}}\sigma_1 \qquad \text{Q.E.D.}$$

This also implies that $E\left(\frac{1}{s^2}\varepsilon_i-\psi_i\right)= 0$, and so

$$E\left(\frac{1}{s^2}\varepsilon_i-\psi_i\right)= 0 \Rightarrow \frac{1}{s^2}\left(E(w)-E(u)\right)-E\left(\psi_{1i}-\psi_{2i}\right)=0$$

$$\Rightarrow \frac{1}{s^2}\left(E(w)-E(u)\right)-\frac{1}{s^2}E(w)+E\left(\psi_{2i}\right)=0$$

$$\boxed{\Rightarrow [3.52]:\ \ E\left(s^2\psi_{2i}\right)=E(u)} \qquad [15]$$

C.4. Eq. [3.36]: The conditional expected value $E\left(e^{-w}\middle|\varepsilon\right)$.

We want to calculate

$$E\left(e^{-w}\middle|\varepsilon\right)=\int_0^{\infty}e^{-w}f_{w|\varepsilon}\left(w\middle|\varepsilon\right)dw=\int_0^{\infty}e^{-w}\left(G_1-G_2\right)^{-1}\frac{2}{\omega_w}\phi\left(\frac{w}{\omega_w}-\frac{\varepsilon}{\omega_1}\right)\Phi\left(\lambda_1\frac{(w-\varepsilon)}{\omega_1}\right)dw \qquad [16]$$

The calculation method here is to transform the integral so that it represents a known moment generating function. To do this we first define the change of variables

$$\tilde{w} \equiv \frac{w}{\omega_w} \Rightarrow \begin{cases} w = \omega_w \tilde{w} \\ dw = \omega_w d\tilde{w} \\ w = 0 \Rightarrow \tilde{w} = 0 \end{cases}$$

and calculate explicitly the Normal density in [16]:

$$E\left(e^{-w} \middle| \varepsilon\right) = \int_0^\infty e^{-\omega_w \tilde{w}} \left(G_1 - G_2\right)^{-1} \frac{2}{\omega_w} \phi\left(\tilde{w} - \frac{\varepsilon}{\omega_1}\right) \Phi\left(\lambda_1 \frac{(\omega_w \tilde{w} - \varepsilon)}{\omega_1}\right) \omega_w d\tilde{w}$$

$$= 2\left(G_1 - G_2\right)^{-1} \int_0^\infty e^{-\omega_w \tilde{w}} e^{-\frac{1}{2}\frac{\varepsilon^2}{\omega_1^2}} e^{\frac{\varepsilon \tilde{w}}{\omega_1}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\tilde{w}^2} \Phi\left(\frac{-\lambda_1 \varepsilon}{\omega_1} + \frac{\lambda_1 \omega_w}{\omega_1} \tilde{w}\right) d\tilde{w}$$

$$= 2\left(G_1 - G_2\right)^{-1} e^{-\frac{1}{2}\frac{\varepsilon^2}{\omega_1^2}} \int_0^\infty e^{\left(\frac{\varepsilon}{\omega_1} - \omega_w\right)\tilde{w}} \phi\left(\tilde{w}\right) \Phi\left(\frac{-\lambda_1 \varepsilon}{\omega_1} + \frac{\lambda_1 \omega_w}{\omega_1} \tilde{w}\right) d\tilde{w}$$

To ease notation we define temporarily $\xi \equiv \frac{\varepsilon}{\omega_1} - \omega_w$, $a \equiv \frac{-\lambda_1 \varepsilon}{\omega_1}$, $b \equiv \frac{\lambda_1 \omega_w}{\omega_1}$

We have

$$E\left(e^{-w} \middle| \varepsilon\right) = 2\left(G_1 - G_2\right)^{-1} \exp\left\{-\tfrac{1}{2}\frac{\varepsilon^2}{\omega_1^2}\right\} \int_0^\infty e^{\xi \tilde{w}} \phi\left(\tilde{w}\right) \Phi\left(a + b\tilde{w}\right) d\tilde{w}$$

Now, since the integrating interval is $\left[0, \infty\right)$, the $\phi\left(\tilde{w}\right)\Phi\left(a + b\tilde{w}\right)$ part of the integrand can be considered as the density kernel of a *truncated* Skew-normal distribution, as defined in Jamalizadeh, Pourmousa & Balakrishnan (2009), their eqs. [7], [8], [9], truncated from below at zero. What is missing to obtain the moment generating function (MGF) is the normalizing constant of the density (given in their eq. [9]). Denote this normalizing constant c^*. We can multiply and divide our expression by c^* to obtain

$$E\left(e^{-w}|\varepsilon\right) = 2\left(G_1 - G_2\right)^{-1} \exp\left\{-\tfrac{1}{2}\frac{\varepsilon^2}{\omega_1^2}\right\}\frac{1}{c^*}\int_0^\infty e^{\xi\tilde{w}}\, c^*\phi\left(\tilde{w}\right)\Phi\left(a + b\tilde{w}\right)d\tilde{w}$$

Now the integral represents the MGF of this truncated Skew-normal distribution, which is given by Theorem 2.2. eq. [13] in Jamalizadeh et al. (2009).

Denoting as earlier by $\Phi_2\left(\cdot\,,\cdot\,;\rho\right)$ the bi-variate standard Normal integral with correlation coefficient ρ, this MGF becomes, for the specific truncation $\left[0,\infty\right)$ and in our notation,

$$MGF\left(\xi\right) = c^* e^{\frac{1}{2}\xi^2}\left\{\Phi_2\left(\frac{b\xi + a}{\sqrt{1+b^2}},\, \infty;\, \rho = \frac{-b}{\sqrt{1+b^2}}\right) - \Phi_2\left(\frac{b\xi + a}{\sqrt{1+b^2}},\, -\xi;\, \rho = \frac{-b}{\sqrt{1+b^2}}\right)\right\}$$

$$= c^* e^{\frac{1}{2}\xi^2}\left\{\Phi\left(\frac{b\xi + a}{\sqrt{1+b^2}}\right) - \Phi_2\left(\frac{b\xi + a}{\sqrt{1+b^2}},\, -\xi;\, \rho = \frac{-b}{\sqrt{1+b^2}}\right)\right\}$$

and so

$$E\left(e^{-w}|\varepsilon\right) = 2\left(G_1 - G_2\right)^{-1} e^{-\frac{1}{2}\frac{\varepsilon^2}{\omega_1^2}} e^{\frac{1}{2}\xi^2}\left\{\Phi\left(\frac{b\xi + a}{\sqrt{1+b^2}}\right) - \Phi_2\left(\frac{b\xi + a}{\sqrt{1+b^2}},\, -\xi;\, \rho = \frac{-b}{\sqrt{1+b^2}}\right)\right\}$$

Note that the first bivariate distribution function collapses to the univariate standard Normal distribution function since one of the variables goes to infinity. We calculate the composite coefficients involved

$$b\xi + a = \frac{\lambda_1\omega_w}{\omega_1}\left(\frac{\varepsilon}{\omega_1} - \omega_w\right) + \frac{-\lambda_1\varepsilon}{\omega_1} = \frac{\lambda_1}{\omega_1}\left(\frac{\omega_w\varepsilon}{\omega_1} - \omega_w^2 - \varepsilon\right) = \frac{\lambda_1}{\omega_1}\left(\left(\frac{\omega_v}{\omega_1} - 1\right)\varepsilon - \omega_w^2\right)$$

Using also previous results we have $b\xi + a = \dfrac{\theta_2}{s_2}\left(-\dfrac{s_2^2}{s^2}\varepsilon - \dfrac{\sigma_1^2 s_2^2}{s^2}\right) = -\dfrac{\theta_2 s_2}{s^2}\left(\varepsilon + \sigma_1^2\right)$

Also $\sqrt{1+b^2} = \dfrac{s_2}{s}\sqrt{1+\theta_1^2}$ from before.

So $\dfrac{b\xi + a}{\sqrt{1+b^2}} = \dfrac{-\dfrac{\theta_2 s_2}{s^2}\left(\varepsilon + \sigma_1^2\right)}{\dfrac{s_2}{s}\sqrt{1+\theta_1^2}} = \dfrac{-\theta_2\left(\varepsilon + \sigma_1^2\right)}{s\sqrt{1+\theta_1^2}} = -\dfrac{\left(\varepsilon + \sigma_1^2\right)}{\omega_2}$

Also $\dfrac{-b}{\sqrt{1+b^2}} = \dfrac{-s^2\omega_w}{s_2^2\omega_2}$ from previously. We also have already calculated that

$\dfrac{s^2\omega_w^2}{s_2^2} = \sigma_1^2 \Rightarrow \dfrac{s^2\omega_w}{s_2^2} = \dfrac{\sigma_1^2}{\omega_w}$. So

$$\frac{-b}{\sqrt{1+b^2}} = \frac{-\sigma_1^2}{\omega_w\omega_2} = \frac{-\sigma_1^2\theta_2}{\dfrac{\sigma_1 s_2}{s} s\sqrt{1+\theta_1^2}} = \frac{-\sigma_1\theta_2}{s_2\sqrt{1+\theta_1^2}} = \frac{-\sigma_1\sigma_2}{s_2\sqrt{\sigma_v^2+\sigma_1^2}} = \frac{-\sigma_1\sigma_2}{s_1 s_2}$$

Finally, $-\dfrac{1}{2}\dfrac{\varepsilon^2}{\omega_1^2} + \dfrac{1}{2}\xi^2 = -\dfrac{1}{2}\dfrac{\varepsilon^2}{\omega_1^2} + \dfrac{1}{2}\left(\dfrac{\varepsilon}{\omega_1} - \omega_w\right)^2 = \dfrac{1}{2}\omega_w^2 - \dfrac{\omega_w}{\omega_1}\varepsilon$

Substituting all into $E\left(e^{-w}\middle|\varepsilon\right)$ we obtain

$$[3.36]:\ E\left(e^{-w}\middle|\varepsilon_i\right) = 2\left(G_{1i} - G_{2i}\right)^{-1}\exp\left\{\frac{1}{2}\omega_w^2 - \frac{\omega_w}{\omega_1}\varepsilon_i\right\}\left\{\Phi\left(-\frac{\left(\varepsilon_i + \sigma_1^2\right)}{\omega_2}\right) - \Phi_2\left(-\frac{\left(\varepsilon_i + \sigma_1^2\right)}{\omega_2},\ \omega_w - \frac{\varepsilon_i}{\omega_1};\ \rho = \frac{-\sigma_1\sigma_2}{s_1 s_2}\right)\right\} \quad [17]$$

which is eq. [3.36] of the main text.

C.5. Eq. [3.37]: The conditional expected value $E\left(e^{w}\middle|\varepsilon\right)$.

We want to calculate

$$E\left(e^{w}\middle|\varepsilon\right)=\int_0^{\infty}e^{w}f_{w|\varepsilon}\left(w\middle|\varepsilon\right)dw=\int_0^{\infty}e^{w}\left(G_1-G_2\right)^{-1}\frac{2}{\omega_w}\phi\left(\frac{w}{\omega_w}-\frac{\varepsilon}{\omega_1}\right)\Phi\left(\lambda_1\frac{\left(w-\varepsilon\right)}{\omega_1}\right)dw \qquad [18]$$

The calculation steps are exactly the same as before, the only changes are those that are brought about by the change in the sign of e^{w}. So we arrive immediately at

$$E\left(e^{w}\middle|\varepsilon\right)=2\left(G_1-G_2\right)^{-1}e^{-\frac{1}{2}\frac{\varepsilon^2}{\omega_1^2}}\int_0^{\infty}e^{\left(\frac{\varepsilon}{\omega_1}+\omega_w\right)\tilde{w}}\phi\left(\tilde{w}\right)\Phi\left(\frac{-\lambda_1\varepsilon}{\omega_1}+\frac{\lambda_1\omega_w}{\omega_1}\tilde{w}\right)d\tilde{w}$$

Setting $\xi\equiv\frac{\varepsilon}{\omega_1}+\omega_w$, $a\equiv\frac{-\lambda_1\varepsilon}{\omega_1}$, $b\equiv\frac{\lambda_1\omega_w}{\omega_1}$

where a and b are the same expressions as in the previous section, we have

$$E\left(e^{w}\middle|\varepsilon\right)=2\left(G_1-G_2\right)^{-1}e^{-\frac{1}{2}\frac{\varepsilon^2}{\omega_1^2}}\int_0^{\infty}e^{\xi\tilde{w}}\phi\left(\tilde{w}\right)\Phi\left(a+b\tilde{w}\right)d\tilde{w}$$

and we arrive immediately at

$$E\left(e^{w}\middle|\varepsilon\right)=2\left(G_1-G_2\right)^{-1}e^{-\frac{1}{2}\frac{\varepsilon^2}{\omega_1^2}}e^{\frac{1}{2}\xi^2}\left\{\Phi\left(\frac{b\xi+a}{\sqrt{1+b^2}}\right)-\Phi_2\left(\frac{b\xi+a}{\sqrt{1+b^2}},-\xi;\rho=\frac{-b}{\sqrt{1+b^2}}\right)\right\}$$

We need to recalculate only the composite coefficients that involve the parameter ξ .

$$b\xi + a = \frac{\lambda_1 \omega_w}{\omega_1}\left(\frac{\varepsilon}{\omega_1} + \omega_w\right) + \frac{-\lambda_1 \varepsilon}{\omega_1} = \frac{\lambda_1}{\omega_1}\left(\frac{\omega_w \varepsilon}{\omega_1} + \omega_w^2 - \varepsilon\right) = \frac{\lambda_1}{\omega_1}\left(\left(\frac{\omega_w}{\omega_1} - 1\right)\varepsilon + \omega_w^2\right)$$

$$\Rightarrow b\xi + a = \frac{\theta_2}{s_2}\left(-\frac{s_2^2}{s^2}\varepsilon + \frac{\sigma_1^2 s_2^2}{s^2}\right) = -\frac{\theta_2 s_2}{s^2}\left(\varepsilon - \sigma_1^2\right)$$

Also $\sqrt{1+b^2} = \frac{s_2}{s}\sqrt{1+\theta_1^2}$ from before.

So $$\frac{b\xi + a}{\sqrt{1+b^2}} = \frac{-\frac{\theta_2 s_2}{s^2}\left(\varepsilon - \sigma_1^2\right)}{\frac{s_2}{s}\sqrt{1+\theta_1^2}} = \frac{-\theta_2\left(\varepsilon - \sigma_1^2\right)}{s\sqrt{1+\theta_1^2}} = -\frac{\left(\varepsilon - \sigma_1^2\right)}{\omega_2}$$

Also $\frac{-b}{\sqrt{1+b^2}} = \frac{-\sigma_1 \sigma_2}{s_1 s_2}$ from before.

Finally, $$-\frac{1}{2}\frac{\varepsilon^2}{\omega_1^2} + \frac{1}{2}\xi^2 = -\frac{1}{2}\frac{\varepsilon^2}{\omega_1^2} + \frac{1}{2}\left(\frac{\varepsilon}{\omega_1} + \omega_w\right)^2 = \frac{1}{2}\omega_w^2 + \frac{\omega_w}{\omega_1}\varepsilon$$

Substituting all into $E\left(e^w \middle| \varepsilon\right)$ we obtain

$$[3.37]:\; E\left(e^w \middle| \varepsilon_i\right) = 2\left(G_{1i} - G_{2i}\right)^{-1} \exp\left\{\frac{1}{2}\omega_w^2 + \frac{\omega_w}{\omega_1}\varepsilon_i\right\}\left[\Phi\left(-\frac{\left(\varepsilon_i - \sigma_1^2\right)}{\omega_2}\right) - \Phi_2\left(-\frac{\left(\varepsilon_i - \sigma_1^2\right)}{\omega_2}; -\left(\omega_w + \frac{\varepsilon_i}{\omega_1}\right); \rho = \frac{-\sigma_1 \sigma_2}{s_1 s_2}\right)\right] \qquad [19]$$

which is eq. [3.37] of the main text.

C.6. Eq. [3.28]: The conditional density $f\left(u \middle| \varepsilon\right)$.

Following the same approach as before, we note that u is independent of $t = v + w$. So the joint distribution of u and t is $f_{u,t}\left(u, t\right) = f_u\left(u\right) f_t\left(t\right)$. Since $\varepsilon = t - u \Rightarrow t = \varepsilon + u$. Then $f_{u,t}\left(u, \varepsilon + w\right) = f_u\left(u\right) f_t\left(\varepsilon + w\right)$ is the joint density of ε and u. The density of u conditional on ε is therefore

$$f_{u|\varepsilon}\left(u|\varepsilon\right)=\frac{f_{u,t}\left(u,\,\varepsilon+u\right)}{f_{\varepsilon}\left(\varepsilon\right)}=\frac{f_{u}\left(u\right)f_{t}\left(\varepsilon+u\right)}{f_{\varepsilon}\left(\varepsilon\right)} \qquad [20]$$

$f_u\left(u\right)$ is a Half-normal density, $f_t\left(\varepsilon+w\right)$ is eq.[11] of the main text, while $f_{\varepsilon}\left(\varepsilon\right)$ is equation [4]. So we have

$$f_{u|\varepsilon}\left(u|\varepsilon\right)=\frac{\sqrt{\frac{2}{\pi}}\sigma_2^{-1}\exp\left\{-\frac{1}{2}\left(\frac{u}{\sigma_2}\right)^2\right\}\frac{2}{s_1}\phi\left(\left(\varepsilon+u\right)/s_1\right)\Phi\left(\theta_1\frac{\left(\varepsilon+u\right)}{s_1}\right)}{\frac{2}{s}\phi\left(\varepsilon/s\right)\left[G_1\ -\ G_2\right]}$$

$$=\left(G_1\ -\ G_2\right)^{-1}\sqrt{\frac{2}{\pi}}\frac{s}{\sigma_2 s_1}\exp\left\{-\frac{1}{2}\left(\frac{u}{\sigma_2}\right)^2-\frac{1}{2}\left(\frac{\left(\varepsilon+u\right)}{s_1}\right)^2+\frac{1}{2}\left(\frac{\varepsilon}{s}\right)^2\right\}\Phi\left(\theta_1\frac{\left(\varepsilon+u\right)}{s_1}\right)$$

where $s_1^2=\sigma_v^2+\sigma_1^2$. Calculating the argument of the exponential we have

$$\left\{-\frac{1}{2}\left(\frac{u}{\sigma_2}\right)^2-\frac{1}{2}\left(\frac{\left(\varepsilon+u\right)}{s_1}\right)^2+\frac{1}{2}\left(\frac{\varepsilon}{s}\right)^2\right\}=-\frac{1}{2}\left\{\frac{u^2}{\sigma_2^2}+\frac{\varepsilon^2}{s_1^2}+2\frac{\varepsilon u}{s_1^2}+\frac{u^2}{s_1^2}-\frac{\varepsilon^2}{s^2}\right\}$$

$$-\frac{1}{2}\left\{\frac{s^2}{\sigma_2^2 s_1^2}u^2+2\frac{\varepsilon u}{s_1^2}+\frac{\sigma_2^2}{s^2 s_1^2}\varepsilon^2\right\}=-\frac{1}{2}\left(\frac{s}{\sigma_2 s_1}u+\frac{\sigma_2}{ss_1}\varepsilon\right)^2$$

Note that $\frac{\sigma_2}{ss_1}=\frac{\sigma_2}{s\sqrt{\sigma_v^2+\sigma_1^2}}=\frac{\sigma_2}{s\sigma_v\sqrt{1+\theta_1^2}}=\frac{\theta_2}{s\sqrt{1+\theta_2^2}}=\frac{1}{\omega_2}$.

Also , we can write

$$\frac{\theta_1}{s_1}=\frac{\theta_1}{s_1}\frac{\omega_2}{\omega_2}=\frac{\theta_1}{s_1}\frac{s\sqrt{1+\theta_1^2}}{\theta_2}\frac{1}{\omega_2}=\frac{\theta_1}{\sigma_v\sqrt{1+\theta_1^2}}\frac{s\sqrt{1+\theta_1^2}}{\theta_2}\frac{1}{\omega_2}=\frac{\theta_1}{\theta_2}\sqrt{1+\theta_1^2+\theta_2^2}\frac{1}{\omega_2}=\lambda_2\frac{1}{\omega_2}$$

Set also $\omega_u \equiv \dfrac{\sigma_2 s_1}{s}$ and insert all into the conditional density while multiplying and dividing by 2 to obtain

$$[3.28]:\ f_{u|\varepsilon}\left(u|\varepsilon\right)=\left(G_1-G_2\right)^{-1}\frac{2}{\omega_u}\phi\left(\frac{u}{\omega_u}+\frac{\varepsilon}{\omega_2}\right)\Phi\left(\lambda_2\frac{\left(u+\varepsilon\right)}{\omega_2}\right) \qquad [21]$$

which is eq. [3.28] of the main text.

C.7. Eq. [3.30]: The conditional expected value $E\left(u|\varepsilon\right)$

We want to calculate

$$E\left(u|\varepsilon\right)=\int_0^\infty u f_{u|\varepsilon}\left(u|\varepsilon\right)du=\int_0^\infty u\left(G_1-G_2\right)^{-1}\frac{2}{\omega_u}\phi\left(\frac{u}{\omega_u}+\frac{\varepsilon}{\omega_2}\right)\Phi\left(\lambda_2\frac{\left(u+\varepsilon\right)}{\omega_2}\right)du \qquad [22]$$

We make the following change of variables:

$$\text{Set } u^* \equiv \frac{u}{\omega_u}+\frac{\varepsilon}{\omega_2} \Rightarrow \begin{cases} u=\omega_u u^*-\left(\omega_u/\omega_2\right)\varepsilon \\ du=\omega_u du^* \\ u=0 \Rightarrow u^*=\left(\varepsilon/\omega_2\right) \end{cases}$$

First we substitute in [22] only for the u outside the density:

$$E\left(u|\varepsilon\right)=\int_0^\infty\left[\omega_u u^*-\left(\omega_u/\omega_2\right)\varepsilon\right]f_{u|\varepsilon}\left(u|\varepsilon\right)du$$

$$=\int_0^\infty \omega_u u^* f_{u|\varepsilon}\left(u|\varepsilon\right)du \;-\; \left(\omega_u/\omega_2\right)\varepsilon\int_0^\infty f_{u|\varepsilon}\left(u|\varepsilon\right)du$$

The 2nd integral equals unity, so we eliminate it and complete the change of variables:

$$E\left(u|\varepsilon\right)=\int_{\varepsilon/\omega_2}^{\infty}\omega_u u^*\left(G_1-G_2\right)^{-1}\frac{2}{\omega_u}\phi\left(u^*\right)\Phi\left(\lambda_2\frac{\left(\omega_u u^*-\left(\omega_u/\omega_2\right)\varepsilon+\varepsilon\right)}{\omega_2}\right)\omega_u du^*-\left(\omega_u/\omega_2\right)\varepsilon$$

$$E\left(u|\varepsilon\right)=\left(G_1-G_2\right)^{-1}2\omega_u\int_{\varepsilon/\omega_2}^{\infty}u^*\phi\left(u^*\right)\Phi\left(\lambda_2\frac{\left(\omega_2-\omega_u\right)}{\omega_2^2}\varepsilon+\frac{\lambda_2\omega_u}{\omega_2}u^*\right)du^*-\left(\omega_u/\omega_2\right)\varepsilon \qquad [23]$$

For the remaining integral we use again the relevant equation from Owen (1980) p. 404, which we remind that evaluated at limits $\left(c,\infty\right)$ is (eq. [11]),

$$\int_c^{\infty}x\phi\left(x\right)\Phi\left(a+bx\right)dx=\frac{b}{\sqrt{1+b^2}}\phi\left(\frac{a}{\sqrt{1+b^2}}\right)\Phi\left(\frac{c+b\left(a+bc\right)}{-\sqrt{1+b^2}}\right)+\phi\left(c\right)\Phi\left(a+bc\right)$$

We now match coefficients between [11] and [23] and calculate the various expressions.

$$a\equiv\frac{\lambda_2\left(\omega_2-\omega_u\right)}{\omega_2^2}\varepsilon\ ,\quad b=\frac{\lambda_2\omega_u}{\omega_2}\ ,\quad c=\varepsilon/\omega_2$$

$$a+bc=\frac{\lambda_2\left(\omega_2-\omega_u\right)}{\omega_2^2}+\frac{\lambda_2\omega_u}{\omega_2}\frac{\varepsilon}{\omega_2}=\frac{\lambda_2}{\omega_2^2}\left(\omega_2-\omega_u+\omega_u\right)\varepsilon=\frac{\lambda_2}{\omega_2}\varepsilon$$

Then $c+b\left(a+bc\right)=\left(\varepsilon/\omega_2\right)+\frac{\lambda_2\omega_u}{\omega_2}\frac{\lambda_2}{\omega_2}\varepsilon=\frac{1}{\omega_2}\left(1+\frac{\lambda_2^2\omega_u}{\omega_2}\right)\varepsilon=\frac{1}{\omega_2}\left(1+\theta_1^2\right)\varepsilon$

The last equality comes from the previous results

$$\frac{\theta_1}{s_1}=\frac{\lambda_2}{\omega_2}\Rightarrow\frac{\lambda_2^2}{\omega_2}=\frac{\theta_1^2\omega_2}{s_1^2}\ \text{ while }\ \omega_u\equiv\frac{\sigma_2 s_1}{s}\ ,\quad\frac{\sigma_2}{ss_1}=\frac{1}{\omega_2}\quad\Rightarrow\omega_u\omega_2=s_1^2$$

We also have $\omega_2=\frac{s\sqrt{1+\theta_1^2}}{\theta_2}$ and so $c+b\left(a+bc\right)=\frac{\theta_2\sqrt{1+\theta_1^2}}{s}\varepsilon$

We turn to

$$\sqrt{1+b^2} = \sqrt{1+\left(\frac{\theta_1\omega_u}{s_1}\right)^2} = \sqrt{1+\theta_1^2\frac{\sigma_2^2}{s^2}} = \frac{1}{s}\sqrt{s^2+\theta_1^2\sigma_2^2} =$$

$$= \frac{1}{s}\sqrt{s_1^2+\left(1+\theta_1^2\right)\sigma_2^2} = \frac{1}{s}\sqrt{s_1^2+\left(\sigma_v^2+\sigma_1^2\right)\frac{\sigma_2^2}{\sigma_v^2}} = \frac{s_1}{s}\sqrt{1+\theta_2^2}$$

So $$\frac{c+b\left(a+bc\right)}{-\sqrt{1+b^2}} = \frac{\frac{\theta_2\sqrt{1+\theta_1^2}}{s}\varepsilon}{-\frac{s_1}{s}\sqrt{1+\theta_2^2}} = -\frac{\theta_2\sqrt{1+\theta_1^2}}{s_1\sqrt{1+\theta_2^2}}\varepsilon$$

We have that $\omega_1 = \dfrac{s\sqrt{1+\theta_2^2}}{\theta_1} \Rightarrow \sqrt{1+\theta_2^2} = \dfrac{\omega_1\theta_1}{s}$ and that $s = \sigma_v\sqrt{1+\theta_1^2+\theta_2^2}$. Inserting both into the expression and rearranging we obtain

$$\frac{c+b\left(bc+a\right)}{-\sqrt{1+b^2}} = -\frac{\theta_2\sqrt{1+\theta_1^2+\theta_2^2}}{\theta_1\,\omega_1}\frac{\sigma_v\sqrt{1+\theta_1^2}}{s_1}\varepsilon$$

But $\dfrac{\theta_2\sqrt{1+\theta_1^2+\theta_2^2}}{\theta_1} = \lambda_1$ while $\dfrac{\sigma_v\sqrt{1+\theta_1^2}}{s_1} = \dfrac{\sqrt{\sigma_v^2+\sigma_1^2}}{s_1} = 1$

So in the end $\dfrac{c+b\left(bc+a\right)}{-\sqrt{1+b^2}} = -\dfrac{\lambda_1}{\omega_1}\varepsilon$.

We also need to calculate $$\frac{a}{\sqrt{1+b^2}} = \frac{\frac{\lambda_2\left(\omega_2-\omega_u\right)}{\omega_2^2}\varepsilon}{\frac{s_1}{s}\sqrt{1+\theta_2^2}}$$

We have that $\omega_2-\omega_u = \dfrac{ss_1}{\sigma_2} - \dfrac{\sigma_2 s_1}{s} = \dfrac{s^2s_1-\sigma_2^2s_1}{s\sigma_2} = \dfrac{s_1^3}{s\sigma_2}$. We also have $\dfrac{\lambda_2}{\omega_2} = \dfrac{\theta_1}{s_1}$. Inserting both and simplifying we obtain

$$\frac{a}{\sqrt{1+b^2}} = \frac{\frac{\theta_1}{s_1\omega_2}\frac{s_1^3}{s\sigma_2}}{\frac{1}{s}s_1\sqrt{1+\theta_2^2}}\varepsilon = \frac{\theta_1 s_1}{\omega_2\sigma_2\sqrt{1+\theta_2^2}}\varepsilon = \frac{\theta_1 s_1}{\frac{ss_1}{\sigma_2}\sigma_2\sqrt{1+\theta_2^2}}\varepsilon = \frac{\theta_1}{s\sqrt{1+\theta_2^2}}\varepsilon = \frac{1}{\omega_1}\varepsilon$$

Finally, $$\frac{b}{\sqrt{1+b^2}} = \frac{\dfrac{\lambda_2\omega_u}{\omega_2}}{\dfrac{1}{s}s_1\sqrt{1+\theta_2^2}} = \frac{s}{s_1^2}\frac{\theta_1}{\sqrt{1+\theta_2^2}}\omega_u = \frac{s^2\omega_u}{s_2^2\omega_1}$$

Using all the results we obtain

$$E\left(u\middle|\varepsilon\right) = \left(G_1 - G_2\right)^{-1} 2\omega_u \left\{ \frac{s^2\omega_u}{s_2^2}\left[\frac{1}{\omega_1}\phi\left(\frac{\varepsilon}{\omega_1}\right)\Phi\left(-\frac{\lambda_1}{\omega_1}\varepsilon\right)\right] + \phi\left(\frac{\varepsilon}{\omega_2}\right)\Phi\left(\frac{\lambda_2}{\omega_2}\varepsilon\right)\right\} - \left(\omega_w/\omega_2\right)\varepsilon$$

We have again arrived at the Skew-normal densities whose distribution functions appear in the density $f_\varepsilon\left(\varepsilon\right)$. Continuing the calculations and using the compact notations introduced for these densities we have

$$E\left(u\middle|\varepsilon\right) = \frac{s^2\omega_u^2}{s_2^2}\psi_1 + \omega_u\omega_2\psi_2 - \left(\omega_u/\omega_2\right)\varepsilon$$

We can further simplify the expression:

$$\left(\omega_u/\omega_2\right) = \frac{\sigma_2^2}{s^2} \quad , \quad \omega_u\omega_2 = \frac{\sigma_2 s_1}{s}\frac{ss_1}{\sigma_2} = s_1^2 = \sigma_v^2 + \sigma_1^2 \quad , \quad \frac{s^2\omega_u^2}{s_1^2} = \frac{s^2\left(\dfrac{\sigma_2 s_1}{s}\right)^2}{s_1^2} = \sigma_2^2$$

So , reinstating also the subscript *i*,

$$E\left(u_i\middle|\varepsilon_i\right) = \sigma_2^2\psi_{1i} + \left(\sigma_v^2 + \sigma_1^2\right)\psi_{2i} - \frac{\sigma_2^2}{s^2}\varepsilon_i = \sigma_2^2\psi_{1i} + \left(s^2 - \sigma_2^2\right)\psi_{2i} - \frac{\sigma_2^2}{s^2}\varepsilon_i$$

$$\boxed{\Rightarrow [3.30]:\ E\left(u_i\middle|\varepsilon_i\right) = s^2\psi_{2i} - \sigma_2^2\left(\frac{1}{s^2}\varepsilon_i - \psi_i\right)} \qquad [24]$$

which is eq. [30] of the main text.

C.8. Eq. [3.38]: The conditional expected value $E\left(e^{-u}\,|\varepsilon\right)$.

We want to calculate

$$E\left(e^{-u}|\varepsilon\right)=\int_0^{\infty}e^{-u}f_{u|\varepsilon}\left(u|\varepsilon\right)du=\int_0^{\infty}e^{-u}\left(G_1-G_2\right)^{-1}\frac{2}{\omega_u}\phi\left(\frac{u}{\omega_u}+\frac{\varepsilon}{\omega_2}\right)\Phi\left(\lambda_2\frac{\left(u+\varepsilon\right)}{\omega_2}\right)du \qquad [25]$$

As previously, the approach is to transform the integral so that it represents a known moment generating function. To do this we first define the change of variables

$$\tilde{u}\equiv\frac{u}{\omega_u}\;\Rightarrow\begin{cases} u=\omega_u\tilde{u} \\ du=\omega_u d\tilde{u} \\ u=0\Rightarrow\tilde{u}=0\end{cases}$$

and calculate explicitly the Normal density:

$$E\left(e^{-u}|\varepsilon\right)=\int_0^{\infty}e^{-\omega_u\tilde{u}}\left(G_1-G_2\right)^{-1}2\phi\left(\tilde{u}+\frac{\varepsilon}{\omega_2}\right)\Phi\left(\lambda_2\frac{\left(\omega_u\tilde{u}+\varepsilon\right)}{\omega_2}\right)d\tilde{u}$$

$$=2\left(G_1-G_2\right)^{-1}\int_0^{\infty}e^{-\omega_u\tilde{u}}e^{-\frac{1}{2}\frac{\varepsilon^2}{\omega_2^2}}e^{-\frac{\varepsilon\tilde{u}}{\omega_2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\tilde{u}^2}\Phi\left(\frac{\lambda_2\varepsilon}{\omega_2}+\frac{\lambda_2\omega_u}{\omega_2}\tilde{u}\right)d\tilde{u}$$

$$=2\left(G_1-G_2\right)^{-1}e^{-\frac{1}{2}\frac{\varepsilon^2}{\omega_2^2}}\int_0^{\infty}e^{-\left(\omega_u+\frac{\varepsilon}{\omega_2}\right)\tilde{u}}\phi\left(\tilde{u}\right)\Phi\left(\frac{\lambda_2\varepsilon}{\omega_2}+\frac{\lambda_2\omega_u}{\omega_2}\tilde{u}\right)d\tilde{u}$$

To ease notation we define temporarily $\xi\equiv-\left(\omega_u+\frac{\varepsilon}{\omega_2}\right),\;\; a\equiv\frac{\lambda_2\varepsilon}{\omega_2},\;\; b\equiv\frac{\lambda_2\omega_u}{\omega_2}$

and we have $E\left(e^{-u}|\varepsilon\right)=2\left(G_1-G_2\right)^{-1}\exp\left\{-\frac{1}{2}\frac{\varepsilon^2}{\omega_2^2}\right\}\int_0^{\infty}e^{\xi\tilde{u}}\phi\left(\tilde{u}\right)\Phi\left(a+b\tilde{u}\right)d\tilde{u}$

Now, since the integrating interval is $\left[0,\infty\right)$, the $\phi(\tilde{u})\Phi(a+b\tilde{u})$ part of the integrand can be considered as the density kernel of a truncated Skew-normal distribution, as previously. Multiplying and dividing our expression by c^* we obtain

$$E\left(e^{-u}\left|\varepsilon\right.\right)=2\left(G_1-G_2\right)^{-1}e^{-\frac{1}{2}\frac{\varepsilon^2}{\omega_2^2}}\frac{1}{c^*}\int_0^{\infty}e^{\xi\tilde{u}}\,c^*\phi(\tilde{u})\Phi(a+b\tilde{u})\,d\tilde{u}$$

The integral now represents the moment generating function of this truncated Skew-normal distribution, and for the specific truncation $\left[0,\infty\right)$ this mgf becomes

$$\begin{aligned}M(\xi)&=c^*e^{\frac{1}{2}\xi^2}\left\{\Phi_2\left(\frac{b\xi+a}{\sqrt{1+b^2}},\,\infty;\,\rho=\frac{-b}{\sqrt{1+b^2}}\right)-\Phi_2\left(\frac{b\xi+a}{\sqrt{1+b^2}},\,-\xi;\,\rho=\frac{-b}{\sqrt{1+b^2}}\right)\right\}\\&=c^*e^{\frac{1}{2}\xi^2}\left\{\Phi\left(\frac{b\xi+a}{\sqrt{1+b^2}}\right)-\Phi_2\left(\frac{b\xi+a}{\sqrt{1+b^2}},\,-\xi;\,\rho=\frac{-b}{\sqrt{1+b^2}}\right)\right\}\end{aligned}$$

and so

$$E\left(e^{-u}\left|\varepsilon\right.\right)=2\left(G_1-G_2\right)^{-1}\exp\left\{-\tfrac{1}{2}\frac{\varepsilon^2}{\omega_2^2}+\tfrac{1}{2}\xi^2\right\}\left\{\Phi\left(\frac{b\xi+a}{\sqrt{1+b^2}}\right)-\Phi_2\left(\frac{b\xi+a}{\sqrt{1+b^2}},\,-\xi;\,\rho=\frac{-b}{\sqrt{1+b^2}}\right)\right\}$$

We calculate the composite coefficients involved,

$$b\xi+a=-\frac{\lambda_2\omega_u}{\omega_2}\left(\omega_u+\frac{\varepsilon}{\omega_2}\right)+\frac{\lambda_2\varepsilon}{\omega_2}=\frac{\lambda_2}{\omega_2}\left(-\frac{\omega_u\varepsilon}{\omega_2}-\omega_u^2+\varepsilon\right)=\frac{\lambda_2}{\omega_2}\left(\left(1-\frac{\omega_u}{\omega_2}\right)\varepsilon-\omega_u^2\right)$$

Using previous results we have

$$b\xi+a=\frac{\theta_1}{s_1}\left(\left(1-\frac{\sigma_2^2}{s^2}\right)\varepsilon-\frac{\sigma_2^2s_1^2}{s^2}\right)=\frac{\theta_1}{s_1}\left(\frac{s_1^2}{s^2}\varepsilon-\frac{\sigma_2^2s_1^2}{s^2}\right)=\frac{\theta_1s_1}{s^2}\left(\varepsilon-\sigma_2^2\right)$$

Also $\sqrt{1+b^2} = \frac{s_1}{s}\sqrt{1+\theta_2^2}$ from previously.

So $\frac{b\xi + a}{\sqrt{1+b^2}} = \frac{\frac{\theta_1 s_1}{s^2}\left(\varepsilon - \sigma_2^2\right)}{\frac{s_1}{s}\sqrt{1+\theta_2^2}} = \frac{\theta_1\left(\varepsilon - \sigma_2^2\right)}{s\sqrt{1+\theta_2^2}} = \frac{\left(\varepsilon - \sigma_2^2\right)}{\omega_1}$

Also $\frac{-b}{\sqrt{1+b^2}} = \frac{-\sigma_1 \sigma_2}{s_1 s_2}$ from previously.

Finally, $-\frac{1}{2}\frac{\varepsilon^2}{\omega_2^2} + \frac{1}{2}\xi^2 = -\frac{1}{2}\frac{\varepsilon^2}{\omega_2^2} + \frac{1}{2}\left(\omega_u + \frac{\varepsilon}{\omega_2}\right)^2 = \frac{1}{2}\omega_u^2 + \frac{\omega_u}{\omega_2}\varepsilon$

So, in the end,

$$\boxed{\begin{aligned}[3.38]:\; E\left(e^{-u}\left|\varepsilon_i\right.\right) = 2\left(G_{1i} - G_{2i}\right)^{-1}\exp\left\{\frac{1}{2}\omega_u^2 + \frac{\omega_u}{\omega_2}\varepsilon_i\right\}\Bigg\{\Phi\left(\frac{\varepsilon_i - \sigma_2^2}{\omega_1}\right) \\ - \Phi_2\left(\frac{\varepsilon_i - \sigma_2^2}{\omega_1}, \frac{\varepsilon_i}{\omega_2} + \omega_u ;\, \rho = \frac{-\sigma_1\sigma_2}{s_1 s_2}\right)\Bigg\}\end{aligned}} \quad [26]$$

which is eq. [3.38] of the main text.

C.9. Eq. [3.39] & [3.40]: The conditional expected value $E\left(e^w e^{-u}\left|\varepsilon\right.\right)$.

Setting $z = w - u$ we want to calculate $E\left(e^z\left|\varepsilon\right.\right)$. To do this we need the conditional density

$$f_{z|\varepsilon}\left(z|\varepsilon\right) = \frac{f_{z,\varepsilon}\left(z, \varepsilon\right)}{f_\varepsilon\left(\varepsilon\right)}$$

Now, v and z are independent so $f_{z,v}\left(z, v\right) = f_z\left(z\right) f_v\left(v\right)$. Moreover $\varepsilon = v + z \Rightarrow v = \varepsilon - z$. Then

$f_{z,\varepsilon}(z,\,\varepsilon) = f_{z,v}(z,\,\varepsilon - z) = f_z(z)\,f_v(\varepsilon - z)$. We have the density $f_z(z)$ from before,

$$f_z(z) = \begin{cases} \dfrac{4}{s_h}\phi(z/s_h)\Phi\left(\dfrac{\sigma_1}{\sigma_2}(z/s_h)\right), & z \le 0 \\ \\ \dfrac{4}{s_h}\phi(z/s_h)\Phi\left(-\dfrac{\sigma_2}{\sigma_1}(z/s_h)\right), & z \ge 0 \end{cases} \qquad s_h = \sqrt{\sigma_1^2 + \sigma_2^2}$$

and $f_v(\varepsilon - z)$ is a Normal density. So

$$E\left(e^z\,|\varepsilon\right) = \int_{-\infty}^{\infty} e^z f_{z|\varepsilon}\left(z|\varepsilon\right)dz = \int_{-\infty}^{\infty} e^z \frac{f_{z,\varepsilon}(z,\,\varepsilon)}{f_\varepsilon(\varepsilon)}dz = \frac{1}{f_\varepsilon(\varepsilon)}\int_{-\infty}^{\infty} e^z f_z(z)\,f_v(\varepsilon - z)\,dz$$

$$E\left(e^z\,|\varepsilon\right) = \int_{-\infty}^{\infty} e^z f_{z|\varepsilon}\left(z|\varepsilon\right)dz = \int_{-\infty}^{\infty} e^z \frac{f_{z,\varepsilon}(z,\,\varepsilon)}{f_\varepsilon(\varepsilon)}dz = \frac{1}{f_\varepsilon(\varepsilon)}\int_{-\infty}^{\infty} e^z f_z(z)\,f_v(\varepsilon - z)\,dz$$

$$= \frac{1}{f_\varepsilon(\varepsilon)}\int_{-\infty}^{0} e^z \frac{4}{s_h}\phi(z/s_h)\Phi\left(\frac{\sigma_1}{\sigma_2}(z/s_h)\right)\frac{1}{\sigma_v}\phi(\varepsilon - z)\,dz$$

$$+\frac{1}{f_\varepsilon(\varepsilon)}\int_{0}^{\infty} e^z \frac{4}{s_h}\phi(z/s_h)\Phi\left(-\frac{\sigma_2}{\sigma_1}(z/s_h)\right)\frac{1}{\sigma_v}\phi(\varepsilon - z)\,dz$$

Comparing the above with eq. [2] of this subsection of the Technical Appendix, we see that this is essentially the same integration procedure we performed in order to obtain $f_\varepsilon(\varepsilon)$, with the added term e^z inside the integrand and $\dfrac{1}{f_\varepsilon(\varepsilon)}$ outside. So we can directly use eq. [3] ,

$$E\left(e^z\,|\varepsilon\right) = \frac{1}{f_\varepsilon(\varepsilon)}C\int_{y=-\infty}^{0}\phi(y)\left[\int_{z=(s_h/\lambda_h)y}^{\infty}\exp\left\{-\frac{1}{2}\frac{s^2}{\sigma_v^2 s_h^2}z^2 + \left(1+\frac{\varepsilon}{\sigma_v^2}\right)z\right\}dz\right.$$

$$\left. -\int_{z=-(s_h\lambda_h)y}^{\infty}\exp\left\{-\frac{1}{2}\frac{s^2}{\sigma_v^2 s_h^2}z^2 + \left(1+\frac{\varepsilon}{\sigma_v^2}\right)z\right\}dz\right]dy \qquad [27]$$

with $C=\dfrac{4(2\pi)^{-1}}{\sigma_v s_h}\exp\left\{-\dfrac{1}{2}\left(\dfrac{\varepsilon}{\sigma_v}\right)^2\right\}$

Using the formula provided in Gradshteyn and Ryzhik (2007) (p.336),

$$\int_m^\infty \exp\left\{-\frac{1}{4\delta}\omega^2-\gamma\omega\right\}d\omega=\sqrt{\pi\delta}\,e^{\delta\gamma^2}\left[1-\operatorname{erf}\left(\gamma\sqrt{\delta}+\frac{m}{2\sqrt{\delta}}\right)\right]$$

the solutions for the two integrals will differ only in their error-function argument due to different lower limits of integration. We match common coefficients:

$$\frac{1}{4\delta}=\frac{1}{2}\frac{s^2}{\sigma_v^2 s_h^2}\Rightarrow\delta=\frac{\sigma_v^2 s_h^2}{2s^2}\quad,\quad\gamma=-\left(1+\frac{\varepsilon}{\sigma_v^2}\right)$$

$$\sqrt{\delta}=\frac{\sigma_v s_h}{s\sqrt{2}},\quad\delta\gamma^2=\frac{\sigma_v^2 s_h^2}{2s^2}\left(-1-\frac{\varepsilon}{\sigma_v^2}\right)^2=\frac{\sigma_v^2 s_h^2}{2s^2}\left(1+\frac{2\varepsilon}{\sigma_v^2}+\frac{\varepsilon^2}{\sigma_v^4}\right)=\frac{\sigma_v^2 s_h^2}{2s^2}+\frac{s_h^2}{s^2}\varepsilon+\frac{s_h^2}{2s^2}\frac{\varepsilon^2}{\sigma_v^2},$$

$$\gamma\sqrt{\delta}=-\left(1+\frac{\varepsilon}{\sigma_v^2}\right)\frac{\sigma_v s_h}{s\sqrt{2}}=-\frac{\sigma_v s_h}{s\sqrt{2}}-\frac{s_h}{s\sigma_v\sqrt{2}}\varepsilon$$

We first calculate the error function arguments.

For the 1st inner integral of [27] we have

$$m=(s_h/\lambda_h)y\Rightarrow 1-\operatorname{erf}\left(\gamma\sqrt{\delta}+\frac{m}{2\sqrt{\delta}}\right)=1-\operatorname{erf}\left(-\frac{\sigma_v s_h}{s\sqrt{2}}-\frac{s_h}{s\sigma_v\sqrt{2}}\varepsilon+\frac{s(s_h/\lambda_h)}{\sqrt{2}\sigma_v s_h}y\right)$$

$$=1-\operatorname{erf}\left(\frac{1}{\sqrt{2}}\left[-\frac{\sigma_v s_h}{s}-\frac{s_h}{s\sigma_v}\varepsilon+\frac{s(s_h/\lambda_h)}{\sigma_v s_h}y\right]\right)$$

$$=1-2\Phi\left(-\frac{\sigma_v s_h}{s}-\frac{s_h}{s\sigma_v}\varepsilon+\frac{(s/\lambda_h)}{\sigma_v}y\right)+1=2\Phi\left(\frac{\sigma_v s_h}{s}+\frac{s_h}{s\sigma_v}\varepsilon-\frac{(s/\lambda_h)}{\sigma_v}y\right)$$

For the 2nd inner integral of [27] we have

$$m=-\left(s_h\lambda_h\right)y \Rightarrow \ 1-\operatorname{erf}\left(\gamma\sqrt{\delta}+\frac{m}{2\sqrt{\delta}}\right)=1-\operatorname{erf}\left(-\frac{\sigma_v s_h}{s\sqrt{2}}-\frac{s_h}{s\sigma_v\sqrt{2}}\varepsilon-\frac{s\left(s_h\lambda_h\right)}{\sqrt{2}\sigma_v s_h}y\right)$$

$$=1-\operatorname{erf}\left(-\frac{1}{\sqrt{2}}\left[\frac{\sigma_v s_h}{s}+\frac{s_h}{s\sigma_v}\varepsilon+\frac{s\lambda_h}{\sigma_v}y\right]\right)$$

$$=1+2\Phi\left(\frac{\sigma_v s_h}{s}+\frac{s_h}{s\sigma_v}\varepsilon+\frac{s\lambda_h}{\sigma_v}y\right)-1=2\Phi\left(\frac{\sigma_v s_h}{s}+\frac{s_h}{s\sigma_v}\varepsilon+\frac{s\lambda_h}{\sigma_v}y\right)$$

Then the solutions to the two inner integrals are

$$\begin{cases}\sqrt{\pi}\dfrac{\sigma_v s_h}{s\sqrt{2}}\exp\left\{\dfrac{\sigma_v^2 s_h^2}{2s^2}+\dfrac{s_h^2}{s^2}\varepsilon+\dfrac{s_h^2}{2s^2}\dfrac{\varepsilon^2}{\sigma_v^2}\right\}2\Phi\left(\dfrac{\sigma_v s_h}{s}+\dfrac{s_h}{s\sigma_v}\varepsilon-\dfrac{\left(s/\lambda_h\right)}{\sigma_v}y\right), & m=\left(s_h/\lambda_h\right)y \\ \sqrt{\pi}\dfrac{\sigma_v s_h}{s\sqrt{2}}\exp\left\{\dfrac{\sigma_v^2 s_h^2}{2s^2}+\dfrac{s_h^2}{s^2}\varepsilon+\dfrac{s_h^2}{2s^2}\dfrac{\varepsilon^2}{\sigma_v^2}\right\}2\Phi\left(\dfrac{\sigma_v s_h}{s}+\dfrac{s_h}{s\sigma_v}\varepsilon+\dfrac{s\lambda_h}{\sigma_v}y\right), & m=-\left(s_h\lambda_h\right)y\end{cases}$$

We bring together all the terms that do not include the remaining integrating variable y

$$C\sqrt{\pi}\frac{\sigma_v s_h}{s\sqrt{2}}2\exp\left\{\frac{\sigma_v^2 s_h^2}{2s^2}+\frac{s_h^2}{s^2}\varepsilon+\frac{s_h^2}{2s^2}\frac{\varepsilon^2}{\sigma_v^2}\right\}$$

$$=\frac{4\left(2\pi\right)^{-1}}{\sigma_v s_h}\exp\left\{-\frac{1}{2}\left(\frac{\varepsilon}{\sigma_v}\right)^2\right\}\sqrt{\pi}\frac{\sigma_v s_h}{s\sqrt{2}}2\exp\left\{\frac{\sigma_v^2 s_h^2}{2s^2}+\frac{s_h^2}{s^2}\varepsilon+\frac{s_h^2}{2s^2}\frac{\varepsilon^2}{\sigma_v^2}\right\}$$

$$=\frac{4}{s\sqrt{2\pi}}\exp\left\{\frac{\sigma_v^2 s_h^2}{2s^2}+\frac{s_h^2}{s^2}\varepsilon\right\}\exp\left\{-\frac{1}{2\sigma_v^2}\left(1-\frac{s_h^2}{s^2}\right)\varepsilon^2\right\}$$

$$=\frac{4}{s\sqrt{2\pi}}\exp\left\{\frac{\sigma_v^2 s_h^2}{2s^2}+\frac{s_h^2}{s^2}\varepsilon\right\}\exp\left\{-\frac{\varepsilon^2}{2s^2}\right\}=\exp\left\{\frac{\sigma_v^2 s_h^2}{2s^2}+\frac{s_h^2}{s^2}\varepsilon\right\}\frac{4}{s}\phi\left(\varepsilon/s\right)$$

So

$$E\left(e^{z}\,|\varepsilon\right)=\frac{1}{f_{\varepsilon}\left(\varepsilon\right)}\exp\left\{\frac{\sigma_v^2 s_h^2}{2s^2}+\frac{s_h^2}{s^2}\,\varepsilon\right\}\frac{4}{s}\phi\left(\varepsilon/s\right)\left[\int_{y=-\infty}^{0}\phi\left(y\right)\Phi\left(\frac{\sigma_v s_h}{s}+\frac{s_h}{s\sigma_v}\varepsilon-\frac{\left(s/\lambda_h\right)}{\sigma_v}y\right)dy\right.$$

$$\left.-\int_{y=-\infty}^{0}\phi\left(y\right)\Phi\left(\frac{\sigma_v s_h}{s}+\frac{s_h}{s\sigma_v}\varepsilon+\frac{s\lambda_h}{\sigma_v}y\right)dy\right]$$

Set temporarily $a\equiv\frac{\sigma_v s_h}{s}+\frac{s_h}{s\sigma_v}\varepsilon\,,\quad \lambda_1\equiv\frac{s}{\lambda_h\sigma_v}\,,\quad \lambda_2\equiv\frac{s\,\lambda_h}{\sigma_v}.$

$$E\left(e^{z}\,|\varepsilon\right)=\frac{1}{f_{\varepsilon}\left(\varepsilon\right)}\exp\left\{\frac{\sigma_v^2 s_h^2}{2s^2}+\frac{s_h^2}{s^2}\,\varepsilon\right\}\frac{4}{s}\phi\left(\varepsilon/s\right)\left[\int_{y=-\infty}^{0}\phi\left(y\right)\Phi\left(a-\lambda_1 y\right)dy\right.$$

$$\left.-\int_{y=-\infty}^{0}\phi\left(y\right)\Phi\left(a+\lambda_2 y\right)dy\right]$$

The solution to the two integrals is given in Owen (1980), p. 403. The general solution adapted to our notation is

$$\int_{-\infty}^{0}\phi\left(y\right)\Phi\left(a\varepsilon+\lambda y\right)dy=\frac{1}{2}\left[\Phi\left(\frac{a\varepsilon}{\sqrt{1+\lambda^2}}\right)-2T\left(\frac{a\varepsilon}{\sqrt{1+\lambda^2}}\,,\,\lambda\right)\right]$$

where $T\left(\cdot\right)$ is Owen's T-function. So

$$E\left(e^{z}\,|\varepsilon\right)=\frac{\left(4/s\right)\phi\left(\varepsilon/s\right)}{f_{\varepsilon}\left(\varepsilon\right)}\exp\left\{\frac{\sigma_v^2 s_h^2}{2s^2}+\frac{s_h^2}{s^2}\,\varepsilon\right\}\left\{\frac{1}{2}\left[\Phi\left(\frac{a}{\sqrt{1+\lambda_1^2}}\right)-2T\left(\frac{a}{\sqrt{1+\lambda_1^2}}\,,\,-\lambda_1\right)\right]\right.$$

$$\left.-\frac{1}{2}\left[\Phi\left(\frac{a}{\sqrt{1+\lambda_2^2}}\right)-2T\left(\frac{a}{\sqrt{1+\lambda_2^2}}\,,\,\lambda_2\right)\right]\right\}$$

The terms in brackets are the distribution functions of Skew-normals with a non-zero location parameter. We work the arguments

$$\frac{a}{\sqrt{1+\lambda_1^2}}=\frac{\dfrac{\sigma_v s_h}{s}+\dfrac{s_h}{s\sigma_v}\varepsilon}{\sqrt{1+\left(\dfrac{s}{\lambda_h\sigma_v}\right)^2}}=\frac{\sigma_v s_h}{\dfrac{s}{\lambda_h\sigma_v}\sqrt{\lambda_h^2\sigma_v^2+s^2}}+\frac{s_h}{\dfrac{s\sigma_v}{\lambda_h\sigma_v}\sqrt{\lambda_h^2\sigma_v^2+s^2}}\varepsilon$$

$$=\frac{\sigma_v^2 s_h\lambda_h}{s\sqrt{\lambda_h^2\sigma_v^2+s^2}}+\frac{s_h\lambda_h}{s\sqrt{\lambda_h^2\sigma_v^2+s^2}}\varepsilon=\frac{s_h\lambda_h}{s\sqrt{\lambda_h^2\sigma_v^2+s^2}}\left(\sigma_v^2+\varepsilon\right)$$

We have

$$\frac{s_h\lambda_h}{s\sqrt{\lambda_h^2\sigma_v^2+s^2}}=\frac{s_h\dfrac{\sigma_1}{\sigma_2}}{s\sqrt{\dfrac{\sigma_1^2}{\sigma_2^2}\sigma_v^2+s^2}}=\frac{\sigma_1 s_h}{s\sqrt{\sigma_1^2\sigma_v^2+\sigma_2^2 s^2}}=\frac{\sigma_1 s_h}{s\sqrt{\sigma_1^2\sigma_v^2+\sigma_2^2 s^2}}=\frac{\sigma_1 s_h}{s\sqrt{\left(\sigma_1^2+\sigma_2^2\right)\left(\sigma_v^2+\sigma_2^2\right)}}$$

$$\frac{\sigma_1 s_h}{s\sqrt{\left(\sigma_1^2+\sigma_2^2\right)\left(\sigma_v^2+\sigma_2^2\right)}}=\frac{\sigma_1 s_h}{ss_h s_2}=\frac{\sigma_1}{ss_2}=\frac{1}{\omega_1}$$

So for the first term we have

$$\Phi\left(\frac{a}{\sqrt{1+\lambda_1^2}}\right)-2T\left(\frac{a}{\sqrt{1+\lambda_1^2}}\,,-\lambda_1\right)=\Phi\left(\frac{\varepsilon-\left(-\sigma_v^2\right)}{\omega_1}\right)-2T\left(\frac{\varepsilon-\left(-\sigma_v^2\right)}{\omega_1}\,,-\lambda_1\right)$$

$$=G\left(\varepsilon;-\sigma_v^2\,,\,\omega_1,-\lambda_1\right)$$

Next,

$$\frac{a}{\sqrt{1+\lambda_2^2}}=\frac{\dfrac{\sigma_v s_h}{s}+\dfrac{s_h}{s\sigma_v}\varepsilon}{\sqrt{1+\left(\dfrac{s\ \lambda_h}{\sigma_v}\right)^2}}=\frac{\sigma_v s_h}{\dfrac{s}{\sigma_v}\sqrt{\lambda_h^2 s^2+\sigma_v^2}}+\frac{s_h}{\dfrac{s\sigma_v}{\sigma_v}\sqrt{\lambda_h^2 s^2+\sigma_v^2}}\varepsilon$$

$$=\frac{\sigma_v^2 s_h}{s\sqrt{\lambda_h^2 s^2+\sigma_v^2}}+\frac{s_h}{s\sqrt{\lambda_h^2 s^2+\sigma_v^2}}\varepsilon=\frac{s_h}{s\sqrt{\lambda_h^2 s^2+\sigma_v^2}}\left(\sigma_v^2+\varepsilon\right)$$

We have

$$\frac{s_h}{s\sqrt{\lambda_h^2 s^2+\sigma_v^2}}=\frac{s_h}{s\sqrt{\dfrac{\sigma_1^2}{\sigma_2^2}s^2+\sigma_v^2}}=\frac{\sigma_2 s_h}{s\sqrt{\sigma_1^2 s^2+\sigma_2^2\sigma_v^2}}=\frac{\sigma_2 s_h}{s\sqrt{\left(\sigma_1^2+\sigma_2^2\right)\left(\sigma_v^2+\sigma_1^2\right)}}$$

$$=\frac{\sigma_2 s_h}{s s_h s_1}=\frac{\sigma_2}{s s_1}=\frac{1}{\omega_2}$$

So for the second term we have

$$\Phi\left(\frac{a}{\sqrt{1+\lambda_2^2}}\right)-2T\left(\frac{a}{\sqrt{1+\lambda_2^2}}\ ,\ \lambda_2\right)=\Phi\left(\frac{\varepsilon-\left(-\sigma_v^2\right)}{\omega_2}\right)-2T\left(\frac{\varepsilon-\left(-\sigma_v^2\right)}{\omega_2}\ ,-\lambda_2\right)$$

$$=G\left(\varepsilon;-\sigma_v^2,\ \omega_2,\ \lambda_2\right)$$

Bringing it all together,

$$E\left(e^z\,|\varepsilon\right)=\frac{(2/s)\phi(\varepsilon/s)}{f_\varepsilon\left(\varepsilon\right)}\exp\left\{\frac{\sigma_v^2 s_h^2}{2s^2}+\frac{s_h^2}{s^2}\varepsilon\right\}\left[G\left(\varepsilon;-\sigma_v^2,\ \omega_1,-\lambda_1\right)-G\left(\varepsilon;-\sigma_v^2,\ \omega_2,\ \lambda_2\right)\right]$$

$$=\exp\left\{\frac{\sigma_v^2\left(\sigma_w^2+\sigma_u^2\right)}{2s^2}+\frac{\sigma_w^2+\sigma_u^2}{s^2}\varepsilon\right\}\frac{\left[G\left(\varepsilon;-\sigma_v^2,\ \omega_1,-\lambda_1\right)-G\left(\varepsilon;-\sigma_v^2,\ \omega_2,\ \lambda_2\right)\right]}{\left[G_1\left(\varepsilon_i;0,\ \omega_1,-\lambda_1\right)-G_2\left(\varepsilon_i;0,\omega_2,\ \lambda_2\right)\right]}$$

or

$$[3.39]:\ E\left(e^{z}\,|\varepsilon\right)=$$
$$=\exp\left\{\frac{\left(\sigma_w^2+\sigma_u^2\right)}{s^2}\left(\frac{\sigma_v^2}{2}+\varepsilon\right)\right\}\frac{\left[G\left(\varepsilon;-\sigma_v^2,\,\omega_1,-\lambda_1\right)-G\left(\varepsilon;-\sigma_v^2,\,\omega_2,\,\lambda_2\right)\right]}{\left[G_1\left(\varepsilon_i\,;0,\,\omega_1,-\lambda_1\right)\ -\ G_2\left(\varepsilon_i\,;0,\omega_2,\,\lambda_2\right)\right]}$$

and in terms of the bivariate Normal integral (simplifying the factor 2)

$$[3.40]:\ E\left(e^{z}\,|\varepsilon\right)=\exp\left\{\frac{\left(\sigma_w^2+\sigma_u^2\right)}{s^2}\left(\frac{\sigma_v^2}{2}+\varepsilon\right)\right\}\times$$
$$\times\frac{\Phi_2\left(\dfrac{\varepsilon_i+\sigma_v^2}{\omega_1},\,0\,;\rho=\dfrac{\lambda_1}{\sqrt{1+\lambda_1^2}}\right)-\Phi_2\left(\dfrac{\varepsilon_i+\sigma_v^2}{\omega_2},\,0\,;\rho=\dfrac{-\lambda_2}{\sqrt{1+\lambda_2^2}}\right)}{\Phi_2\left(\dfrac{\varepsilon_i}{\omega_1},\,0\,;\rho=\dfrac{\lambda_1}{\sqrt{1+\lambda_1^2}}\right)-\Phi_2\left(\dfrac{\varepsilon_i}{\omega_2},\,0\,;\rho=\dfrac{-\lambda_2}{\sqrt{1+\lambda_2^2}}\right)}$$

D. The score of the log likelihood function.

We have

$$\tilde{L}\left(\boldsymbol{\varepsilon}|\mathbf{y},\mathbf{X},\mathbf{q}\right)=n\ln\left(\frac{2}{\sqrt{2\pi}}\right)-n\ln s-\frac{1}{2s^2}\sum_{i=1}^{n}\left(y_i-\mathbf{x}_i'\boldsymbol{\beta}\right)^2+\sum_{i=1}^{n}\ln\left(G_{1i}-G_{2i}\right)$$

The derivatives of the log-likelihood w.r.t the parameters are, in matrix notation where convenient,

$$\frac{\partial\tilde{L}}{\partial\boldsymbol{\beta}}=\frac{1}{s^2}\left(\mathbf{X}'\mathbf{y}-\mathbf{X}'\mathbf{X}\boldsymbol{\beta}\right)-\mathbf{X}'\boldsymbol{\psi} \qquad [28]$$

$$\frac{\partial\tilde{L}}{\partial s}=\frac{1}{s}\left(\frac{1}{s^2}\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}-n-\boldsymbol{\varepsilon}'\boldsymbol{\psi}\right)=\frac{1}{s}\left(\frac{1}{s^2}\sum_{i=1}^{n}\varepsilon_i^2-n-\sum_{i=1}^{n}\varepsilon_i\psi_i\right) \qquad [29]$$

$$\frac{\partial \tilde{L}}{\partial \theta_1} = \frac{\theta_1}{\lambda_2\left(1+\theta_1^2\right)} \sum_{i=1}^{n} \left\{ \sqrt{\frac{2}{\pi}} \phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s} \varepsilon_i \right) \left(G_{1i}-G_{2i}\right)^{-1} + \lambda_2 \left(\frac{1+\theta_1^2}{\theta_1^2} \psi_{1i} + \psi_{2i} \right) \varepsilon_i \right\} \qquad [30]$$

$$\frac{\partial \tilde{L}}{\partial \theta_2} = \frac{\theta_2}{\lambda_1\left(1+\theta_2^2\right)} \sum_{i=1}^{n} \left\{ \sqrt{\frac{2}{\pi}} \phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s} \varepsilon_i \right) \left(G_{1i}-G_{2i}\right)^{-1} - \lambda_1 \left(\psi_{1i} + \frac{1+\theta_2^2}{\theta_2^2} \psi_{2i} \right) \varepsilon_i \right\} \qquad [31]$$

where we have kept $\varepsilon_i = y_i - \mathbf{x}_i'\boldsymbol{\beta}$ for compactness.

D.1. Derivatives w.r.t. regressor coefficients.

For the coefficient of the *k*-th regressor we have

$$\frac{\partial}{\partial \beta_k} \tilde{L} = \frac{1}{s^2} \sum_{i=1}^{n} \left(y_i - \mathbf{x}_i'\boldsymbol{\beta}\right) x_{ik} + \sum_{i=1}^{n} \left(G_{1i}-G_{2i}\right)^{-1} \left(g_{1i} \frac{\partial \varepsilon_i}{\partial \beta_k} - g_{2i} \frac{\partial \varepsilon_i}{\partial \beta_k} \right)$$

$$= \frac{1}{s^2} \sum_{i=1}^{n} \left(y_i - \mathbf{x}_i'\boldsymbol{\beta}\right) x_{ik} + \sum_{i=1}^{n} \psi_i \frac{\partial \varepsilon_i}{\partial \beta_k} \quad = \frac{1}{s^2} \sum_{i=1}^{n} \left(y_i - \mathbf{x}_i'\boldsymbol{\beta}\right) x_{ik} - \sum_{i=1}^{n} \psi_i x_{ik}$$

$$= \frac{1}{s^2} \sum_{i=1}^{n} \left(y_i - \mathbf{x}_i'\boldsymbol{\beta}\right) x_{ik} - \sum_{i=1}^{n} \psi_i x_{ik}$$

For the whole regressor coefficient vector this aggregates to

$$\frac{\partial \tilde{L}}{\partial \boldsymbol{\beta}} = \frac{1}{s^2} \left(\mathbf{X}'\mathbf{y} - \mathbf{X}'\mathbf{X}\boldsymbol{\beta} \right) - \mathbf{X}'\boldsymbol{\psi}$$

D.2. The derivative $\dfrac{\partial \tilde{L}}{\partial s}$.

$$\frac{\partial \tilde{L}}{\partial s} = -\frac{n}{s} + \frac{1}{s^3}\sum_{i=1}^{n}\left(y_i - \mathbf{x}_i'\boldsymbol{\beta}\right)^2 +$$

$$+ \sum_{i=1}^{n}\left(G_{1i} - G_{2i}\right)^{-1}\frac{\partial}{\partial s}\left\{\int_{-\infty}^{\varepsilon_i/\omega_1} 2\phi(t)\Phi(-\lambda_1 t)\,dt - \int_{-\infty}^{\varepsilon_i/\omega_2} 2\phi(t)\Phi(\lambda_2 t)\,dt\right\}$$

Remember that the skew parameters do not depend on s under the chosen parametrization. Then

$$\frac{\partial \tilde{L}}{\partial s} = -\frac{n}{s} + \frac{1}{s^3}\sum_{i=1}^{n}\left(y_i - \mathbf{x}_i'\boldsymbol{\beta}\right)^2$$

$$+ \sum_{i=1}^{n}\left(G_{1i} - G_{2i}\right)^{-1} 2\phi(\varepsilon_i/\omega_1)\Phi(-\lambda_1\varepsilon_i/\omega_1)\frac{\partial(\varepsilon_i/\omega_1)}{\partial s}$$

$$- \sum_{i=1}^{n}\left(G_{1i} - G_{2i}\right)^{-1} 2\phi(\varepsilon_i/\omega_2)\Phi(\lambda_2\varepsilon_i/\omega_1)\frac{\partial(\varepsilon_i/\omega_2)}{\partial s}$$

Now , $\omega_1 \equiv \dfrac{s\sqrt{1+\theta_2^2}}{\theta_1}$, $\omega_2 \equiv \dfrac{s\sqrt{1+\theta_1^2}}{\theta_2}$ so

$$\frac{\partial(\varepsilon_i/\omega_1)}{\partial s} = \varepsilon_i\frac{\partial}{\partial s}\left(\frac{\theta_1}{s\sqrt{1+\theta_2^2}}\right) = -\varepsilon_i\frac{\theta_1}{\sqrt{1+\theta_2^2}}\frac{1}{s^2} = -(\varepsilon_i/\omega_1)\frac{1}{s}$$

$$\frac{\partial(\varepsilon_i/\omega_2)}{\partial s} = \varepsilon_i\frac{\partial}{\partial s}\left(\frac{\theta_2}{s\sqrt{1+\theta_1^2}}\right) = -\varepsilon_i\frac{\theta_2}{\sqrt{1+\theta_1^2}}\frac{1}{s^2} = -(\varepsilon_i/\omega_2)\frac{1}{s}$$

Inserting into the expression we have

$$\frac{\partial\tilde{L}}{\partial s} = -\frac{n}{s} + \frac{1}{s^3}\sum_{i=1}^{n}\left(y_i - \mathbf{x}_i'\boldsymbol{\beta}\right)^2$$

$$-\sum_{i=1}^{n}\left(G_{1i}-G_{2i}\right)^{-1}2\phi\left(\varepsilon_i/\omega_1\right)\Phi\left(-\lambda_1\varepsilon_i/\omega_1\right)\left(\varepsilon_i/\omega_1\right)\frac{1}{s}$$

$$+\sum_{i=1}^{n}\left(G_{1i}-G_{2i}\right)^{-1}2\phi\left(\varepsilon_i/\omega_2\right)\Phi\left(\lambda_2\varepsilon_i/\omega_1\right)\left(\varepsilon_i/\omega_2\right)\frac{1}{s}$$

$$\Rightarrow \frac{\partial\tilde{L}}{\partial s} = -\frac{n}{s} + \frac{1}{s^3}\sum_{i=1}^{n}\left(y_i - \mathbf{x}_i'\boldsymbol{\beta}\right)^2 - \frac{1}{s}\sum_{i=1}^{n}\varepsilon_i\left(G_{1i}-G_{2i}\right)^{-1}g_{1i} + \frac{1}{s}\sum_{i=1}^{n}\varepsilon_i\left(G_{1i}-G_{2i}\right)^{-1}g_{2i}$$

$$= -\frac{n}{s} + \frac{1}{s^3}\sum_{i=1}^{n}\left(y_i - \mathbf{x}_i'\boldsymbol{\beta}\right)^2 - \frac{1}{s}\sum_{i=1}^{n}\varepsilon_i\psi_{1i} + \frac{1}{s}\sum_{i=1}^{n}\varepsilon_i\psi_{2i}$$

$$\Rightarrow \frac{\partial\tilde{L}}{\partial s} = \frac{1}{s}\left(\frac{1}{s^2}\sum_{i=1}^{n}\left(y_i - \mathbf{x}_i'\boldsymbol{\beta}\right)^2 - n - \sum_{i=1}^{n}\left(y_i - \mathbf{x}_i'\boldsymbol{\beta}\right)\psi_i\right) \qquad [32]$$

D.3. $\dfrac{\partial\tilde{L}}{\partial\theta_1}$.

The parameters θ_1, θ_2 appear only in the term $\ln\left(G_{1i}-G_{2i}\right)$ of the log-likelihood.

$$\frac{\partial\tilde{L}_i}{\partial\theta_1} = \frac{\partial}{\partial\theta_1}\ln\left(G_{1i}-G_{2i}\right) = \left(G_{1i}-G_{2i}\right)^{-1}\frac{\partial}{\partial\theta_1}\left(G_{1i}-G_{2i}\right) \qquad [33]$$

We calculate each partial derivative in [33] separately.

$$\frac{\partial}{\partial\theta_1}G_{1i} = \frac{\partial}{\partial\theta_1}\int_{-\infty}^{\varepsilon_i/\omega_1}2\phi(t)\Phi\left(-\lambda_1 t\right)dt$$

$$= 2\phi\left(\varepsilon_i/\omega_1\right)\Phi\left(-\lambda_1\varepsilon_i/\omega_1\right)\frac{\partial\left(\varepsilon_i/\omega_1\right)}{\partial\theta_1} + \int_{-\infty}^{\varepsilon_i/\omega_1}2\phi(t)\frac{\partial\Phi\left(-\lambda_1 t\right)}{\partial\theta_1}dt$$

Now $\quad \dfrac{\partial(\varepsilon_i/\omega_1)}{\partial\theta_1} = \varepsilon_i \dfrac{\partial}{\partial\theta_1}\left(\dfrac{\theta_1}{s\sqrt{1+\theta_2^2}}\right) = \varepsilon_i \dfrac{1}{s\sqrt{1+\theta_2^2}} = (\varepsilon_i/\omega_1)\dfrac{1}{\theta_1}$

and so $\quad \dfrac{\partial}{\partial\theta_1}G_{1i} = \dfrac{\varepsilon_i}{\theta_1}g_{1i} + \int_{-\infty}^{\varepsilon_i/\omega_1} 2\phi(t)\dfrac{\partial\Phi(-\lambda_1 t)}{\partial\theta_1}dt \qquad [34]$

Also, $\quad \dfrac{\partial\Phi(-\lambda_1 t)}{\partial\theta_1} = \phi(\lambda_1 t)\dfrac{\partial(-\lambda_1 t)}{\partial\theta_1} = -t\phi(\lambda_1 t)\dfrac{\partial\lambda_1}{\partial\theta_1}$

therefore

$$\frac{\partial}{\partial\theta_1}G_{1i} = \frac{\varepsilon_i}{\theta_1}g_{1i} - 2\frac{\partial\lambda_1}{\partial\theta_1}\int_{-\infty}^{\varepsilon_i/\omega_1} t\phi(t)\phi(\lambda_1 t)\,dt$$

$$= \frac{\varepsilon_i}{\theta_1}g_{1i} - \frac{2}{(1+\lambda_1^2)\sqrt{2\pi}}\frac{\partial\lambda_1}{\partial\theta_1}\int_{-\infty}^{\varepsilon_i\sqrt{1+\lambda_1^2}/\omega_1}\left(t\sqrt{1+\lambda_1^2}\right)\phi\left(t\sqrt{1+\lambda_1^2}\right)d\left(t\sqrt{1+\lambda_1^2}\right)$$

$$= \frac{\varepsilon_i}{\theta_1}g_{1i} - \frac{2}{(1+\lambda_1^2)\sqrt{2\pi}}\frac{\partial\lambda_1}{\partial\theta_1}\int_{-\infty}^{\varepsilon_i\sqrt{1+\lambda_1^2}/\omega_1}\tilde{t}\phi(\tilde{t})\,d\tilde{t} \Rightarrow \frac{\partial}{\partial\theta_1}G_{1i} = \frac{\varepsilon_i}{\theta_1}g_{1i} + \sqrt{\frac{2}{\pi}}\frac{\phi\left(\varepsilon_i\sqrt{1+\lambda_1^2}/\omega_1\right)}{(1+\lambda_1^2)}\frac{\partial\lambda_1}{\partial\theta_1}$$

We have

$$1+\lambda_1^2 = 1+\frac{\theta_2^2}{\theta_1^2}\left(1+\theta_1^2+\theta_2^2\right) = \frac{\theta_1^2+\theta_2^2\left(1+\theta_1^2+\theta_2^2\right)}{\theta_1^2} = \frac{\left(\theta_1^2+\theta_2^2\right)\left(1+\theta_2^2\right)}{\theta_1^2} = \left(\theta_1^2+\theta_2^2\right)\frac{\omega_1^2}{s^2}$$

Also

$$\frac{\partial\lambda_1}{\partial\theta_1} = \frac{\partial\left(\dfrac{\theta_2}{\theta_1}\sqrt{1+\theta_1^2+\theta_2^2}\right)}{\partial\theta_1} = \theta_2\left\{-\frac{\sqrt{1+\theta_1^2+\theta_2^2}}{\theta_1^2} + \frac{1}{\theta_1}\left(\frac{1}{2}\frac{2\theta_1\sqrt{1+\theta_1^2+\theta_2^2}}{1+\theta_1^2+\theta_2^2}\right)\right\}$$

$$=\theta_2\sqrt{1+\theta_1^2+\theta_2^2}\left\{\frac{-\left(1+\theta_1^2+\theta_2^2\right)+\theta_1^2}{\theta_1^2\left(1+\theta_1^2+\theta_2^2\right)}\right\}=-\frac{\left(1+\theta_2^2\right)}{\theta_1\left(\theta_1/\theta_2\right)\sqrt{1+\theta_1^2+\theta_2^2}}=-\frac{\left(1+\theta_2^2\right)}{\theta_1\lambda_2}=-\frac{\theta_1}{\lambda_2}\frac{\omega_1^2}{s^2}$$

Therefore

$$\frac{\partial}{\partial\theta_1}G_{1i}=\frac{\varepsilon_i}{\theta_1}g_{1i}-\sqrt{\frac{2}{\pi}}\frac{\phi\left(\varepsilon_i\sqrt{1+\lambda_1^2}/\omega_1\right)}{\left(\theta_1^2+\theta_2^2\right)\frac{\omega_1^2}{s^2}}\frac{\theta_1}{\lambda_2}\frac{\omega_1^2}{s^2}=\frac{\varepsilon_i}{\theta_1}g_{1i}-\sqrt{\frac{2}{\pi}}\frac{1}{\lambda_2}\frac{\theta_1}{\left(\theta_1^2+\theta_2^2\right)}\phi\left(\varepsilon_i\sqrt{1+\lambda_1^2}/\omega_1\right)$$

Moreover

$$\frac{\sqrt{1+\lambda_1^2}}{\omega_1}=\frac{\sqrt{1+\frac{\theta_2^2}{\theta_1^2}\left(1+\theta_1^2+\theta_2^2\right)}}{s\sqrt{1+\theta_2^2}/\theta_1}=\frac{\sqrt{\theta_1^2+\theta_2^2+\theta_2^2\theta_1^2+\theta_2^2\theta_2^2}}{s\sqrt{1+\theta_2^2}}=\frac{\sqrt{\left(\theta_1^2+\theta_2^2\right)\left(1+\theta_2^2\right)}}{s\sqrt{1+\theta_2^2}}=\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}$$

So finally

$$\frac{\partial}{\partial\theta_1}G_{1i}=\frac{\varepsilon_i}{\theta_1}g_{1i}-\sqrt{\frac{2}{\pi}}\frac{1}{\lambda_2}\frac{\theta_1}{\left(\theta_1^2+\theta_2^2\right)}\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\varepsilon_i\right) \qquad [35]$$

We turn now to $\frac{\partial}{\partial\theta_1}G_{2i}$.

$$\frac{\partial}{\partial\theta_1}G_{2i}=\frac{\partial}{\partial\theta_1}\int_{-\infty}^{\varepsilon_i/\omega_2}2\phi(t)\Phi(\lambda_2 t)\,dt$$

$$=2\phi\left(\varepsilon_i/\omega_2\right)\Phi\left(\lambda_2\varepsilon_i/\omega_2\right)\frac{\partial\left(\varepsilon_i/\omega_2\right)}{\partial\theta_1}+\int_{-\infty}^{\varepsilon_i/\omega_2}2\phi(t)\frac{\partial\Phi\left(\lambda_2 t\right)}{\partial\theta_1}dt$$

Now

$$\frac{\partial\left(\varepsilon_i/\omega_2\right)}{\partial\theta_1}=\varepsilon_i\frac{\partial}{\partial\theta_1}\left(\frac{\theta_2}{s\sqrt{1+\theta_1^2}}\right)=\varepsilon_i\frac{\theta_2}{s}\frac{\partial}{\partial\theta_1}\left(1+\theta_1^2\right)^{-1/2}$$

$$=\varepsilon_i\frac{\theta_2}{s}\left(-\frac{1}{2}\frac{2\theta_1}{1+\theta_1^2}\frac{1}{\sqrt{1+\theta_1^2}}\right)=-\left(\varepsilon_i/\omega_2\right)\frac{\theta_1}{1+\theta_1^2}$$

and so

$$\frac{\partial}{\partial\theta_1}G_{2i}=-\frac{\theta_1}{1+\theta_1^2}\varepsilon_i g_{2i}+\int_{-\infty}^{\varepsilon_i/\omega_2}2\phi(t)\frac{\partial\Phi\left(\lambda_2 t\right)}{\partial\theta_1}dt \qquad [36]$$

Also, $\dfrac{\partial\Phi\left(\lambda_2 t\right)}{\partial\theta_1}=\phi\left(\lambda_2 t\right)\dfrac{\partial\left(\lambda_2 t\right)}{\partial\theta_1}=t\phi\left(\lambda_2 t\right)\dfrac{\partial\lambda_2}{\partial\theta_1}$ so

$$\frac{\partial}{\partial\theta_1}G_{2i}=-\frac{\theta_1}{1+\theta_1^2}\varepsilon_i g_{2i}+2\frac{\partial\lambda_2}{\partial\theta_1}\int_{-\infty}^{\varepsilon_i/\omega_2}t\phi\left(\lambda_2 t\right)\phi(t)\,dt$$

$$=-\frac{\theta_1}{1+\theta_1^2}\varepsilon_i g_{2i}+\frac{2}{\left(1+\lambda_2^2\right)\sqrt{2\pi}}\frac{\partial\lambda_2}{\partial\theta_1}\int_{-\infty}^{\varepsilon_i\sqrt{1+\lambda_2^2}/\omega_2}\left(t\sqrt{1+\lambda_2^2}\right)\phi\left(t\sqrt{1+\lambda_2^2}\right)d\left(t\sqrt{1+\lambda_2^2}\right)$$

$$=-\frac{\theta_1}{1+\theta_1^2}\varepsilon_i g_{2i}+\frac{2}{\left(1+\lambda_2^2\right)\sqrt{2\pi}}\frac{\partial\lambda_2}{\partial\theta_1}\int_{-\infty}^{\varepsilon_i\sqrt{1+\lambda_2^2}/\omega_2}\tilde{t}\phi\left(\tilde{t}\right)d\left(\tilde{t}\right)$$

$$=-\frac{\theta_1}{1+\theta_1^2}\varepsilon_i g_{2i}-\frac{2}{\left(1+\lambda_2^2\right)\sqrt{2\pi}}\frac{\partial\lambda_2}{\partial\theta_1}\phi\left(\varepsilon_i\sqrt{1+\lambda_2^2}/\omega_2\right)$$

We have

$$1+\lambda_2^2=1+\frac{\theta_1^2}{\theta_2^2}\left(1+\theta_1^2+\theta_2^2\right)=\frac{\theta_2^2+\theta_1^2\left(1+\theta_1^2+\theta_2^2\right)}{\theta_2^2}=\frac{\left(\theta_1^2+\theta_2^2\right)\left(1+\theta_1^2\right)}{\theta_2^2}=\left(\theta_1^2+\theta_2^2\right)\frac{\omega_2^2}{s^2}$$

Also,

$$\frac{\partial \lambda_2}{\partial \theta_1} = \frac{\partial\left(\dfrac{\theta_1}{\theta_2}\sqrt{1+\theta_1^2+\theta_2^2}\right)}{\partial \theta_1} = \frac{1}{\theta_2}\left\{\sqrt{1+\theta_1^2+\theta_2^2} + \theta_1\left(\frac{1}{2}\frac{2\theta_1\sqrt{1+\theta_1^2+\theta_2^2}}{1+\theta_1^2+\theta_2^2}\right)\right\}$$

$$=\left(\frac{1}{\theta_2}\sqrt{1+\theta_1^2+\theta_2^2} + \frac{1}{\theta_2}\frac{\theta_1^2}{\sqrt{1+\theta_1^2+\theta_2^2}}\right) = \left(\frac{\lambda_2}{\theta_1} + \frac{\theta_1}{\lambda_1}\right) = \frac{\lambda_1\lambda_2+\theta_1^2}{\theta_1\lambda_1}$$

and $\sqrt{1+\lambda_2^2}\,/\,\omega_2 = \dfrac{\sqrt{\theta_1^2+\theta_2^2}}{s}$. Putting it all together

$$\frac{\partial}{\partial \theta_1}G_{2i} = -\left[\frac{\theta_1}{1+\theta_1^2}\varepsilon_i g_{2i} + \sqrt{\frac{2}{\pi}}\frac{s^2}{\left(\theta_1^2+\theta_2^2\right)\omega_2^2}\frac{\lambda_1\lambda_2+\theta_1^2}{\theta_1\lambda_1}\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\varepsilon_i\right)\right]$$

So

$$\frac{\partial \tilde{L}_i}{\partial \theta_1} = \left(G_{1i}-G_{2i}\right)^{-1}\frac{\partial}{\partial \theta_1}\left(G_{1i}-G_{2i}\right)$$

$$= \left(G_{1i}-G_{2i}\right)^{-1}\left[\frac{\varepsilon_i}{\theta_1}g_{1i} - \sqrt{\frac{2}{\pi}}\frac{1}{\lambda_2}\frac{\theta_1}{\left(\theta_1^2+\theta_2^2\right)}\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\varepsilon_i\right)\right]$$

$$\left(G_{1i}-G_{2i}\right)^{-1}\left[\frac{\theta_1}{1+\theta_1^2}\varepsilon_i g_{2i} + \sqrt{\frac{2}{\pi}}\frac{s^2}{\left(\theta_1^2+\theta_2^2\right)\omega_2^2}\frac{\lambda_1\lambda_2+\theta_1^2}{\theta_1\lambda_1}\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\varepsilon_i\right)\right]$$

and re-arranging,

$$\frac{\partial \tilde{L}_i}{\partial \theta_1} = \frac{\varepsilon_i}{\theta_1}\psi_{1i} + \frac{\theta_1}{1+\theta_1^2}\varepsilon_i\psi_{2i} + \left(G_{1i}-G_{2i}\right)^{-1}\sqrt{\frac{2}{\pi}}\frac{1}{\left(\theta_1^2+\theta_2^2\right)}\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\varepsilon_i\right)\left[\frac{s^2}{\omega_2^2}\frac{\lambda_1\lambda_2+\theta_1^2}{\theta_1\lambda_1} - \frac{\theta_1}{\lambda_2}\right]$$

The term $\left(\frac{\lambda_1\lambda_2+\theta_1^2}{\theta_1\lambda_1}\right)\frac{s^2}{\omega_2^2}-\frac{\theta_1}{\lambda_2}$ can be simplified. We have

$$\left(\frac{\lambda_1\lambda_2+\theta_1^2}{\theta_1\lambda_1}\right)\frac{s^2}{\omega_2^2}-\frac{\theta_1}{\lambda_2}=\left(\frac{\lambda_1\lambda_2+\theta_1^2}{\theta_2\sqrt{1+\theta_1^2+\theta_2^2}}\right)\frac{\theta_2^2}{1+\theta_1^2}-\frac{\theta_2}{\sqrt{1+\theta_1^2+\theta_2^2}}$$

$$=\frac{\theta_2}{\sqrt{1+\theta_1^2+\theta_2^2}}\left(\frac{\lambda_1\lambda_2+\theta_1^2}{1+\theta_1^2}-1\right)=\frac{\theta_2}{\sqrt{1+\theta_1^2+\theta_2^2}}\left(\frac{1+\theta_1^2+\theta_2^2+\theta_1^2\;-1-\theta_1^2}{1+\theta_1^2}\right)$$

$$=\frac{\theta_2}{\sqrt{1+\theta_1^2+\theta_2^2}}\left(\frac{\theta_1^2+\theta_2^2}{1+\theta_1^2}\right)=\frac{\theta_1}{1+\theta_1^2}\frac{\theta_1^2+\theta_2^2}{\lambda_2}$$

So

$$\frac{\partial\tilde{L}_i}{\partial\theta_1}=\frac{\varepsilon_i}{\theta_1}\psi_{1i}+\frac{\theta_1}{1+\theta_1^2}\varepsilon_i\psi_{2i}+\left(G_{1i}-G_{2i}\right)^{-1}\sqrt{\frac{2}{\pi}}\frac{1}{\lambda_2}\frac{\theta_1}{1+\theta_1^2}\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\varepsilon_i\right)$$

$$\frac{\partial\tilde{L}_i}{\partial\theta_1}=\frac{\theta_1}{1+\theta_1^2}\left[\left(\frac{1+\theta_1^2}{\theta_1^2}\psi_{1i}+\psi_{2i}\right)\varepsilon_i+\left(G_{1i}-G_{2i}\right)^{-1}\sqrt{\frac{2}{\pi}}\frac{1}{\lambda_2}\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\varepsilon_i\right)\right]$$

or

$$\boxed{\frac{\partial\tilde{L}_i}{\partial\theta_1}=\frac{\theta_1}{\lambda_2\left(1+\theta_1^2\right)}\left[\lambda_2\left(\frac{1+\theta_1^2}{\theta_1^2}\psi_{1i}+\psi_{2i}\right)\varepsilon_i+\left(G_{1i}-G_{2i}\right)^{-1}\sqrt{\frac{2}{\pi}}\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\varepsilon_i\right)\right]}\qquad[37]$$

Summing over n we obtain the sample log-likelihood w.r.t. θ_1.

D.4. $\dfrac{\partial \tilde{L}}{\partial \theta_2}$ **.**

We follow the same steps as before, to calculate

$$\frac{\partial \tilde{L}_i}{\partial \theta_2} = \frac{\partial}{\partial \theta_2}\ln\left(G_{1i} - G_{2i}\right) = \left(G_{1i} - G_{2i}\right)^{-1}\frac{\partial}{\partial \theta_2}\left(G_{1i} - G_{2i}\right)$$

$$\frac{\partial}{\partial \theta_2}G_{1i} = \frac{\partial}{\partial \theta_2}\int_{-\infty}^{\varepsilon_i/\omega_1} 2\phi(t)\Phi(-\lambda_1 t)\,dt$$

$$= 2\phi(\varepsilon_i/\omega_1)\Phi(-\lambda_1\varepsilon_i/\omega_1)\frac{\partial(\varepsilon_i/\omega_1)}{\partial \theta_2} + \int_{-\infty}^{\varepsilon_i/\omega_1} 2\phi(t)\frac{\partial \Phi(-\lambda_1 t)}{\partial \theta_2}\,dt$$

Now,

$$\frac{\partial(\varepsilon_i/\omega_1)}{\partial \theta_2} = \varepsilon_i\frac{\theta_1}{s}\frac{\partial}{\partial \theta_2}\left(1+\theta_2^2\right)^{-1/2} = \varepsilon_i\frac{\theta_1}{s}\left(-\frac{1}{2}\frac{2\theta_2}{1+\theta_2^2}\frac{1}{\sqrt{1+\theta_2^2}}\right) = -\left(\varepsilon_i/\omega_1\right)\frac{\theta_2}{1+\theta_2^2}$$

and so

$$\frac{\partial}{\partial \theta_2}G_{1i} = -\frac{\theta_2}{1+\theta_2^2}\varepsilon_i g_{1i} + \int_{-\infty}^{\varepsilon_i/\omega_1} 2\phi(t)\frac{\partial \Phi(-\lambda_1 t)}{\partial \theta_2}\,dt \qquad [38]$$

Also,

$$\frac{\partial \Phi(-\lambda_1 t)}{\partial \theta_2} = \phi(\lambda_1 t)\frac{\partial(-\lambda_1 t)}{\partial \theta_2} = -t\phi(\lambda_1 t)\frac{\partial\left(\frac{\theta_2}{\theta_1}\sqrt{1+\theta_1^2+\theta_2^2}\right)}{\partial \theta_2}$$

$$= -t\phi(\lambda_1 t)\frac{1}{\theta_1}\left\{\sqrt{1+\theta_1^2+\theta_2^2} + \frac{\theta_2^2}{\sqrt{1+\theta_1^2+\theta_2^2}}\right\}$$

$$= -t\phi(\lambda_1 t)\left(\frac{\lambda_1}{\theta_2} + \frac{\theta_2}{\lambda_2}\right)$$

Inserting into the integral of [38] we have

$$\int_{-\infty}^{\varepsilon_i/\omega_1} 2\phi(t)\frac{\partial\Phi(-\lambda_1 t)}{\partial\theta_2}dt = -\int_{-\infty}^{\varepsilon_i/\omega_1} 2\phi(t)t\phi(\lambda_1 t)\left(\frac{\lambda_1}{\theta_2}+\frac{\theta_2}{\lambda_2}\right)dt$$

$$=-2\left(\frac{\lambda_1}{\theta_2}+\frac{\theta_2}{\lambda_2}\right)\int_{-\infty}^{\varepsilon_i/\omega_1} t\phi(t)\phi(\lambda_1 t)dt = -\sqrt{\frac{2}{\pi}}\left(\frac{\lambda_1}{\theta_2}+\frac{\theta_2}{\lambda_2}\right)\int_{-\infty}^{\varepsilon_i/\omega_1} t\phi\left(\sqrt{1+\lambda_1^2}\,t\right)dt$$

We have previously calculated the value of $\sqrt{1+\lambda_1^2}=\frac{\omega_1}{s}\sqrt{\theta_1^2+\theta_2^2}$. Manipulating the integral we have

$$\int_{-\infty}^{\varepsilon_i/\omega_1} 2\phi(t)\frac{\partial\Phi(-\lambda_1 t)}{\partial\theta_2}dt = -\sqrt{\frac{2}{\pi}}\left(\frac{\lambda_1}{\theta_2}+\frac{\theta_2}{\lambda_2}\right)\frac{1}{\omega_1^2}\int_{-\infty}^{\varepsilon_i} t\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}t\right)dt$$

$$=-\sqrt{\frac{2}{\pi}}\left(\frac{\lambda_1}{\theta_2}+\frac{\theta_2}{\lambda_2}\right)\frac{s}{\omega_1^2}\frac{1}{\sqrt{\theta_1^2+\theta_2^2}}\int_{-\infty}^{\varepsilon_i} t\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}t\right)dt$$

$$=-\sqrt{\frac{2}{\pi}}\left(\frac{\lambda_1}{\theta_2}+\frac{\theta_2}{\lambda_2}\right)\frac{1}{\omega_1^2}\frac{s^2}{\theta_1^2+\theta_2^2}\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\varepsilon_i\right)$$

and inserting back into [38] we get

$$\frac{\partial}{\partial\theta_2}G_{1i}=-\frac{\theta_2}{1+\theta_2^2}\varepsilon_i g_{1i}+\sqrt{\frac{2}{\pi}}\left(\frac{\lambda_1}{\theta_2}+\frac{\theta_2}{\lambda_2}\right)\frac{1}{\omega_1^2}\frac{s^2}{\theta_1^2+\theta_2^2}\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\varepsilon_i\right) \qquad [39]$$

We turn to $\frac{\partial}{\partial\theta_2}G_{2i}$.

$$\frac{\partial}{\partial\theta_2}G_{2i} = \frac{\partial}{\partial\theta_2}\int_{-\infty}^{\varepsilon_i/\omega_2} 2\phi(t)\Phi(\lambda_2 t)\,dt$$

$$= 2\phi(\varepsilon_i/\omega_2)\Phi(\lambda_2\varepsilon_i/\omega_2)\frac{\partial(\varepsilon_i/\omega_2)}{\partial\theta_2} + \int_{-\infty}^{\varepsilon_i/\omega_2} 2\phi(t)\frac{\partial\Phi(\lambda_2 t)}{\partial\theta_2}dt$$

Now,

$$\frac{\partial(\varepsilon_i/\omega_2)}{\partial\theta_2} = \varepsilon_i\frac{\partial}{\partial\theta_2}\left(\frac{\theta_2}{s\sqrt{1+\theta_1^2}}\right) = \varepsilon_i\frac{1}{s\sqrt{1+\theta_1^2}} = (\varepsilon_i/\omega_2)\frac{1}{\theta_2}$$

and so $$\frac{\partial}{\partial\theta_2}G_{2i} = \frac{1}{\theta_2}\varepsilon_i g_{2i} + \int_{-\infty}^{\varepsilon_i/\omega_2} 2\phi(t)\frac{\partial\Phi(\lambda_2 t)}{\partial\theta_2}dt \qquad [40]$$

Also, $$\frac{\partial\Phi(\lambda_2 t)}{\partial\theta_2} = \phi(\lambda_2 t)\frac{\partial(\lambda_2 t)}{\partial\theta_2} = t\phi(\lambda_2 t)\frac{\partial\left(\frac{\theta_1}{\theta_2}\sqrt{1+\theta_1^2+\theta_2^2}\right)}{\partial\theta_2}$$

$$= t\phi(\lambda_2 t)\theta_1\left\{-\frac{\sqrt{1+\theta_1^2+\theta_2^2}}{\theta_2^2} + \frac{1}{\theta_2}\left(\frac{1}{2}\frac{2\theta_2\sqrt{1+\theta_1^2+\theta_2^2}}{1+\theta_1^2+\theta_2^2}\right)\right\} = t\phi(\lambda_2 t)\left(-\frac{\lambda_2}{\theta_2} + \frac{\theta_2}{\lambda_1}\right)$$

Inserting into the integral of [40] we have

$$\int_{-\infty}^{\varepsilon_i/\omega_2} 2\phi(t)t\phi(\lambda_2 t)\left(-\frac{\lambda_2}{\theta_2} + \frac{\theta_2}{\lambda_1}\right)dt = \sqrt{\frac{2}{\pi}}\left(-\frac{\lambda_2}{\theta_2} + \frac{\theta_2}{\lambda_1}\right)\int_{-\infty}^{\varepsilon_i/\omega_2} t\phi\left(\sqrt{1+\lambda_2}\,t\right)dt$$

Using the previous result $\sqrt{1+\lambda_2} = \frac{\omega_2\sqrt{\theta_1^2+\theta_2^2}}{s}$ and eliminating the scaling of ε_i we have

$$\sqrt{\frac{2}{\pi}}\left(-\frac{\lambda_2}{\theta_2}+\frac{\theta_2}{\lambda_1}\right)\int_{-\infty}^{\varepsilon_i/\omega_2} t\phi\left(\sqrt{1+\lambda_2}\,t\right)dt$$

$$=\sqrt{\frac{2}{\pi}}\left(-\frac{\lambda_2}{\theta_2}+\frac{\theta_2}{\lambda_1}\right)\frac{s}{\sqrt{\theta_1^2+\theta_2^2}}\frac{1}{\omega_2^2}\int_{-\infty}^{\varepsilon_i} t\,\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\,\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\,t\right)dt$$

$$=-\sqrt{\frac{2}{\pi}}\left(-\frac{\lambda_2}{\theta_2}+\frac{\theta_2}{\lambda_1}\right)\frac{1}{\omega_2^2}\frac{s^2}{\theta_1^2+\theta_2^2}\,\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\,\varepsilon_i\right)$$

Inserting this into [40] we have

$$\frac{\partial}{\partial\theta_2}G_{2i}=\frac{1}{\theta_2}\varepsilon_i g_{2i}\;-\sqrt{\frac{2}{\pi}}\left(-\frac{\lambda_2}{\theta_2}+\frac{\theta_2}{\lambda_1}\right)\frac{1}{\omega_2^2}\frac{s^2}{\theta_1^2+\theta_2^2}\,\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\,\varepsilon_i\right) \qquad [41]$$

Bringing together the two derivatives [39] and [41] we have

$$\frac{\partial}{\partial\theta_2}G_{1i}-\frac{\partial}{\partial\theta_2}G_{2i}=-\frac{\theta_2}{1+\theta_2^2}\varepsilon_i g_{1i}\;+\sqrt{\frac{2}{\pi}}\left(\frac{\lambda_1}{\theta_2}+\frac{\theta_2}{\lambda_2}\right)\frac{1}{\omega_1^2}\frac{s^2}{\theta_1^2+\theta_2^2}\,\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}\,\varepsilon_i}{s}\right)$$

$$-\frac{1}{\theta_2}\varepsilon_i g_{2i}\;+\sqrt{\frac{2}{\pi}}\left(-\frac{\lambda_2}{\theta_2}+\frac{\theta_2}{\lambda_1}\right)\frac{1}{\omega_2^2}\frac{s^2}{\theta_1^2+\theta_2^2}\,\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}\,\varepsilon_i}{s}\right)$$

$$=-\left(\frac{\theta_2}{1+\theta_2^2}g_{1i}+\frac{1}{\theta_2}g_{2i}\right)\varepsilon_i$$

$$+\sqrt{\frac{2}{\pi}}\frac{s^2}{\theta_1^2+\theta_2^2}\,\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\,\varepsilon_i\right)\left\{\left(-\frac{\lambda_2}{\theta_2}+\frac{\theta_2}{\lambda_1}\right)\frac{1}{\omega_2^2}+\left(\frac{\lambda_1}{\theta_2}+\frac{\theta_2}{\lambda_2}\right)\frac{1}{\omega_1^2}\right\}$$

We can simplify the coefficient expression,

$$\left(-\frac{\lambda_2}{\theta_2}+\frac{\theta_2}{\lambda_1}\right)\frac{1}{\omega_2^2}=\frac{-\lambda_1\lambda_2+\theta_2^2}{\theta_2\lambda_1}\frac{1}{\omega_2^2}=-\frac{\left(1+\theta_1^2\right)\theta_1}{\theta_2^2\sqrt{1+\theta_1^2+\theta_2^2}}\frac{\theta_2^2}{s^2\left(1+\theta_1^2\right)}=-\frac{\theta_1}{s^2\sqrt{1+\theta_1^2+\theta_2^2}}$$

and

$$\left(\frac{\lambda_1}{\theta_2}+\frac{\theta_2}{\lambda_2}\right)\frac{1}{\omega_1^2}=\frac{\lambda_1\lambda_2+\theta_2^2}{\theta_2\lambda_2}\frac{1}{\omega_1^2}=\frac{1+\theta_1^2+2\theta_2^2}{\theta_1\sqrt{1+\theta_1^2+\theta_2^2}}\frac{\theta_1^2}{s^2\left(1+\theta_2^2\right)}=\frac{1+\theta_1^2+2\theta_2^2}{\left(1+\theta_2^2\right)}\frac{\theta_1}{s^2\sqrt{1+\theta_1^2+\theta_2^2}}$$

So

$$\left\{\left(-\frac{\lambda_2}{\theta_2}+\frac{\theta_2}{\lambda_1}\right)\frac{1}{\omega_2^2}+\left(\frac{\lambda_1}{\theta_2}+\frac{\theta_2}{\lambda_2}\right)\frac{1}{\omega_1^2}\right\}=-\frac{\theta_1}{s^2\sqrt{1+\theta_1^2+\theta_2^2}}+\frac{1+\theta_1^2+2\theta_2^2}{\left(1+\theta_2^2\right)}\frac{\theta_1}{s^2\sqrt{1+\theta_1^2+\theta_2^2}}$$

$$=\frac{\theta_1}{s^2\sqrt{1+\theta_1^2+\theta_2^2}}\left(\frac{1+\theta_1^2+2\theta_2^2}{1+\theta_2^2}-1\right)=\frac{\left(\theta_1^2+\theta_2^2\right)}{s^2}\frac{\theta_2}{\left(1+\theta_2^2\right)\lambda_1}$$

Then $\displaystyle \frac{\partial}{\partial\theta_2}\left(G_{1i}-G_{2i}\right)=-\left(\frac{\theta_2}{1+\theta_2^2}g_{1i}+\frac{1}{\theta_2}g_{2i}\right)\varepsilon_i+\sqrt{\frac{2}{\pi}}\frac{\theta_2}{\left(1+\theta_2^2\right)\lambda_1}\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\varepsilon_i\right)$

and $\displaystyle \frac{\partial\tilde{L}_i}{\partial\theta_2}=\frac{\partial}{\partial\theta_2}\ln\left(G_{1i}-G_{2i}\right)=\left(G_{1i}-G_{2i}\right)^{-1}\frac{\partial}{\partial\theta_2}\left(G_{1i}-G_{2i}\right)$

$$\frac{\partial\tilde{L}_i}{\partial\theta_2}=\frac{\theta_2}{1+\theta_2^2}\left\{\sqrt{\frac{2}{\pi}}\frac{1}{\lambda_1}\phi\left(\sqrt{\theta_1^2+\theta_2^2}\frac{\varepsilon_i}{s}\right)\left(G_{1i}-G_{2i}\right)^{-1}-\left(\psi_{1i}+\frac{1+\theta_2^2}{\theta_2^2}\psi_{2i}\right)\varepsilon_i\right\}$$

or

$$\boxed{\frac{\partial\tilde{L}_i}{\partial\theta_2}=\frac{\theta_2}{\lambda_1\left(1+\theta_2^2\right)}\left\{\sqrt{\frac{2}{\pi}}\phi\left(\sqrt{\theta_1^2+\theta_2^2}\frac{\varepsilon_i}{s}\right)\left(G_{1i}-G_{2i}\right)^{-1}-\lambda_1\left(\psi_{1i}+\frac{1+\theta_2^2}{\theta_2^2}\psi_{2i}\right)\varepsilon_i\right\}}\qquad[42]$$

Summing over n we obtain the sample log-likelihood w.r.t. θ_2.

E. Concavity of the log-likelihood.

See Bergstrom & Bagnoli (2005) and also Brascamp & Lieb (1975), for the various log-concavity results we draw upon.

E1. Concavity w.r.t. the variable.

The log-likelihood for the typical observation (suppressing the subscript *i*) is

$$\tilde{L}\left(\varepsilon \middle| y,\mathbf{x},\mathbf{q}\right)=\ln\left(\frac{2}{\sqrt{2\pi}}\right)-\ln s-\frac{1}{2}\left(\varepsilon/s\right)^{2}+\ln\left\{G_{1}\left(\varepsilon;\,0,\,\omega_{1},\,-\lambda_{1}\right)-\,G_{2}\left(\varepsilon;\,0,\omega_{2},\,\lambda_{2}\right)\right\} \qquad [43]$$

and as a whole it is concave in the variable ε as mentioned in the main text. The first three terms are just the components of a usual Normal specification (with different constant). But we need to prove that the last term is separately concave in the variable, in order to subsequently show that concavity in the parameters also holds. We have already shown that the distribution function of a Skew-normal random variable can be expressed in term of a bivariate standard Normal integral,

$$G\left(\varepsilon,\,\xi,\,\sigma,\lambda\right)=2\Phi_{2}\left(\frac{\varepsilon-\xi}{\sigma},\,0;\;\rho=\frac{-\lambda}{\sqrt{1+\lambda^{2}}}\right)$$

The order of integration in the bivariate double integral can be reversed. Namely we can write,

$$\Phi_{2}\left(\frac{\varepsilon}{\sigma},\,0;\;\rho\right)=\int_{-\infty}^{0}\int_{-\infty}^{\varepsilon/\sigma}\frac{1}{2\pi\sqrt{1-\rho^{2}}}\exp\left\{-\frac{1}{2\left(1-\rho^{2}\right)}\left(v^{2}+w^{2}-2\rho vw\right)\right\}dvdw$$

where we have also set $\xi=0$ since this is our case.

Considering the inner integral here we have, again after swapping the limits and multiplying by minus one,

$$I_{IN} = \int_{-\varepsilon/\sigma}^{\infty} \exp\left\{-\frac{v^2}{2\left(1-\rho^2\right)} - \frac{\rho w}{1-\rho^2} v\right\} dv$$

This integral has also been solved previously. Performing the relevant steps, we arrive at

$$\Phi_2\left(\frac{\varepsilon}{\sigma}, 0;\ \rho\right) = \int_{-\infty}^{0} \phi(w)\Phi\left(\frac{\varepsilon}{\sigma\left(\sqrt{1-\rho^2}\right)} - \frac{\rho w}{\left(\sqrt{1-\rho^2}\right)}\right) dw \qquad [44]$$

We match this general expression to our case. For the Skew-normal distribution function $G_1(\varepsilon;\ 0,\ \omega_1,\ -\lambda_1)$ we have

$\dfrac{-\rho}{\left(\sqrt{1-\rho^2}\right)} = -\lambda_1$. Note also that $\dfrac{1}{\sqrt{1-\rho^2}} = \sqrt{1+\lambda_1^2}$. So

$$\frac{1}{\sigma\left(\sqrt{1-\rho^2}\right)} = \frac{\sqrt{1+\lambda_1^2}}{\omega_1} = \frac{\sqrt{1+\dfrac{\sigma_2^2 s^2}{\sigma_1^2\sigma_u^2}}}{\omega_1} = \frac{\sqrt{\sigma_1^2\sigma_u^2 + \sigma_2^2\left(\sigma_u^2+\sigma_1^2+\sigma_2^2\right)}}{\sigma_1\sigma_u\omega_1} = \frac{s_2\sqrt{\sigma_1^2+\sigma_2^2}}{\sigma_1\sigma_u\omega_1}$$

$$= \frac{s_2\sqrt{\sigma_1^2+\sigma_2^2}}{\sigma_1\sigma_u\dfrac{s\sqrt{1+\theta_2^2}}{\theta_1}} = \frac{s_2\sqrt{\sigma_1^2+\sigma_2^2}}{\sigma_u s\sqrt{\sigma_u^2+\sigma_2^2}} = \frac{s_2\sqrt{\sigma_1^2+\sigma_2^2}}{\sigma_u s s_2} = \frac{\sqrt{\theta_1^2+\theta_2^2}}{s}$$

So $G_1(\varepsilon;\ 0,\ \omega_1,\ -\lambda_1) = 2\int_{-\infty}^{0} \phi(w)\Phi\left(\dfrac{\sqrt{\theta_1^2+\theta_2^2}}{s}\varepsilon - \lambda_1 w\right) dw$

Performing the same steps for the second Skew-normal distribution function $G_2(\varepsilon;\ 0, \omega_2,\ \lambda_2)$ we have $\dfrac{-\rho}{\left(\sqrt{1-\rho^2}\right)} = \lambda_2$ and $\dfrac{1}{\sqrt{1-\rho^2}} = \sqrt{1+\lambda_2^2}$. So here

$$\frac{1}{\sigma\left(\sqrt{1-\rho^2}\right)} = \frac{\sqrt{1+\lambda_2^2}}{\omega_2} = \frac{\sqrt{1+\dfrac{\sigma_1^2 s^2}{\sigma_2^2\sigma_u^2}}}{\omega_1} = \frac{\sqrt{\sigma_2^2\sigma_u^2+\sigma_1^2\left(\sigma_u^2+\sigma_1^2+\sigma_2^2\right)}}{\sigma_2\sigma_u\omega_1} = \frac{s_1\sqrt{\sigma_1^2+\sigma_2^2}}{\sigma_2\sigma_u\omega_1}$$

$$= \frac{s_1\sqrt{\sigma_1^2+\sigma_2^2}}{\sigma_2\sigma_u\dfrac{s\sqrt{1+\theta_1^2}}{\theta_2}} = \frac{s_1\sqrt{\sigma_1^2+\sigma_2^2}}{\sigma_u s\sqrt{\sigma_u^2+\sigma_1^2}} = \frac{s_1\sqrt{\sigma_1^2+\sigma_2^2}}{\sigma_u s s_1} = \frac{\sqrt{\theta_1^2+\theta_2^2}}{s}$$

i.e. we end up with the same expression as before. So

$$G_2\left(\varepsilon;\,0,\,\omega_2,\,\lambda_2\right) = 2\int_{-\infty}^{0}\phi(w)\Phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\varepsilon+\lambda_2 w\right)dw$$

Combining we obtain

$$G_1\left(\varepsilon;\,0,\,\omega_1,\,-\lambda_1\right) - G_2\left(\varepsilon;\,0,\omega_2,\,\lambda_2\right) = \qquad [45]$$

$$= 2\int_{-\infty}^{0}\phi(w)\left\{\Phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\varepsilon-\lambda_1 w\right) - \Phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\varepsilon+\lambda_2 w\right)\right\}dw$$

Define for clarity

$$h_1\left(\varepsilon,w\right) \equiv \frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\varepsilon-\lambda_1 w \quad , \quad h_2\left(\varepsilon,w\right) \equiv \frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\varepsilon+\lambda_2 w$$

and note that since the variable w varies in $\left(-\infty\,,\,0\right]$, we have $h_1\left(\varepsilon,w\right) > h_2\left(\varepsilon,w\right)\ \forall\left(\varepsilon,w\right)$ in the domain $\mathbb{R}\times\mathbb{R}_+$, which is a convex set and so concavity can be defined. We are now looking at

$$G_1\left(\varepsilon;\,0,\,\omega_1,\,-\lambda_1\right) - G_2\left(\varepsilon;\,0,\omega_2,\,\lambda_2\right) = 2\int_{-\infty}^{0}\phi(w)\left\{\Phi\left(h_1\left(\varepsilon,w\right)\right)-\Phi\left(h_2\left(\varepsilon,w\right)\right)\right\}dw \qquad [46]$$

Pratt (1981) has proven that for a univariate continuous distribution function $F(z)$ that has a log-concave density, the bivariate function

$$H(z_1,\, z_2) = F(z_1) - F(z_2) \quad , \quad z_1 > z_2 \text{ is log-concave in } (z_1,\, z_2).$$

In our case the distribution function is the standard Normal which has a log-concave density and $h_1(\varepsilon, w) \; > \; h_2(\varepsilon, w) \;\; \forall\, (\varepsilon, w)$. So the bivariate function

$$H\big(h_1(\varepsilon,w),\, h_2(\varepsilon,w)\big) = \Phi\big(h_1(\varepsilon,w)\big) - \Phi\big(h_2(\varepsilon,w)\big)$$

is log-concave in $\big(h_1(\varepsilon,w),\, h_2(\varepsilon,w)\big)$. Moreover, since this holds for arbitrarily chosen ε and w, and since both $h_1(\varepsilon,w)$ and $h_2(\varepsilon,w)$ are linear functions of (ε,w), we obtain that

$$F(\varepsilon,\, w) = \Phi\big(h_1(\varepsilon,w)\big) - \Phi\big(h_2(\varepsilon,w)\big) \text{ is log-concave in } (\varepsilon, w).$$

Re-writing eq.[46], we then have

$$G_1(\varepsilon;\, 0,\, \omega_1,\, -\lambda_1) \; - \; G_2(\varepsilon;\, 0, \omega_2,\, \lambda_2) \; = \; 2\int_{-\infty}^{0} \phi(w) F(\varepsilon,\, w)\, dw \qquad [47]$$

Now, the product of two log-concave functions is a log-concave function. Furthermore, if a function is log-concave in two variables, then integrating out one of the two produces a function that is log-concave in the remaining variable. So

$G_1(\varepsilon;\, 0,\, \omega_1,\, -\lambda_1) \; - \; G_2(\varepsilon;\, 0, \omega_2,\, \lambda_2)$ is log-concave in ε. QED.

E2. Concavity w.r.t. the parameters.

To transform the log-likelihood [43] into a function in the coefficients of interest we proceed as follows:

We define the $(1+K)\times 1$ vectors $\mathbf{z} = \begin{bmatrix} y \\ -\mathbf{x} \end{bmatrix}$ and $\tilde{\boldsymbol{\beta}} = \left[1 \;\; \boldsymbol{\beta}'\right]'$ and so we can write

$\varepsilon = \tilde{\boldsymbol{\beta}}'\mathbf{z}$. Also, defining $\gamma_0 = \dfrac{1}{s}$, $\boldsymbol{\delta}_0 = \gamma_0 \boldsymbol{\beta}$, $\mathbf{a}_0 = \left[\gamma_0 \;\; \boldsymbol{\delta}_0\right]' = \gamma_0 \tilde{\boldsymbol{\beta}}$, we have

$(\varepsilon/s) = \gamma_0 \tilde{\boldsymbol{\beta}}'\mathbf{z} = \mathbf{a}_0'\mathbf{z}$.

We turn now to the last component of the log-likelihood,

$$\tilde{H}(\varepsilon) \equiv \ln\left\{G_1(\varepsilon;\, 0,\, \omega_1,\, -\lambda_1) \;-\; G_2(\varepsilon;\, 0, \omega_2,\, \lambda_2)\right\}.$$

The variable ε enters the RHS in four positions,

$$\ln\left\{G_1\left(\varepsilon/\omega_1, (-\lambda_1/\omega_1)\varepsilon\right) - G_2\left(\varepsilon/\omega_2, (\lambda_2/\omega_2)\varepsilon\right)\right\}.$$

If we define $\gamma_1 = \dfrac{1}{\omega_1}$, $\gamma_2 = \dfrac{1}{\omega_2}$ and $\Gamma = \left[\gamma_1 \; -\lambda_1\gamma_1 \;\; \gamma_2 \;\; \lambda_2\gamma_2\right]$

then we have $\tilde{H}(\varepsilon) = N(\varepsilon\Gamma)$, and since we have previously proven that $\tilde{H}$ is concave in ε, N is concave in its argument since it represents a composition of $\tilde{H}$ with an affine mapping of ε. But then N is also concave in Γ since $\varepsilon\Gamma$ can equivalently be seen as an affine mapping of Γ. So we have $\tilde{H}(\varepsilon) = N(\varepsilon\Gamma) = P(\Gamma)$ and P is concave in its argument.

Defining $\Lambda = \left[[1 \;\; 0] \; [-\lambda_1 \;\; 0] \;\; [0 \;\; 1] \; [0 \;\; \lambda_2]\right]$ and , $\boldsymbol{\gamma}_{12} = \left[\gamma_1 \; \gamma_2\right]'$, Γ can be written as $\Gamma = \Lambda \otimes \boldsymbol{\gamma}_{12}$ where $\otimes$ denotes the Kronecker product. So $P(\Lambda \otimes \boldsymbol{\gamma}_{12})$ is concave in its argument. It is easy to verify that linear combinations of $\boldsymbol{\gamma}_{12}$ produce at most a subset of linear combinations of Γ, and so concavity holds also w.r.t. $\boldsymbol{\gamma}_{12}$, i.e. $\tilde{H}(\varepsilon) = N(\varepsilon\Gamma) = P(\Gamma) = P(\Lambda \otimes \boldsymbol{\gamma}_{12}) = F(\boldsymbol{\gamma}_{12})$ and F is concave in $\boldsymbol{\gamma}_{12}$.

Define now the vector $\boldsymbol{\xi} = [\gamma_0, \gamma_1, \gamma_2, \boldsymbol{\delta}_0]'$ which is a one-to-one reparametrization of the coefficients of interest $\mathbf{q} = [s, \theta_1, \theta_2, \boldsymbol{\beta}]'$, and note that collecting all the above results, we can write a transformed log-likelihood in $\boldsymbol{\xi}$ as

$$\bar{\tilde{L}}(\boldsymbol{\xi}) = \ln\left(\frac{2}{\sqrt{2\pi}}\right) + \ln\gamma_0 - \frac{1}{2}(\mathbf{a}_0'\mathbf{z})^2 + F(\boldsymbol{\gamma}_{12})$$

Now $\gamma_0, \mathbf{a}_0, \boldsymbol{\gamma}_{12}$ can all be derived as linear transformations of $\boldsymbol{\xi}$ multiplied by an appropriate for each case vector or matrix of constants. Specifically, setting

$$\boldsymbol{\tau}_1 = [1\ 0\ 0\ 0]', \quad \boldsymbol{\tau}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}', \quad \boldsymbol{\tau}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}' \text{ we have}$$

$$\bar{\tilde{L}}(\boldsymbol{\xi}) = \frac{1}{2}\ln\left(\frac{2}{\pi}\right) + \ln(\boldsymbol{\tau}_1'\boldsymbol{\xi}) - \frac{1}{2}(\boldsymbol{\xi}'\boldsymbol{\tau}_2\mathbf{z})^2 + F(\boldsymbol{\tau}_3'\boldsymbol{\xi})$$

Note that all elements of $\boldsymbol{\xi}$ appear in $\bar{\tilde{L}}(\boldsymbol{\xi})$.

Now, the natural logarithm is concave in its argument, the function $\left(-\frac{1}{2}(\cdot)^2\right)$ is concave in its argument, and so is F. Since all these arguments are affine transformations of $\boldsymbol{\xi}$, all three components are concave also in $\boldsymbol{\xi}$, and so their sum which is the transformed log-likelihood (plus a constant), is concave in $\boldsymbol{\xi}$. With concavity established and the other regularity conditions, the ML estimator associated with $\bar{\tilde{L}}(\boldsymbol{\xi})$ is consistent for $\boldsymbol{\xi}$. Since $\boldsymbol{\xi}$ is a one-to-one reparametrization of $\mathbf{q} = [s, \theta_1, \theta_2, \boldsymbol{\beta}]'$ then, the ML estimator associated with $\tilde{L}(\mathbf{q})$ will be consistent for the vector $\mathbf{q}$, given the MLE's invariance-under-reparametrization property. The same property establishes consistency of the ML estimator of the original parameters $(\sigma_u, \sigma_1, \sigma_2, \boldsymbol{\beta})$.

F. Bias of the OLS Estimator and orthogonality conditions.

F.1. Bias of the OLS Estimator.

The OLS estimator for the beta coefficients is

$$\hat{\boldsymbol{\beta}}_{OLS} = \left(n^{-1}\mathbf{X}'\mathbf{X}\right)^{-1}\left(n^{-1}\mathbf{X}'\mathbf{y}\right) = \boldsymbol{\beta} \; + \; \left(n^{-1}\mathbf{X}'\mathbf{X}\right)^{-1}\left(n^{-1}\mathbf{X}'\boldsymbol{\varepsilon}\right)$$

$$\Rightarrow E\left(\hat{\boldsymbol{\beta}}_{OLS}\right) - \boldsymbol{\beta} = \left[E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right]^{-1} E\left(\mathbf{x}_i\varepsilon_i\right)$$

The error term is assumed mean-independent from the regressors, and so uncorrelated,

$E\left(\mathbf{x}_i\varepsilon_i\right) = E\left(\mathbf{x}_i\right)E\left(\varepsilon_i\right)$. Therefore $E\left(\hat{\boldsymbol{\beta}}_{OLS}\right) - \boldsymbol{\beta} = \left[E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right]^{-1} E\left(\mathbf{x}_i\right)E\left(\varepsilon_i\right)$.

The sample is assumed i.i.d. and so ergodic-stationary, implying that sample averages estimate consistently the corresponding theoretical expected values. So $\left(n^{-1}\mathbf{X}'\mathbf{X}\right)^{-1}\left(n^{-1}\mathbf{X}'\cdot\mathbf{1}\right)$ estimates consistently $\left[E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right]^{-1} E\left(\mathbf{x}_i\right)$.

Let X be the matrix of regressors without the constant term, and so $\mathbf{X} = \left[\mathbf{1} \quad X\right]$. Then

$$n^{-1}\mathbf{X}'\mathbf{X} = n^{-1}\begin{bmatrix}\mathbf{1}' \\ X'\end{bmatrix}\left[\mathbf{1} \quad X\right] = n^{-1}\begin{bmatrix} n & \mathbf{1}'\cdot X \\ X'\cdot\mathbf{1} & X'X\end{bmatrix} = \begin{bmatrix} 1 & \bar{S}_X' \\ \bar{S}_X & n^{-1}X'X\end{bmatrix}$$

$$n^{-1}\mathbf{X}'\cdot\mathbf{1} = \begin{bmatrix} 1 \\ \bar{S}_X\end{bmatrix}$$

where $\bar{S}_X$ denotes the column vector holding the sample means of the non-constant regressors. Applying block-matrix inversion using the Schur complement of unity in this matrix, $M^{-1} = \left(n^{-1}X'X - \bar{S}_X\bar{S}_X'\right)^{-1}$, we have

$$\left(n^{-1}\mathbf{X}'\mathbf{X}\right)^{-1}=\begin{bmatrix}1 & \bar{S}_X' \\ \bar{S}_X & n^{-1}X'X\end{bmatrix}^{-1}=\begin{bmatrix}1+\bar{S}_X' M^{-1}\bar{S}_X & -\bar{S}_X' M^{-1} \\ -M^{-1}\bar{S}_X & M^{-1}\end{bmatrix}$$

Therefore

$$\left(n^{-1}\mathbf{X}'\mathbf{X}\right)^{-1}\left(n^{-1}\mathbf{X}'\cdot\mathbf{1}\right)=\begin{bmatrix}1+\bar{S}_X' M^{-1}\bar{S}_X & -\bar{S}_X' M^{-1} \\ -M^{-1}\bar{S}_X & M^{-1}\end{bmatrix}\begin{bmatrix}1 \\ \bar{S}_X\end{bmatrix}=\begin{bmatrix}1+\bar{S}_X' M^{-1}\bar{S}_X-\bar{S}_X' M^{-1}\bar{S}_X \\ -M^{-1}\bar{S}_X+M^{-1}\bar{S}_X\end{bmatrix}$$

$$\Rightarrow\left(n^{-1}\mathbf{X}'\mathbf{X}\right)^{-1}\left(n^{-1}\mathbf{X}'\cdot\mathbf{1}\right)=\begin{bmatrix}1 \\ \mathbf{0}\end{bmatrix}$$

It follows that $E\left(\hat{\boldsymbol{\beta}}_{OLS}\right)-\boldsymbol{\beta}=\begin{bmatrix}E\left(\varepsilon_i\right) \\ \mathbf{0}\end{bmatrix}$ and so only the OLS estimator of the constant term is biased and inconsistent.

F.2. An orthogonality condition from $\dfrac{\partial\tilde{L}_i}{\partial\theta_2},\dfrac{\partial\tilde{L}_i}{\partial\theta_2}$

We have obtained previously

$$\frac{\partial\tilde{L}}{\partial\theta_1}=\frac{\theta_1}{\lambda_2\left(1+\theta_1^2\right)}\sum_{i=1}^{n}\left\{\sqrt{\frac{2}{\pi}}\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\varepsilon_i\right)\left(G_{1i}-G_{2i}\right)^{-1}+\lambda_2\left(\frac{1+\theta_1^2}{\theta_1^2}\psi_{1i}+\psi_{2i}\right)\varepsilon_i\right\} \qquad [48]$$

$$\frac{\partial\tilde{L}}{\partial\theta_2}=\frac{\theta_2}{\lambda_1\left(1+\theta_2^2\right)}\sum_{i=1}^{n}\left\{\sqrt{\frac{2}{\pi}}\phi\left(\frac{\sqrt{\theta_1^2+\theta_2^2}}{s}\varepsilon_i\right)\left(G_{1i}-G_{2i}\right)^{-1}-\lambda_1\left(\psi_{1i}+\frac{1+\theta_2^2}{\theta_2^2}\psi_{2i}\right)\varepsilon_i\right\} \qquad [49]$$

The MLE will set these equal to zero, so we can ignore the constant factor outside the sums. Also note that the first components in each sum are identical. So we arrive at

$$\frac{\partial\tilde{L}}{\partial\theta_1}=\frac{\partial\tilde{L}}{\partial\theta_2}=0\Rightarrow\sum_{i=1}^{n}\left(\frac{\lambda_2\left(1+\theta_1^2\right)}{\theta_1^2}\psi_{1i}+\lambda_2\psi_{2i}\right)\varepsilon_i=-\sum_{i=1}^{n}\left(\lambda_1\psi_{1i}+\frac{\lambda_1\left(1+\theta_2^2\right)}{\theta_2^2}\psi_{2i}\right)\varepsilon_i$$

Re-arranging we get

$$\sum_{i=1}^{n}\left(\left[\frac{\lambda_2\left(1+\theta_1^2\right)}{\theta_1^2}+\lambda_1\right]\psi_{1i}+\left[\lambda_2+\frac{\lambda_1\left(1+\theta_2^2\right)}{\theta_2^2}\right]\psi_{2i}\right)\varepsilon_i=0$$

By how the lambdas are defined we have the relations

$$\lambda_1=\lambda_2\frac{\theta_2^2}{\theta_1^2}\Rightarrow\lambda_2=\lambda_1\frac{\theta_1^2}{\theta_2^2}\Rightarrow\frac{\lambda_2}{\theta_1^2}=\frac{\lambda_1}{\theta_2^2}$$

Substituting

$$\sum_{i=1}^{n}\left(\left[\frac{\lambda_2\left(1+\theta_1^2\right)}{\theta_1^2}+\lambda_2\frac{\theta_2^2}{\theta_1^2}\right]\psi_{1i}+\left[\lambda_1\frac{\theta_1^2}{\theta_2^2}+\frac{\lambda_1\left(1+\theta_2^2\right)}{\theta_2^2}\right]\psi_{2i}\right)\varepsilon_i=0$$

$$\Rightarrow\sum_{i=1}^{n}\left(\frac{\lambda_2}{\theta_1^2}\left(1+\theta_1^2+\theta_2^2\right)\psi_{1i}+\frac{\lambda_1}{\theta_2^2}\left(1+\theta_1^2+\theta_2^2\right)\psi_{2i}\right)\varepsilon_i=0$$

Since $\frac{\lambda_2}{\theta_1^2}=\frac{\lambda_1}{\theta_2^2}$, the coefficient factors are the same, so we are left with

$$\boxed{\sum_{i=1}^{n}\left(\psi_{1i}+\psi_{2i}\right)\varepsilon_i=0\Rightarrow\quad [3.50]:\ E\left[\left(\psi_{1i}+\psi_{2i}\right)\varepsilon_i\right]=0}$$

which is an orthogonality condition that is imposed by the MLE at the optimum.

G. Correlation between a variable and its conditional expected value.

We want to show that Pearson's correlation coefficient between a random variable Y and its conditional expectation related to another variable W, $E(Y|W)$ is

$$\operatorname{corr}(Y, E(Y|W)) = SD[E(Y|W)]/SD(Y)$$

Since by definition $\operatorname{corr}(Y, E(Y|W)) = \dfrac{\operatorname{Cov}[Y, E(Y|W)]}{SD(Y) \cdot SD[E(Y|W)]}$

it suffices to show that $\operatorname{Cov}[Y, E(Y|W)] = \operatorname{Var}[E(Y|W)]$.

We have $\operatorname{Cov}[Y, E(Y|W)] = E[Y \cdot E(Y|W)] - E[E(Y|W)]E(Y)$

We also have $E(Y) = E[E(Y|W)]$ and $Y \equiv E(Y|W) + e_{YW}$, where e_{YW} is the conditional expectation function error and by construction $E(e_{YW}|W) = 0$.

Using these we have

$$\operatorname{Cov}[Y, E(Y|W)] = E[(E(Y|W) + e_{YW}) \cdot E(Y|W)] - E[E(Y|W)] \cdot E[E(Y|W)]$$

$$= E(E(Y|W))^2 + E[e_{YW} \cdot E(Y|W)] - (E[E(Y|W)])^2$$

$$= \operatorname{Var}[E(Y|W)] + E[e_{YW} \cdot E(Y|W)]$$

Applying the Law of Iterated Expectations on the second term we have

$$E[e_{YW} \cdot E(Y|W)] = E\{E[e_{YW} \cdot E(Y|W)|W]\} = E\{E(Y|W)E[e_{YW}|W]\} = E\{E(Y|W) \cdot 0\} = 0$$

QED. --

TECHNICAL APPENDIX 3.III.

A Corrected OLS/Method of Moments estimator for 2TSF models without closed-form densities.

A. Central Moments and Cumulants of the composite error term.

We need to obtain the expressions for the 2nd, 3d, 4th and 5th central moment of $\varepsilon = v + w - u$, under the following general assumptions:

1. The random variable v has zero mean, is symmetric around zero and so it has all odd moments equal to zero,

2. The random variables w and u have non-zero mean and non zero 2nd and higher central moments,

3. The three random variables are jointly independent.

First, we set $\tilde{w} = w - E(w),\ \tilde{u} = u - E(u),\ \tilde{z} = \tilde{w} - \tilde{u}$ and so $E(\tilde{w}) = E(\tilde{u}) = E(\tilde{z}) = 0$ With this notation, $\left(\varepsilon - E(\varepsilon)\right) = \left(v + \tilde{z}\right)$, and $\tilde{z}$ is independent of v. We can use the convenient relations between central moments and cumulants κ_r

$$E\left(\tilde{x}^2\right) = \kappa_2 = \mathrm{Var}\left(x\right), \quad E\left(\tilde{x}^3\right) = \kappa_3, \quad E\left(\tilde{x}^4\right) = \kappa_4 + 3\kappa_2^2, \quad E\left(\tilde{x}^5\right) = \kappa_5 + 10\kappa_3\kappa_2 .$$

Cumulants have the properties
$$\begin{cases} \kappa_1\left(X + c\right) = c + \kappa_1\left(X\right) \\ \kappa_r\left(X + c\right) = \kappa_r\left(X\right), \quad r \geq 2 \end{cases}$$

and $\kappa_r\left(cx\right) = c^r \kappa_r\left(x\right)$ and so $\kappa_r\left(-X\right) = \left(-1\right)^r \kappa_r\left(X\right), \quad r \geq 2$.

Also, for independent random variables

$$\kappa_r\left(X + Y\right) = \kappa_r\left(X\right) + \kappa_r\left(Y\right), \ \ \kappa_r\left(X - Y\right) = \kappa_r\left(X + \left(-Y\right)\right) = \kappa_r\left(X\right) + \left(-1\right)^r \kappa_r\left(Y\right), \ r \geq 2$$

These will be used repeatedly below.

A.1. 2nd central moment.

$$E\left(\varepsilon - E(\varepsilon)\right)^2 = \mathrm{Var}\left(\varepsilon\right) = \kappa_2\left(\varepsilon\right) = \kappa_2\left(v+\tilde{z}\right) = \kappa_2\left(v\right) + \kappa_2\left(\tilde{z}\right) = \kappa_2\left(v\right) + \kappa_2\left(\tilde{w}\right) + \left(-1\right)^2 \kappa_2\left(\tilde{u}\right)$$

$$\Rightarrow E\left(\tilde{\varepsilon}^2\right) = \kappa_2\left(v\right) + \kappa_2\left(\tilde{w}\right) + \kappa_2\left(\tilde{u}\right) = \mathrm{Var}(v) + \mathrm{Var}(w) + \mathrm{Var}(u)$$

A.2 3d central moment.

$$E\left(\varepsilon - E(\varepsilon)\right)^3 = \kappa_3\left(\varepsilon\right) = \kappa_3\left(v+\tilde{z}\right) = \kappa_3\left(v\right) + \kappa_3\left(\tilde{z}\right) = 0 + \kappa_3\left(\tilde{z}\right)$$

given the assumptions on the random variable v. Further,

$\kappa_3\left(\tilde{z}\right) = \kappa_3\left(\tilde{w}-\tilde{u}\right) = \kappa_3\left(w\right) - \kappa_3\left(u\right)$, so $E\left(\tilde{\varepsilon}^3\right) = \kappa_3\left(w\right) - \kappa_3\left(u\right)$

A.3 4th central moment.

$$E\left(\varepsilon - E(\varepsilon)\right)^4 = \kappa_4\left(\varepsilon\right) + 3\left(\kappa_2\left(\varepsilon\right)\right)^2$$

$$\Rightarrow E\left(\tilde{\varepsilon}^4\right) - 3\left[E\left(\tilde{\varepsilon}^2\right)\right]^2 = \kappa_4\left(\varepsilon\right) = \kappa_4\left(v+\tilde{z}\right) = \kappa_4\left(v\right) + \kappa_4\left(w\right) + \kappa_4\left(u\right)$$

We note that for the Normal distribution $\kappa_4 = 0$.

A.4 5th central moment.

$$E\left(\varepsilon - E(\varepsilon)\right)^5 = \kappa_5\left(\varepsilon\right) + 10\kappa_3\left(\varepsilon\right)\kappa_2\left(\varepsilon\right)$$

$$\Rightarrow E\left(\tilde{\varepsilon}^5\right) - 10E\left(\tilde{\varepsilon}^3\right)E\left(\tilde{\varepsilon}^2\right) = \kappa_5\left(\varepsilon\right) = \kappa_5\left(v+\tilde{z}\right) = \kappa_5\left(v\right) + \kappa_5\left(w\right) - \kappa_5\left(u\right)$$

But $\kappa_5\left(v\right) = E\left(v^5\right) - 10\kappa_3\left(v\right)\kappa_2\left(v\right)$ and $E\left(v^5\right) = \kappa_3\left(v\right) = 0$.

So $E\left(\tilde{\varepsilon}^5\right) - 10E\left(\tilde{\varepsilon}^3\right)E\left(\tilde{\varepsilon}^2\right) = \kappa_5\left(w\right) - \kappa_5\left(u\right)$.

B. Unbiased estimation of the central moments and the cumulants of the composite error term.

We consider the linear regression model, in matrix form,

$$\mathrm{y} = \mathrm{X}\beta + \varepsilon, \quad \varepsilon = v + w - u$$

where X is a $n \times K$ matrix including a constant term. We assume the error term is 5th-order independent from the regressors, $E\left(\varepsilon_i^r \,|\mathrm{X}\right) = E\left(\varepsilon_i^r\right)$, $r = 1,2,3,4,5$. We also assume as everywhere an i.i.d. sample. The usual regularity conditions apply.

We define the matrices $\mathrm{P} = \mathrm{X}\left(\mathrm{X}'\mathrm{X}\right)^{-1}\mathrm{X}'$, with typical element $\left[h_{ji}\right]$ and $\mathrm{M} = I_{\mathrm{n}} - \mathrm{P}$ with typical element $\left[m_{ji}\right]$, and $m_{ji} = -h_{ji}$, $m_{ii} = 1 - h_{ii}$. Both these matrices are symmetric and idempotent. P is the orthogonal projection matrix for X (the "hat" matrix) and M is the "residual-maker" matrix, because $\mathrm{My} = \hat{\varepsilon}_{OLS}$.

B.1 Properties of the M and P matrices.

The following properties hold, (some because we include a constant term in X):

For the P matrix,

$0 \le h_{ii} \le 1$, and if the matrix includes a constant term then $\dfrac{1}{n} \le h_{ii} \le 1$,

$-\dfrac{1}{2} \le h_{ji} \le \dfrac{1}{2}$, and if the matrix includes a constant term $\dfrac{1}{n} - \dfrac{1}{2} \le h_{ji} \le \dfrac{1}{2}$,

and the bounds are attained in special cases (see Mohammadi 2016).

We also obtain easily,

$$\mathrm{PX} = \mathrm{X} \;\Rightarrow\; \sum\nolimits_{i=1}^{n} h_{ji} = 1, \qquad j = 1,\ldots,n \qquad [1]$$

$$\mathrm{MX} = \mathbf{0} \Rightarrow \sum\nolimits_{i=1}^{n} m_{ji} = 0, \qquad j = 1,\ldots,n \ , \qquad [2]$$

the second equalities because we have included a constant term in the regressor matrix. With a constant term included we have $\mathrm{M}\cdot\mathbf{1} = \mathbf{0}$, so we obtain

$$\mathrm{My} = \mathrm{M\varepsilon} = \mathrm{M}\big(\varepsilon - E(\varepsilon)\cdot\mathbf{1}\big) \equiv \mathrm{M}\tilde{\varepsilon} = \hat{\varepsilon}_{OLS} \ ,$$

and we will drop in the sequence the OLS-subscript.

The idempotent property implies, for $\mathbf{m}_j, \mathbf{h}_\ell$ being row/column vectors of P and M respectively

$$\mathbf{h}_j'\mathbf{h}_\ell = h_{j\ell} \Rightarrow \mathbf{h}_j'\mathbf{h}_j = h_{jj}$$
$$\mathbf{m}_j'\mathbf{m}_\ell = m_{j\ell} \Rightarrow \mathbf{m}_j'\mathbf{m}_j = m_{jj} = 1 - h_{jj} \ .$$

Due to symmetry the above imply also

$$\sum_{\ell=1}^{n} h_{j\ell}^2 = h_{jj} \Rightarrow \sum_{\ell=1,\ell\neq j}^{n} h_{j\ell}^2 = h_{jj}\left(1 - h_{jj}\right), \qquad \sum_{\ell=1}^{n} m_{j\ell}^2 = m_{jj} \ .$$

Regarding the trace of these matrices we have, by applying the cyclic property of the trace,

$$\mathrm{tr}\left\{\mathrm{P}\right\} = \sum_{i=1}^{n} h_{ii} = K, \qquad \mathrm{tr}\left\{\mathrm{M}\right\} = \sum_{i=1}^{n} m_{ii} = n - K$$

Combining the above, we have $\sum_{j=1}^{n}\sum_{i=1}^{n} m_{ji}^2 = n - K$.

This can be also obtained by the use of the Hadamard product (element-wise multiplication), $\mathrm{M}^{(2)} \equiv \mathrm{M} \circ \mathrm{M}$, which we will exploit in the sequence. One of its properties is that the sum of the elements of the Hadamard product of two matrices A,B equals the trace of AB': $\mathbf{1}'(\mathrm{A} \circ \mathrm{B})\mathbf{1} = \operatorname{tr}\{\mathrm{AB}'\}$ (see e.g. Styan 1973, eq. 2.8).

We then have

$$\sum_{j=1}^{n}\sum_{i=1}^{n} m_{ji}^2 = \mathbf{1}'(\mathrm{M} \circ \mathrm{M})\mathbf{1} = \operatorname{tr}\{\mathrm{MM}'\} = \operatorname{tr}\{\mathrm{MM}\} = \operatorname{tr}\{\mathrm{M}\} = n - K\ ,$$

where we have used the symmetry and idempotency of M. In words, the sum of all squared elements of M equals its trace. The same hold for the other matrix,

$$\sum_{j=1}^{n}\sum_{i=1}^{n} h_{ji}^2 = \mathbf{1}'(\mathrm{P} \circ \mathrm{P})\mathbf{1} = \operatorname{tr}\{\mathrm{PP}\} = \operatorname{tr}\{\mathrm{P}\} = K\ .$$

Finally we show that $\mathrm{P} \to \mathbf{0} \ \Rightarrow \mathrm{M} \to I$. We have

$$\mathrm{P} = \mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}' \ = \frac{1}{n}\mathrm{X}\left(\frac{1}{n}\mathrm{X}'\mathrm{X}\right)^{-1}\mathrm{X}'$$

By the standard regularity conditions ,

$$\left(\frac{1}{n}\mathrm{X}'\mathrm{X}\right)^{-1} \xrightarrow{p} \Omega_{K\times K} = \begin{bmatrix} \omega_{11} & \cdots & \omega_{1K} \\ \vdots & \ddots & \vdots \\ \omega_{K1} & \cdots & \omega_{KK} \end{bmatrix}_{K\times K}$$

a finite, non-stochastic well-defined matrix. With an ergodic-stationary sample, we then get

$$\operatorname{plim}\mathrm{P} = \operatorname{plim}\left[\frac{1}{n}\mathrm{X}\Omega\mathrm{X}'\right].$$ Analyze,

$$\mathrm{X}=\begin{bmatrix} x_{11} & \cdots & x_{1K} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nK} \end{bmatrix}_{n\times K} \quad , \quad \mathrm{X}'=\begin{bmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{1K} & \cdots & x_{nK} \end{bmatrix}_{K\times n}$$

Then,

$$\mathrm{X\Omega X}' = \begin{bmatrix} x_{11} & \cdots & x_{1K} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nK} \end{bmatrix}_{n\times K} \begin{bmatrix} \omega_{11} & \cdots & \omega_{1K} \\ \vdots & \ddots & \vdots \\ \omega_{K1} & \cdots & \omega_{KK} \end{bmatrix}_{K\times K} \begin{bmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{1K} & \cdots & x_{nK} \end{bmatrix}_{K\times n}$$

Consider as an example, the upper leftmost element of this matrix product. It will be

$$[1,1] = \left(\sum_{j=1}^{K} x_{1j}\omega_{j1}\right)x_{11} + \left(\sum_{j=1}^{K} x_{1j}\omega_{j2}\right)x_{12} + \ldots + \left(\sum_{j=1}^{K} x_{1j}\omega_{jK}\right)x_{1K}$$

This sum of sums has K^2 number of terms in total, and they do not increase in number as sample size grows (what increases is the dimension of the P matrix, which is $n\times n$). The same holds for all elements of $\mathrm{X\Omega X}'$. So each divided by n, it will go to zero. Therefore

$$\mathrm{P} \xrightarrow{p} \mathbf{0} \Rightarrow \mathrm{M} \xrightarrow{p} I.$$

The fact that all elements of P tend to zero individually does not imply that the same will happen to a matrix product in which P may appear (since then we will obtain sums of n elements).

B.2 Unbiased estimators for central moment and cumulants.

B.2.1. 2nd central moment.

It is an established result that, under our assumptions, we have

$$E\left(\frac{1}{n}\sum_{i=1}^{n}\hat{\varepsilon}_i^2 | \mathrm{X}\right)=\frac{1}{n}E\left[\left(\mathrm{M}\tilde{\varepsilon}\right)'\left(\mathrm{M}\tilde{\varepsilon}\right)|\mathrm{X}\right]=\frac{1}{n}E\left(\tilde{\varepsilon}'\mathrm{M}\tilde{\varepsilon}|\mathrm{X}\right)=\mathrm{Var}\left(\varepsilon_i|\mathrm{X}\right)\frac{1}{n}\sum_{i=1}^{n}m_{ii}=\frac{n-K}{n}\sigma_\varepsilon^2$$

Therefore, an unbiased estimator for the 2nd central moment/cumulant/ variance is

$$\boxed{[3.57]:\quad \hat{\mu}_2\left(\varepsilon\right)=\hat{\kappa}_2\left(\varepsilon\right)=\left(c_2\right)^{-1}\frac{1}{n}\sum_{i=1}^{n}\hat{\varepsilon}_i^2,\qquad c_2=\frac{1}{n}\sum_{i=1}^{n}m_{ii}=\frac{n-K}{n}\to 1} \qquad [3]$$

B.2.2. 3d central moment.

We have

$$\hat{\varepsilon}_j^3=\hat{\varepsilon}_j^2\hat{\varepsilon}_j=\left(\mathbf{m}_j'\tilde{\varepsilon}\right)^2\left(\mathbf{m}_j'\tilde{\varepsilon}\right)=\mathbf{m}_j'\tilde{\varepsilon}\tilde{\varepsilon}'\mathbf{m}_j\left(\mathbf{m}_j'\tilde{\varepsilon}\right)$$

$$=\mathbf{m}_j'\begin{bmatrix}\tilde{\varepsilon}_1^2\left(\mathbf{m}_j'\tilde{\varepsilon}\right) & \tilde{\varepsilon}_1\tilde{\varepsilon}_2\left(\mathbf{m}_j'\tilde{\varepsilon}\right) & \cdots & \tilde{\varepsilon}_1\tilde{\varepsilon}_n\left(\mathbf{m}_j'\tilde{\varepsilon}\right)\\ \tilde{\varepsilon}_2\tilde{\varepsilon}_1\left(\mathbf{m}_j'\tilde{\varepsilon}\right) & \ddots & & \vdots\\ \vdots & & \ddots & \vdots\\ \tilde{\varepsilon}_n\tilde{\varepsilon}_1\left(\mathbf{m}_j'\tilde{\varepsilon}\right) & \tilde{\varepsilon}_n\tilde{\varepsilon}_2\left(\mathbf{m}_j'\tilde{\varepsilon}\right) & \cdots & \tilde{\varepsilon}_n^2\left(\mathbf{m}_j'\tilde{\varepsilon}\right)\end{bmatrix}\mathbf{m}_j$$

For say, the elements of the first row we have

diagonal: $\tilde{\varepsilon}_1^2\left(\mathbf{m}_j'\tilde{\varepsilon}\right)=m_{j1}\tilde{\varepsilon}_1^3+\tilde{\varepsilon}_1^2\sum_{i\neq 1}m_{ji}\tilde{\varepsilon}_i \;\Rightarrow E\left[\tilde{\varepsilon}_1^2\left(\mathbf{m}_j'\tilde{\varepsilon}\right)|\mathrm{X}\right]=\; m_{j1}E\left(\tilde{\varepsilon}_1^3|\mathrm{X}\right)+0$

and

$$\textit{off-diagonal: } \tilde{\varepsilon}_1\,\tilde{\varepsilon}_2\left(\mathbf{m}'_j\tilde{\varepsilon}\right) = \tilde{\varepsilon}_1\,\tilde{\varepsilon}_2\sum_{i=1}^{n} m_{ji}\tilde{\varepsilon}_i \;\Rightarrow E\left[\tilde{\varepsilon}_1\,\tilde{\varepsilon}_2\left(\mathbf{m}'_j\tilde{\varepsilon}\right)|\mathrm{X}\right] = 0$$

since in all elements, a 1st power of the centered error term will remain.

So

$$E\left(\hat{\varepsilon}_j^3|\mathrm{X}\right) = \mathbf{m}'_j\begin{bmatrix} m_{j1}E\left(\tilde{\varepsilon}_1^3|\mathrm{X}\right) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & m_{jn}E\left(\tilde{\varepsilon}_n^3|\mathrm{X}\right)\end{bmatrix}\mathbf{m}_j = E\left(\tilde{\varepsilon}_j^3\right)\mathbf{m}'_j\begin{bmatrix} m_{j1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & m_{jn}\end{bmatrix}\mathbf{m}_j$$

$$= E\left(\tilde{\varepsilon}_j^3\right)\left[m_{j1}^2 \quad \cdots \quad m_{jn}^2\right]\begin{bmatrix} m_{j1} \\ \vdots \\ m_{jn}\end{bmatrix} = E\left(\tilde{\varepsilon}_j^3\right)\sum_{i=1}^{n} m_{ji}^3 \qquad [4]$$

Therefore

$$E\left(\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^3|\mathrm{X}\right) = c_3E\left(\tilde{\varepsilon}_j^3\right), \quad c_3 = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{n} m_{ji}^3$$

and since c_3 is purely a function of the regressors,

$$\left(c_3\right)^{-1}E\left(\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^3|X\right) = E\left(\tilde{\varepsilon}_j^3\right) \Rightarrow E\left(\left(c_3\right)^{-1}\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^3\right) = E\left(\tilde{\varepsilon}_j^3\right)$$

So an unbiased estimator for the 3d central moment/cumulant is

$$\boxed{[3.58]:\; \hat{\mu}_3\left(\varepsilon\right) = \hat{\kappa}_3\left(\varepsilon\right) = \left(c_3\right)^{-1}\frac{1}{n}\sum_{i=1}^{n}\hat{\varepsilon}_i^3, \qquad c_3 = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{n} m_{ji}^3} \qquad [5]$$

Using the Hadamard product, c_3 can also be written as

$$c_3 = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{n} m_{ji}^3 = \frac{1}{n}\mathbf{1}'\left(\mathrm{M}\circ\mathrm{M}\circ\mathrm{M}\right)\mathbf{1} = \frac{1}{n}\mathbf{1}'\left[\left(\mathrm{M}\circ\mathrm{M}\right)\circ\mathrm{M}\right]\mathbf{1} = \frac{1}{n}\mathrm{tr}\left\{\mathrm{M}^{(2)}\mathrm{M}\right\}$$

We now show analytically that $c_3 \xrightarrow{p} 1$. We have

$$c_3 = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{n} m_{ji}^3 = \frac{1}{n}\sum_{i=1}^{n} m_{ii}^3 + \frac{1}{n}\sum_{j\neq i} m_{ji}^3 \quad \text{where } \sum_{j\neq i} m_{ji}^3 \text{ is a double sum.}$$

Continuing,

$c_3 = \frac{1}{n}\sum_{i=1}^{n}\left(1-h_{ii}\right)^3 + \frac{1}{n}\sum_{j\neq i} m_{ji}^3$. For the first term we have

$$\frac{1}{n}\sum_{i=1}^{n}\left(1-h_{ii}\right)^3 = \frac{1}{n}\sum_{i=1}^{n}\left(1-2h_{ii}+h_{ii}^2\right)\left(1-h_{ii}\right) = \frac{1}{n}\sum_{i=1}^{n}\left(1-h_{ii}-2h_{ii}+2h_{ii}^2+h_{ii}^2-h_{ii}^3\right)$$

$$=1 - \frac{3}{n}\sum_{i=1}^{n} h_{ii} + \frac{3}{n}\sum_{i=1}^{n} h_{ii}^2 - \frac{1}{n}\sum_{i=1}^{n} h_{ii}^3$$

Since $h_{ii} \leq 1$, and not all h_{ii} can be equal to 1 since $\sum_{i=1}^{n} h_{ii} = K < n$, we have $\sum_{i=1}^{n} h_{ii}^2 < \sum_{i=1}^{n} h_{ii} = K$, $\sum_{i=1}^{n} h_{ii}^3 < \sum_{i=1}^{n} h_{ii} = K$. So all these sums are bounded and divided by n they go to zero. It follows that $\frac{1}{n}\sum_{i=1}^{n}\left(1-h_{ii}\right)^3 \rightarrow 1$.

For the other component of c_3 we have $\frac{1}{n}\sum_{j\neq i} m_{ji}^3 = -\frac{1}{n}\sum_{j\neq i} h_{ji}^3$. Now, since the off-diagonal elements of the P matrix are strictly smaller than unity in absolute terms, we have, for a single row

$$-h_{ji}^2 < -h_{ji}^3 < h_{ji}^2 \Rightarrow -\sum_{j=1,\, j\neq i}^{n} h_{ji}^2 < -\sum_{j=1,\, j\neq i}^{n} h_{ji}^3 < \sum_{j=1,\, j\neq i}^{n} h_{ji}^2$$

But the sum of the off-diagonal elements-squared in any row, are equal to $h_{ii}\left(1-h_{ii}\right)$ so we obtain

$$-h_{ii}\left(1-h_{ii}\right) < -\sum_{j=1,\, j\neq i}^{n} h_{ji}^3 < h_{ii}\left(1-h_{ii}\right)$$

and for all rows divided by n

$$-\frac{1}{n}\sum_{i=1}^{n} h_{ii}\left(1-h_{ii}\right) < \frac{1}{n}\sum_{j\neq i} m_{ji}^3 < \frac{1}{n}\sum_{i=1}^{n} h_{ii}\left(1-h_{ii}\right) \Rightarrow \frac{1}{n}\sum_{j\neq i} m_{ji}^3 < \left|\frac{1}{n}\sum_{i=1}^{n} h_{ii}\left(1-h_{ii}\right)\right|$$

$$\Rightarrow \frac{1}{n}\sum_{j\neq i} m_{ji}^3 < \left|\frac{K}{n} - \frac{1}{n}\sum_{i=1}^{n} h_{ii}^2\right|$$

Both terms in the absolute value go to zero, so $\frac{1}{n}\sum_{j\neq i} m_{ji}^3 \to 0$. Therefore $c_3 \xrightarrow{p} 1$.

The above analysis verified that we can use directly the relevant probability limits,

$$c_3 = \frac{1}{n}\operatorname{tr}\left\{\mathrm{M}^{(2)}\mathrm{M}\right\} = \frac{1}{n}\operatorname{tr}\left\{\left(\mathrm{M}\circ\mathrm{M}\right)\mathrm{M}\right\} \xrightarrow{p} \operatorname{plim}\frac{1}{n}\operatorname{tr}\left\{\left(\mathrm{I}_n\circ\mathrm{I}_n\right)\mathrm{I}_n\right\} = \operatorname{plim}\frac{1}{n}\operatorname{tr}\left\{\mathrm{I}_n\right\} = 1$$

which is what we will use for the subsequent bias-correction factors.

B.2.3. 4th central moment.

We have

$$\hat{\varepsilon}_j^4=\hat{\varepsilon}_j^2\hat{\varepsilon}_j^2=\mathbf{m}'_j\tilde{\varepsilon}\tilde{\varepsilon}'\mathbf{m}_j\left(\mathbf{m}'_j\tilde{\varepsilon}\right)^2=\mathbf{m}'_j\begin{bmatrix}\tilde{\varepsilon}_1^2 & \tilde{\varepsilon}_1\tilde{\varepsilon}_2 & \cdots & \tilde{\varepsilon}_1\tilde{\varepsilon}_n\\ \tilde{\varepsilon}_2\tilde{\varepsilon}_1 & \ddots & & \vdots\\ \vdots & & \ddots & \vdots\\ \tilde{\varepsilon}_n\tilde{\varepsilon}_1 & \tilde{\varepsilon}_n\tilde{\varepsilon}_2 & \cdots & \tilde{\varepsilon}_n^2\end{bmatrix}\mathbf{m}_j\left(\mathbf{m}'_j\tilde{\varepsilon}\right)^2$$

$$=\mathbf{m}'_j\begin{bmatrix}\tilde{\varepsilon}_1^2\left(\mathbf{m}'_j\tilde{\varepsilon}\right)^2 & \tilde{\varepsilon}_1\tilde{\varepsilon}_2\left(\mathbf{m}'_j\tilde{\varepsilon}\right)^2 & \cdots & \tilde{\varepsilon}_1\tilde{\varepsilon}_n\left(\mathbf{m}'_j\tilde{\varepsilon}\right)^2\\ \tilde{\varepsilon}_2\tilde{\varepsilon}_1\left(\mathbf{m}'_j\tilde{\varepsilon}\right)^2 & \ddots & & \vdots\\ \vdots & & \ddots & \vdots\\ \tilde{\varepsilon}_n\tilde{\varepsilon}_1\left(\mathbf{m}'_j\tilde{\varepsilon}\right)^2 & \tilde{\varepsilon}_n\tilde{\varepsilon}_2\left(\mathbf{m}'_j\tilde{\varepsilon}\right)^2 & \cdots & \tilde{\varepsilon}_n^2\left(\mathbf{m}'_j\tilde{\varepsilon}\right)^2\end{bmatrix}\mathbf{m}_j\equiv\mathbf{m}'_j\mathbf{A}\mathbf{m}_j$$

Now, $\left(\mathbf{m}'_j\tilde{\varepsilon}\right)^2=\left(\sum_{i=1}^n m_{ji}\tilde{\varepsilon}_i\right)^2=\sum_{i=1}^n m_{ji}^2\tilde{\varepsilon}_i^2+\sum_{k\neq\ell}m_{jk}m_{j\ell}\tilde{\varepsilon}_k\tilde{\varepsilon}_\ell$

where $\sum_{k\neq\ell}$ denotes a double sum, so it contains $n(n-1)$ elements (we will keep this notational convention in what follows).

We will use μ_4 for the 4th central moment and σ^4 for the squared variance. Then

a) Diagonal terms of the matrix.

$$[1,1]:E\left(\tilde{\varepsilon}_1^2\left(\mathbf{m}'_j\tilde{\varepsilon}\right)^2|\mathrm{X}\right)=E\left[\tilde{\varepsilon}_1^2\sum_{i=1}^n m_{ji}^2\tilde{\varepsilon}_i^2+\tilde{\varepsilon}_1^2\sum_{k\neq\ell}m_{jk}m_{j\ell}\tilde{\varepsilon}_k\tilde{\varepsilon}_\ell|\mathrm{X}\right]=m_{j1}^2\mu_4+\sigma^4\sum_{i\neq1}^n m_{ji}^2+0$$

$$[2,2]:E\left(\tilde{\varepsilon}_2^2\left(\mathbf{m}'_j\tilde{\varepsilon}\right)^2|\mathrm{X}\right)=E\left[\tilde{\varepsilon}_2^2\sum_{i=1}^n m_{ji}^2\tilde{\varepsilon}_i^2+\tilde{\varepsilon}_2^2\,2\sum_{k\neq\ell}m_{jk}m_{j\ell}\tilde{\varepsilon}_k\tilde{\varepsilon}_\ell|\mathrm{X}\right]=m_{j2}^2\mu_4+\sigma^4\sum_{i\neq2}^n m_{ji}^2+0$$

etc. We have $\sum_{i=1}^n m_{ji}^2=m_{jj}$, So we manipulate into

$$[1,1]:\ m_{j1}^2\mu_4-m_{j1}^2\sigma^4+\sigma^4\sum_{i=1}^n m_{ji}^2=\left(\mu_4-\sigma^4\right)m_{j1}^2+\sigma^4m_{jj}\quad\text{etc}$$

Also, we apply add-and subtract and obtain

$$[1,1]:\left(\mu_4-\sigma^4\right)m_{j1}^2\,+\,\sigma^4 m_{jj}=\left(\mu_4-\sigma^4\right)m_{j1}^2\,+\,\sigma^4 m_{jj}+\left(2\sigma^4 m_{j1}^2-2\sigma^4 m_{j1}^2\right)$$

$$=\left(\mu_4-3\sigma^4\right)m_{j1}^2\,+\,\sigma^4 m_{jj}\,+\,2\sigma^4 m_{j1}^2$$

b) Off-diagonal terms.

$$[k,\ell]:E\left(\tilde{\varepsilon}_k\tilde{\varepsilon}_\ell\left(\mathbf{m}_j'\tilde{\varepsilon}\right)^2|\mathrm{X}\right)=E\left[\tilde{\varepsilon}_k\tilde{\varepsilon}_\ell\sum_{i=1}^{n}m_{ji}^2\tilde{\varepsilon}_i^2\,+\,\tilde{\varepsilon}_k\tilde{\varepsilon}_\ell\sum_{k\neq\ell}m_{jk}m_{j\ell}\tilde{\varepsilon}_k\tilde{\varepsilon}_\ell\,|\mathrm{X}\right]=0+2\sigma^4 m_{jk}m_{j\ell}$$

We decompose the matrix $E\left(\mathbf{A}|\mathrm{X}\right)$ in three additive matrices, the first being diagonal and the second the identity matrix,

$$E\left(\mathrm{A}_1|\mathrm{X}\right)=\left(\mu_4-3\sigma^4\right)\begin{bmatrix} m_{j1}^2 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & m_{jn}^2\end{bmatrix},\qquad E\left(\mathrm{A}_2|\mathrm{X}\right)=\sigma^4 m_{jj}\mathrm{I}_{\mathrm{n}}$$

and

$$E\left(\mathrm{A}_3|\mathrm{X}\right)=2\sigma^4\begin{bmatrix} m_{j1}^2 & m_{j1}m_{j2} & \cdots & m_{j1}m_{jn}\\ m_{j2}m_{j1} & & & \vdots\\ & \ddots & & m_{j,n-1}m_{jn}\\ m_{jn}m_{j1} & m_{jn}m_{j2} & \cdots & m_{jn}^2\end{bmatrix}=2\sigma^4\mathbf{m}_j\mathbf{m}_j'$$

So

$$E\left(\hat{\varepsilon}_j^4|\mathrm{X}\right)=\mathbf{m}_j'E\left(\mathbf{A}|\mathrm{X}\right)\mathbf{m}_j=\mathbf{m}_j'\left[E\left(\mathrm{A}_1|\mathrm{X}\right)+E\left(\mathrm{A}_2|\mathrm{X}\right)+E\left(\mathrm{A}_3|\mathrm{X}\right)\right]\mathbf{m}_j$$

$$=\left(\mu_4-3\sigma^4\right)\mathbf{m}_j'\begin{bmatrix} m_{j1}^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & m_{jn}^2 \end{bmatrix}\mathbf{m}_j \;+\; \sigma^4 m_{jj}\mathbf{m}_j' I_n \mathbf{m}_j \;+\; 2\sigma^4 \mathbf{m}_j'\mathbf{m}_j\mathbf{m}_j'\mathbf{m}_j$$

and using $\mathbf{m}_j'\mathbf{m}_j = m_{jj}$

$$\Rightarrow E\left(\hat{\varepsilon}_j^4 \,|\mathrm{X}\right)=\left(\mu_4-3\sigma^4\right)\sum_{i=1}^{n} m_{ji}^4 \;+\sigma^4 m_{jj}^2 \;+2\sigma^4 m_{jj}^2$$

$$\Rightarrow E\left(\hat{\varepsilon}_j^4 \,|\mathrm{X}\right)=\mu_4\sum_{i=1}^{n} m_{ji}^4 \;+\; \sigma^4\left(3m_{jj}^2-3\sum_{i=1}^{n} m_{ji}^4\right) \qquad [6]$$

So

$$E\left(\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^4 \,|\mathrm{X}\right)=\mu_4 c_{41} \;+\; \sigma^4\left(3c_{42}-3c_{41}\right)$$

$$c_{41}=\frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{n} m_{ji}^4\,,\quad c_{42}=\frac{1}{n}\sum_{j=1}^{n} m_{jj}^2 \qquad [7]$$

Using the Hadamard product

$$c_{41}=\frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{n} m_{ji}^4 \;=\frac{1}{n}\mathbf{1}'\left(\mathrm{M}\circ\mathrm{M}\circ\mathrm{M}\circ\mathrm{M}\right)\mathbf{1}=\frac{1}{n}\mathbf{1}'\left[\left(\mathrm{M}\circ\mathrm{M}\right)\circ\left(\mathrm{M}\circ\mathrm{M}\right)\right]\mathbf{1}=\frac{1}{n}\operatorname{tr}\left\{\mathrm{M}^{(2)}\mathrm{M}^{(2)}\right\}$$

while $\quad c_{42}=\dfrac{1}{n}\displaystyle\sum_{j=1}^{n} m_{jj}^2=\dfrac{1}{n}\operatorname{tr}\left\{\mathrm{M}^{(2)}\right\}$

To arrive at an unbiased estimator we must find a suitable estimator for the squared variance, σ^4. Consider the product

$$[\xi \neq v]: \hat{\varepsilon}_{\xi}^{2}\hat{\varepsilon}_{v}^{2} = \left(\mathbf{m}'_{\xi}\tilde{\varepsilon}\right)^{2}\left(\mathbf{m}'_{v}\tilde{\varepsilon}\right)^{2} = \mathbf{m}'_{\xi}\tilde{\varepsilon}\tilde{\varepsilon}'\mathbf{m}_{\xi}\left(\mathbf{m}'_{v}\tilde{\varepsilon}\right)^{2}$$

$$= \mathbf{m}'_{\xi}\begin{bmatrix} \tilde{\varepsilon}_{1}^{2}\left(\mathbf{m}'_{v}\tilde{\varepsilon}\right)^{2} & \tilde{\varepsilon}_{1}\tilde{\varepsilon}_{2}\left(\mathbf{m}'_{v}\tilde{\varepsilon}\right)^{2} & \cdots & \tilde{\varepsilon}_{1}\tilde{\varepsilon}_{n}\left(\mathbf{m}'_{v}\tilde{\varepsilon}\right)^{2} \\ \tilde{\varepsilon}_{2}\tilde{\varepsilon}_{1}\left(\mathbf{m}'_{v}\tilde{\varepsilon}\right)^{2} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \tilde{\varepsilon}_{n}\tilde{\varepsilon}_{1}\left(\mathbf{m}'_{v}\tilde{\varepsilon}\right)^{2} & \tilde{\varepsilon}_{n}\tilde{\varepsilon}_{2}\left(\mathbf{m}'_{v}\tilde{\varepsilon}\right)^{2} & \cdots & \tilde{\varepsilon}_{n}^{2}\left(\mathbf{m}'_{v}\tilde{\varepsilon}\right)^{2} \end{bmatrix}\mathbf{m}_{\xi} \equiv \mathbf{m}'_{\xi}\mathbf{A}_{v}\mathbf{m}_{\xi}$$

This is the same matrix as previously, the only difference is that the outside vectors are from a different observation. So we will end up with

$$E\left(\hat{\varepsilon}_{\xi}^{2}\hat{\varepsilon}_{v}^{2}\,|\mathrm{X}\right) = \left(\mu_{4} - 3\sigma^{4}\right)\mathbf{m}'_{\xi}\begin{bmatrix} m_{v1}^{2} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & m_{vn}^{2} \end{bmatrix}\mathbf{m}_{\xi} \; + \; \sigma^{4}m_{vv}\mathbf{m}'_{\xi}I_{n}\mathbf{m}_{\xi} \; + \; 2\sigma^{4}\mathbf{m}'_{\xi}\mathbf{m}_{v}\mathbf{m}'_{v}\mathbf{m}_{\xi}$$

$$= \left(\mu_{4} - 3\sigma^{4}\right)\sum_{i=1}^{n} m_{\xi i}^{2}m_{vi}^{2} \; + \; \sigma^{4}m_{vv}m_{\xi\xi} \; + \; 2\sigma^{4}m_{\xi v}m_{v\xi}$$

$$= \; \mu_{4}\sum_{i=1}^{n} m_{\xi i}^{2}m_{vi}^{2} + \sigma^{4}\left(m_{vv}m_{\xi\xi} + 2m_{\xi v}m_{v\xi} - 3\sum_{i=1}^{n} m_{\xi i}^{2}m_{vi}^{2}\right)$$

So

$$E\left(\frac{1}{n(n-1)}\sum_{\xi \neq v}\hat{\varepsilon}_{\xi}^{2}\hat{\varepsilon}_{v}^{2}\,|\mathrm{X}\right) = \mu_{4}c_{43} \; + \; \sigma^{4}\left(c_{44} - 3c_{43}\right)$$

$$c_{43} = \frac{1}{n(n-1)}\sum_{\xi \neq v}\sum_{i=1}^{n} m_{\xi i}^{2}m_{vi}^{2}\;, \quad c_{44} = \frac{1}{n(n-1)}\sum_{\xi \neq v}\left(m_{vv}m_{\xi\xi} + 2m_{\xi v}m_{v\xi}\right)$$

Note that $\sum_{\xi \neq v}\sum_{i=1}^{n} m_{\xi i}^{2}m_{vi}^{2}$ is the sum of the elements of the matrix product $\left(\mathrm{M}\circ\mathrm{M}\right)\left(\mathrm{M}\circ\mathrm{M}\right) = \mathrm{M}^{(2)}\mathrm{M}^{(2)}$, minus the sum of the diagonal terms i.e. its trace. So we can write

$$c_{43} = \frac{1}{n(n-1)} \sum_{\xi \neq v} \sum_{i=1}^{n} m_{\xi i}^2 m_{vi}^2 = \frac{\mathbf{1}' \mathrm{M}^{(2)} \mathrm{M}^{(2)} \mathbf{1} - \mathrm{tr}\left\{\mathrm{M}^{(2)} \mathrm{M}^{(2)}\right\}}{n(n-1)}$$

For c_{44} we have that the first component of the double sum, $\sum_{\xi \neq v} m_{vv} m_{\xi\xi}$, can be obtained as

$$\sum_{\xi \neq v} m_{vv} m_{\xi\xi} = \mathbf{1}' \left(\sum_{j=1}^{n} m_{jj} \, \mathrm{diag}\left\{m_{ii}\right\} \right) \mathbf{1} - \sum_{j=1}^{n} m_{jj}^2 = \mathbf{1}' \left(\mathrm{diag}\left\{m_{ii}\right\} \sum_{j=1}^{n} m_{jj} \right) \mathbf{1} - \sum_{j=1}^{n} m_{jj}^2$$

$$= \mathbf{1}' \left(\mathrm{diag}\left\{m_{ii}\right\} (n-K) \right) \mathbf{1} - \sum_{j=1}^{n} m_{jj}^2 = (n-K) \mathbf{1}' \left(\mathrm{diag}\left\{m_{ii}\right\} \right) \mathbf{1} - \sum_{j=1}^{n} m_{jj}^2 = (n-K)^2 - \sum_{j=1}^{n} m_{jj}^2$$

The 2nd component of the double sum, $2\sum_{\xi \neq v} m_{\xi v} m_{v\xi}$, due to symmetry can be obtained as

$$2\sum_{\xi \neq v} m_{\xi v} m_{v\xi} = 2\left(\mathbf{1}' (\mathrm{M} \circ \mathrm{M}) \mathbf{1}\right) - 2\sum_{j=1}^{n} m_{jj}^2 = 2\,\mathrm{tr}(\mathrm{MM}) - 2\sum_{j=1}^{n} m_{jj}^2 = 2\,\mathrm{tr}(\mathrm{M}) - 2\sum_{j=1}^{n} m_{jj}^2$$

$$= 2(n-K) - 2\sum_{j=1}^{n} m_{jj}^2$$

So

$$c_{44} = \frac{1}{n(n-1)} \sum_{\xi \neq v} \left(m_{vv} m_{\xi\xi} + 2 m_{\xi v} m_{v\xi} \right) = \frac{1}{n(n-1)} \left[(n-K)^2 - \sum_{j=1}^{n} m_{jj}^2 + 2(n-K) - 2\sum_{j=1}^{n} m_{jj}^2 \right]$$

$$\Rightarrow c_{44} = \frac{(n-K)(n-K+2) - 3\sum_{j=1}^{n} m_{jj}^2}{n(n-1)} = \frac{(n-K)(n-K+2) - 3\,\mathrm{tr}\left\{\mathrm{M}^{(2)}\right\}}{n(n-1)}$$

Consider now the expression

$$\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^4 - \frac{1}{n(n-1)}\sum_{\xi\neq v}\hat{\varepsilon}_\xi^2\hat{\varepsilon}_v^2 \Rightarrow E\left(\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^4 - \frac{1}{n(n-1)}\sum_{\xi\neq v}\hat{\varepsilon}_\xi^2\hat{\varepsilon}_v^2 \,|\, X\right) =$$

$$= \mu_4 c_{41} + \sigma^4\left(3c_{42} - 3c_{41}\right) - \mu_4 c_{43} - \sigma^4\left(c_{44} - 3c_{43}\right)$$

This guides us for the adjustments we have to make. Consider the estimator

$$\frac{c_{44} - 3c_{43}}{c_{41}c_{44} - 3c_{42}c_{43}}\left[\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^4 - \frac{\left(3c_{42} - 3c_{41}\right)}{\left(c_{44} - 3c_{43}\right)}\frac{1}{n(n-1)}\sum_{\xi\neq v}\hat{\varepsilon}_\xi^2\hat{\varepsilon}_v^2\right]$$

Its conditional expected value is

$$E\left\{\frac{c_{44} - 3c_{43}}{c_{41}c_{44} - 3c_{42}c_{43}}\left[\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^4 - \frac{\left(3c_{42} - 3c_{41}\right)}{\left(c_{44} - 3c_{43}\right)}\frac{1}{n(n-1)}\sum_{\xi\neq v}\hat{\varepsilon}_\xi^2\hat{\varepsilon}_v^2\right]|\mathrm{X}\right\}$$

$$= \frac{c_{44} - 3c_{43}}{c_{41}c_{44} - 3c_{42}c_{43}}E\left\{\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^4 - \frac{\left(3c_{42} - 3c_{41}\right)}{\left(c_{44} - 3c_{43}\right)}\frac{1}{n(n-1)}\sum_{\xi\neq v}\hat{\varepsilon}_\xi^2\hat{\varepsilon}_v^2 \,|\mathrm{X}\right\}$$

$$= \frac{c_{44} - 3c_{43}}{c_{41}c_{44} - 3c_{42}c_{43}}\Big\{\mu_4 c_{41} + \sigma^4\left(3c_{42} - 3c_{41}\right)$$

$$-\mu_4 c_{43}\frac{\left(3c_{42} - 3c_{41}\right)}{\left(c_{44} - 3c_{43}\right)} - \sigma^4\left(c_{44} - 3c_{43}\right)\frac{\left(3c_{42} - 3c_{41}\right)}{\left(c_{44} - 3c_{43}\right)}\Big\}$$

$$= \frac{c_{44} - 3c_{43}}{c_{41}c_{44} - 3c_{42}c_{43}}\left[\mu_4 c_{41} - \mu_4 c_{43}\frac{\left(3c_{42} - 3c_{41}\right)}{\left(c_{44} - 3c_{43}\right)}\right] = \mu_4$$

Therefore an unbiased estimator for the 4th central moment is

$$[3.59],[3.61]: \quad \hat{\mu}_4 = \frac{c_{44}-3c_{43}}{c_{41}c_{44}-3c_{42}c_{43}}\left[\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^4 \;-\; \frac{(3c_{42}-3c_{41})}{(c_{44}-3c_{43})}\frac{1}{n(n-1)}\sum_{\xi\neq v}\hat{\varepsilon}_\xi^2\hat{\varepsilon}_v^2\right]$$

$$c_{41}=\frac{1}{n}\operatorname{tr}\left\{\mathrm{M}^{(2)}\mathrm{M}^{(2)}\right\}, \quad c_{42}=\frac{1}{n}\operatorname{tr}\left\{\mathrm{M}^{(2)}\right\}, \quad c_{43}=\frac{\mathbf{1}'\mathrm{M}^{(2)}\mathrm{M}^{(2)}\mathbf{1}-nc_{41}}{n(n-1)} \qquad [8]$$

$$c_{44}=\frac{(n-K)(n-K+2)-3nc_{42}}{n(n-1)}$$

At the limit we have $c_{41}\to 1$, $c_{42}\to 1$, $c_{43}\to 0$, $c_{44}\to 1$ so

$$\hat{\mu}_4 \to \frac{1-0}{1-0}\left[\operatorname{plim}\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^4 \;-\; \frac{0}{1-0}\operatorname{plim}\frac{1}{n(n-1)}\sum_{\xi\neq v}\hat{\varepsilon}_\xi^2\hat{\varepsilon}_v^2\right]=\mu_4$$

We turn to obtain an unbiased estimator for the 4th cumulant $\kappa_4=\mu^4-3\sigma^4$. Rewrite

$$\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^4 \;-\; \frac{1}{n(n-1)}\sum_{\xi\neq v}\hat{\varepsilon}_\xi^2\hat{\varepsilon}_v^2 : E\left(\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^4 - \frac{1}{n(n-1)}\sum_{\xi\neq v}\hat{\varepsilon}_\xi^2\hat{\varepsilon}_v^2 \,|\mathrm{X}\right)=$$

$$=\left(\mu_4-3\sigma^4\right)c_{41} + \sigma^4 3c_{42} - \left(\mu_4-3\sigma^4\right)c_{43} - \sigma^4 c_{44}$$

$$=\kappa_4 c_{41} + \sigma^4 3c_{42} - \kappa_4 c_{43} - \sigma^4 c_{44}$$

Here consider the estimator

$$\hat{\kappa}_4(\varepsilon)=\frac{c_{44}}{c_{41}c_{44}-3c_{42}c_{43}}\left[\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^4 \;-\; \frac{3c_{42}}{c_{44}}\frac{1}{n(n-1)}\sum_{\xi\neq v}\hat{\varepsilon}_\xi^2\hat{\varepsilon}_v^2\right]$$

Its conditional expected value is

$$E\left\{\frac{c_{44}}{c_{41}c_{44}-3c_{42}c_{43}}\left[\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^4 \;-\frac{3c_{42}}{c_{44}}\frac{1}{n(n-1)}\sum_{\xi\neq v}\hat{\varepsilon}_\xi^2\hat{\varepsilon}_v^2\right]\middle|\mathrm{X}\right\}$$

$$=\frac{c_{44}}{c_{41}c_{44}-3c_{42}c_{43}}E\left\{\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^4 \;-\frac{3c_{42}}{c_{44}}\frac{1}{n(n-1)}\sum_{\xi\neq v}\hat{\varepsilon}_\xi^2\hat{\varepsilon}_v^2\,\middle|X\right\}$$

$$=\frac{c_{44}}{c_{41}c_{44}-3c_{42}c_{43}}E\left\{\kappa_4c_{41}\;+\;\sigma^4 3c_{42}\;-\;\kappa_4c_{43}\frac{3c_{42}}{c_{44}}\;-\;\sigma^4c_{44}\frac{3c_{42}}{c_{44}}\middle|\mathrm{X}\right\}$$

$$=\frac{c_{44}}{c_{41}c_{44}-3c_{42}c_{43}}E\left[\kappa_4c_{41}-\kappa_4c_{43}\frac{3c_{42}}{c_{44}}\middle|\mathrm{X}\right]=\kappa_4$$

Therefore, the fourth kapa-statistic

$$[3.60]:\quad \hat{\kappa}_4(\varepsilon)=\frac{c_{44}}{c_{41}c_{44}-3c_{42}c_{43}}\left[\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^4 \;-\frac{3c_{42}}{c_{44}}\frac{1}{n(n-1)}\sum_{\xi\neq v}\hat{\varepsilon}_\xi^2\hat{\varepsilon}_v^2\right] \qquad [9]$$

is an unbiased estimator for the fourth cumulant.

Consistency is also preserved since

$$\hat{\kappa}_4(\varepsilon)\rightarrow\frac{1}{1-0}\left[\kappa_4\;+\;3\sigma^4\;-3\left(\kappa_4\cdot 0\;+\sigma^4\cdot 1\right)\right]=\kappa_4$$

B.2.4. 5th central moment.

We have

$$\hat{\varepsilon}_j^5=\hat{\varepsilon}_j^2\hat{\varepsilon}_j^3=\mathbf{m}'_j\begin{bmatrix}\tilde{\varepsilon}_1^2 & \tilde{\varepsilon}_1\tilde{\varepsilon}_2 & \cdots & \tilde{\varepsilon}_1\tilde{\varepsilon}_n\\ \tilde{\varepsilon}_2\tilde{\varepsilon}_1 & \ddots & & \vdots\\ \vdots & & \ddots & \vdots\\ \tilde{\varepsilon}_n\tilde{\varepsilon}_1 & \tilde{\varepsilon}_n\tilde{\varepsilon}_2 & \cdots & \tilde{\varepsilon}_n^2\end{bmatrix}\mathbf{m}_j\left(\mathbf{m}'_j\tilde{\varepsilon}\right)^3$$

$$= \mathbf{m}_j' \begin{bmatrix} \tilde{\varepsilon}_1^2 \left(\mathbf{m}_j'\tilde{\varepsilon}\right)^3 & \tilde{\varepsilon}_1 \tilde{\varepsilon}_2 \left(\mathbf{m}_j'\tilde{\varepsilon}\right)^3 & \cdots & \tilde{\varepsilon}_1 \tilde{\varepsilon}_n \left(\mathbf{m}_j'\tilde{\varepsilon}\right)^3 \\ \tilde{\varepsilon}_2 \tilde{\varepsilon}_1 \left(\mathbf{m}_j'\tilde{\varepsilon}\right)^3 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \tilde{\varepsilon}_n \tilde{\varepsilon}_1 \left(\mathbf{m}_j'\tilde{\varepsilon}\right)^3 & \tilde{\varepsilon}_n \tilde{\varepsilon}_2 \left(\mathbf{m}_j'\tilde{\varepsilon}\right)^3 & \cdots & \tilde{\varepsilon}_n^2 \left(\mathbf{m}_j'\tilde{\varepsilon}\right)^3 \end{bmatrix} \mathbf{m}_j \equiv \mathbf{m}_j' \mathrm{A} \mathbf{m}_j$$

Now, $\left(\mathbf{m}_j'\tilde{\varepsilon}\right)^3 = \left(\sum_{i=1}^{n} m_{ji}\tilde{\varepsilon}_i\right)^2 \left(\sum_{i=1}^{n} m_{ji}\tilde{\varepsilon}_i\right) = \left(\sum_{i=1}^{n} m_{ji}^2\tilde{\varepsilon}_i^2 + \sum\sum_{k\neq\ell} m_{jk} m_{j\ell}\tilde{\varepsilon}_k\tilde{\varepsilon}_\ell\right)\left(\sum_{i=1}^{n} m_{ji}\tilde{\varepsilon}_i\right)$

$$= \sum_{i=1}^{n} m_{ji}^3\tilde{\varepsilon}_i^3 + \sum_{k\neq\ell}^{n}\left(m_{jk}\tilde{\varepsilon}_k \sum_{\ell\neq k}^{n} m_{j\ell}^2\tilde{\varepsilon}_\ell^2\right) + \sum_{i=1}^{n}\left(m_{ji}\tilde{\varepsilon}_i \sum\sum_{k\neq\ell} m_{jk} m_{j\ell}\tilde{\varepsilon}_k\tilde{\varepsilon}_\ell\right)$$

We will use μ_2, μ_3, μ_5 for the 2nd, 3d and 5th central moment.

We calculate the elements of the matrix A

a) Diagonal terms, $[\xi,\xi]$: $E\left(\tilde{\varepsilon}_\xi^2 \left(\mathbf{m}_j'\tilde{\varepsilon}\right)^3 | \mathrm{X}\right)$

First,

$$E\left(\tilde{\varepsilon}_\xi^2 \sum_{i=1}^{n} m_{ji}^3\tilde{\varepsilon}_i^3 | \mathrm{X}\right) = m_{j\xi}^3\mu_5 + \mu_2\mu_3\sum_{i\neq\xi}^{n} m_{ji}^3 = m_{j\xi}^3\left(\mu_5 - \mu_2\mu_3\right) + \mu_2\mu_3\sum_{i=1}^{n} m_{ji}^3 \qquad [10]$$

Second,

$E\left(\tilde{\varepsilon}_\xi^2 \sum_{k\neq\ell}^{n}\left(m_{jk}\tilde{\varepsilon}_k \sum_{\ell\neq k}^{n} m_{j\ell}^2\tilde{\varepsilon}_\ell^2\right) | \mathrm{X}\right)$: since $\tilde{\varepsilon}_k$ never pairs in index with the inner sum, the only non-zero terms here will be those for which $\xi = k$ and so necessarily $\xi = k \neq \ell$. So

$$E\left(\tilde{\varepsilon}_\xi^2 \sum_{k\neq\ell}^{n}\left(m_{jk}\tilde{\varepsilon}_k \sum_{\ell\neq k}^{n} m_{j\ell}^2\tilde{\varepsilon}_\ell^2\right) | \mathrm{X}\right) = 0 + E\left(m_{j\xi}\tilde{\varepsilon}_\xi^3 \sum_{\ell\neq\xi}^{n} m_{j\ell}^2\tilde{\varepsilon}_\ell^2 | \mathrm{X}\right) = \mu_2\mu_3 m_{j\xi}\sum_{\ell\neq\xi}^{n} m_{j\ell}^2 \qquad [11]$$

Third,

$$E\left[\tilde{\varepsilon}_\xi^2 \sum_{i=1}^{n}\left(m_{ji}\tilde{\varepsilon}_i \sum\sum_{k\neq\ell} m_{jk}m_{j\ell}\tilde{\varepsilon}_k\tilde{\varepsilon}_\ell \,|\mathrm{X}\right)\right]$$: here we will have four $\tilde{\varepsilon}$'s in each final element of the sum product. The non-zero terms will be those where two index-pairs appear, so that no 1st power is present. Moreover, since the inner double sum contains products with different indexes always, then we have only the cases $\xi = k \neq \ell = i$ and $\xi = \ell \neq k = i$. So

$$\xi = k \neq \ell = i: \quad E\left[\sum_{i\neq\xi}^{n}\left(m_{ji}^2\tilde{\varepsilon}_i^2 m_{j\xi}\tilde{\varepsilon}_\xi^3 \,|\mathrm{X}\right)\right] = \mu_2\mu_3 m_{j\xi}\sum_{i\neq\xi}^{n} m_{ji}^2 \qquad (n-1) \text{ terms}$$

$$\xi = \ell \neq k = i: \quad E\left[\sum_{i\neq\xi}^{n}\left(m_{ji}^2\tilde{\varepsilon}_i^2 m_{j\xi}\tilde{\varepsilon}_\xi^3 \,|\mathrm{X}\right)\right] = \mu_2\mu_3 m_{j\xi}\sum_{i\neq\xi}^{n} m_{ji}^2 \qquad (n-1) \text{ terms}$$

and therefore

$$E\left[\tilde{\varepsilon}_\xi^2 \sum_{i=1}^{n}\left(m_{ji}\tilde{\varepsilon}_i \sum\sum_{k\neq\ell} m_{jk}m_{j\ell}\tilde{\varepsilon}_k\tilde{\varepsilon}_\ell \,|\mathrm{X}\right)\right] = 2\mu_2\mu_3 m_{j\xi}\sum_{i\neq\xi}^{n} m_{ji}^2 \qquad [12]$$

Putting it all together ([10],[11], [12]) we have for the diagonal terms

$$[\xi,\xi]: \; E\left(\tilde{\varepsilon}_\xi^2\left(\mathbf{m}'_j\tilde{\varepsilon}\right)^3 |\mathrm{X}\right) = m_{j\xi}^3\left(\mu_5 - \mu_2\mu_3\right) + \; \mu_2\mu_3\sum_{i=1}^{n} m_{ji}^3 + \mu_2\mu_3 m_{j\xi}\sum_{\ell\neq\xi}^{n} m_{j\ell}^2$$
$$+ \; 2\mu_2\mu_3 m_{j\xi}\sum_{i\neq\xi}^{n} m_{ji}^2$$

$$= m_{j\xi}^3\left(\mu_5 - \mu_2\mu_3\right) + \; \mu_2\mu_3\sum_{i=1}^{n} m_{ji}^3 + 3\mu_2\mu_3 m_{j\xi}\sum_{\ell\neq\xi}^{n} m_{j\ell}^2$$

$$= m_{j\xi}^3\left(\mu_5 - \mu_2\mu_3\right) + \; \mu_2\mu_3\sum_{i=1}^{n} m_{ji}^3 + 3\mu_2\mu_3 m_{j\xi}\left(\sum_{\ell=1}^{n} m_{j\ell}^2 - m_{j\xi}^2\right)$$

and since $\sum_{\ell=1}^{n} m_{j\ell}^2 = m_{jj}$ we arrive at

$$[\xi,\xi]:\ E\left(\tilde{\varepsilon}_{\xi}^{2}\left(\mathbf{m}_{j}'\tilde{\varepsilon}\right)^{3}|\mathrm{X}\right)=m_{j\xi}^{3}\left(\mu_{5}-\mu_{2}\mu_{3}\right)+\ \mu_{2}\mu_{3}\sum_{i=1}^{n}m_{ji}^{3}+3\mu_{2}\mu_{3}m_{j\xi}m_{jj}\ -\ 3\mu_{2}\mu_{3}m_{j\xi}^{3} \qquad [13]$$

b) Off-diagonal terms, $[\xi,v]:E\left(\tilde{\varepsilon}_{\xi}\tilde{\varepsilon}_{v}\left(\mathbf{m}_{j}'\tilde{\varepsilon}\right)^{3}|\mathrm{X}\right)$

First, $$E\left(\tilde{\varepsilon}_{\xi}\tilde{\varepsilon}_{v}\sum_{i=1}^{n}m_{ji}^{3}\tilde{\varepsilon}_{i}^{3}|\mathrm{X}\right)=0 \qquad [14]$$

since a 1st power is always present.

Second,

$E\left(\tilde{\varepsilon}_{\xi}\tilde{\varepsilon}_{v}\sum_{k\neq\ell}^{n}\left(m_{jk}\tilde{\varepsilon}_{k}\sum_{\ell\neq k}^{n}m_{j\ell}^{2}\tilde{\varepsilon}_{\ell}^{2}\right)|\mathrm{X}\right)$: we need to form pairs of indexes to obtain non-zero elements. So we have the cases $\xi=k,v=\ell$ or $\xi=\ell,v=k$. Then

$$\xi=k,v=\ell:\ E\left(\tilde{\varepsilon}_{\xi}\tilde{\varepsilon}_{v}\sum_{k\neq\ell}^{n}\left(m_{jk}\tilde{\varepsilon}_{k}\sum_{\ell\neq k}^{n}m_{j\ell}^{2}\tilde{\varepsilon}_{\ell}^{2}\right)|\mathrm{X}\right)=E\left(m_{j\xi}m_{jv}^{2}\tilde{\varepsilon}_{\xi}^{2}\tilde{\varepsilon}_{v}^{3}|\mathrm{X}\right)=\mu_{2}\mu_{3}m_{j\xi}m_{jv}^{2}$$

$$\xi=\ell,v=k:\ E\left(\tilde{\varepsilon}_{\xi}\tilde{\varepsilon}_{v}\sum_{k\neq\ell}^{n}\left(m_{jk}\tilde{\varepsilon}_{k}\sum_{\ell\neq k}^{n}m_{j\ell}^{2}\tilde{\varepsilon}_{\ell}^{2}\right)|\mathrm{X}\right)=E\left(m_{j\xi}^{2}m_{jv}\tilde{\varepsilon}_{\xi}^{3}\tilde{\varepsilon}_{v}^{2}|\mathrm{X}\right)=\mu_{2}\mu_{3}m_{j\xi}^{2}m_{jv}$$

So $$E\left(\tilde{\varepsilon}_{\xi}\tilde{\varepsilon}_{v}\sum_{k\neq\ell}^{n}\left(m_{jk}\tilde{\varepsilon}_{k}\sum_{\ell\neq k}^{n}m_{j\ell}^{2}\tilde{\varepsilon}_{\ell}^{2}\right)|\mathrm{X}\right)=\mu_{2}\mu_{3}\left(m_{j\xi}^{2}m_{jv}+m_{j\xi}m_{jv}^{2}\right) \qquad [15]$$

Third,

$E\left(\tilde{\varepsilon}_{\xi}\tilde{\varepsilon}_{v}\sum_{i=1}^{n}\left(m_{ji}\tilde{\varepsilon}_{i}\sum\sum_{k\neq\ell}m_{jk}m_{j\ell}\tilde{\varepsilon}_{k}\tilde{\varepsilon}_{\ell}\right)|\mathrm{X}\right)$: here (ξ,v) must pair to (k,ℓ) to have non-zero elements. Due also to symmetry we get

$$E\left(\tilde{\varepsilon}_{\xi}\tilde{\varepsilon}_{v}\sum_{i=1}^{n}\left(m_{ji}\tilde{\varepsilon}_{i}\sum\sum_{k\neq\ell}m_{jk}m_{j\ell}\tilde{\varepsilon}_{k}\tilde{\varepsilon}_{\ell}\right)|\mathrm{X}\right)=E\left(\sum_{i=1}^{n}m_{ji}\tilde{\varepsilon}_{i}\left(2m_{j\xi}m_{jv}\tilde{\varepsilon}_{\xi}^{2}\tilde{\varepsilon}_{v}^{2}\right)|\mathrm{X}\right)$$

$$= 2E\left(m_{j\xi}^2 m_{jv}\tilde{\varepsilon}_\xi^3\tilde{\varepsilon}_v^2 + m_{j\xi}m_{jv}^2\tilde{\varepsilon}_\xi^2\tilde{\varepsilon}_v^3 \middle| \mathrm{X}\right) = 2\mu_2\mu_3\left(m_{j\xi}^2 m_{jv} + m_{j\xi}m_{jv}^2\right)$$

So overall

$$[\xi, v]:\ E\left(\tilde{\varepsilon}_\xi\tilde{\varepsilon}_v\left(\mathbf{m}'_j\tilde{\varepsilon}\right)^3 \middle| \mathrm{X}\right) = 3\mu_2\mu_3\left(m_{j\xi}^2 m_{jv} + m_{j\xi}m_{jv}^2\right) \qquad [16]$$

Now we want to calculate $\mathbf{m}'_j E\left(\mathrm{A}|\mathrm{X}\right)\mathbf{m}_j$ where the diagonal of $E\left(\mathrm{A}|\mathrm{X}\right)$ is given by [13],

$$[\xi, \xi]:\ m_{j\xi}^3\left(\mu_5 - \mu_2\mu_3\right) + \mu_2\mu_3\sum_{i=1}^{n} m_{ji}^3 + 3\mu_2\mu_3\left(m_{j\xi}m_{jj} - m_{j\xi}^3\right)$$

and the off-diagonal elements by [16],

$$[\xi, v] = 3\mu_2\mu_3\left(m_{j\xi}^2 m_{jv} + m_{j\xi}m_{jv}^2\right)$$

As before, we break initially the matrix in a sum of three matrices with the first being diagonal and the second the identity matrix,

$$E\left(\hat{\varepsilon}_j^5 \middle| \mathrm{X}\right) = \mathbf{m}'_j E\left(\mathrm{A}|\mathrm{X}\right)\mathbf{m}_j = \mathbf{m}'_j\left[E\left(\mathrm{A}_1|\mathrm{X}\right) + E\left(\mathrm{A}_2|\mathrm{X}\right) + 3\mu_2\mu_3 E\left(\mathrm{A}_3|\mathrm{X}\right)\right]\mathbf{m}_j$$

$$= \left(\mu_5 - \mu_2\mu_3\right)\mathbf{m}'_j\begin{bmatrix} m_{j1}^3 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & m_{jn}^3 \end{bmatrix}\mathbf{m}_j + \mu_2\mu_3\left(\sum_{i=1}^{n} m_{ji}^3\right)\mathbf{m}'_j I_n \mathbf{m}_j + 3\mu_2\mu_3\mathbf{m}'_j E\left(\mathrm{A}_3|\mathrm{X}\right)\mathbf{m}_j$$

$$E\left(\mathrm{A}_3|\mathrm{X}\right) = \begin{bmatrix} m_{j1}m_{jj} - m_{j1}^3 & \cdots & m_{j1}^2 m_{jn} + m_{j1}m_{jn}^2 \\ m_{j2}^2 m_{j1} + m_{j2}m_{j1}^2 & \ddots & \vdots \\ \vdots & \cdots & m_{j1}m_{jj} - m_{j1}^3 \end{bmatrix}$$

$$= \left(\mu_5 - \mu_2\mu_3\right)\sum_{i=1}^{n} m_{ji}^5 + \mu_2\mu_3 m_{jj}\left(\sum_{i=1}^{n} m_{ji}^3\right) + 3\mu_2\mu_3\mathbf{m}'_j E\left(\mathrm{A}_3|\mathrm{X}\right)\mathbf{m}_j \qquad [17]$$

We turn to $E\left(\mathrm{A}_3 \middle| \mathrm{X}\right)$ which we decompose into

$$\begin{bmatrix} m_{j1}m_{jj} - m_{j1}^3 & m_{j1}^2 m_{j2} + m_{j1} m_{j2}^2 & \cdots & m_{j1}^2 m_{jn} + m_{j1} m_{jn}^2 \\ m_{j2}^2 m_{j1} + m_{j2} m_{j1}^2 & \ddots & & m_{j2}^2 m_{jn} + m_{j2} m_{jn}^2 \\ \vdots & & \ddots & \vdots \\ m_{jn}^2 m_{j1} + m_{jn} m_{j1}^2 & & & m_{j1}m_{jj} - m_{j1}^3 \end{bmatrix} = m_{jj}\,\mathrm{diag}\left\{m_{ji}\right\} - \mathrm{diag}\left\{m_{ji}^3\right\}$$

$$+ \begin{bmatrix} 0 & m_{j1}^2 m_{j2} & \cdots & m_{j1}^2 m_{jn} \\ m_{j2}^2 m_{j1} & 0 & m_{j2}^2 m_{j3} & \cdots \\ & & \ddots & \\ m_{jn}^2 m_{j1} & m_{jn}^2 m_{j2} & & 0 \end{bmatrix} + \begin{bmatrix} 0 & m_{j1} m_{j2}^2 & \cdots & m_{j1} m_{jn}^2 \\ m_{j2} m_{j1}^2 & 0 & m_{j2} m_{j3}^2 & \cdots \\ & & \ddots & \\ m_{jn} m_{j1}^2 & m_{jn} m_{j2}^2 & & 0 \end{bmatrix}$$

$$= m_{jj}\,\mathrm{diag}\left\{m_{ji}\right\} - \mathrm{diag}\left\{m_{ji}^3\right\} - \mathrm{diag}\left\{m_{ji}^3\right\} - \mathrm{diag}\left\{m_{ji}^3\right\}$$

$$+ \begin{bmatrix} m_{j1}^3 & m_{j1}^2 m_{j2} & \cdots & m_{j1}^2 m_{jn} \\ m_{j2}^2 m_{j1} & m_{j2}^3 & m_{j2}^2 m_{j3} & \cdots \\ & & m_{j3}^3 & \\ m_{jn}^2 m_{j1} & m_{jn}^2 m_{j2} & & m_{jn}^3 \end{bmatrix} + \begin{bmatrix} m_{j1}^3 & m_{j1} m_{j2}^2 & \cdots & m_{j1} m_{jn}^2 \\ m_{j2} m_{j1}^2 & m_{j2}^3 & m_{j2} m_{j3}^2 & \cdots \\ & & m_{j3}^3 & \\ m_{jn} m_{j1}^2 & m_{jn} m_{j2}^2 & & m_{jn}^3 \end{bmatrix}$$

$$\Rightarrow E\left(\mathrm{A}_3 \middle| \mathrm{X}\right) = m_{jj}\,\mathrm{diag}\left\{m_{ji}\right\} - 3\,\mathrm{diag}\left\{m_{ji}^3\right\} + \mathbf{m}_j^{(2)}\mathbf{m}_j' + \mathbf{m}_j\left(\mathbf{m}_j^{(2)}\right)' \qquad [18]$$

Therefore

$$E\left(\hat{\varepsilon}_j^5 \middle| \mathrm{X}\right) = \left(\mu_5 - \mu_2\mu_3\right)\sum_{i=1}^n m_{ji}^5 + \mu_2\mu_3 m_{jj}\left(\sum_{i=1}^n m_{ji}^3\right)$$
$$+ 3\mu_2\mu_3\mathbf{m}_j'\left[m_{jj}\,\mathrm{diag}\left\{m_{ji}\right\} - 3\,\mathrm{diag}\left\{m_{ji}^3\right\} + \mathbf{m}_j^{(2)}\mathbf{m}_j' + \mathbf{m}_j\left(\mathbf{m}_j^{(2)}\right)'\right]\mathbf{m}_j$$

$$= \left(\mu_5 - \mu_2\mu_3\right)\sum_{i=1}^n m_{ji}^5 + \mu_2\mu_3 m_{jj}\left(\sum_{i=1}^n m_{ji}^3\right) + 3\mu_2\mu_3 m_{jj}\left(\sum_{i=1}^n m_{ji}^3\right) - 9\mu_2\mu_3\sum_{i=1}^n m_{ji}^5$$
$$+ 3\mu_2\mu_3\mathbf{m}_j'\mathbf{m}_j^{(2)}\mathbf{m}_j'\mathbf{m}_j + 3\mu_2\mu_3\mathbf{m}_j'\mathbf{m}_j\left(\mathbf{m}_j^{(2)}\right)'\mathbf{m}_j$$

We have $\mathbf{m}_j'\mathbf{m}_j = m_{jj}$, $\quad \mathbf{m}_j'\mathbf{m}_j^{(2)} = \left(\mathbf{m}_j^{(2)}\right)'\mathbf{m}_j = \sum_{i=1}^{n} m_{ji}^3$

So

$$E\left(\hat{\varepsilon}_j^5 \middle| \mathrm{X}\right) = \left(\mu_5 - \mu_2\mu_3\right)\sum_{i=1}^{n} m_{ji}^5 + \mu_2\mu_3 m_{jj}\left(\sum_{i=1}^{n} m_{ji}^3\right) + 3\mu_2\mu_3 m_{jj}\left(\sum_{i=1}^{n} m_{ji}^3\right) - 9\mu_2\mu_3\sum_{i=1}^{n} m_{ji}^5$$
$$+ 3\mu_2\mu_3 m_{jj}\left(\sum_{i=1}^{n} m_{ji}^3\right) + 3\mu_2\mu_3 m_{jj}\left(\sum_{i=1}^{n} m_{ji}^3\right)$$

$$\Rightarrow E\left(\hat{\varepsilon}_j^5 \middle| \mathrm{X}\right) = \left(\mu_5 - 10\mu_2\mu_3\right)\sum_{i=1}^{n} m_{ji}^5 + 10\mu_2\mu_3 m_{jj}\left(\sum_{i=1}^{n} m_{ji}^3\right) \qquad [19]$$

or also

$$E\left(\hat{\varepsilon}_j^5 \middle| \mathrm{X}\right) = \mu_5\sum_{i=1}^{n} m_{ji}^5 + 10\mu_2\mu_3\left[m_{jj}\left(\sum_{i=1}^{n} m_{ji}^3\right) - \sum_{i=1}^{n} m_{ji}^5\right]$$

and therefore

$$E\left(\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^5 \middle| X\right) = \mu_5 c_{51} + \mu_2\mu_3\left(10c_{52} - 10c_{51}\right)$$

$$c_{51} = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{n} m_{ji}^5, \quad c_{52} = \frac{1}{n}\sum_{j=1}^{n}\left[m_{jj}\left(\sum_{i=1}^{n} m_{ji}^3\right)\right] \qquad [20]$$

We can also write

$$c_{51} = \frac{\mathbf{1}'\left(\mathrm{M}\circ\mathrm{M}\circ\mathrm{M}\circ\mathrm{M}\circ\mathrm{M}\right)\mathbf{1}}{n} = \frac{\mathbf{1}'\left(\left(\mathrm{M}\circ\mathrm{M}\circ\mathrm{M}\right)\circ\left(\mathrm{M}\circ\mathrm{M}\right)\right)\mathbf{1}}{n} = \frac{\mathrm{tr}\left\{\mathrm{M}^{(3)}\mathrm{M}^{(2)}\right\}}{n}$$

$$c_{52} = \frac{\mathbf{1}'\mathrm{M}^{(3)}\mathbf{m}_d}{n}$$

where $\mathbf{m}_d$ is a column vector holding the diagonal elements of M.

Now to arrive at an unbiased estimator, we need an estimator for $\mu_2\mu_3$. Consider

$$[\xi \neq v]: \hat{\varepsilon}_\xi^2 \hat{\varepsilon}_v^3 = \left(\mathbf{m}'_\xi \tilde{\varepsilon}\right)^2 \left(\mathbf{m}'_v \tilde{\varepsilon}\right)^3 = \mathbf{m}'_\xi \tilde{\varepsilon}\tilde{\varepsilon}' \mathbf{m}_\xi \left(\mathbf{m}'_v \tilde{\varepsilon}\right)^3$$

$$= \mathbf{m}'_\xi \begin{bmatrix} \tilde{\varepsilon}_1^2 \left(\mathbf{m}'_v \tilde{\varepsilon}\right)^3 & \tilde{\varepsilon}_1 \tilde{\varepsilon}_2 \left(\mathbf{m}'_v \tilde{\varepsilon}\right)^3 & \cdots & \tilde{\varepsilon}_1 \tilde{\varepsilon}_n \left(\mathbf{m}'_v \tilde{\varepsilon}\right)^3 \\ \tilde{\varepsilon}_2 \tilde{\varepsilon}_1 \left(\mathbf{m}'_v \tilde{\varepsilon}\right)^3 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \tilde{\varepsilon}_n \tilde{\varepsilon}_1 \left(\mathbf{m}'_v \tilde{\varepsilon}\right)^3 & \tilde{\varepsilon}_n \tilde{\varepsilon}_2 \left(\mathbf{m}'_v \tilde{\varepsilon}\right)^3 & \cdots & \tilde{\varepsilon}_n^2 \left(\mathbf{m}'_v \tilde{\varepsilon}\right)^3 \end{bmatrix} \mathbf{m}_\xi \equiv \mathbf{m}'_\xi \mathrm{A}_v \mathbf{m}_\xi$$

This is the same matrix as before, the only difference is that the outside vectors are from a different observation. So we will arrive at

$$E\left(\hat{\varepsilon}_\xi^2 \hat{\varepsilon}_v^3 \middle| \mathrm{X}\right) = \left(\mu_5 - \mu_2\mu_3\right) \mathbf{m}'_\xi \begin{bmatrix} m_{v1}^3 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & m_{vn}^3 \end{bmatrix} \mathbf{m}_\xi + \mu_2\mu_3 \left(\sum_{i=1}^n m_{vi}^3\right) \mathbf{m}'_\xi I_n \mathbf{m}_\xi + 3\mu_2\mu_3 \mathbf{m}'_\xi E\left(\mathrm{A}_3 \middle| \mathrm{X}\right) \mathbf{m}_\xi$$

with $$E\left(\mathrm{A}_3 \middle| \mathrm{X}\right) = m_{vv} \operatorname{diag}\{m_{vi}\} - 3\operatorname{diag}\left\{m_{vi}^3\right\} + \mathbf{m}_v^{(2)} \mathbf{m}'_v + \mathbf{m}_v \left(\mathbf{m}_v^{(2)}\right)'$$

So

$$E\left(\hat{\varepsilon}_\xi^2 \hat{\varepsilon}_v^3 \middle| \mathrm{X}\right) = \left(\mu_5 - \mu_2\mu_3\right) \sum_{i=1}^n m_{\xi i}^2 m_{vi}^3 + \mu_2\mu_3 \left(\sum_{i=1}^n m_{vi}^3\right) m_{\xi\xi} + 3\mu_2\mu_3 \mathbf{m}'_\xi m_{vv} \operatorname{diag}\{m_{vi}\} \mathbf{m}_\xi$$

$$-3\mu_2\mu_3 \mathbf{m}'_\xi 3\operatorname{diag}\left\{m_{vi}^3\right\} \mathbf{m}_\xi + 3\mu_2\mu_3 \mathbf{m}'_\xi \mathbf{m}_v^{(2)} \mathbf{m}'_v \mathbf{m}_\xi + 3\mu_2\mu_3 \mathbf{m}'_\xi \mathbf{m}_v \left(\mathbf{m}_v^{(2)}\right)' \mathbf{m}_\xi$$

$$= \left(\mu_5 - \mu_2\mu_3\right) \sum_{i=1}^n m_{\xi i}^2 m_{vi}^3 + \mu_2\mu_3 \left(\sum_{i=1}^n m_{vi}^3\right) m_{\xi\xi} + 3\mu_2\mu_3 m_{vv} \sum_{i=1}^n m_{\xi i}^2 m_{vi}$$

$$-9\mu_2\mu_3 \sum_{i=1}^n m_{\xi i}^2 m_{vi}^3 + 3\mu_2\mu_3 m_{\xi v} \left(\sum_{i=1}^n m_{\xi i} m_{vi}^2\right) + 3\mu_2\mu_3 m_{\xi v} \left(\sum_{i=1}^n m_{\xi i} m_{vi}^2\right)$$

$$=\left(\mu_5-10\mu_2\mu_3\right)\sum_{i=1}^{n}m_{\xi i}^2 m_{vi}^3 \;+\; \mu_2\mu_3\left(\sum_{i=1}^{n}m_{vi}^3\right)m_{\xi\xi} \;+3\mu_2\mu_3\, m_{vv}\sum_{i=1}^{n}m_{\xi i}^2 m_{vi}$$

$$+6\mu_2\mu_3\, m_{\xi v}\left(\sum_{i=1}^{n}m_{\xi i}m_{vi}^2\right)$$

$$\Rightarrow E\left(\hat{\varepsilon}_\xi^2\hat{\varepsilon}_v^3\,|X\right)=\mu_5\sum_{i=1}^{n}m_{\xi i}^2 m_{vi}^3 \;+\; \mu_2\mu_3\left[\left(\sum_{i=1}^{n}m_{vi}^3\right)m_{\xi\xi}+3m_{vv}\sum_{i=1}^{n}m_{\xi i}^2 m_{vi}+\;6\,m_{\xi v}\left(\sum_{i=1}^{n}m_{\xi i}m_{vi}^2\right)\right.$$

$$\left.-10\sum_{i=1}^{n}m_{\xi i}^2 m_{vi}^3\right]$$

Therefore

$$E\left(\frac{1}{n(n-1)}\sum_{\xi\neq v}\hat{\varepsilon}_\xi^2\hat{\varepsilon}_v^3\,|\mathrm{X}\right)=\mu_5 c_{53} \;+\; \mu_2\mu_3\left(c_{54}-10c_{53}\right)$$

$$c_{53}=\frac{1}{n(n-1)}\sum_{\xi\neq v}\sum_{i=1}^{n}m_{\xi i}^2 m_{vi}^3 \qquad [21]$$

$$c_{54}=\frac{1}{n(n-1)}\left[\sum_{\xi\neq v}m_{\xi\xi}\left(\sum_{i=1}^{n}m_{vi}^3\right)+3\sum_{\xi\neq v}m_{vv}\sum_{i=1}^{n}m_{\xi i}^2 m_{vi}\;+\;6\sum_{\xi\neq v}m_{\xi v}\left(\sum_{i=1}^{n}m_{\xi i}m_{vi}^2\right)\right]$$

Analogously with c_{43} previously, we have

$$c_{53}=\frac{\mathbf{1}'\mathrm{M}^{(3)}\mathrm{M}^{(2)}\mathbf{1}-\mathrm{tr}\left\{\mathrm{M}^{(3)}\mathrm{M}^{(2)}\right\}}{n(n-1)}$$

Regarding c_{54}, we have component-by-component

$$c_{54}(\mathrm{I}):\sum_{\xi\neq v}m_{\xi\xi}\left(\sum_{i=1}^{n}m_{vi}^3\right)=\;\sum_{j=1}^{n}m_{jj}\left(\sum_{i=1}^{n}m_{1i}^3+\ldots+\sum_{i=1}^{n}m_{ni}^3\right)\;-\;\sum_{j=1}^{n}m_{jj}\left(\sum_{i=1}^{n}m_{ji}^3\right)$$

$$=\left(\sum_{i=1}^{n}m_{1i}^3+\ldots+\sum_{i=1}^{n}m_{ni}^3\right)\sum_{j=1}^{n}m_{jj}-nc_{52}\;=\left(n-K\right)\left[\mathbf{1}'\left(\mathrm{M}\circ\mathrm{M}\circ\mathrm{M}\right)\mathbf{1}\right]-nc_{52}$$

$$\Rightarrow c_{54}(\mathrm{I}) = (n-K)\,\mathrm{tr}\left\{\mathrm{M}^{(2)}\mathrm{M}\right\} - nc_{52} \qquad [22]$$

$$c_{54}(\mathrm{II}): 3\sum_{\xi\neq v} m_{vv}\sum_{i=1}^{n} m_{\xi i}^{2} m_{vi} = 3\sum_{j=1}^{n} m_{jj}\left(\sum_{i=1}^{n} m_{ji}m_{1i}^{2} + \ldots + \sum_{i=1}^{n} m_{ji}m_{ni}^{2}\right) - 3\sum_{j=1}^{n} m_{jj}\left(\sum_{i=1}^{n} m_{ji}^{3}\right)$$

$$= 3\sum_{j=1}^{n} m_{jj}\left(\mathbf{1}'\mathrm{M}^{(2)}\mathbf{m}_j\right) - 3nc_{52} = 3\cdot\mathbf{1}'\mathrm{M}^{(2)}\left(\sum_{j=1}^{n} m_{jj}\mathbf{m}_j\right) - 3nc_{52}$$

$$= 3\cdot\mathbf{1}'\mathrm{M}^{(2)}\left(\mathrm{M}\,\mathrm{diag}\left\{m_{jj}\right\}\mathbf{1}\right) - 3nc_{52}$$

$$\Rightarrow c_{54}(\mathrm{II}) = 3\cdot\mathbf{1}'\mathrm{M}^{(2)}\mathrm{M}\mathbf{m}_d - 3nc_{52} \qquad [23]$$

$$c_{54}(\mathrm{III}) := 6\sum_{\xi\neq v} m_{\xi v}\left(\sum_{i=1}^{n} m_{\xi i}m_{vi}^{2}\right) =$$

$$6m_{12}\left(\sum_{i=1}^{n} m_{1i}m_{2i}^{2}\right) + \ldots + 6m_{1n}\left(\sum_{i=1}^{n} m_{1i}m_{ni}^{2}\right) + 6m_{11}\left(\sum_{i=1}^{n} m_{1i}^{3}\right) - 6m_{11}\left(\sum_{i=1}^{n} m_{1i}^{3}\right)$$

$$+ \quad 6m_{21}\left(\sum_{i=1}^{n} m_{2i}m_{1i}^{2}\right) + \ldots + 6m_{2n}\left(\sum_{i=1}^{n} m_{2i}m_{ni}^{2}\right) + 6m_{22}\left(\sum_{i=1}^{n} m_{2i}^{3}\right) - 6m_{22}\left(\sum_{i=1}^{n} m_{2i}^{3}\right)$$

$$\vdots$$

$$+ \quad 6m_{n1}\left(\sum_{i=1}^{n} m_{ni}m_{1i}^{2}\right) + \ldots + 6m_{n,n-1}\left(\sum_{i=1}^{n} m_{ni}m_{n-1,i}^{2}\right) + 6m_{nn}\left(\sum_{i=1}^{n} m_{ni}^{3}\right) - 6m_{nn}\left(\sum_{i=1}^{n} m_{ni}^{3}\right)$$

$$= \quad 6\mathbf{m}_1'\left(\mathbf{m}_1^{(2)}m_{11} + \mathbf{m}_2^{(2)}m_{12} + \ldots + \mathbf{m}_n^{(2)}m_{1n}\right) - 6m_{11}\left(\sum_{i=1}^{n} m_{1i}^{3}\right)$$

$$+ \quad 6\mathbf{m}_2'\left(\mathbf{m}_1^{(2)}m_{21} + \mathbf{m}_2^{(2)}m_{22} + \ldots + \mathbf{m}_n^{(2)}m_{2n}\right) - 6m_{22}\left(\sum_{i=1}^{n} m_{2i}^{3}\right)$$

$$\vdots$$

$$+ \quad 6\mathbf{m}_n'\left(\mathbf{m}_1^{(2)}m_{n1} + \mathbf{m}_2^{(2)}m_{n2} + \ldots + \mathbf{m}_n^{(2)}m_{nn}\right) - 6m_{nn}\left(\sum_{i=1}^{n} m_{ni}^{3}\right)$$

$$\Rightarrow c_{54}(\mathrm{III}) = 6\cdot\mathbf{1}'\left[\left(\mathrm{M}\mathrm{M}^{(2)}\right)\circ\mathrm{M}\right]\mathbf{1} - 6nc_{52} = 6\,\mathrm{tr}\left\{\mathrm{M}\mathrm{M}^{(2)}\mathrm{M}\right\} - 6nc_{52} = 6\,\mathrm{tr}\left\{\mathrm{M}^{(2)}\mathrm{M}\right\} - 6nc_{52} \qquad [24]$$

where we used the cyclic property of the trace, $\mathrm{tr}\left\{\mathrm{M}\mathrm{M}^{(2)}\mathrm{M}\right\} = \mathrm{tr}\left\{\mathrm{M}^{(2)}\mathrm{M}\mathrm{M}\right\} = \mathrm{tr}\left\{\mathrm{M}^{(2)}\mathrm{M}\right\}$.

Collecting results,

$$c_{54}=\frac{1}{n(n-1)}\Big[(n-K)\operatorname{tr}\left\{\mathrm{M}^{(2)}\mathrm{M}\right\}+3\left(\mathbf{1}'\mathrm{M}^{(2)}\mathrm{M}\mathbf{m}_d\right)+\ 6\operatorname{tr}\left\{\mathrm{M}^{(2)}\mathrm{M}\right\}-10nc_{52}\Big]$$

$$=\frac{1}{n(n-1)}\Big[(n-K+6)\operatorname{tr}\left\{\mathrm{M}^{(2)}\mathrm{M}\right\}+3\left(\mathbf{1}'\mathrm{M}^{(2)}\mathrm{M}\mathbf{m}_d\right)-10nc_{52}\Big]$$

Applying the same steps as for the 4th central moment we obtain the unbiased estimator

$$[3.62]:\ \hat{\mu}_5(\varepsilon)=\frac{c_{54}-10c_{53}}{c_{51}c_{54}-10c_{53}c_{52}}\left(\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^5\ -\ \frac{10c_{52}\ -10\,c_{51}}{c_{54}-10c_{53}}\frac{1}{n(n-1)}\sum_{\xi\neq v}^{n}\hat{\varepsilon}_\xi^2\hat{\varepsilon}_v^3\right)$$

$$c_{51}=\frac{\operatorname{tr}\left\{\mathrm{M}^{(3)}\mathrm{M}^{(2)}\right\}}{n},\quad c_{52}=\frac{\mathbf{1}'\mathrm{M}^{(3)}\mathbf{m}_d}{n},\qquad c_{53}=\frac{\mathbf{1}'\mathrm{M}^{(3)}\mathrm{M}^{(2)}\mathbf{1}\ -\ nc_{51}}{n(n-1)},$$

$$c_{54}=\frac{1}{n(n-1)}\Big[(n-K+6)\operatorname{tr}\left\{\mathrm{M}^{(2)}\mathrm{M}\right\}+3\left(\mathbf{1}'\mathrm{M}^{(2)}\mathrm{M}\mathbf{m}_d\right)-10nc_{52}\Big]$$

We have $c_{51}\to 1$, $c_{52}\to 1,$ $c_{53}\to 0,$ $c_{54}\to 1,$ so

$$\hat{\mu}_5(\varepsilon)\to\frac{1-0}{1-0}\left(\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^5\ -\ \frac{10\ -10}{1-0}\frac{1}{n(n-1)}\sum_{\xi\neq v}^{n}\hat{\varepsilon}_\xi^2\hat{\varepsilon}_v^3\right)=\mu_5(\varepsilon)$$

The unbiased estimator for the 5th cumulant.

Here we re-write

$$E\left(\frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_j^5\,|\mathrm{X}\right)=\ \kappa_5c_{51}\ +\ \mu_2\mu_3 10c_{52},\quad E\left(\frac{1}{n(n-1)}\sum_{\xi\neq v}^{n}\hat{\varepsilon}_\xi^2\hat{\varepsilon}_v^3\,|\mathrm{X}\right)=\kappa_5c_{53}\ +\ \mu_2\mu_3c_{54}$$

which leads to

$$\boxed{[3.63]:\ \hat{\kappa}_5(\varepsilon) = \frac{c_{54}}{c_{51}c_{54} - 10c_{53}c_{52}} \left(\frac{1}{n}\sum_{j=1}^{n} \hat{\varepsilon}_j^5 \ - \ \frac{10c_{52}}{c_{54}} \frac{1}{n(n-1)} \sum_{\xi \neq v}^{n} \hat{\varepsilon}_\xi^2 \hat{\varepsilon}_v^3 \right)} \qquad [25]$$

and

$$\hat{\kappa}_5(\varepsilon) \to \frac{1}{1\cdot 1 - 10\cdot 0\cdot 1} \left(\frac{1}{n}\sum_{j=1}^{n} \hat{\varepsilon}_j^5 \ - \ \frac{10\cdot 1}{1} \frac{1}{n(n-1)} \sum_{\xi \neq v}^{n} \hat{\varepsilon}_\xi^2 \hat{\varepsilon}_v^3 \right) = \mu_5(\varepsilon) - 10\mu_2\mu_3 = \kappa_5(\varepsilon).$$

C. The limiting distribution of the COLS/MM estimator.

We are interested in the limiting joint distribution of the vector

$$n^{1/2}\hat{\mathbf{h}}_N(\mathbf{q}_0) = \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[f_{ji}(\mathrm{y}_n, \mathrm{X}_n) - \kappa_j(\varepsilon) \right], \quad j = 2,3,4,5 \right\}$$

where $f_{ji}(\mathrm{y}_n, \mathrm{X}_n)$ are functions of the OLS residuals and of the regressor matrix through the bias-correction terms.

We will prove first that each element of the vector obeys a Normal Central Limit Theorem. We will invoke Theorem 2.1/Corollary 2.1 in Withers (1981). Adapting the notation where necessary, let the sum $S_N = \sum_{i=1}^{n_N} X_{iN}$ where $\{X_{iN} : i = 1,2,\ldots,n_N,\ \ N = 1,2,\ldots\}$ is a *triangular array* of real dependent random variables, and typically $n_N = N$ (to obtain the triangular shape). We need the framework of triangular arrays because as the sample size increases the distribution of all elements of the sum changes. X_{iN} represents $f_{ji}(\mathrm{y}_n, \mathrm{X}_n) - \kappa_j(\varepsilon)$ (examining each $j = 2,3,4,5$ separately), so these variables are zero-mean, identically distributed, equicorrelated, and asymptotically independent. Then, as we have mentioned in the main text, this Triangular Array is *mixing* under any mixing concept, like for example strong-mixing or uniform mixing. It follows that it is also strongly "ℓ–mixing" as the concept is defined in Withers (1981), see p. 513.

For triangular arrays the definition of the mixing concept involves three "rounds" of maximization, see Withers (1981), eq. 1.2, 1.9, 1.10., instead of the usual two for a single stochastic process. But the criterion for a triangular array to be mixing is, in the end, the same, requiring the examination of an infinite sequence and whether the mixing coefficient goes to zero at the limit. But whenever our sequences become infinite the X_{iN} variables become independent and the mixing coefficient becomes zero. Now, let

$$S_N(a,b)=\sum_{i=a+1}^{a+b} X_{iN}, \quad 0\le a, \quad 1\le b\le n_N-a$$

and $\tilde{c}_N(k)=\sup\left\{E\left(X_{\gamma N}, X_{\delta N}\right)\right\}, \quad 0\le k\le n_N,$ where the supremum is over $\left\{\gamma,\delta : |\gamma-\delta|\ge k, 1\le\gamma\le n_N, 1\le\delta\le n_N\right\}$. In our case where the variables are equicorrelated for given n_N, the distance between the variables does not matter and so $\forall k, \ \tilde{c}(k)=c_{n_N}$, the common covariance in each row of the triangular array given n_N, while we have $c_{n_N}\to 0$ as n_N (and hence necessarily also N) goes to infinity, or $c_\infty=0$. Define $C_N(0)=(n_N-1)c_{n_N}$.

Then, **Theorem 2.1/Corollary 2.1. of Withers (1981)** can be re-stated as follows: If

1. The triangular array is ℓ–mixing

2. $\sup_{a,N}\left(E\left[\left|S_N(a,b)\right|^{2+\varepsilon}\right]\right)^{\frac{1}{2+\varepsilon}}=O\left(\sqrt{b}\right)$ as $b\to\infty$, for some $\varepsilon>0$

3. $1+C_N(0)=O\left(\sigma_N^2/n_N\right)$ as $N\to\infty$, $\sigma_N^2=\mathrm{Var}\left(S_N(a,b)\right)$,

then $$\frac{S_N}{\sqrt{\mathrm{Var}(S_N)}}\xrightarrow{d} N(0,1) \quad \text{as} \quad N\to\infty$$

The **1st condition** is satisfied, as we have already discussed.

Regarding the **2nd condition**, note that since the variables in $S_N(a,b)$ are identically distributed, (for each N), maximizing over a given b results in $a^*=0$ so that we include as many variables as possible in $S_N(a,b)$, if their covariance is positive, and the opposite holds if their covariance is negative. Regarding maximization over the row N of the triangular array, some N will be chosen, as long as b is treated as finite. But once we consider $b \to \infty$, then necessarily we have to consider only sums $S_N(a,b)$ of infinite length, where the variables present in the sum become independent. So for the purposes of verifying that the condition holds, we can set $a=0$ and $b=n_N$ and examine

$n_N^{-1/2}\left(E\left[\left|S_N\right|^{2+\varepsilon}\right]\right)^{\frac{1}{2+\varepsilon}}$, as $n_N \to \infty$. We can also set $\varepsilon = 2$ and we get

$$n_N^{-1/2}\left(E\left[\left|S_N\right|^4\right]\right)^{\frac{1}{4}} = \left(E\left[\left|n_N^{-1/2}S_N\right|^4\right]\right)^{\frac{1}{4}} = \left(E\left[\left|n_N^{-1/2}S_N\right|^2\left|n_N^{-1/2}S_N\right|^2\right]\right)^{\frac{1}{4}}$$

$$= \left[E\left(\left|n_N^{-1}\sum_{i=1}^{n_N} X_{iN}^2 + n_N^{-1}\sum_i\sum_{k\neq i} X_{iN}X_{kN}\right|\left|n_N^{-1}\sum_{i=1}^{n_N} X_i^2 + n_N^{-1}\sum_i\sum_{k\neq i} X_{iN}X_{kN}\right|\right)\right]^{\frac{1}{4}}$$

As $n_N \to \infty$ we have $n_N^{-1}\sum_{i=1}^{n_N} X_{iN}^2 \xrightarrow{p} E\left(X_\infty^2\right) \equiv \sigma_X^2 < \infty$, and

$$\operatorname{plim} n_N^{-1}\sum_i\sum_{k\neq i} X_{iN}X_{kN} \xrightarrow{p} \lim n_N^{-1}\sum_i\sum_{k\neq i} E\left(\operatorname{plim} X_{iN}X_{kN}\right) = \lim n_N^{-1}\sum_i\sum_{k\neq i} E\left(X_{i\infty}X_{k\infty}\right)$$

which, due to uniform integrability becomes

$$\ldots = \lim n_N^{-1}\sum_i\sum_{k\neq i} c_\infty \ = \lim n_N^{-1}\cdot 0 = 0, \quad \text{so} \quad n_N^{-1/2}\left(E\left[\left|S_N\right|^4\right]\right)^{\frac{1}{4}} \to \sigma_X < \infty, \qquad \text{as } n_N \to \infty$$

and the 2nd condition is satisfied.

For the **3d condition** we want

$$\frac{1+C_N(0)}{\sigma_N^2/n_N} < \infty \ \Rightarrow 0 \ < \frac{1+\left(n_N-1\right)c_{n_N}}{\sigma_X^2+\left(n_N-1\right)c_{n_N}} < \infty \qquad \text{as} \quad N \to \infty,$$

which holds. So the 3d condition is also satisfied. Therefore,

$$\frac{S_N}{\sqrt{\operatorname{Var}\left(S_N\right)}} = \frac{n_N^{-1/2}S_N}{\sqrt{\sigma_{X,N}^2+\left(n_N-1\right)c_{n_N}}} \xrightarrow{d} N(0,1) \quad \text{as} \quad N \to \infty$$

But here $n_N^{-1/2} S_N$ represents any one of the elements of the vector $n^{1/2}\hat{\mathbf{h}}_N(\mathbf{q}_0)$, so we conclude that each one of them obeys the Normal Central Limit Theorem.

Multivariate Normality. We invoke the Cramér-Wold theorem (or "device"), see for example Lehmann (1999), p. 284, and we examine an arbitrary linear combination of the elements of $n^{1/2}\hat{\mathbf{h}}_N(\mathbf{q}_0)$. Let non-zero $\mathbf{b} = (b_2, b_3, b_4, b_5)' \in \mathbb{R}^4$. Using notation established in the main text (while suppressing the presence of the true coefficient vector $\mathbf{q}_0$), we have

$$\mathbf{b}'\left(n^{1/2}\hat{\mathbf{h}}_N\right) = n^{-1/2}\left(b_2\sum_{i=1}^{n}\hat{h}_{2,i} + b_3\sum_{i=1}^{n}\hat{h}_{3,i} + b_4\sum_{i=1}^{n}\hat{h}_{4,i} + b_5\sum_{i=1}^{n}\hat{h}_{5,i}\right)$$

Analyzing the sums we have $$\mathbf{b}'\left(n^{1/2}\hat{\mathbf{h}}_N\right) = n^{-1/2}\begin{bmatrix} b_2\hat{h}_{2,1} + b_2\hat{h}_{2,2} + \ldots + b_2\hat{h}_{2,n} \\ + \; b_3\hat{h}_{3,1} + b_3\hat{h}_{3,2} + \ldots + b_3\hat{h}_{3,n} \\ + \; b_4\hat{h}_{4,1} + b_4\hat{h}_{4,2} + \ldots + b_4\hat{h}_{4,n} \\ + \; b_5\hat{h}_{5,1} + b_5\hat{h}_{5,2} + \ldots + b_5\hat{h}_{5,n} \end{bmatrix}$$

Summing vertically, we get $$\mathbf{b}'\left(n^{1/2}\hat{\mathbf{h}}_N\right) = n^{-1/2}\left(\mathbf{b}'\sum_{j=2}^{5}\hat{h}_{j,1} + \mathbf{b}'\sum_{j=2}^{5}\hat{h}_{j,2} + \ldots + \mathbf{b}'\sum_{j=2}^{5}\hat{h}_{j,n}\right)$$

$$\Rightarrow \mathbf{b}'\left(n^{1/2}\hat{\mathbf{h}}_N\right) = n^{-1/2}\left(\mathbf{b}'\hat{\mathbf{h}}_1 + \mathbf{b}'\hat{\mathbf{h}}_2 + \ldots + \mathbf{b}'\hat{\mathbf{h}}_n\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{b}'\hat{\mathbf{h}}_i \ .$$

So we have encapsulated the dependence that will remain at the limit *inside* the elements $\mathbf{b}'\hat{\mathbf{h}}_i$. These elements have finite moments, are zero-mean due to the use of the kapa-statistics, identically distributed, and equicorrelated, with the dependence vanishing at the limit. In other words we have the same situation as before, and so here too the CLT result from Withers (1981) applies on $\mathbf{b}'\left(n^{1/2}\hat{\mathbf{h}}_N\right)$. Then, by the Cramér-Wold theorem, it follows that $n^{1/2}\hat{\mathbf{h}}_N$ obeys a multivariate Normal Law at the limit.

In the Addendum to this section (to be found at p. 523) we examine the variance of the limiting distribution.

TECHNICAL APPENDIX 3.IV:
The semi-Gamma 2TSF specification.

5.1. The Gamma-Exponential specification.

5.1.1. The composite error density.

We want the density of $\varepsilon = v + w - u$ where

$$v \sim N\left(0, \sigma_v^2\right), \qquad f_w\left(w\right) = \frac{w^{k-1}}{\Gamma\left(k\right)\theta^k}\exp\left\{-w/\theta\right\}, \qquad f_u\left(u\right) = \frac{1}{\sigma_u}\exp\left\{-u/\sigma_u\right\}.$$

We consider first the difference $z \equiv w - u \Rightarrow u = w - z$. Since $0 \le u$ it follows that we must have $0 \le w - z \Rightarrow z \le w$, which will be a binding constraint for $z > 0$.

So we have

A) $z \le 0$

$$f_z\left(z\right) = \int_0^\infty \frac{w^{k-1}}{\Gamma\left(k\right)\theta^k}\exp\left\{-w/\theta\right\}\frac{1}{\sigma_u}\exp\left\{-\left(w-z\right)/\sigma_u\right\}dw$$

$$= \frac{\exp\left\{z/\sigma_u\right\}}{\Gamma\left(k\right)\theta^k\sigma_u}\int_0^\infty w^{k-1}\exp\left\{-\left(\frac{\sigma_u + \theta}{\theta\sigma_u}\right)w\right\}dw$$

We can transform the integral into a Gamma function,

$$f_z(z)=\frac{\exp\{z/\sigma_u\}}{\Gamma(k)\theta^k\sigma_u}\left(\frac{\theta\sigma_u}{\sigma_u+\theta}\right)^k\int_0^\infty\left(\frac{\sigma_u+\theta}{\theta\sigma_u}w\right)^{k-1}\exp\left\{-\left(\frac{\sigma_u+\theta}{\theta\sigma_u}\right)w\right\}d\left(\frac{\sigma_u+\theta}{\theta\sigma_u}w\right)$$

$$=\frac{\exp\{z/\sigma_u\}}{\Gamma(k)\theta^k\sigma_u}\left(\frac{\theta\sigma_u}{\sigma_u+\theta}\right)^k\int_0^\infty y^{k-1}\exp\{-y\}\,dy=\frac{\exp\{z/\sigma_u\}}{\Gamma(k)\theta^k\sigma_u}\left(\frac{\theta\sigma_u}{\sigma_u+\theta}\right)^k\Gamma(k)$$

$$\Rightarrow z\le 0,\quad f_z(z)=\frac{\sigma_u^{k-1}}{(\sigma_u+\theta)^k}\exp\{z/\sigma_u\} \qquad [1]$$

B) $z>0$

$$f_z(z)=\int_z^\infty\frac{w^{k-1}}{\Gamma(k)\theta^k}\exp\{-w/\theta\}\frac{1}{\sigma_u}\exp\{-(w-z)/\sigma_u\}\,dw$$
$$=\frac{\exp\{z/\sigma_u\}}{\Gamma(k)\theta^k\sigma_u}\int_z^\infty w^{k-1}\exp\left\{-\left(\frac{\sigma_u+\theta}{\theta\sigma_u}\right)w\right\}dw$$

Set $\delta\equiv\sigma_u\theta/(\sigma_u+\theta)$ and manipulate as before,

$$f_z(z)=\frac{\exp\{z/\sigma_u\}}{\Gamma(k)\theta^k\sigma_u}\left(\frac{\theta\sigma_u}{\sigma_u+\theta}\right)^k\int_{z/\delta}^\infty\left(\frac{w}{\delta}\right)^{k-1}\exp\left\{-\frac{w}{\delta}\right\}d(w/\delta)$$

$$=\frac{\exp\{z/\sigma_u\}}{\Gamma(k)}\frac{\sigma_u^{k-1}}{(\sigma_u+\theta)^k}\left[\int_0^\infty y^{k-1}\exp\{-y\}\,dy-\int_0^{z/\delta}y^{k-1}\exp\{-y\}\,dy\right]$$

$$=\frac{\exp\{z/\sigma_u\}}{\Gamma(k)}\frac{\sigma_u^{k-1}}{(\sigma_u+\theta)^k}\Gamma(k)\;-\;\frac{\sigma_u^{k-1}\exp\{z/\sigma_u\}}{(\sigma_u+\theta)^k}\frac{\int_{z/\delta}^\infty y^{k-1}\exp\{-y\}\,dy}{\Gamma(k)}$$

The remaining Integral is the lower incomplete Gamma function $\gamma(k,z/\delta)$, so we arrive at

$$z>0, \quad f_z(z)=\frac{\sigma_u^{k-1}\exp\{z/\sigma_u\}}{(\sigma_u+\theta)^k}\left[1-\frac{\gamma(k,z/\delta)}{\Gamma(k)}\right]$$

But $\dfrac{\gamma(k,z/\delta)}{\Gamma(k)}$ is the distribution function of a Gamma random variable with shape parameter k and scale parameter δ. Set then $\dfrac{\gamma(k,z/\delta)}{\Gamma(k)}\equiv F_G(z;k,\delta)$ to arrive at

$$[3.80]: \quad f_z(z)=\begin{cases}\dfrac{\sigma_u^{k-1}}{(\sigma_u+\theta)^k}\exp\{z/\sigma_u\} & z\le 0\\[2ex] \dfrac{\sigma_u^{k-1}}{(\sigma_u+\theta)^k}\exp\{z/\sigma_u\}\left[1-F_G(z;k,\delta)\right] & z>0\end{cases} \qquad [2]$$

Moving to $\varepsilon=v+z$ we then have

$$f_\varepsilon(\varepsilon)=\int_{-\infty}^{0}\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\tfrac{1}{2}\frac{(\varepsilon-z)^2}{\sigma_v^2}\right\}\frac{\sigma_u^{k-1}}{(\sigma_u+\theta)^k}\exp\{z/\sigma_u\}\,dz$$

$$+\int_{0}^{\infty}\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\tfrac{1}{2}\frac{(\varepsilon-z)^2}{\sigma_v^2}\right\}\frac{\sigma_u^{k-1}}{(\sigma_u+\theta)^k}\exp\{z/\sigma_u\}\left[1-F_G(z;k,\delta)\right]dz$$

$$=\int_{-\infty}^{\infty}\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\tfrac{1}{2}\frac{(\varepsilon-z)^2}{\sigma_v^2}\right\}\frac{\sigma_u^{k-1}}{(\sigma_u+\theta)^k}\exp\{z/\sigma_u\}\,dz$$

$$-\int_{0}^{\infty}\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\tfrac{1}{2}\frac{(\varepsilon-z)^2}{\sigma_v^2}\right\}\frac{\sigma_u^{k-1}}{(\sigma_u+\theta)^k}\exp\{z/\sigma_u\}F_G(z;k,\delta)\,dz$$

$$= \frac{\exp\left\{-\frac{1}{2}\frac{\varepsilon^2}{\sigma_v^2}\right\}}{\sigma_v\sqrt{2\pi}} \frac{\sigma_u^{k-1}}{\left(\sigma_u+\theta\right)^k} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\frac{z^2}{\sigma_v^2}\right\} \exp\left\{\left(\frac{\varepsilon}{\sigma_v^2}+\frac{1}{\sigma_u}\right)z\right\} dz$$

$$-\frac{\sigma_u^{k-1}}{\left(\sigma_u+\theta\right)^k} \int_0^{\infty} e^{z/\sigma_u} \frac{1}{\sigma_v}\phi\left(\frac{\varepsilon-z}{\sigma_v}\right) F_G\left(z;k,\delta\right) dz$$

For the first integral, in Gradshteyn and Ryzhik (2007) eq. 3.323(2), p. 337 we find

$$\int_{-\infty}^{\infty} \exp\left\{-p^2 x \pm qx\right\} dx = \exp\left\{\frac{q^2}{4p^2}\right\}\frac{\sqrt{\pi}}{p}$$

In our case,

$$p^2 = \frac{1}{2\sigma_v^2},\quad q = \left(\frac{\varepsilon}{\sigma_v^2}+\frac{1}{\sigma_u}\right) \ \Rightarrow\ q^2 = \left(\frac{\varepsilon}{\sigma_v^2}\right)^2 + 2\frac{\varepsilon}{\sigma_v^2\sigma_u} + \frac{1}{\sigma_u^2}$$

So we obtain

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\frac{z^2}{\sigma_v^2}\right\} \exp\left\{\left(\frac{\varepsilon}{\sigma_v^2}+\frac{1}{\sigma_u}\right)z\right\} dz = \exp\left\{\frac{\left(\frac{\varepsilon}{\sigma_v^2}\right)^2 + 2\frac{\varepsilon}{\sigma_v^2\sigma_u} + \frac{1}{\sigma_u^2}}{2/\sigma_v^2}\right\}\frac{\sqrt{\pi}}{1/\sqrt{2}\sigma_v}$$

$$= \exp\left\{\frac{1}{2}\left(\varepsilon^2/\sigma_v^2\right) + \frac{\varepsilon}{\sigma_u} + \frac{\sigma_v^2}{2\sigma_u^2}\right\}\sigma_v\sqrt{2\pi}\ .$$

Plugging back in,

$$f_{\varepsilon}(\varepsilon)=\frac{\exp\left\{-\frac{1}{2}\left(\varepsilon^2/\sigma_v^2\right)\right\}}{\sigma_v\sqrt{2\pi}}\frac{\sigma_u^{k-1}}{(\sigma_u+\theta)^k}\exp\left\{\frac{1}{2}\left(\varepsilon^2/\sigma_v^2\right)+\frac{\varepsilon}{\sigma_u}+\frac{\sigma_v^2}{2\sigma_u^2}\right\}\sigma_v\sqrt{2\pi}$$

$$-\frac{\sigma_u^{k-1}}{(\sigma_u+\theta)^k}\int_0^{\infty}e^{z/\sigma_u}\frac{1}{\sigma_v}\phi\left(\frac{\varepsilon-z}{\sigma_v}\right)F_G(z;k,\delta)\,dz$$

and simplifying,

$$[3.81]:\quad f_{\varepsilon}(\varepsilon)=\frac{\sigma_u^{k-1}}{(\sigma_u+\theta)^k}\left[\exp\left\{\frac{\varepsilon}{\sigma_u}+\frac{\sigma_v^2}{2\sigma_u^2}\right\}-\int_0^{\infty}e^{z/\sigma_u}\frac{1}{\sigma_v}\phi\left(\frac{\varepsilon-z}{\sigma_v}\right)F_G(z;k,\delta)\,dz\right].$$

The remaining integral will have to be numerically evaluated.

5.1.2. Conditional densities and Individual measures.

We are after $E(w|\varepsilon)$, $E(e^{\pm w}|\varepsilon)$, $E(u|\varepsilon), E(e^{\pm u}|\varepsilon)$, and $E(e^{w-u}|\varepsilon)$.

A. *w*-variable.

$$E(g(w)|\varepsilon)=\int_0^{\infty}g(w)f_{w|\varepsilon}(w|\varepsilon)\,dw=\frac{1}{f_{\varepsilon}(\varepsilon)}\int_0^{\infty}g(w)f_{w\varepsilon}(w,\varepsilon)\,dw$$

Define $\xi=v-u$. The joint distribution of w and ξ is

$f_{w,\xi}(w,\,\xi)=f_w(w)\,f_{\xi}(\xi)$. Since $\varepsilon=\xi+w\Rightarrow\xi=\varepsilon-w$ we have that

$$f_{w,\xi}(w,\,\varepsilon-w)=f_w(w)\,f_{\xi}(\varepsilon-w)=f_{w\varepsilon}(w,\varepsilon)$$

Now ξ is essentially the composite error term in a one-tier Exponential production SF model, and its density is (see Kumbhakar and Lovell 2000, eq. 3.2.37 p.80)

$$f_{\xi}(\xi) = \frac{1}{\sigma_u} \exp\left\{ \frac{\xi}{\sigma_u} + \frac{\sigma_v^2}{2\sigma_u^2} \right\} \Phi\left(-\frac{\xi}{\sigma_v} - \frac{\sigma_v}{\sigma_u} \right)$$

So

$$f_{\xi}(\varepsilon - w) = \frac{1}{\sigma_u} \exp\left\{ \frac{\varepsilon - w}{\sigma_u} + \frac{\sigma_v^2}{2\sigma_u^2} \right\} \Phi\left(-\frac{\varepsilon - w}{\sigma_v} - \frac{\sigma_v}{\sigma_u} \right)$$

while $f_w(w) = \dfrac{1}{\Gamma(k)\theta^k} w^{k-1} \exp(-w/\theta) = f_G(w; k, \theta)$

So $f_{w\varepsilon}(w, \varepsilon) = f_w(w) f_{\xi}(\varepsilon - w) = f_G(w; k, \theta) \dfrac{1}{\sigma_u} \exp\left\{ \dfrac{\varepsilon - w}{\sigma_u} + \dfrac{\sigma_v^2}{2\sigma_u^2} \right\} \Phi\left(-\dfrac{\varepsilon - w}{\sigma_v} - \dfrac{\sigma_v}{\sigma_u} \right)$

Therefore

$$E\left(g(w)|\varepsilon\right) = \frac{1}{f_{\varepsilon}(\varepsilon)} \int_0^{\infty} g(w) f_G(w; k, \theta) \frac{1}{\sigma_u} \exp\left\{ \frac{\varepsilon - w}{\sigma_u} + \frac{\sigma_v^2}{2\sigma_u^2} \right\} \Phi\left(-\frac{\varepsilon - w}{\sigma_v} - \frac{\sigma_v}{\sigma_u} \right) dw$$

or

[3.86]:

$$E\left(g(w)|\varepsilon\right) = \frac{\exp\left\{ \dfrac{\varepsilon}{\sigma_u} + \dfrac{\sigma_v^2}{2\sigma_u^2} \right\}}{\sigma_u f_{\varepsilon}(\varepsilon)} \int_0^{\infty} g(w) \exp\left\{ \frac{-w}{\sigma_u} \right\} f_G(w; k, \theta) \frac{1}{\sigma_u} \Phi\left(-\frac{\varepsilon - w}{\sigma_v} - \frac{\sigma_v}{\sigma_u} \right) dw$$

where w is just the dummy variable of integration.

B. *u*-variable.

$$E\left(g\left(u\right)\middle|\varepsilon\right)=\int_0^\infty g\left(u\right)f_{u|\varepsilon}\left(u\middle|\varepsilon\right)dw=\frac{1}{f_\varepsilon\left(\varepsilon\right)}\int_0^\infty g\left(u\right)f_{u\varepsilon}\left(u,\varepsilon\right)du$$

Define $t=v+w$. The joint distribution of u and t is

$$f_{u,t}\left(u,\,t\right)=f_u\left(u\right)f_t\left(t\right)\ .$$

Since $\varepsilon=t-u\Rightarrow t=\varepsilon+u$ we have that

$$f_{u,t}\left(u,\,\varepsilon+u\right)=f_u\left(u\right)f_t\left(\varepsilon+u\right)=f_{u\varepsilon}\left(u,\varepsilon\right)$$

Now

$$f_t\left(t\right)=\int_0^\infty f_v\left(t-w\right)f_w\left(w\right)dw$$

So

$$E\left(g\left(u\right)\middle|\varepsilon\right)=\frac{1}{f_\varepsilon\left(\varepsilon\right)}\int_0^\infty g\left(u\right)f_u\left(u\right)f_t\left(\varepsilon+u\right)du$$

$$=\frac{1}{f_\varepsilon\left(\varepsilon\right)}\int_0^\infty g\left(u\right)f_u\left(u\right)\int_0^\infty f_v\left(\varepsilon+u-w\right)f_w\left(w\right)dwdu$$

To be left with a single integral, we change the order of integration, which is straightforward since the limits of integration in the two integrals do not involve the integrating variables.

$$E\left(g\left(u\right)\middle|\varepsilon\right)=\frac{1}{f_\varepsilon\left(\varepsilon\right)}\int_0^\infty f_w\left(w\right)\int_0^\infty g\left(u\right)f_u\left(u\right)f_v\left(\varepsilon+u-w\right)dudw$$

Using the distributional assumptions, we have

$$f_v(\varepsilon+u-w)=\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma_v^2}(\varepsilon+u-w)^2\right\}$$

$$=\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma_v^2}(\varepsilon+u)^2+\ \frac{1}{\sigma_v^2}(\varepsilon+u)w\ -\frac{w^2}{2\sigma_v^2}\right\}$$

$$=\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\ \frac{w^2}{2\sigma_v^2}+\frac{\varepsilon}{\sigma_v^2}w\right\}\exp\left\{-\frac{1}{2\sigma_v^2}\left(\varepsilon^2+2\varepsilon u+u^2\right)+\ \frac{w}{\sigma_v^2}u\right\}$$

$$=\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\frac{\varepsilon^2}{2\sigma_v^2}\right\}\exp\left\{-\ \frac{w^2}{2\sigma_v^2}+\frac{\varepsilon}{\sigma_v^2}w\right\}\exp\left\{-\frac{1}{2\sigma_v^2}u^2-\left(\frac{\varepsilon-w}{\sigma_v^2}\right)u\right\}$$

$$=\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma_v^2}(\varepsilon-w)^2\right\}\exp\left\{-\frac{1}{2\sigma_v^2}u^2-\left(\frac{\varepsilon-w}{\sigma_v^2}\right)u\right\}$$

So the inner integral of $E\left(g(u)|\varepsilon\right)$ becomes

$$\int_0^\infty g(u)\,f_u(u)\,f_v(\varepsilon+u-w)\,du$$

$$=\exp\left\{-\frac{1}{2\sigma_v^2}(\varepsilon-w)^2\right\}\frac{1}{\sigma_u\sigma_v\sqrt{2\pi}}\int_0^\infty g(u)\exp\left\{\frac{-u}{\sigma_u}\right\}\exp\left\{-\frac{1}{2\sigma_v^2}u^2-\left(\frac{\varepsilon-w}{\sigma_v^2}\right)u\right\}du$$

$$=\exp\left\{-\frac{1}{2\sigma_v^2}(\varepsilon-w)^2\right\}\frac{1}{\sigma_u\sigma_v\sqrt{2\pi}}\int_0^\infty g(u)\exp\left\{-\frac{1}{2\sigma_v^2}u^2-\left(\frac{\varepsilon-w}{\sigma_v^2}+\frac{1}{\sigma_u}\right)u\right\}du$$

$$=\exp\left\{-\frac{1}{2\sigma_v^2}(\varepsilon-w)^2\right\}\frac{1}{\sigma_u\sigma_v\sqrt{2\pi}}\cdot I_g$$

This will have a different solution, depending on $g(u)$.

A. $g(u)=u$

Here we have

$$E(u|\varepsilon)=\frac{1}{f_\varepsilon(\varepsilon)}\int_0^\infty f_w(w)\frac{1}{\sigma_u\sigma_v\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma_v^2}(\varepsilon-w)^2\right\}\cdot I_g dw$$

$$I_g=\int_0^\infty u\exp\left\{-\frac{1}{2\sigma_v^2}u^2-\left(\frac{\varepsilon-w}{\sigma_v^2}+\frac{1}{\sigma_u}\right)u\right\}du\equiv\int_0^\infty u\exp\left\{-au^2-bu\right\}du$$

Combining Erdelyi et al. (1954), 6.3(13), p. 313 with Gradshteyn and Ryzhik (2007) eq. 9.254(2), p. 1030, this has general solution

$$I_g=\frac{1}{2a}\left[1-b\int_0^\infty \exp\left\{-au^2-bu\right\}du\right]=\frac{1}{2a}\left[1-b\frac{\sqrt{\pi}}{\sqrt{a}}\exp\left\{\frac{b^2}{4a}\right\}\Phi\left(-\frac{b}{\sqrt{2a}}\right)\right]$$

$$I_g=\sigma_v^2\left[1-\left(\frac{\varepsilon-w}{\sigma_v^2}+\frac{1}{\sigma_u}\right)\sigma_v\sqrt{2\pi}\exp\left\{\frac{1}{2}\left(\frac{\varepsilon-w}{\sigma_v}+\frac{\sigma_v}{\sigma_u}\right)^2\right\}\Phi\left(-\left(\frac{\varepsilon-w}{\sigma_v}+\frac{\sigma_v}{\sigma_u}\right)\right)\right]$$

$$I_g=\sigma_v^2\left[1-\left(\frac{\varepsilon-w}{\sigma_v}+\frac{\sigma_v}{\sigma_u}\right)\sqrt{2\pi}\exp\left\{\frac{1}{2}\left(\frac{\varepsilon-w}{\sigma_v}+\frac{\sigma_v}{\sigma_u}\right)^2\right\}\Phi\left(-\left(\frac{\varepsilon-w}{\sigma_v}+\frac{\sigma_v}{\sigma_u}\right)\right)\right]$$

So (switching notation to u from w)

$$E(u|\varepsilon)=\frac{(\sigma_v/\sigma_u)}{f_\varepsilon(\varepsilon)}\int_0^\infty f_G(u;k,\theta)\phi\left(\frac{\varepsilon-u}{\sigma_v}\right)\left[1-\psi\sqrt{2\pi}\exp\left\{\frac{1}{2}\psi^2\right\}\Phi(-\psi)\right]du$$

$$\psi=\frac{\varepsilon-u}{\sigma_v}+\frac{\sigma_v}{\sigma_u}$$

$$[3.88]:\ E(u|\varepsilon)=\frac{(\sigma_v/\sigma_u)}{f_\varepsilon(\varepsilon)}\int_0^\infty f_G(u;k,\theta)\phi\left(\frac{\varepsilon-u}{\sigma_v}\right)\left[1-\frac{\psi\,\Phi(-\psi)}{\phi(\psi)}\right]du,\qquad \psi=\frac{\varepsilon-u}{\sigma_v}+\frac{\sigma_v}{\sigma_u}$$

B. $g(u)=\exp\{\pm u\}$

Here

$$I_g=\int_0^\infty \exp\left\{-\frac{1}{2\sigma_v^2}u^2-\left(\frac{\varepsilon-w}{\sigma_v^2}+\frac{1}{\sigma_u}\mp 1\right)u\right\}du\equiv\int_0^\infty \exp\left\{-au^2-bu\right\}du$$

Combining Erdelyi et al. (1954), 6.3(13), p. 313 with Gradshteyn and Ryzhik (2007) eq. 9.254(1), p. 1030, this has general solution

$$\int_0^\infty \exp\left\{-au^2-bu\right\}du=\frac{\sqrt{\pi}}{\sqrt{a}}\exp\left\{\frac{b^2}{4a}\right\}\Phi\left(-\frac{b}{\sqrt{2a}}\right)$$

so

$$I_g=\frac{\sqrt{\pi}}{\sqrt{\left(1/2\sigma_v^2\right)}}\exp\left\{\frac{\left(\frac{\varepsilon-w}{\sigma_v^2}+\frac{1}{\sigma_u}\mp 1\right)^2}{\frac{4}{2\sigma_v^2}}\right\}\Phi\left(-\frac{\left(\frac{\varepsilon-w}{\sigma_v^2}+\frac{1}{\sigma_u}\mp 1\right)}{\sqrt{\left(2/2\sigma_v^2\right)}}\right)$$

$$=\sigma_v\sqrt{2\pi}\exp\left\{\frac{1}{2}\left(\frac{\varepsilon-w}{\sigma_v}+\frac{\sigma_v}{\sigma_u}\mp\sigma_v\right)^2\right\}\Phi\left(-\left(\frac{\varepsilon-w}{\sigma_v}+\frac{\sigma_v}{\sigma_u}\mp\sigma_v\right)\right)$$

Therefore,

$$\int_0^\infty \exp\{\pm u\}\,f_u(u)\,f_v(\varepsilon+u-w)\,du$$

$$=\exp\left\{-\frac{1}{2\sigma_v^2}(\varepsilon-w)^2\right\}\frac{1}{\sigma_u}\exp\left\{\frac{1}{2}\left(\frac{\varepsilon-w}{\sigma_v}+\frac{\sigma_v}{\sigma_u}\mp\sigma_v\right)^2\right\}\Phi\left(-\left(\frac{\varepsilon-w}{\sigma_v}+\frac{\sigma_v}{\sigma_u}\mp\sigma_v\right)\right)$$

$$=\frac{1}{\sigma_u}\exp\left\{\frac{1}{2}\left(\frac{\sigma_v}{\sigma_u}\mp\sigma_v\right)^2\right\}\exp\left\{\left(\frac{\sigma_v}{\sigma_u}\mp\sigma_v\right)\left(\frac{\varepsilon-w}{\sigma_v}\right)\right\}\Phi\left(-\left(\frac{\varepsilon-w}{\sigma_v}+\frac{\sigma_v}{\sigma_u}\mp\sigma_v\right)\right)$$

and so

$$E\left(\exp\{\pm u\}\left|\varepsilon\right.\right)=\frac{1}{f_\varepsilon\left(\varepsilon\right)}\int_0^\infty f_w\left(w\right)\int_0^\infty \exp\{\pm u\}\; f_u\left(u\right) f_v\left(\varepsilon+u-w\right)dudw$$

$$=\frac{1}{f_\varepsilon\left(\varepsilon\right)}\int_0^\infty\left[f_G\left(w;k,\theta\right)\frac{1}{\sigma_u}\exp\left\{\frac{1}{2}\left(\frac{\sigma_v}{\sigma_u}\mp\sigma_v\right)^2\right\}\exp\left\{\left(\frac{\sigma_v}{\sigma_u}\mp\sigma_v\right)\left(\frac{\varepsilon-w}{\sigma_v}\right)\right\}\right.$$

$$\left.\times\,\Phi\left(-\left(\frac{\varepsilon-w}{\sigma_v}+\frac{\sigma_v}{\sigma_u}\mp\sigma_v\right)\right)\right]dw$$

[3.89]: $E\left(\exp\{\pm u\}\left|\varepsilon\right.\right)=$

$$=\frac{\exp\left\{\frac{\sigma_v^2}{2}\left(\sigma_u^{-1}\mp1\right)^2+\left(\sigma_u^{-1}\mp1\right)\varepsilon\right\}}{\sigma_u f_\varepsilon\left(\varepsilon\right)}\int_0^\infty f_G\left(w;k,\theta\right)\exp\left\{-\left(\sigma_u^{-1}\mp1\right)w\right\}\Phi\left(-\left(\frac{\varepsilon-w}{\sigma_v}+\frac{\sigma_v}{\sigma_u}\mp\sigma_v\right)\right)dw$$

C. The net effect $E\left(e^{w-u}\left|\varepsilon\right.\right)$.

$$E\left(e^{w-u}\left|\varepsilon\right.\right)=E\left(e^z\left|\varepsilon\right.\right)=\frac{1}{f_\varepsilon\left(\varepsilon\right)}\int_{-\infty}^\infty e^z f_{\varepsilon,z}\left(\varepsilon,z\right)dz$$

The joint density of v and $z=w-u$ is

$$f_{v,z}\left(v,z\right)=f_v\left(v\right)f_z\left(z\right)=f_v\left(\varepsilon-z\right)f_z\left(z\right)=f_{\varepsilon,z}\left(\varepsilon,z\right)$$

So

$$E\left(e^{w-u}\left|\varepsilon\right.\right)=E\left(e^{z}\left|\varepsilon\right.\right)=\frac{1}{f_{\varepsilon}\left(\varepsilon\right)}\int_{-\infty}^{\infty}e^{z}f_{v}\left(\varepsilon-z\right)f_{z}\left(z\right)dz$$

$$=\frac{1}{f_{\varepsilon}\left(\varepsilon\right)}\int_{-\infty}^{0}e^{z}\frac{1}{\sigma_{v}\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma_{v}^{2}}\left(\varepsilon-z\right)^{2}\right\}\frac{\sigma_{u}^{k-1}}{\left(\sigma_{u}+\theta\right)^{k}}\exp\left\{z/\sigma_{u}\right\}dz$$

$$+\frac{1}{f_{\varepsilon}\left(\varepsilon\right)}\int_{0}^{\infty}e^{z}\frac{1}{\sigma_{v}\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma_{v}^{2}}\left(\varepsilon-z\right)^{2}\right\}\frac{\sigma_{u}^{k-1}}{\left(\sigma_{u}+\theta\right)^{k}}\exp\left\{z/\sigma_{u}\right\}\left[1-F_{G}\left(z;k,\delta\right)\right]dz$$

$$=\frac{1}{f_{\varepsilon}\left(\varepsilon\right)}\int_{-\infty}^{\infty}\frac{e^{z}}{\sigma_{v}\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma_{v}^{2}}\left(\varepsilon-z\right)^{2}\right\}\frac{\sigma_{u}^{k-1}}{\left(\sigma_{u}+\theta\right)^{k}}\exp\left\{z/\sigma_{u}\right\}dz$$

$$-\frac{1}{f_{\varepsilon}\left(\varepsilon\right)}\int_{0}^{\infty}\frac{e^{z}}{\sigma_{v}\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma_{v}^{2}}\left(\varepsilon-z\right)^{2}\right\}\frac{\sigma_{u}^{k-1}}{\left(\sigma_{u}+\theta\right)^{k}}\exp\left\{z/\sigma_{u}\right\}F_{G}\left(z;k,\delta\right)dz$$

$$=\frac{\sigma_{u}^{k-1}\exp\left\{-\frac{1}{2\sigma_{v}^{2}}\varepsilon^{2}\right\}}{f_{\varepsilon}\left(\varepsilon\right)\left(\sigma_{u}+\theta\right)^{k}\sigma_{v}\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left\{-\frac{z^{2}}{2\sigma_{v}^{2}}+\left(\frac{\varepsilon}{\sigma_{v}^{2}}+\frac{1}{\sigma_{u}}+1\right)z\right\}dz$$

$$-\frac{\sigma_{u}^{k-1}}{f_{\varepsilon}\left(\varepsilon\right)\left(\sigma_{u}+\theta\right)^{k}\sigma_{v}}\int_{0}^{\infty}\exp\left\{\left(1+\sigma_{u}^{-1}\right)z\right\}\phi\left(\frac{\varepsilon-z}{\sigma_{v}}\right)F_{G}\left(z;k,\delta\right)dz$$

The first integral can be solved, using $\int\limits_{-\infty}^{\infty}\exp\left\{-p^{2}x\pm qx\right\}dx=\exp\left\{\frac{q^{2}}{4p^{2}}\right\}\frac{\sqrt{\pi}}{p}$

and we get

$$\int_{-\infty}^{\infty}\exp\left\{-\frac{z^{2}}{2\sigma_{v}^{2}}+\left(\frac{\varepsilon}{\sigma_{v}^{2}}+\frac{1}{\sigma_{u}}+1\right)z\right\}dz=\exp\left\{\frac{\left(\frac{\varepsilon}{\sigma_{v}^{2}}+\frac{1}{\sigma_{u}}+1\right)^{2}}{2/\sigma_{v}^{2}}\right\}\sigma_{v}\sqrt{2\pi}$$

$$= \exp\left\{\frac{1}{2}\left(\frac{\varepsilon}{\sigma_v} + \frac{\sigma_v}{\sigma_u} + \sigma_v\right)^2\right\}\sigma_v\sqrt{2\pi}$$

So

$$E\left(e^{w-u}\left|\varepsilon\right.\right) = \frac{\sigma_u^{k-1}\exp\left\{-\frac{1}{2\sigma_v^2}\varepsilon^2\right\}}{f_\varepsilon\left(\varepsilon\right)\left(\sigma_u+\theta\right)^k\sigma_v\sqrt{2\pi}}\exp\left\{\frac{1}{2}\left(\frac{\varepsilon}{\sigma_v} + \frac{\sigma_v}{\sigma_u} + \sigma_v\right)^2\right\}\sigma_v\sqrt{2\pi}$$

$$-\frac{\sigma_u^{k-1}}{f_\varepsilon\left(\varepsilon\right)\left(\sigma_u+\theta\right)^k\sigma_v}\int_0^\infty \exp\left\{\left(1+\sigma_u^{-1}\right)z\right\}\phi\left(\frac{\varepsilon - z}{\sigma_v}\right)F_G\left(z;k,\delta\right)dz$$

$$E\left(e^{w-u}\left|\varepsilon\right.\right) = \frac{\sigma_u^{k-1}}{f_\varepsilon\left(\varepsilon\right)\left(\sigma_u+\theta\right)^k}\exp\left\{\frac{\sigma_v^2}{2}\left(1+\sigma_u^{-1}\right)^2 + \left(1+\sigma_u^{-1}\right)\varepsilon\right\}$$

$$-\frac{\sigma_u^{k-1}}{f_\varepsilon\left(\varepsilon\right)\left(\sigma_u+\theta\right)^k\sigma_v}\int_0^\infty \exp\left\{\left(1+\sigma_u^{-1}\right)z\right\}\phi\left(\frac{\varepsilon - z}{\sigma_v}\right)F_G\left(z;k,\delta\right)dz$$

$$\boxed{\begin{aligned}[3.90]:\quad E\left(e^{w-u}\left|\varepsilon\right.\right) = \frac{\sigma_u^{k-1}}{f_\varepsilon\left(\varepsilon\right)\left(\sigma_u+\theta\right)^k}\Bigg[&\exp\left\{\frac{\sigma_v^2}{2}\left(1+\sigma_u^{-1}\right)^2 + \left(1+\sigma_u^{-1}\right)\varepsilon\right\}\\ &-\frac{1}{\sigma_v}\int_0^\infty \exp\left\{\left(1+\sigma_u^{-1}\right)z\right\}\phi\left(\frac{\varepsilon - z}{\sigma_v}\right)F_G\left(z;k,\delta\right)dz\Bigg]\end{aligned}}$$

5.1.3. The Gamma-Exponential 2TSF moment equations.

1) For the symmetric normal distribution we only need its second central moment, the variance, $E\left(v^2\right)=\sigma_v^2$.

2) For the Gamma distribution, the cumulant generating function is

$K(t)=-k\ln(1-\theta t)$. This can provide us also with the needed moments of the Exponential distribution by setting $k=1$.

Derivative	Gamma	Exponential ($k=1$)	# Cumulant
$\frac{dK(t)}{dt}=\frac{k\theta}{1-\theta t}$	$t=0\rightarrow k\theta$	$t=0\rightarrow\theta=\sigma_u$	κ_1
$\frac{d^2K(t)}{dt^2}=\frac{k\theta^2}{(1-\theta t)^2}$	$t=0\rightarrow k\theta^2$	$t=0\rightarrow\sigma_u^2$	κ_2
$\frac{d^3K(t)}{dt^3}=\frac{2k\theta^3}{(1-\theta t)^3}$	$t=0\rightarrow 2k\theta^3$	$t=0\rightarrow 2\sigma_u^3$	κ_3
$\frac{d^4K(t)}{dt^4}=\frac{6k\theta^4}{(1-\theta t)^4}$	$t=0\rightarrow 6k\theta^4$	$t=0\rightarrow 6\sigma_u^4$	κ_4
$\frac{d^5K(t)}{dt^5}=\frac{24k\theta^5}{(1-\theta t)^5}$	$t=0\rightarrow 24k\theta^5$	$t=0\rightarrow 24\sigma_u^5$	κ_5

5.1.4. The determinant of the Jacobian of the moment conditions.

We have the system of equations

$$\begin{cases} \hat{\kappa}_2(\varepsilon) & -\left(\sigma_v^2+k\theta^2+\sigma_u^2\right)=0 \\ (1/2)\hat{\kappa}_3(\varepsilon) & -\left(k\theta^3-\sigma_u^3\right)=0 \\ (1/6)\hat{\kappa}_4(\varepsilon) & -\left(k\theta^4+\sigma_u^4\right)=0 \\ (1/24)\hat{\kappa}_5(\varepsilon) & -\left(k\theta^5-\sigma_u^5\right)=0 \end{cases}$$

For the vector $\mathbf{q}=\left(\sigma_v,k,\theta,\sigma_u\right)'$ the Jacobian matrix is

$$
\mathrm{J} = \begin{bmatrix} -2\sigma_v & -\theta^2 & -2k\theta & -2\sigma_u \\ 0 & -\theta^3 & -3k\theta^2 & 3\sigma_u^2 \\ 0 & -\theta^4 & -4k\theta^3 & -4\sigma_u^3 \\ 0 & -\theta^5 & -5k\theta^4 & 5\sigma_u^4 \end{bmatrix}
$$

Its determinant is

$$
|\mathrm{J}| = \begin{vmatrix} -2\sigma_v & -\theta^2 & -2k\theta & -2\sigma_u \\ 0 & -\theta^3 & -3k\theta^2 & 3\sigma_u^2 \\ 0 & -\theta^4 & -4k\theta^3 & -4\sigma_u^3 \\ 0 & -\theta^5 & -5k\theta^4 & 5\sigma_u^4 \end{vmatrix} = -2\sigma_v \begin{vmatrix} -\theta^3 & -3k\theta^2 & 3\sigma_u^2 \\ -\theta^4 & -4k\theta^3 & -4\sigma_u^3 \\ -\theta^5 & -5k\theta^4 & 5\sigma_u^4 \end{vmatrix} = 2\sigma_v \begin{vmatrix} \theta^3 & 3k\theta^2 & -3\sigma_u^2 \\ \theta^4 & 4k\theta^3 & 4\sigma_u^3 \\ \theta^5 & 5k\theta^4 & -5\sigma_u^4 \end{vmatrix}
$$

$$
= 2\sigma_v \theta^3 k\theta^2 \sigma_u^2 \begin{vmatrix} 1 & 3 & -3 \\ \theta & 4\theta & 4\sigma_u \\ \theta^2 & 5\theta^2 & -5\sigma_u^2 \end{vmatrix} = 2\sigma_v k\theta^5 \sigma_u^2 \left[1 \cdot \begin{vmatrix} 4\theta & 4\sigma_u \\ 5\theta^2 & -5\sigma_u^2 \end{vmatrix} - 3 \begin{vmatrix} \theta & 4\sigma_u \\ \theta^2 & -5\sigma_u^2 \end{vmatrix} - 3 \begin{vmatrix} \theta & 4\theta \\ \theta^2 & 5\theta^2 \end{vmatrix} \right]
$$

$$
= 2\sigma_v k\theta^5 \sigma_u^2 \left[20\theta\sigma_u \begin{vmatrix} 1 & 1 \\ \theta & -\sigma_u \end{vmatrix} - 3\theta\sigma_u \begin{vmatrix} 1 & 4 \\ \theta & -5\sigma_u \end{vmatrix} - 3\theta\theta^2 \begin{vmatrix} 1 & 4 \\ 1 & 5 \end{vmatrix} \right]
$$

$$
= 2\sigma_v k\theta^6 \sigma_u^2 \left[-20\sigma_u \left(\sigma_u + \theta \right) + 3\sigma_u \left(5\sigma_u + 4\theta \right) - 3\theta^2 \right]
$$

$$
= 2\sigma_v k\theta^6 \sigma_u^2 \left(-5\sigma_u^2 - 8\sigma_u \theta - 3\theta^2 \right)
$$

$$
\Rightarrow |\mathrm{J}| = -2\sigma_v k\theta^6 \sigma_u^2 \left(5\sigma_u^2 + 8\sigma_u \theta + 3\theta^2 \right)
$$

Evidently, this determinant can be zero only if one of the parameters involved is also zero.

5.2 The Exponential-Gamma 2TSF specification.

5.2.1. The composite error density.

Here we assume that the error components obey the following laws:

$$v \sim N\left(0,\sigma_v^2\right),\ w \sim Exp\left(\sigma_w\right),\quad u \sim Gamma\left(k,\theta\right) \qquad [3]$$

where for the Gamma distribution we adopt as before the shape-scale parametrization,

$$f_u\left(u\right)=\frac{1}{\Gamma\left(k\right)\theta^k}u^{k-1}\exp\left\{-u/\theta\right\} \qquad [4]$$

We consider first the difference $z \equiv w-u \Rightarrow w=z+u$. Since $0 \le w$ it follows that we must have $0 \le z+u \Rightarrow -z \le u$, which will be a binding constraint in the convolution for $z \le 0$.

So we have

A) $z \le 0$

$$f_z\left(z\right)=\int_{-z}^{\infty}\frac{u^{k-1}}{\Gamma\left(k\right)\theta^k}\exp\left\{-u/\theta\right\}\frac{1}{\sigma_w}\exp\left\{-\left(z+u\right)/\sigma_w\right\}du$$

$$=\frac{\exp\left\{-z/\sigma_w\right\}}{\Gamma\left(k\right)\theta^k\sigma_w}\int_{-z}^{\infty}u^{k-1}\exp\left\{-\left(\frac{\sigma_w+\theta}{\theta\sigma_w}\right)u\right\}du$$

We can transform the integral into a Gamma quantile function. Multiply and divide by $\left(\dfrac{\sigma_w+\theta}{\theta\sigma_w}\right)^k$,

$$f_z(z) = \frac{\exp\{-z/\sigma_w\}}{\theta^k \sigma_w}\left(\frac{\sigma_w+\theta}{\theta\sigma_w}\right)^{-k} \int_{-z}^{\infty} \frac{1}{\Gamma(k)}\left(\frac{\sigma_w+\theta}{\theta\sigma_w}\right)^{k} u^{k-1} \exp\left\{-\left(\frac{\sigma_w+\theta}{\theta\sigma_w}\right)u\right\}du$$

Set $\delta \equiv \sigma_w\theta/(\sigma_w+\theta)$ and so write

$$f_z(z) = \frac{\sigma_w^{k-1}}{(\sigma_w+\theta)^k}\exp\{-z/\sigma_w\}\int_{-z}^{\infty}\frac{1}{\Gamma(k)\delta^k}u^{k-1}\exp\left\{-\frac{u}{\delta}\right\}du$$

The integrand is a now a Gamma density (and $-z \geq 0$) so we arrive at

$$f_z(z) = \frac{\sigma_w^{k-1}}{(\sigma_w+\theta)^k}\exp\{-z/\sigma_w\}\left[1-F_G(-z;k,\delta)\right] \qquad [5]$$

B) $z>0$

$$f_z(z)=\int_0^{\infty}\frac{u^{k-1}}{\Gamma(k)\theta^k}\exp\{-u/\theta\}\frac{1}{\sigma_w}\exp\{-(z+u)/\sigma_w\}du$$

$$= \frac{\exp\{-z/\sigma_w\}}{\Gamma(k)\theta^k\sigma_w}\int_0^{\infty}u^{k-1}\exp\left\{-\left(\frac{\sigma_w+\theta}{\theta\sigma_w}\right)u\right\}du$$

$$f_z(z) = \frac{\exp\{-z/\sigma_w\}}{\theta^k \sigma_w}\left(\frac{\sigma_w+\theta}{\theta\sigma_w}\right)^{-k} \int_{0}^{\infty} \frac{1}{\Gamma(k)}\left(\frac{\sigma_w+\theta}{\theta\sigma_w}\right)^{k} u^{k-1} \exp\left\{-\left(\frac{\sigma_w+\theta}{\theta\sigma_w}\right)u\right\}du$$

The integral here equals unity since the integrand is a Gamma density and it is integrated over $[0,\infty)$ So here

$$f_z(z) = \frac{\sigma_w^{k-1}}{(\sigma_w+\theta)^k}\exp\{-z/\sigma_w\}$$

and in all

$$[3.103]: \quad f_z(z)=\begin{cases}\dfrac{\sigma_w^{k-1}}{(\sigma_w+\theta)^k}\exp\{-z/\sigma_w\}\left[1-F_G(-z;k,\delta)\right] & z\le 0\\[2ex] \dfrac{\sigma_w^{k-1}}{(\sigma_w+\theta)^k}\exp\{-z/\sigma_w\} & z>0\end{cases} \qquad [6]$$

Moving to $\varepsilon = v+z$ we then have

$$f_\varepsilon(\varepsilon)=\int_{-\infty}^{0}\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\tfrac{1}{2}\frac{(\varepsilon-z)^2}{\sigma_v^2}\right\}\frac{\sigma_w^{k-1}}{(\sigma_w+\theta)^k}\exp\{-z/\sigma_w\}\left[1-F_G(-z;k,\delta)\right]dz$$

$$+\int_{0}^{\infty}\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\tfrac{1}{2}\frac{(\varepsilon-z)^2}{\sigma_v^2}\right\}\frac{\sigma_w^{k-1}}{(\sigma_w+\theta)^k}\exp\{-z/\sigma_w\}dz$$

$$=\int_{-\infty}^{\infty}\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\tfrac{1}{2}\frac{(\varepsilon-z)^2}{\sigma_v^2}\right\}\frac{\sigma_w^{k-1}}{(\sigma_w+\theta)^k}\exp\{-z/\sigma_w\}dz$$

$$-\int_{-\infty}^{0}\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\tfrac{1}{2}\frac{(\varepsilon-z)^2}{\sigma_v^2}\right\}\frac{\sigma_w^{k-1}}{(\sigma_w+\theta)^k}\exp\{-z/\sigma_w\}F_G(-z;k,\delta)dz$$

$$=\frac{\exp\left\{-\dfrac{1}{2}\dfrac{\varepsilon^2}{\sigma_v^2}\right\}}{\sigma_v\sqrt{2\pi}}\frac{\sigma_w^{k-1}}{(\sigma_w+\theta)^k}\int_{-\infty}^{\infty}\exp\left\{-\tfrac{1}{2}\frac{z^2}{\sigma_v^2}\right\}\exp\left\{\left(\frac{\varepsilon}{\sigma_v^2}-\frac{1}{\sigma_w}\right)z\right\}dz$$

$$-\frac{\sigma_w^{k-1}}{(\sigma_w+\theta)^k}\int_{-\infty}^{0}e^{-z/\sigma_w}\frac{1}{\sigma_v}\phi\left(\frac{\varepsilon-z}{\sigma_v}\right)F_G(-z;k,\delta)dz$$

Before dealing with the first integral, we want to switch the limits of integration to the second. We get

$$f_\varepsilon(\varepsilon)=\frac{\exp\left\{-\frac{1}{2}\frac{\varepsilon^2}{\sigma_v^2}\right\}}{\sigma_v\sqrt{2\pi}}\frac{\sigma_w^{k-1}}{(\sigma_w+\theta)^k}\int_{-\infty}^{\infty}\exp\left\{-\tfrac{1}{2}\frac{z^2}{\sigma_v^2}\right\}\exp\left\{\left(\frac{\varepsilon}{\sigma_v^2}-\frac{1}{\sigma_w}\right)z\right\}dz$$

$$-\frac{\sigma_w^{k-1}}{(\sigma_w+\theta)^k}\int_0^\infty e^{z/\sigma_w}\frac{1}{\sigma_v}\phi\left(\frac{\varepsilon+z}{\sigma_v}\right)F_G(z;k,\delta)\,dz$$

For the first integral, as before we use Gradshteyn and Ryzhik (2007) eq. 3.323(2), p. 337

$$\int_{-\infty}^{\infty}\exp\left\{-p^2x\pm qx\right\}dx=\exp\left\{\frac{q^2}{4p^2}\right\}\frac{\sqrt{\pi}}{p}$$

In our case,

$$p^2=\frac{1}{2\sigma_v^2},\quad q=\left(\frac{\varepsilon}{\sigma_v^2}-\frac{1}{\sigma_w}\right)\ \Rightarrow\ q^2=\left(\frac{\varepsilon}{\sigma_v^2}\right)^2-\ 2\frac{\varepsilon}{\sigma_v^2\sigma_w}+\ \frac{1}{\sigma_w^2}$$

So we obtain

$$\int_{-\infty}^{\infty}\exp\left\{-\tfrac{1}{2}\frac{z^2}{\sigma_v^2}\right\}\exp\left\{\left(\frac{\varepsilon}{\sigma_v^2}-\frac{1}{\sigma_w}\right)z\right\}dz=\exp\left\{\frac{\left(\frac{\varepsilon}{\sigma_v^2}\right)^2-\ 2\frac{\varepsilon}{\sigma_v^2\sigma_w}+\ \frac{1}{\sigma_w^2}}{2/\sigma_v^2}\right\}\frac{\sqrt{\pi}}{1/\sqrt{2}\sigma_v}$$

$$=\exp\left\{\tfrac{1}{2}\left(\varepsilon^2/\sigma_v^2\right)-\frac{\varepsilon}{\sigma_w}+\frac{\sigma_v^2}{2\sigma_w^2}\right\}\sigma_v\sqrt{2\pi}$$

Plugging back in,

$$f_\varepsilon(\varepsilon)=\frac{\exp\left\{-\frac{1}{2}\frac{\varepsilon^2}{\sigma_v^2}\right\}}{\sigma_v\sqrt{2\pi}}\frac{\sigma_w^{k-1}}{(\sigma_w+\theta)^k}\exp\left\{\tfrac{1}{2}\left(\varepsilon^2/\sigma_v^2\right)-\frac{\varepsilon}{\sigma_w}+\frac{\sigma_v^2}{2\sigma_w^2}\right\}\sigma_v\sqrt{2\pi}$$

$$-\frac{\sigma_w^{k-1}}{(\sigma_w+\theta)^k}\int_0^\infty e^{z/\sigma_w}\frac{1}{\sigma_v}\phi\left(\frac{\varepsilon+z}{\sigma_v}\right)F_G(z;k,\delta)\,dz$$

and simplifying and compacting,

$$\boxed{[3.104]:\quad f_\varepsilon(\varepsilon)=\frac{\sigma_w^{k-1}}{(\sigma_w+\theta)^k}\left[\exp\left\{-\frac{\varepsilon}{\sigma_w}+\frac{\sigma_v^2}{2\sigma_w^2}\right\}-\int_0^\infty e^{z/\sigma_w}\frac{1}{\sigma_v}\phi\left(\frac{\varepsilon+z}{\sigma_v}\right)F_G(z;k,\delta)\,dz\right]}$$

The second integral will have to be numerically evaluated.

5.2.2. JLMS measures.

A. *w*-variable.

$$E\left(g(w)|\varepsilon\right)=\int_0^\infty g(w)f_{w|\varepsilon}(w|\varepsilon)\,dw=\frac{1}{f_\varepsilon(\varepsilon)}\int_0^\infty g(w)f_{\varepsilon,w}(\varepsilon,w)\,dw$$

Define $\xi=v-u\Rightarrow v=\xi+u$. The joint distribution of w and ξ is

$$f_{w,\xi}(w,\,\xi)=f_w(w)\,f_\xi(\xi)\quad\text{and}\quad f_\xi(\xi)=\int_0^\infty f_v(\xi+u)\,f_u(u)\,du$$

Since $\varepsilon=\xi+w\Rightarrow\xi=\varepsilon-w$ we have that

$$f_{w,\varepsilon}(w,\,\varepsilon)=f_{w,\xi}(w,\,\varepsilon-w)=f_w(w)\,f_\xi(\varepsilon-w)$$

$$\Rightarrow f_{w,\varepsilon}(w,\,\varepsilon)=f_{w,\xi}(w,\,\varepsilon-w)=f_w(w)\int_0^\infty f_v(\varepsilon-w+u)\,f_u(u)\,du$$

So

$$E\left(g(w)|\varepsilon\right)=\frac{1}{f_\varepsilon(\varepsilon)}\int_0^\infty g(w)f_w(w)\int_0^\infty f_v(\varepsilon-w+u)\,f_u(u)\,du\,dw$$

To eventually be left with a single integral, we change the order of integration

$$E\left(g(w)\middle|\varepsilon\right)=\frac{1}{f_{\varepsilon}(\varepsilon)}\int_0^{\infty} f_u(u)\int_0^{\infty} g(w)\, f_w(w)\, f_v(\varepsilon-w+u)\, f_u(u)\,dwdu$$

We have

$$f_v(\varepsilon+u-w)=\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma_v^2}(\varepsilon+u-w)^2\right\}=$$

$$=\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma_v^2}(\varepsilon+u)^2\right\}\exp\left\{-\frac{w^2}{2\sigma_v^2}+\frac{(\varepsilon+u)}{\sigma_v^2}w\right\}$$

and since $f_w(w)$ is here an Exponential density, the inner integral becomes

$$\int_0^{\infty} g(w)\, f_v(\varepsilon+u-w)\, f_w(w)\,dw=$$

$$=\frac{1}{\sigma_v\sigma_w\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma_v^2}(\varepsilon+u)^2\right\}\int_0^{\infty} g(w)\exp\left\{-\frac{w^2}{2\sigma_v^2}-\left(\frac{1}{\sigma_w}-\frac{(\varepsilon+u)}{\sigma_v^2}\right)w\right\}dw$$

$$=\frac{1}{\sigma_v\sigma_w}\phi\left(\frac{\varepsilon+u}{\sigma_v}\right)\int_0^{\infty} g(w)\exp\left\{-\frac{w^2}{2\sigma_v^2}-\left(\frac{1}{\sigma_w}-\frac{(\varepsilon+u)}{\sigma_v^2}\right)w\right\}dw=\frac{1}{\sigma_v\sigma_w}\phi\left(\frac{\varepsilon+u}{\sigma_v}\right)\cdot I_g$$

The integral now will have different solutions, depending on $g(w)$.

A. $g(w)=w$

Here we have the general form

$$I_g=\int_0^{\infty} w\exp\left\{-\frac{w^2}{2\sigma_v^2}-\left(\frac{1}{\sigma_w}-\frac{(\varepsilon+u)}{\sigma_v^2}\right)w\right\}dw\equiv\int_0^{\infty} w\exp\left\{-aw^2-bw\right\}dw$$

This has the general solution

$$I_g = \frac{1}{2a}\left[1 - b\int_0^{\infty}\exp\left\{-aw^2 - bw\right\}dw\right] = \frac{1}{2a}\left[1 - b\frac{\sqrt{\pi}}{\sqrt{a}}\exp\left\{\frac{b^2}{4a}\right\}\Phi\left(-\frac{b}{\sqrt{2a}}\right)\right]$$

In our case

$$I_g = \sigma_v^2\left[1 - \left(\frac{1}{\sigma_w} - \frac{(\varepsilon+u)}{\sigma_v^2}\right)\sigma_v\sqrt{2\pi}\exp\left\{\frac{\left(\left(\frac{1}{\sigma_w} - \frac{(\varepsilon+u)}{\sigma_v^2}\right)\right)^2}{2/\sigma_v^2}\right\}\Phi\left(-\frac{\left(\frac{1}{\sigma_w} - \frac{(\varepsilon+u)}{\sigma_v^2}\right)}{1/\sigma_v}\right)\right]$$

$$I_g = \sigma_v^2\left[1 - \left(\frac{1}{\sigma_w} - \frac{(\varepsilon+u)}{\sigma_v^2}\right)\sigma_v\sqrt{2\pi}\exp\left\{\frac{1}{2}\left(\frac{\sigma_v}{\sigma_w} - \frac{(\varepsilon+u)}{\sigma_v}\right)^2\right\}\Phi\left(-\frac{(\varepsilon+u)}{\sigma_v} + \frac{\sigma_v}{\sigma_w}\right)\right]$$

$$E(w|\varepsilon) = \frac{(\sigma_v/\sigma_w)}{f_\varepsilon(\varepsilon)}\int_0^{\infty} f_G(u;k,\theta)\phi\left(\frac{\varepsilon+u}{\sigma_v}\right)\left[1 - \psi\sqrt{2\pi}\exp\left\{\frac{1}{2}\psi^2\right\}\Phi(\psi)\right]du$$

$$\psi = \frac{\sigma_v}{\sigma_w} - \frac{(\varepsilon+u)}{\sigma_v}$$

or

$$[3.109]:\ E(w|\varepsilon) = \frac{(\sigma_v/\sigma_w)}{f_\varepsilon(\varepsilon)}\int_0^{\infty} f_G(u;k,\theta)\phi\left(\frac{\varepsilon+u}{\sigma_v}\right)\left[1 - \frac{\psi\,\Phi(\psi)}{\phi(\psi)}\right]du, \qquad \psi = \frac{\sigma_v}{\sigma_w} - \frac{(\varepsilon+u)}{\sigma_v}$$

B. $g(w) = \exp\{\pm w\}$

Here we have the general form

$$I_g = \int_0^{\infty}\exp\left\{-\frac{w^2}{2\sigma_v^2} - \left(\frac{1}{\sigma_w} \mp 1 - \frac{(\varepsilon+u)}{\sigma_v^2}\right)w\right\}dw \equiv \int_0^{\infty}\exp\left\{-aw^2 - bw\right\}dw = \frac{\sqrt{\pi}}{\sqrt{a}}\exp\left\{\frac{b^2}{4a}\right\}\Phi\left(-\frac{b}{\sqrt{2a}}\right)$$

In our case

$$I_g = \sigma_v \sqrt{2\pi}\exp\left\{\frac{\left(\left(\frac{1}{\sigma_w} \mp 1 - \frac{(\varepsilon+u)}{\sigma_v^2}\right)\right)^2}{2/\sigma_v^2}\right\}\Phi\left(-\frac{\left(\left(\frac{1}{\sigma_w} \mp 1 - \frac{(\varepsilon+u)}{\sigma_v^2}\right)\right)}{1/\sigma_v}\right)$$

$$I_g = \sigma_v \sqrt{2\pi}\exp\left\{\frac{1}{2}\left(\frac{\sigma_v}{\sigma_w} \mp \sigma_v - \frac{(\varepsilon+u)}{\sigma_v}\right)^2\right\}\Phi\left(-\left(\frac{\sigma_v}{\sigma_w} \mp \sigma_v - \frac{(\varepsilon+u)}{\sigma_v}\right)\right)$$

So here

$$\int_0^\infty g(w)\, f_w(w)\, f_v(\varepsilon - w + u)\, f_u(u)\, dw = \frac{1}{\sigma_v \sigma_w}\phi\left(\frac{\varepsilon+u}{\sigma_v}\right)\times$$

$$\times \sigma_v \sqrt{2\pi}\exp\left\{\frac{1}{2}\left(\frac{\sigma_v}{\sigma_w} \mp \sigma_v - \frac{(\varepsilon+u)}{\sigma_v}\right)^2\right\}\Phi\left(-\left(\frac{\sigma_v}{\sigma_w} \mp \sigma_v - \frac{(\varepsilon+u)}{\sigma_v}\right)\right)$$

$$= \frac{1}{\sigma_w}\exp\left\{\frac{\sigma_v^2}{2}\left(\sigma_w^{-1} \mp 1\right)^2\right\}\exp\left\{-\left(\sigma_w^{-1} \mp 1\right)(\varepsilon+u)\right\}\Phi\left(-\left(\frac{\sigma_v}{\sigma_w} \mp \sigma_v - \frac{(\varepsilon+u)}{\sigma_v}\right)\right)$$

and so

$$[3.110]:\quad E\left(\exp\{\pm w\}|\varepsilon\right) = \frac{\exp\left\{\frac{\sigma_v^2}{2}\left(\sigma_w^{-1} \mp 1\right)^2 - \left(\sigma_w^{-1} \mp 1\right)\varepsilon\right\}}{\sigma_w f_\varepsilon(\varepsilon)}\times$$
$$\times \int_0^\infty \exp\left\{-\left(\sigma_w^{-1} \mp 1\right)u\right\} f_G(u;k,\theta)\Phi\left(-\left(\frac{\sigma_v}{\sigma_w} \mp \sigma_v - \frac{(\varepsilon+u)}{\sigma_v}\right)\right)du$$

B. *u*-variable.

$$E\left(g(u)|\varepsilon\right) = \int_0^\infty g(u) f_{u|\varepsilon}(u|\varepsilon)\, du = \frac{1}{f_\varepsilon(\varepsilon)}\int_0^\infty g(u) f_{u\varepsilon}(u,\varepsilon)\, du$$

Define $t = v + w$. The joint distribution of u and t is

$$f_{u,t}\left(u,\, t\right) = f_u\left(u\right) f_t\left(t\right) \ .$$

Since $\varepsilon = t - u \Rightarrow t = \varepsilon + u$ we have that

$$f_{u,t}\left(u,\, \varepsilon + u\right) = f_u\left(u\right) f_t\left(\varepsilon + u\right) = f_{u\varepsilon}\left(u, \varepsilon\right)$$

Now

$$f_t\left(t\right) = \int_0^\infty f_v\left(t - w\right) f_w\left(w\right) dw$$

So

$$E\left(g\left(u\right) \middle| \varepsilon\right) = \frac{1}{f_\varepsilon\left(\varepsilon\right)} \int_0^\infty g\left(u\right) f_u\left(u\right) f_t\left(\varepsilon + u\right) du$$

$$= \frac{1}{f_\varepsilon\left(\varepsilon\right)} \int_0^\infty g\left(u\right) f_u\left(u\right) \int_0^\infty f_v\left(\varepsilon + u - w\right) f_w\left(w\right) dw du$$

We have

$$f_v\left(\varepsilon + u - w\right) = \frac{1}{\sigma_v \sqrt{2\pi}} \exp\left\{ -\frac{1}{2\sigma_v^2} \left(\varepsilon + u - w\right)^2 \right\} =$$

$$= \frac{1}{\sigma_v \sqrt{2\pi}} \exp\left\{ -\frac{1}{2\sigma_v^2} \left(\varepsilon + u\right)^2 \right\} \exp\left\{ -\frac{w^2}{2\sigma_v^2} + \frac{\left(\varepsilon + u\right)}{\sigma_v^2} w \right\}$$

and since $f_w\left(w\right)$ is here an Exponential density, the inner integral becomes

$$\int_0^\infty f_v\left(\varepsilon+u-w\right) f_w\left(w\right) dw =$$

$$= \frac{1}{\sigma_v \sigma_w \sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma_v^2}\left(\varepsilon+u\right)^2\right\} \int_0^\infty \exp\left\{-\frac{w^2}{2\sigma_v^2} - \left(\frac{1}{\sigma_w} - \frac{\left(\varepsilon+u\right)}{\sigma_v^2}\right) w\right\} dw$$

The integral now has general solution

$$\int_0^\infty \exp\left\{-au^2 - bu\right\} du = \frac{\sqrt{\pi}}{\sqrt{a}} \exp\left\{\frac{b^2}{4a}\right\} \Phi\left(-\frac{b}{\sqrt{2a}}\right)$$

So

$$\int_0^\infty \exp\left\{-\frac{w^2}{2\sigma_v^2} - \left(\frac{1}{\sigma_w} - \frac{\left(\varepsilon+u\right)}{\sigma_v^2}\right) w\right\} dw = \sigma_v \sqrt{2\pi} \exp\left\{\frac{\left(\frac{1}{\sigma_w} - \frac{\left(\varepsilon+u\right)}{\sigma_v^2}\right)^2}{2/\sigma_v^2}\right\} \Phi\left(-\frac{\frac{1}{\sigma_w} - \frac{\left(\varepsilon+u\right)}{\sigma_v^2}}{1/\sigma_v}\right)$$

$$= \sigma_v \sqrt{2\pi} \exp\left\{\frac{1}{2}\frac{\left(\varepsilon+u\right)^2}{\sigma_v^2} - \frac{\left(\varepsilon+u\right)}{\sigma_w} + \frac{\sigma_v^2}{2\sigma_w^2}\right\} \Phi\left(\frac{\left(\varepsilon+u\right)}{\sigma_v} - \frac{\sigma_v}{\sigma_w}\right)$$

Plugging back in,

$$\int_0^\infty f_v\left(\varepsilon+u-w\right) f_w\left(w\right) dw =$$

$$= \frac{1}{\sigma_v \sigma_w \sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma_v^2}\left(\varepsilon+u\right)^2\right\} \sigma_v \sqrt{2\pi} \exp\left\{\frac{1}{2}\frac{\left(\varepsilon+u\right)^2}{\sigma_v^2} - \frac{\left(\varepsilon+u\right)}{\sigma_w} + \frac{\sigma_v^2}{2\sigma_w^2}\right\} \Phi\left(\frac{\left(\varepsilon+u\right)}{\sigma_v} - \frac{\sigma_v}{\sigma_w}\right)$$

$$= \frac{1}{\sigma_w} \exp\left\{-\frac{\left(\varepsilon+u\right)}{\sigma_w} + \frac{\sigma_v^2}{2\sigma_w^2}\right\} \Phi\left(\frac{\left(\varepsilon+u\right)}{\sigma_v} - \frac{\sigma_v}{\sigma_w}\right)$$

So

$$E\left(g(u)|\varepsilon\right)=\frac{1}{f_{\varepsilon}(\varepsilon)}\int_{0}^{\infty}g(u)f_{u}(u)\,f_{t}(\varepsilon+u)\,du$$

$$=\frac{1}{f_{\varepsilon}(\varepsilon)}\int_{0}^{\infty}g(u)f_{u}(u)\frac{1}{\sigma_{w}}\exp\left\{-\frac{(\varepsilon+u)}{\sigma_{w}}+\frac{\sigma_{v}^{2}}{2\sigma_{w}^{2}}\right\}\Phi\left(\frac{(\varepsilon+u)}{\sigma_{v}}-\frac{\sigma_{v}}{\sigma_{w}}\right)du$$

$$\boxed{[3.111]: E\left(g(u)|\varepsilon\right)=\frac{\exp\left\{-\frac{\varepsilon}{\sigma_{w}}+\frac{\sigma_{v}^{2}}{2\sigma_{w}^{2}}\right\}}{\sigma_{w}f_{\varepsilon}(\varepsilon)}\int_{0}^{\infty}g(u)\exp\left\{\frac{-u}{\sigma_{w}}\right\}f_{G}(u;k,\theta)\,\Phi\left(\frac{(\varepsilon+u)}{\sigma_{v}}-\frac{\sigma_{v}}{\sigma_{w}}\right)du}$$

The integral will have to be evaluated numerically.

C. The net effect $E\left(e^{w-u}|\varepsilon\right)$.

What changes from the previous case is the distribution of z.

$$E\left(e^{w-u}|\varepsilon\right)=\frac{1}{f_{\varepsilon}(\varepsilon)}\int_{-\infty}^{0}\frac{e^{z}}{\sigma_{v}\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma_{v}^{2}}(\varepsilon-z)^{2}\right\}\frac{\sigma_{w}^{k-1}}{(\sigma_{w}+\theta)^{k}}\exp\{-z/\sigma_{w}\}\left[1-F_{G}(-z;k,\delta)\right]dz$$

$$+\frac{1}{f_{\varepsilon}(\varepsilon)}\int_{0}^{\infty}\frac{e^{z}}{\sigma_{v}\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma_{v}^{2}}(\varepsilon-z)^{2}\right\}\frac{\sigma_{w}^{k-1}}{(\sigma_{w}+\theta)^{k}}\exp\{-z/\sigma_{w}\}\,dz$$

$$=\frac{\exp\left\{-\frac{\varepsilon^{2}}{2\sigma_{v}^{2}}\right\}}{f_{\varepsilon}(\varepsilon)\sigma_{v}\sqrt{2\pi}}\frac{\sigma_{w}^{k-1}}{(\sigma_{w}+\theta)^{k}}\int_{-\infty}^{\infty}\exp\left\{-\frac{z^{2}}{2\sigma_{v}^{2}}+\left(\frac{\varepsilon}{\sigma_{v}^{2}}+1-\sigma_{w}^{-1}\right)z\right\}dz$$

$$-\frac{1}{f_{\varepsilon}(\varepsilon)}\frac{\sigma_{w}^{k-1}}{(\sigma_{w}+\theta)^{k}}\int_{-\infty}^{0}\exp\left\{\left(1-\sigma_{w}^{-1}\right)z\right\}\frac{1}{\sigma_{v}}\phi\left(\frac{\varepsilon-z}{\sigma_{v}}\right)F_{G}(-z;k,\delta)\,dz$$

The first integral can be solved, using $\int\limits_{-\infty}^{\infty} \exp\left\{-p^2 x \pm qx\right\} dx = \exp\left\{\frac{q^2}{4p^2}\right\}\frac{\sqrt{\pi}}{p}$

and we get

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{z^2}{2\sigma_v^2} + \left(\frac{\varepsilon}{\sigma_v^2} + 1 - \sigma_w^{-1}\right)z\right\} dz = \exp\left\{\frac{\left(\frac{\varepsilon}{\sigma_v^2} + 1 - \sigma_w^{-1}\right)^2}{2/\sigma_v^2}\right\}\sigma_v\sqrt{2\pi}$$

$$= \exp\left\{\frac{1}{2}\left(\frac{\varepsilon}{\sigma_v} + \sigma_v - \frac{\sigma_v}{\sigma_w}\right)^2\right\}\sigma_v\sqrt{2\pi} = \sigma_v\sqrt{2\pi}\exp\left\{\frac{1}{2}\frac{\varepsilon^2}{\sigma_v^2}\right\}\exp\left\{\frac{\sigma_v^2}{2}\left(1-\sigma_w^{-1}\right)^2 + \left(1-\sigma_w^{-1}\right)\varepsilon\right\}$$

So

$$E\left(e^{w-u}\left|\varepsilon\right.\right) = \frac{\exp\left\{-\frac{\varepsilon^2}{2\sigma_v^2}\right\}}{f_\varepsilon\left(\varepsilon\right)\sigma_v\sqrt{2\pi}}\frac{\sigma_w^{k-1}}{\left(\sigma_w+\theta\right)^k}\sigma_v\sqrt{2\pi}\exp\left\{\frac{1}{2}\frac{\varepsilon^2}{\sigma_v^2}\right\}\exp\left\{\frac{\sigma_v^2}{2}\left(1-\sigma_w^{-1}\right)^2 + \left(1-\sigma_w^{-1}\right)\varepsilon\right\}$$

$$-\frac{1}{f_\varepsilon\left(\varepsilon\right)}\frac{\sigma_w^{k-1}}{\left(\sigma_w+\theta\right)^k}\int_{-\infty}^{0}\exp\left\{\left(1-\sigma_w^{-1}\right)z\right\}\frac{1}{\sigma_v}\phi\left(\frac{\varepsilon-z}{\sigma_v}\right)F_G\left(-z;k,\delta\right)dz$$

and simplifying and switching the limits of integration we get

$$[3.112]:\ E\left(e^{w-u}\left|\varepsilon\right.\right) = \frac{1}{f_\varepsilon\left(\varepsilon\right)}\frac{\sigma_w^{k-1}}{\left(\sigma_w+\theta\right)^k}\left[\exp\left\{\frac{\sigma_v^2}{2}\left(1-\sigma_w^{-1}\right)^2 + \left(1-\sigma_w^{-1}\right)\varepsilon\right\}\right.$$
$$\left. - \int_0^{\infty}\exp\left\{-\left(1-\sigma_w^{-1}\right)z\right\}\frac{1}{\sigma_v}\phi\left(\frac{\varepsilon+z}{\sigma_v}\right)F_G\left(z;k,\delta\right)dz\right]$$

5.2.3. The determinant of the Jacobian of the moment conditions.

We have the system of equations

$$\begin{cases} \hat{\kappa}_2(\varepsilon) & -\left(\sigma_v^2 + \sigma_w^2 + k\theta^2\right) = 0 \\ (1/2)\hat{\kappa}_3(\varepsilon) & -\left(\sigma_w^3 - k\theta^3\right) = 0 \\ (1/6)\hat{\kappa}_4(\varepsilon) & -\left(\sigma_w^4 + k\theta^4\right) = 0 \\ (1/24)\hat{\kappa}_5(\varepsilon) & -\left(\sigma_w^5 - k\theta^5\right) = 0 \end{cases}$$

For the vector $\mathbf{q} = \left(\sigma_v, \sigma_w, k, \theta\right)'$ the Jacobian matrix is

$$\mathbf{J} = \begin{bmatrix} -2\sigma_v & -2\sigma_w & -\theta^2 & -2k\theta \\ 0 & -3\sigma_w^2 & \theta^3 & 3k\theta^2 \\ 0 & -4\sigma_w^3 & -\theta^4 & -4k\theta^3 \\ 0 & -5\sigma_w^4 & \theta^5 & 5k\theta^4 \end{bmatrix}$$

Its determinant is

$$|\mathrm{J}| = \begin{vmatrix} -2\sigma_v & -2\sigma_w & -\theta^2 & -2k\theta \\ 0 & -3\sigma_w^2 & \theta^3 & 3k\theta^2 \\ 0 & -4\sigma_w^3 & -\theta^4 & -4k\theta^3 \\ 0 & -5\sigma_w^4 & \theta^5 & 5k\theta^4 \end{vmatrix} = -2\sigma_v \begin{vmatrix} -3\sigma_w^2 & \theta^3 & 3k\theta^2 \\ -4\sigma_w^3 & -\theta^4 & -4k\theta^3 \\ -5\sigma_w^4 & \theta^5 & 5k\theta^4 \end{vmatrix} = 2\sigma_v \begin{vmatrix} 3\sigma_w^2 & \theta^3 & 3k\theta^2 \\ 4\sigma_w^3 & -\theta^4 & -4k\theta^3 \\ 5\sigma_w^4 & \theta^5 & 5k\theta^4 \end{vmatrix}$$

$$= 2\sigma_v\sigma_w^2\theta^3k\theta^2 \begin{vmatrix} 3 & 1 & 3 \\ 4\sigma_w & -\theta & -4\theta \\ 5\sigma_w^2 & \theta^2 & 5\theta^2 \end{vmatrix} = 2\sigma_v\sigma_w^2\theta^5k\left[3\begin{vmatrix} -\theta & -4\theta \\ \theta^2 & 5\theta^2 \end{vmatrix} - 1\begin{vmatrix} 4\sigma_w & -4\theta \\ 5\sigma_w^2 & 5\theta^2 \end{vmatrix} + 3\begin{vmatrix} 4\sigma_w & -\theta \\ 5\sigma_w^2 & \theta^2 \end{vmatrix}\right]$$

$$= 2\sigma_v\sigma_w^2\theta^5k\left[3\theta^3\begin{vmatrix} -1 & -4 \\ 1 & 5 \end{vmatrix} - 20\sigma_w\theta\begin{vmatrix} 1 & -1 \\ \sigma_w & \theta \end{vmatrix} + 3\begin{vmatrix} 4\sigma_w & -\theta \\ 5\sigma_w^2 & \theta^2 \end{vmatrix}\right]$$

$$= 2\sigma_v \sigma_w^2 \theta^5 k\left[-3\theta^3 - 20\sigma_w \theta\left(\theta + \sigma_w\right) + 3\left(4\sigma_w \theta^2 + 5\sigma_w^2 \theta\right)\right]$$

$$= 2\sigma_v \sigma_w^2 \theta^5 k\left[-3\theta^3 - 20\sigma_w \theta\left(\theta + \sigma_w\right) + 3\sigma_w \theta\left(4\theta + 5\sigma_w\right)\right]$$

$$= 2\sigma_v \sigma_w^2 \theta^5 k\left[-3\theta^3 - \sigma_w \theta\left[20\left(\theta + \sigma_w\right) - 3\left(4\theta + 5\sigma_w\right)\right]\right]$$

$$= 2\sigma_v \sigma_w^2 \theta^5 k\left[-3\theta^3 - \sigma_w \theta\left[20\left(\theta + \sigma_w\right) - 3\left(4\theta + 5\sigma_w\right)\right]\right]$$

$$= 2\sigma_v \sigma_w^2 \theta^5 k\left[-3\theta^3 - \sigma_w \theta\left(8\theta + 5\sigma_w\right)\right] = 2\sigma_v \sigma_w^2 \theta^6 k\left[-3\theta^2 - \sigma_w\left(8\theta + 5\sigma_w\right)\right]$$

$$\Rightarrow \left|\mathrm{J}\right| = -2\sigma_v \sigma_w^2 \theta^6 k\left[3\theta^2 + \sigma_w\left(8\theta + 5\sigma_w\right)\right]$$

Again, this determinant can be zero only if one of the parameters involved is also zero.

TECHNICAL APPENDIX 3.V.

The Generalized Exponential 2TSF specification.

We are considering the composite error term $\varepsilon = v + w - u$ with

$$v \sim \mathrm{N}\left(0, \sigma_v^2\right),\ w \sim GE\left(2, \theta_w, 0\right),\ \ u \sim GE\left(2, \theta_u, 0\right)$$

Writing $f_E\left(w; \theta_w\right)$ for the density of an Exponential random variable with scale parameter θ_w, note that

$$w \sim GE\left(2, \theta_w, 0\right) \Rightarrow f_w\left(w\right) = 2 f_E\left(w; \theta_w\right) - f_E\left(w; \theta_w/2\right)$$

$$u \sim GE\left(2, \theta_u, 0\right) \Rightarrow f_u\left(u\right) = 2 f_E\left(u; \theta_u\right) - f_E\left(u; \theta_u/2\right)$$

A. The $GE\left(2, \theta_w, 0\right)$ distribution.

Droping the w, u subscripts, to find the **mode of the distribution**, we calculate

$$\frac{\partial}{\partial w} f_w\left(w\right) = \frac{\partial}{\partial w}\left[\frac{2}{\theta}\exp\left\{-w/\theta\right\} - \frac{2}{\theta}\exp\left\{-2w/\theta\right\}\right]$$

$$= \left[-\frac{2}{\theta^2}\exp\left\{-w/\theta\right\} + \frac{4}{\theta^2}\exp\left\{-2w/\theta\right\}\right] = 0$$

$$\Rightarrow \frac{2}{\theta^2}\exp\left\{-w/\theta\right\}\left[-1 + 2\exp\left\{-w/\theta\right\}\right] = 0$$

$$\Rightarrow \exp\{-w/\theta\} = \frac{1}{2} \;\Rightarrow\; -\frac{w}{\theta} = \ln 1/2 \;\Rightarrow \text{mode} = \theta \ln 2$$

Median.

The distribution function is $F_w(w) = \left(1-\exp\{-w/\theta\}\right)^2$

The median of the distribution is

$$F_w(w^*) : \left(1-\exp\{-w^*/\theta\}\right)^2 = \frac{1}{2} \;\Rightarrow\; \exp\{-w^*/\theta\} = 1-\frac{1}{\sqrt{2}} \;\Rightarrow \text{median} = -\theta \ln\left(1-1/\sqrt{2}\right)$$

A.1. The Moment Generating function of $f_w(w)$ and its moments.

The MGF of w is

$$E\left(\exp\{tw\}\right) = \int_0^\infty \exp\{tw\} f_w(w) \;= 2\int_0^\infty \exp\{tw\} f_E(w;\theta)\,dw - \int_0^\infty \exp\{tw\} f_E(w;\theta/2)\,dw$$

$$= 2\frac{1/\theta}{1/\theta - t} \;-\; \frac{2/\theta}{2/\theta - t} \;=\; \frac{2}{1-\theta t} \;-\; \frac{2}{2-\theta t} \;=\; 2\frac{2-\theta t - 1 + \theta t}{(1-\theta t)(2-\theta t)}$$

$$\Rightarrow E\left(\exp\{tw\}\right) = \frac{2}{(1-\theta t)(2-\theta t)} = \frac{2}{2-3\theta t + \theta^2 t^2}$$

Solving the 2nd-degree polynomial in the denominator we have the roots

$$\Rightarrow E\left(\exp\{tw\}\right) = MGF_w(t) = \frac{2}{(1-\theta t)(2-\theta t)} = \frac{2}{2-3\theta t+\theta^2 t^2} = 2\left(2-3\theta t+\theta^2 t^2\right)^{-1}$$

$$r(t)_{1,2} = \frac{3\theta \pm \sqrt{9\theta^2 - 4\cdot\theta^2\cdot 2}}{2\theta^2} = \frac{3\theta \pm \theta}{2\theta^2} = \begin{cases} 1/\theta \\ 2/\theta \end{cases}$$

For the polynomial to be positive t must be outside the interval determined by the roots, so we must have $\{t < 1/\theta\} \cup \{t > 2/\theta\}$

Mean.

$$\frac{\partial}{\partial t}MGF_w(t)=2\left[-\left(2-3\theta t+\theta^2t^2\right)^{-2}\left(-3\theta+2\theta^2t\right)\right]\Big|_{t=0}=\frac{6\theta}{4}=\frac{3}{2}\theta=E(w)$$

2nd raw moment and variance.

$$\frac{\partial^2}{\partial t^2}MGF_w(t)=2\left[2\left(2-3\theta t+\theta^2t^2\right)^{-3}\left(-3\theta+2\theta^2t\right)^2-2\theta^2\left(2-3\theta t+\theta^2t^2\right)^{-2}\right]\Big|_{t=0}=$$

$$=2\left[2(2)^{-3}(-3\theta)^2-2\theta^2(2)^{-2}\right]=2\frac{18-4}{8}\theta^2$$

$$\Rightarrow E\left(w^2\right)=\frac{14}{4}\theta^2$$

The variance therefore is $\mathrm{Var}(w)=E\left(w^2\right)-\left[E(w)\right]^2=\frac{14}{4}\theta^2-\frac{9}{4}\theta^2=\frac{5}{4}\theta^2$

3d raw moment and skewness.

$$\frac{\partial^3}{\partial t^3}MGF_w(t)=2\left[-6\left(2-3\theta t+\theta^2t^2\right)^{-4}\left(-3\theta+2\theta^2t\right)^3+2\left(-3\theta+2\theta^2t\right)2\theta^2 2\left(2-3\theta t+\theta^2t^2\right)^{-3}\right]$$

$$-2\left[-2\left(2-3\theta t+\theta^2t^2\right)^{-3}\left(-3\theta+2\theta^2t\right)2\theta^2\right]\Big|_{t=0}$$

$$=2\left[-6(2)^{-4}(-3\theta)^3+2(-3\theta)2\theta^2 2(2)^{-3}\right]-2\left[-2(2)^{-3}(-3\theta)2\theta^2\right]$$

$$=2\left[\frac{6\cdot 27}{16}\theta^3-3\theta^3\right]-3\theta^3=\left(\frac{324}{16}-9\right)\theta^3=\left(\frac{81}{4}-\frac{36}{4}\right)\theta^3$$

$$\Rightarrow E\left(w^3\right)=\frac{45}{4}\theta^3$$

The skewness coefficient is

$$\gamma_1 = \frac{\kappa_3(w)}{\left[\kappa_2(w)\right]^{3/2}} = \frac{E\left(w^3\right) - 3E(w)\operatorname{Var}(w) - \left[E(w)\right]^3}{\left[\operatorname{Var}(w)\right]^{3/2}} = \frac{\frac{45}{4} - 3\frac{3}{2}\frac{5}{4} - \left(\frac{3}{2}\right)^2}{\left(\frac{5}{4}\right)^{3/2}}\theta^3$$

$$= \frac{\frac{45}{4} - \frac{45}{8} - \frac{9}{4}}{(5/4)^{3/2}}\theta^3 = \frac{\frac{90}{8} - \frac{45}{8} - \frac{18}{8}}{(5/4)^{3/2}}\theta^3 = \frac{27/8}{(5/4)^{3/2}}\theta^3 = \frac{27\sqrt{64}}{8\sqrt{125}}\theta^3$$

$$\Rightarrow \gamma_1(w) = \frac{27}{\sqrt{125}}\theta^3 \approx 2.415\theta^3 \quad \text{while we have obtained } \kappa_3(w) = \frac{27}{8}\theta^3$$

A.2. Exponentiated mode, $\operatorname{mode}\left(\exp\{\pm w\}\right)$.

To obtain $\operatorname{mode}\left(\exp\{w\}\right)$ we need to obtain its density.

We have $q \equiv \exp\{w\} \Rightarrow \ln q = w \Rightarrow \dfrac{\partial w}{\partial q} = \dfrac{1}{q}, \quad q \geq 1$.

So

$$f_q(q) = \frac{1}{q} f_w(\ln q) \;\Rightarrow\; f_q(q) = \frac{1}{q}\frac{2}{\theta_w}\exp\{-\ln q/\theta_w\} - \frac{1}{q}\frac{2}{\theta_w}\exp\{-2\ln q/\theta_w\}$$

$$\Rightarrow f_q(q) = \frac{1}{q}\frac{2}{\theta_w}\left[q^{-1/\theta_w} - q^{-2/\theta_w}\right] \Rightarrow f_q(q) = \frac{2}{\theta_w}\left[q^{-1/\theta_w - 1} - q^{-2/\theta_w - 1}\right]$$

Then

$$\frac{d}{dq} f_q(q) \propto -\left(\frac{1}{\theta_w} + 1\right) q^{-1/\theta_w - 2} + \left(\frac{2}{\theta_w} + 1\right) q^{-2/\theta_w - 2} = 0$$

$$\Rightarrow \left(\frac{1+\theta_w}{\theta_w}\right) q^{-1/\theta_w - 2} = \left(\frac{2+\theta_w}{\theta_w}\right) q^{-2/\theta_w - 2} \Rightarrow q^{1/\theta_w} = \frac{2+\theta_w}{1+\theta_w}$$

$$\boxed{[3.132]: \;\Rightarrow\; \text{mode}\left(\exp\{w\}\right) = \left(\frac{2+\theta_w}{1+\theta_w}\right)^{\theta_w} .}$$

For $q \equiv \exp\{-w\} \Rightarrow -\ln q = w \Rightarrow \dfrac{\partial w}{\partial q} = -\dfrac{1}{q}, \quad q \geq 1$

we have

$$f_q(q) = \frac{1}{q} f_w(-\ln q) \;\Rightarrow\; f_q(q) = \frac{1}{q}\frac{2}{\theta_w}\exp\{\ln q/\theta_w\} - \frac{1}{q}\frac{2}{\theta_w}\exp\{2\ln q/\theta_w\}$$

$$\Rightarrow f_q(q) = \frac{1}{q}\frac{2}{\theta_w}\left[q^{1/\theta_w} - q^{2/\theta_w}\right] \Rightarrow f_q(q) = \frac{2}{\theta_w}\left[q^{1/\theta_w - 1} - q^{2/\theta_w - 1}\right]$$

Then

$$\frac{d}{dq} f_q(q) \propto \left(\frac{1}{\theta_w} - 1\right) q^{1/\theta_w - 2} - \left(\frac{2}{\theta_w} - 1\right) q^{2/\theta_w - 2} = 0$$

$$\Rightarrow (1-\theta_w) q^{1/\theta_w} = (2-\theta_w) q^{2/\theta_w} \Rightarrow q^{1/\theta_w} = \frac{1-\theta_w}{2-\theta_w}$$

$$\boxed{[3.132]: \;\Rightarrow\; \text{mode}\left(\exp\{-w\}\right) = \left(\frac{1-\theta_w}{2-\theta_w}\right)^{\theta_w} .}$$

B. The distribution of $z = w - u$.

Since the variables are assumed independent, then the integral of the convolution is

$$f_z(z) = \int_{u_{\min}(z)}^{\infty} \left[2 f_E(z+u;\theta_w) - f_E(z+u;\theta_w/2)\right]\left[2 f_E(u;\theta_u) - f_E(u;\theta_u/2)\right] du$$

where the lower limit of integration will depend on z. This integral breaks into four integrals due to linearity, and in all of them the evaluation of the upper limit of integration is zero. So we have

$$f_z(z) = 4\int_{u_{\min}(z)}^{\infty} f_E(z+u;\theta_w)\, f_E(u;\theta_u)\,du \;-\; 2\int_{u_{\min}(z)}^{\infty} f_E(z+u;\theta_w)\, f_E(u;\theta_u/2)\,du$$

$$-2\int_{u_{\min}(z)}^{\infty} f_E(z+u;\theta_w/2)\, f_E(u;\theta_u) \;+\; \int_{u_{\min}(z)}^{\infty} f_E(z+u;\theta_w/2)\, f_E(u;\theta_u/2)\,du$$

$$= \frac{4}{\theta_w\theta_u}\left[\frac{\theta_w\theta_u}{\theta_w+\theta_u}\exp\{-z/\theta_w\}\exp\left\{-\left(\frac{\theta_w+\theta_u}{\theta_w\theta_u}\right)u_{\min}(z)\right\}\right.$$

$$-\frac{\theta_w\theta_u}{2\theta_w+\theta_u}\exp\{-z/\theta_w\}\exp\left\{-\left(\frac{2\theta_w+\theta_u}{\theta_w\theta_u}\right)u_{\min}(z)\right\}$$

$$-\frac{\theta_w\theta_u}{\theta_w+2\theta_u}\exp\{-2z/\theta_w\}\exp\left\{-\left(\frac{\theta_w+2\theta_u}{\theta_w\theta_u}\right)u_{\min}(z)\right\}$$

$$\left.+\frac{\theta_w\theta_u}{2\theta_w+2\theta_u}\exp\{-2z/\theta_w\}\exp\left\{-\left(\frac{2\theta_w+2\theta_u}{\theta_w\theta_u}\right)u_{\min}(z)\right\}\right]$$

$$= 4\left[\exp\{-z/\theta_w\}\left(\frac{\exp\left\{-\left(\frac{\theta_w+\theta_u}{\theta_w\theta_u}\right)u_{\min}(z)\right\}}{\theta_w+\theta_u} - \frac{\exp\left\{-\left(\frac{2\theta_w+\theta_u}{\theta_w\theta_u}\right)u_{\min}(z)\right\}}{2\theta_w+\theta_u}\right)\right.$$

$$\left.+\exp\{-2z/\theta_w\}\left(\frac{\exp\left\{-\left(\frac{2\theta_w+2\theta_u}{\theta_w\theta_u}\right)u_{\min}(z)\right\}}{2\theta_w+2\theta_u} - \frac{\exp\left\{-\left(\frac{\theta_w+2\theta_u}{\theta_w\theta_u}\right)u_{\min}(z)\right\}}{\theta_w+2\theta_u}\right)\right]$$

Since $z = w - u \Rightarrow z + u = w \geq 0 \Rightarrow \; u \geq -z$. This will hold always when $z > 0$ but it will restrict the limit of integration when $z \leq 0$. So

$$u_{\min}(z) = \begin{cases} -z & z \leq 0 \\ 0 & z > 0 \end{cases}$$

We have

$z \leq 0$

$$f_z(z) = 4\left[\exp\{-z/\theta_w\}\left(\frac{\exp\left\{\left(\dfrac{\theta_w+\theta_u}{\theta_w\theta_u}\right)z\right\}}{\theta_w+\theta_u} - \frac{\exp\left\{\left(\dfrac{2\theta_w+\theta_u}{\theta_w\theta_u}\right)z\right\}}{2\theta_w+\theta_u}\right)\right.$$

$$\left. + \exp\{-2z/\theta_w\}\left(\frac{\exp\left\{\left(\dfrac{2\theta_w+2\theta_u}{\theta_w\theta_u}\right)z\right\}}{2\theta_w+2\theta_u} - \frac{\exp\left\{\left(\dfrac{\theta_w+2\theta_u}{\theta_w\theta_u}\right)z\right\}}{\theta_w+2\theta_u}\right)\right]$$

$$f_z(z) = 4\left[\left(\frac{\exp\left\{\left(\dfrac{\theta_w+\theta_u}{\theta_w\theta_u} - \dfrac{1}{\theta_w}\right)z\right\}}{\theta_w+\theta_u} - \frac{\exp\left\{\left(\dfrac{2\theta_w+\theta_u}{\theta_w\theta_u} - \dfrac{1}{\theta_w}\right)z\right\}}{2\theta_w+\theta_u}\right)\right.$$

$$\left. + \left(\frac{\exp\left\{\left(\dfrac{2\theta_w+2\theta_u}{\theta_w\theta_u} - \dfrac{2}{\theta_w}\right)z\right\}}{2\theta_w+2\theta_u} - \frac{\exp\left\{\left(\dfrac{\theta_w+2\theta_u}{\theta_w\theta_u} - \dfrac{2}{\theta_w}\right)z\right\}}{\theta_w+2\theta_u}\right)\right]$$

$$f_z(z) = 4\left[\frac{\exp\{z/\theta_u\}}{\theta_w+\theta_u} - \frac{\exp\{2z/\theta_u\}}{2\theta_w+\theta_u} + \frac{\exp\{2z/\theta_u\}}{2\theta_w+2\theta_u} - \frac{\exp\{z/\theta_u\}}{\theta_w+2\theta_u}\right]$$

$$f_z(z) = 4\left[\left(\frac{1}{\theta_w+\theta_u} - \frac{1}{\theta_w+2\theta_u}\right)\exp\{z/\theta_u\} - \left(\frac{1}{2\theta_w+\theta_u} - \frac{1}{2\theta_w+2\theta_u}\right)\exp\{2z/\theta_u\}\right]$$

$$f_z(z) = 4\left[\frac{\theta_u\exp\{z/\theta_u\}}{(\theta_w+\theta_u)(\theta_w+2\theta_u)} - \frac{\theta_u\exp\{2z/\theta_u\}}{(2\theta_w+\theta_u)(2\theta_w+2\theta_u)}\right]$$

$$\Rightarrow z \le 0, \quad f_z(z) = \frac{2\theta_u}{(\theta_w+\theta_u)}\left[\frac{2\exp\{z/\theta_u\}}{(\theta_w+2\theta_u)} - \frac{\exp\{2z/\theta_u\}}{(2\theta_w+\theta_u)}\right]$$

For $z > 0$ we have

$$f_z(z) = 4\left[\exp\{-z/\theta_w\}\left(\frac{1}{\theta_w+\theta_u} - \frac{1}{2\theta_w+\theta_u}\right) + \exp\{-2z/\theta_w\}\left(\frac{1}{2\theta_w+2\theta_u} - \frac{1}{\theta_w+2\theta_u}\right)\right]$$

$$= 4\left[\exp\{-z/\theta_w\}\left(\frac{\theta_w}{(\theta_w+\theta_u)(2\theta_w+\theta_u)}\right) - \exp\{-2z/\theta_w\}\left(\frac{\theta_w}{(2\theta_w+2\theta_u)(\theta_w+2\theta_u)}\right)\right]$$

$$\Rightarrow z > 0, \quad f_z(z) = \frac{2\theta_w}{\theta_w+\theta_u}\left[\frac{2\exp\{-z/\theta_w\}}{2\theta_w+\theta_u} - \frac{\exp\{-2z/\theta_w\}}{\theta_w+2\theta_u}\right]$$

So in all for $z = w - u$

$$[3.125]: \quad f_z(z) = \begin{cases} \dfrac{2\theta_u}{\theta_w+\theta_u}\left[\dfrac{2\exp\{z/\theta_u\}}{\theta_w+2\theta_u} - \dfrac{\exp\{2z/\theta_u\}}{2\theta_w+\theta_u}\right] & z \le 0 \\[2ex] \dfrac{2\theta_w}{\theta_w+\theta_u}\left[\dfrac{2\exp\{-z/\theta_w\}}{2\theta_w+\theta_u} - \dfrac{\exp\{-2z/\theta_w\}}{\theta_w+2\theta_u}\right] & z > 0 \end{cases}$$

B.1. The moment generating function of $z = w - u$.

The MGF of z is immediately obtained due to independence as

$$MGF_z(t) = E\left(\exp\{tz\}\right) = E\left(\exp\{tw\}\right)E\left(\exp\{-tu\}\right) = \frac{4}{(1-\theta_w t)(2-\theta_w t)(1+\theta_u t)(2+\theta_u t)}$$

B.2. The distribution function of $z = w - u$.

We have

$$z \le 0, \quad F_z(z) = \int_{-\infty}^{z} \frac{2\theta_u}{\theta_w + \theta_u}\left[\frac{2\exp\{s/\theta_u\}}{\theta_w + 2\theta_u} \quad - \quad \frac{\exp\{2s/\theta_u\}}{2\theta_w + \theta_u}\right] ds$$

$$= \frac{2\theta_u}{\theta_w + \theta_u}\int_{-\infty}^{z} \frac{2\exp\{s/\theta_u\}}{(\theta_w + 2\theta_u)} ds \quad - \quad \frac{2\theta_u}{\theta_w + \theta_u}\int_{-\infty}^{z} \frac{\exp\{2s/\theta_u\}}{2\theta_w + \theta_u} ds$$

$$\Rightarrow z \le 0, \quad F_z(z) = \frac{4\theta_u^2}{\theta_w + \theta_u}\frac{\exp\{z/\theta_u\}}{(\theta_w + 2\theta_u)} \quad - \quad \frac{\theta_u^2}{\theta_w + \theta_u}\frac{\exp\{2z/\theta_u\}}{(2\theta_w + \theta_u)}$$

Then $\Rightarrow F_z(0) = \dfrac{4\theta_u^2}{\theta_w + \theta_u}\dfrac{1}{(\theta_w + 2\theta_u)} \quad - \quad \dfrac{\theta_u^2}{\theta_w + \theta_u}\dfrac{1}{(2\theta_w + \theta_u)}$

$$z > 0, \quad F_z(z) = F_z(0) + \int_{0}^{z} \frac{2\theta_w}{\theta_w + \theta_u}\left[\frac{2\exp\{-z/\theta_w\}}{2\theta_w + \theta_u} \quad - \quad \frac{\exp\{-2z/\theta_w\}}{\theta_w + 2\theta_u}\right] ds$$

$$= 1 - \int_{z}^{\infty} \frac{2\theta_w}{\theta_w + \theta_u}\left[\frac{2\exp\{-z/\theta_w\}}{2\theta_w + \theta_u} \quad - \quad \frac{\exp\{-2z/\theta_w\}}{\theta_w + 2\theta_u}\right] ds$$

$$=1 - \frac{2\theta_w}{\theta_w+\theta_u}\int_z^{\infty}\frac{2\exp\{-z/\theta_w\}}{2\theta_w+\theta_u}ds + \frac{2\theta_w}{\theta_w+\theta_u}\int_z^{\infty}\frac{\exp\{-2z/\theta_w\}}{\theta_w+2\theta_u}ds$$

$$=1 - \frac{2\theta_w}{\theta_w+\theta_u}\frac{2\theta_w}{2\theta_w+\theta_u}\int_z^{\infty}\frac{1}{\theta_w}\exp\{-z/\theta_w\}ds + \frac{2\theta_w}{\theta_w+\theta_u}\frac{\theta_w}{2(\theta_w+2\theta_u)}\int_z^{\infty}\frac{2}{\theta_w}\exp\{-2z/\theta_w\}ds$$

$$=1 - \frac{2\theta_w}{\theta_w+\theta_u}\frac{2\theta_w}{2\theta_w+\theta_u}\exp\{-z/\theta_w\} + \frac{2\theta_w}{\theta_w+\theta_u}\frac{\theta_w}{2(\theta_w+2\theta_u)}\exp\{-2z/\theta_w\}$$

So in all

$$[3.126]:\quad F_z(z)=\begin{cases}\dfrac{4\theta_u^2}{\theta_w+\theta_u}\dfrac{\exp\{z/\theta_u\}}{(\theta_w+2\theta_u)} - \dfrac{\theta_u^2}{\theta_w+\theta_u}\dfrac{\exp\{2z/\theta_u\}}{(2\theta_w+\theta_u)} & z\le 0\\[2ex] 1 - \dfrac{4\theta_w^2}{\theta_w+\theta_u}\dfrac{\exp\{-z/\theta_w\}}{2\theta_w+\theta_u} + \dfrac{\theta_w^2}{\theta_w+\theta_u}\dfrac{\exp\{-2z/\theta_w\}}{(\theta_w+2\theta_u)} & z>0\end{cases}$$

B.2.1. The probability $\Pr(w>u)$.

$$\Pr(w>u)=\Pr(z>0)=1-\Pr(z\le 0)=1-F_z(0)$$

$$=1-\left(\frac{4\theta_u^2}{\theta_w+\theta_u}\frac{1}{(\theta_w+2\theta_u)} - \frac{\theta_u^2}{\theta_w+\theta_u}\frac{1}{(2\theta_w+\theta_u)}\right)=1-\frac{2\theta_u^2(8\theta_w+4\theta_u-\theta_w-2\theta_u)}{(\theta_w+\theta_u)(\theta_w+2\theta_u)(2\theta_w+\theta_u)}$$

$$=1-\frac{8\theta_u^2\theta_w+4\theta_u^3-\theta_u^2\theta_w-2\theta_u^3}{(\theta_w+\theta_u)(\theta_w+2\theta_u)(2\theta_w+\theta_u)}=1-\frac{7\theta_u^2\theta_w+2\theta_u^3}{(\theta_w+\theta_u)(\theta_w+2\theta_u)(2\theta_w+\theta_u)}$$

$$\Rightarrow \Pr(w>u)=1-\frac{\theta_u^2(7\theta_w+2\theta_u)}{(\theta_w+\theta_u)(\theta_w+2\theta_u)(2\theta_w+\theta_u)}$$

B.3. The mode.

$f_z(z)$ is a unimodal density with two branches, but continuous everywhere. First we calculate the argmax of each branch separately. The density was obtained previously,

$$f_z(z) = \begin{cases} \dfrac{2\theta_u}{\theta_w+\theta_u}\left[\dfrac{2\exp\{z/\theta_u\}}{\theta_w+2\theta_u} - \dfrac{\exp\{2z/\theta_u\}}{2\theta_w+\theta_u}\right] & z \le 0 \\[2ex] \dfrac{2\theta_w}{\theta_w+\theta_u}\left[\dfrac{2\exp\{-z/\theta_w\}}{2\theta_w+\theta_u} - \dfrac{\exp\{-2z/\theta_w\}}{\theta_w+2\theta_u}\right] & z > 0 \end{cases}$$

and we have, ignoring multiplicative constants,

$$\frac{d}{dz} f_{z\le 0}(z) \propto \frac{2}{\theta_u}\frac{\exp\{z/\theta_u\}}{\theta_w+2\theta_u} - \frac{2}{\theta_u}\frac{\exp\{2z/\theta_u\}}{2\theta_w+\theta_u} = 0 \Rightarrow \frac{\exp\{z/\theta_u\}}{\theta_w+2\theta_u} = \frac{\exp\{2z/\theta_u\}}{2\theta_w+\theta_u}$$

$$\Rightarrow \frac{2\theta_w+\theta_u}{\theta_w+2\theta_u} = \exp\{z/\theta_u\} \Rightarrow \ \mathrm{mode}(z)_{z\le 0} = \theta_u \ln\left(\frac{2\theta_w+\theta_u}{\theta_w+2\theta_u}\right)$$

For the other branch,

$$\frac{d}{dz} f_{z\le 0}(z) \propto -\frac{2}{\theta_w}\frac{\exp\{-z/\theta_w\}}{2\theta_w+\theta_u} + \frac{2}{\theta_w}\frac{\exp\{-2z/\theta_w\}}{\theta_w+2\theta_u} = 0 \Rightarrow \frac{\exp\{z/\theta_w\}}{2\theta_w+\theta_u} = \frac{1}{\theta_w+2\theta_u}$$

$$\Rightarrow \ \mathrm{mode}(z)_{z>0} = \theta_w \ln\left(\frac{2\theta_w+\theta_u}{\theta_w+2\theta_u}\right).$$

Comparing the two expressions we see that if $\theta_w > \theta_u$ then $\mathrm{mode}(z)_{z\le 0}$ falls outside the prescribed negative domain (i.e. it is a positive number), so it is excluded, while $\mathrm{mode}(z)_{z>0}$ is also a positive number and is permitted. The reverse holds when $\theta_w < \theta_u$. Moreover, since the function is continuous, each of the two maximization procedures includes also the case $z=0$ where we change branch, so we do not need to compare the value of the density at $z=0$ with the one remaining mode. It follows that

$$\boxed{[3.127]:\ \ \mathrm{mode}(z)=\ln\left(\frac{2\theta_w+\theta_u}{\theta_w+2\theta_u}\right)\cdot\max\{\theta_w,\theta_u\}\ .}$$

B.4. The median.

a) $\mathrm{med}(z)<0$ **. Here**

$$\mathrm{med}(z):\frac{4\theta_u^2}{\theta_w+\theta_u}\frac{\exp\{\mathrm{med}(z)/\theta_u\}}{(\theta_w+2\theta_u)} - \frac{\theta_u^2}{\theta_w+\theta_u}\frac{\exp\{2\,\mathrm{med}(z)/\theta_u\}}{(2\theta_w+\theta_u)}=\frac{1}{2}$$

b) $\mathrm{med}(z)>0$ **. Here**

$$\mathrm{med}(z):1 - \frac{4\theta_w^2}{\theta_w+\theta_u}\frac{\exp\{-\mathrm{med}(z)/\theta_w\}}{2\theta_w+\theta_u} + \frac{\theta_w^2}{\theta_w+\theta_u}\frac{\exp\{-2\,\mathrm{med}(z)/\theta_w\}}{(\theta_w+2\theta_u)}=\frac{1}{2}$$

$$\Rightarrow \mathrm{med}(z):\ \frac{4\theta_w^2}{\theta_w+\theta_u}\frac{\exp\{-\mathrm{med}(z)/\theta_w\}}{2\theta_w+\theta_u} - \frac{\theta_w^2}{\theta_w+\theta_u}\frac{\exp\{-2\,\mathrm{med}(z)/\theta_w\}}{(\theta_w+2\theta_u)}=\frac{1}{2}$$

B.4.1. Exponentiated mode, $\mathrm{mode}\left[\exp\{w-u\}\right]=\mathrm{mode}\left(\exp\{z\}\right)$.

We have $q\equiv\exp\{z\}\Rightarrow \ln q=z\Rightarrow \dfrac{\partial z}{\partial q}=\dfrac{1}{q}$. Also, $z\le 0\Rightarrow 0<q\le 1,\ z>0\Rightarrow q>1$

Therefore

$$f_q(q)=\frac{1}{q}f_z(\ln q)=\begin{cases}\dfrac{2\theta_u}{\theta_w+\theta_u}\dfrac{1}{q}\left[\dfrac{2q^{1/\theta_u}}{\theta_w+2\theta_u} - \dfrac{q^{2/\theta_u}}{2\theta_w+\theta_u}\right] & 0<q\le 1\\[2ex] \dfrac{2\theta_w}{\theta_w+\theta_u}\dfrac{1}{q}\left[\dfrac{2q^{-1/\theta_w}}{2\theta_w+\theta_u} - \dfrac{q^{-2/\theta_w}}{\theta_w+2\theta_u}\right] & q>1\end{cases}$$

$$
\Rightarrow f_q(q) = \begin{cases} \dfrac{2\theta_u}{\theta_w+\theta_u}\left[\dfrac{2q^{1/\theta_u-1}}{\theta_w+2\theta_u} - \dfrac{q^{2/\theta_u-1}}{2\theta_w+\theta_u}\right] & 0<q\le 1 \\[2ex] \dfrac{2\theta_w}{\theta_w+\theta_u}\left[\dfrac{2q^{-1/\theta_w-1}}{2\theta_w+\theta_u} - \dfrac{q^{-2/\theta_w-1}}{\theta_w+2\theta_u}\right] & q>1 \end{cases}
$$

Note that since $\ln q$ and $1/q$ are continuous functions in $(0,\infty)$ it follows that $f_q(q)$, being the product of $1/q$ with the composition of $f_z(z)$ with $\ln q$ will also be continuous.

Ignoring multiplicative constants we find the argmax for each branch.

$$
0 \le q \le 1,\ \frac{d}{dq} f_q(q) \propto \left(\frac{1}{\theta_u}-1\right)\frac{2q^{1/\theta_u-2}}{\theta_w+2\theta_u} - \left(\frac{2}{\theta_u}-1\right)\frac{q^{2/\theta_u-2}}{2\theta_w+\theta_u} = 0
$$

$$
\Rightarrow \left(\frac{1-\theta_u}{\theta_u}\right)\frac{2q^{1/\theta_u}}{\theta_w+2\theta_u} = \left(\frac{2-\theta_u}{\theta_u}\right)\frac{q^{2/\theta_u}}{2\theta_w+\theta_u} \Rightarrow \frac{1-\theta_u}{2-\theta_u}\frac{4\theta_w+2\theta_u}{\theta_w+2\theta_u} = q^{1/\theta_u}
$$

$$
\Rightarrow \mathrm{mode}\left(\exp\{q\}\right)_{q\le 1} = \left(\frac{1-\theta_u}{2-\theta_u}\frac{4\theta_w+2\theta_u}{\theta_w+2\theta_u}\right)^{\theta_u}
$$

For the second branch,

$$
q>1,\ \frac{d}{dq} f_q(q) \propto -\left(\frac{1}{\theta_w}+1\right)\frac{2q^{-1/\theta_w-2}}{2\theta_w+\theta_u} + \left(\frac{2}{\theta_w}+1\right)\frac{q^{-2/\theta_w-2}}{\theta_w+2\theta_u} = 0
$$

$$
\Rightarrow \left(\frac{1}{\theta_w}+1\right)\frac{2q^{-1/\theta_w-2}}{2\theta_w+\theta_u} = \left(\frac{2}{\theta_w}+1\right)\frac{q^{-2/\theta_w-2}}{\theta_w+2\theta_u} \Rightarrow (1+\theta_w)\frac{2q^{-1/\theta_w}}{2\theta_w+\theta_u} = (2+\theta_w)\frac{q^{-2/\theta_u}}{\theta_w+2\theta_u}
$$

$$
\Rightarrow q^{1/\theta_w} = \frac{2+\theta_w}{1+\theta_w}\frac{2\theta_w+\theta_u}{2\theta_w+4\theta_u} \Rightarrow \mathrm{mode}\left(\exp\{q\}\right)_{q>1} = \left(\frac{2+\theta_w}{1+\theta_w}\frac{2\theta_w+\theta_u}{2\theta_w+4\theta_u}\right)^{\theta_w}
$$

Here the situation is not as clear-cut is with the mode of the difference. We have to calculate the two modes and check if one of them is not permissible, or compare their values

We can write a compact expression as

$$[3.133]: \quad \text{mode}\left(\exp\{w-u\}\right) = \max\left\{q_0 \cdot I\{q_0 \le 1\},\ q_1 \cdot I\{q_1 > 1\}\right\},$$

$$q_0 = \left(\frac{1-\theta_u}{2-\theta_u}\frac{4\theta_w + 2\theta_u}{\theta_w + 2\theta_u}\right)^{\theta_u}, \quad q_1 = \left(\frac{2+\theta_w}{1+\theta_w}\frac{2\theta_w + \theta_u}{2\theta_w + 4\theta_u}\right)^{\theta_w}$$

C. The distribution of the composite error term.

C.1 The density of the composite error term.

We want the density of $\varepsilon = v + w - u = v + z$. Under independence we have a long but straightforward convolution, taking into account the branches of $f_z(z)$,

$$f_\varepsilon(\varepsilon) = \int_{-\infty}^{0} \frac{1}{\sigma_v}\phi\left(\frac{\varepsilon - z}{\sigma_v}\right)\frac{2\theta_u}{\theta_w + \theta_u}\left[\frac{2\exp\{z/\theta_u\}}{\theta_w + 2\theta_u} - \frac{\exp\{2z/\theta_u\}}{2\theta_w + \theta_u}\right]dz$$

$$+ \int_{0}^{\infty} \frac{1}{\sigma_v}\phi\left(\frac{\varepsilon - z}{\sigma_v}\right)\frac{2\theta_w}{\theta_w + \theta_u}\left[\frac{2\exp\{-z/\theta_w\}}{2\theta_w + \theta_u} - \frac{\exp\{-2z/\theta_w\}}{\theta_w + 2\theta_u}\right]dz$$

Taking constants out and switching limits of integration,

$$
\begin{aligned}
&= \frac{4\theta_u}{\sigma_v\left(\theta_w+\theta_u\right)\left(\theta_w+2\theta_u\right)}\int_0^\infty \phi\left(\frac{\varepsilon+z}{\sigma_v}\right)\exp\left\{-z/\theta_u\right\}dz \\
&\quad -\frac{2\theta_u}{\sigma_v\left(\theta_w+\theta_u\right)\left(2\theta_w+\theta_u\right)}\int_0^\infty \phi\left(\frac{\varepsilon+z}{\sigma_v}\right)\exp\left\{-2z/\theta_u\right\}dz \\
&\quad +\frac{4\theta_w}{\sigma_v\left(\theta_w+\theta_u\right)\left(2\theta_w+\theta_u\right)}\int_0^\infty \phi\left(\frac{\varepsilon-z}{\sigma_v}\right)\exp\left\{-z/\theta_w\right\}dz \\
&\quad -\frac{2\theta_w}{\sigma_v\left(\theta_w+\theta_u\right)\left(\theta_w+2\theta_u\right)}\int_0^\infty \phi\left(\frac{\varepsilon-z}{\sigma_v}\right)\exp\left\{-2z/\theta_w\right\}dz
\end{aligned}
$$

Apply the transformations

$$
y=\frac{\varepsilon+z}{\sigma_v}\Rightarrow\begin{cases} z=\sigma_v y-\varepsilon \\ dz=\sigma_v dy \\ z=0\Rightarrow y=\varepsilon/\sigma_v \\ z=\infty\Rightarrow y=\infty \end{cases}, \qquad y=\frac{\varepsilon-z}{\sigma_v}\Rightarrow\begin{cases} z=\varepsilon-\sigma_v y \\ dz=-\sigma_v dy \\ z=0\Rightarrow y=\varepsilon/\sigma_v \\ z=\infty\Rightarrow y=-\infty \end{cases}
$$

1st integral is transformed into (ignoring the constant terms)

$$
\int_0^\infty \phi\left(\frac{\varepsilon+z}{\sigma_v}\right)\exp\left\{-z/\theta_u\right\}dz \quad = \quad \sigma_v\int_{\varepsilon/\sigma_v}^\infty \phi\left(y\right)\exp\left\{-\left(\sigma_v y-\varepsilon\right)/\theta_u\right\}dy
$$

$$
= \quad \sigma_v\exp\left\{\varepsilon/\theta_u\right\}\int_{\varepsilon/\sigma_v}^\infty \phi\left(y\right)\exp\left\{\left(-\sigma_v/\theta_u\right)y\right\}dy
$$

Applying Owen (1980) formula 100,010 p. 409

We have

$$\sigma_v \exp\{\varepsilon/\theta_u\} \int_{\varepsilon/\sigma_v}^{\infty} \phi(y) \exp\{(-\sigma_v/\theta_u)y\} dy =$$

$$= \sigma_v \exp\{\varepsilon/\theta_u\} \exp\left\{(-\sigma_v/\theta_u)^2/2\right\}\left[\Phi\left(\infty + (\sigma_v/\theta_u)\right) - \Phi\left(\varepsilon/\sigma_v + (\sigma_v/\theta_u)\right)\right]$$

$$= \sigma_v \exp\left\{\frac{\varepsilon}{\theta_u} + \frac{\sigma_v^2}{2\theta_u^2}\right\}\Phi\left[-\left(\frac{\varepsilon}{\sigma_v} + \frac{\sigma_v}{\theta_u}\right)\right]$$

So **1st component of** $f_\varepsilon(\varepsilon)$: $\dfrac{4\theta_u}{(\theta_w + \theta_u)(\theta_w + 2\theta_u)} \exp\left\{\dfrac{\varepsilon}{\theta_u} + \dfrac{\sigma_v^2}{2\theta_u^2}\right\}\Phi\left[-\left(\dfrac{\varepsilon}{\sigma_v} + \dfrac{\sigma_v}{\theta_u}\right)\right]$

2nd integral is transformed into (ignoring the constant terms)

$$\int_0^{\infty} \phi\left(\frac{\varepsilon + z}{\sigma_v}\right) \exp\{-2z/\theta_u\} dz \quad = \quad \sigma_v \int_{\varepsilon/\sigma_v}^{\infty} \phi(y) \exp\{-2(\sigma_v y - \varepsilon)/\theta_u\} dy$$

$$= \quad \sigma_v \exp\{2\varepsilon/\theta_u\} \int_{\varepsilon/\sigma_v}^{\infty} \phi(y) \exp\{(-2\sigma_v/\theta_u)y\} dy$$

Applying Owen (1980) formula 100,010 p. 409 we have

$$\sigma_v \exp\{2\varepsilon/\theta_u\} \int_{\varepsilon/\sigma_v}^{\infty} \phi(y) \exp\{(-2\sigma_v/\theta_u)y\} dy =$$

$$= \sigma_v \exp\{2\varepsilon/\theta_u\} \exp\left\{(-2\sigma_v/\theta_u)^2/2\right\}\left[\Phi\left(\infty + (2\sigma_v/\theta_u)\right) - \Phi\left(\varepsilon/\sigma_v + (2\sigma_v/\theta_u)\right)\right]$$

$$= \sigma_v \exp\left\{\frac{2\varepsilon}{\theta_u} + \frac{2\sigma_v^2}{\theta_u^2}\right\}\Phi\left[-\left(\frac{\varepsilon}{\sigma_v} + \frac{2\sigma_v}{\theta_u}\right)\right]$$

So **2nd component of** $f_\varepsilon(\varepsilon)$: $-\dfrac{2\theta_u}{(\theta_w + \theta_u)(2\theta_w + \theta_u)} \exp\left\{\dfrac{2\varepsilon}{\theta_u} + \dfrac{2\sigma_v^2}{\theta_u^2}\right\}\Phi\left[-\left(\dfrac{\varepsilon}{\sigma_v} + \dfrac{2\sigma_v}{\theta_u}\right)\right]$

3d integral is transformed into (ignoring the constant terms)

$$\int_{0}^{\infty}\phi\left(\frac{\varepsilon-z}{\sigma_v}\right)\exp\{-z/\theta_w\}\,dz=-\sigma_v\int_{\varepsilon/\sigma_v}^{-\infty}\phi(y)\exp\{-(\varepsilon-\sigma_v y)/\theta_w\}\,dy$$

$$=\sigma_v\exp\{-\varepsilon/\theta_w\}\int_{-\infty}^{\varepsilon/\sigma_v}\phi(y)\exp\{\sigma_v y/\theta_w\}\,dy$$

$$=\sigma_v\exp\{-\varepsilon/\theta_w\}\exp\left\{\frac{\sigma_v^2}{2\theta_w^2}\right\}\left[\Phi\left(\frac{\varepsilon}{\sigma_v}-\frac{\sigma_v}{\theta_w}\right)-\Phi\left(-\infty-\frac{\sigma_v}{\theta_w}\right)\right]$$

$$=\sigma_v\exp\left\{\frac{\sigma_v^2}{2\theta_w^2}-\frac{\varepsilon}{\theta_w}\right\}\Phi\left(\frac{\varepsilon}{\sigma_v}-\frac{\sigma_v}{\theta_w}\right)$$

So **3d component of** $f_\varepsilon(\varepsilon)$**:** $\dfrac{4\theta_w}{(\theta_w+\theta_u)(2\theta_w+\theta_u)}\exp\left\{\dfrac{\sigma_v^2}{2\theta_w^2}-\dfrac{\varepsilon}{\theta_w}\right\}\Phi\left(\dfrac{\varepsilon}{\sigma_v}-\dfrac{\sigma_v}{\theta_w}\right)$

4th integral is transformed into (ignoring the constant terms)

$$\int_{0}^{\infty}\phi\left(\frac{\varepsilon-z}{\sigma_v}\right)\exp\{-2z/\theta_w\}\,dz=-\sigma_v\int_{\varepsilon/\sigma_v}^{-\infty}\phi(y)\exp\{-2(\varepsilon-\sigma_v y)/\theta_w\}\,dy$$

$$=\sigma_v\exp\{-2\varepsilon/\theta_w\}\int_{-\infty}^{\varepsilon/\sigma_v}\phi(y)\exp\{2\sigma_v y/\theta_w\}\,dy$$

$$=\sigma_v\exp\{-2\varepsilon/\theta_w\}\exp\left\{\frac{2\sigma_v^2}{\theta_w^2}\right\}\left[\Phi\left(\frac{\varepsilon}{\sigma_v}-\frac{2\sigma_v}{\theta_w}\right)-\Phi\left(-\infty-\frac{2\sigma_v}{\theta_w}\right)\right]$$

$$=\sigma_v\exp\left\{\frac{2\sigma_v^2}{\theta_w^2}-\frac{2\varepsilon}{\theta_w}\right\}\Phi\left(\frac{\varepsilon}{\sigma_v}-\frac{2\sigma_v}{\theta_w}\right)$$

So **4th component of** $f_\varepsilon(\varepsilon)$**:** $-\dfrac{2\theta_w}{(\theta_w+\theta_u)(\theta_w+2\theta_u)}\exp\left\{\dfrac{2\sigma_v^2}{\theta_w^2}-\dfrac{2\varepsilon}{\theta_w}\right\}\Phi\left(\dfrac{\varepsilon}{\sigma_v}-\dfrac{2\sigma_v}{\theta_w}\right)$

Bringing results together,

$$f_\varepsilon(\varepsilon)=\frac{4\theta_u}{(\theta_w+\theta_u)(\theta_w+2\theta_u)}\exp\left\{\frac{\varepsilon}{\theta_u}+\frac{\sigma_v^2}{2\theta_u^2}\right\}\Phi\left[-\left(\frac{\varepsilon}{\sigma_v}+\frac{\sigma_v}{\theta_u}\right)\right]$$

$$-\frac{2\theta_u}{(\theta_w+\theta_u)(2\theta_w+\theta_u)}\exp\left\{\frac{2\varepsilon}{\theta_u}+\frac{2\sigma_v^2}{\theta_u^2}\right\}\Phi\left[-\left(\frac{\varepsilon}{\sigma_v}+\frac{2\sigma_v}{\theta_u}\right)\right]$$

$$+\frac{4\theta_w}{(\theta_w+\theta_u)(2\theta_w+\theta_u)}\exp\left\{\frac{\sigma_v^2}{2\theta_w^2}-\frac{\varepsilon}{\theta_w}\right\}\Phi\left(\frac{\varepsilon}{\sigma_v}-\frac{\sigma_v}{\theta_w}\right)$$

$$-\frac{2\theta_w}{(\theta_w+\theta_u)(\theta_w+2\theta_u)}\exp\left\{\frac{2\sigma_v^2}{\theta_w^2}-\frac{2\varepsilon}{\theta_w}\right\}\Phi\left(\frac{\varepsilon}{\sigma_v}-\frac{2\sigma_v}{\theta_w}\right)$$

For compactness, set

$$a_u=\frac{\varepsilon}{\theta_u}+\frac{\sigma_v^2}{2\theta_u^2}\quad\Rightarrow\frac{2\varepsilon}{\theta_u}+\frac{2\sigma_v^2}{\theta_u^2}=2a_u+\frac{\sigma_v^2}{\theta_u^2},\quad b_u=-\left(\frac{\varepsilon}{\sigma_v}+\frac{\sigma_v}{\theta_u}\right)\Rightarrow-\left(\frac{\varepsilon}{\sigma_v}+\frac{2\sigma_v}{\theta_u}\right)=b_u-\frac{\sigma_v}{\theta_u}$$

$$a_w=\frac{\sigma_v^2}{2\theta_w^2}-\frac{\varepsilon}{\theta_w}\quad\Rightarrow\frac{2\sigma_v^2}{\theta_w^2}-\frac{2\varepsilon}{\theta_w}=2a_w+\frac{\sigma_v^2}{\theta_w^2},\quad b_w=\frac{\varepsilon}{\sigma_v}-\frac{\sigma_v}{\theta_w}\Rightarrow\frac{\varepsilon}{\sigma_v}-\frac{2\sigma_v}{\theta_w}=b_w-\frac{\sigma_v}{\theta_w}$$

Then,

$$f_\varepsilon(\varepsilon)=\frac{4\theta_u \exp\{a_u\}\Phi(b_u)}{(\theta_w+\theta_u)(\theta_w+2\theta_u)} - \frac{2\theta_u \exp\left\{2a_u+(\sigma_v/\theta_u)^2\right\}\Phi\left(b_u-\dfrac{\sigma_v}{\theta_u}\right)}{(\theta_w+\theta_u)(2\theta_w+\theta_u)}$$

$$+\frac{4\theta_w \exp\{a_w\}\Phi(b_w)}{(\theta_w+\theta_u)(2\theta_w+\theta_u)} - \frac{2\theta_w \exp\left\{2a_w+(\sigma_v/\theta_w)^2\right\}\Phi\left(b_w-\dfrac{\sigma_v}{\theta_w}\right)}{(\theta_w+\theta_u)(\theta_w+2\theta_u)}$$

and finally

$$[3.123]:\quad f_\varepsilon(\varepsilon)=\frac{2}{\theta_w+\theta_u}\left[\frac{2\theta_u \exp\{a_u\}\Phi(b_u)}{\theta_w+2\theta_u} - \frac{\theta_u \exp\left\{2a_u+(\sigma_v/\theta_u)^2\right\}\Phi(b_u-\sigma_v/\theta_u)}{2\theta_w+\theta_u}\right.$$

$$\left.+\frac{2\theta_w \exp\{a_w\}\Phi(b_w)}{2\theta_w+\theta_u} - \frac{\theta_w \exp\left\{2a_w+(\sigma_v/\theta_w)^2\right\}\Phi(b_w-\sigma_v/\theta_w)}{\theta_w+2\theta_u}\right]$$

C.2 The distribution function of the composite error term.

$$F_\varepsilon(\varepsilon)=\int_{-\infty}^{\varepsilon} f_\varepsilon(s)ds=\frac{2}{\theta_w+\theta_u}\left[\frac{2\theta_u}{\theta_w+2\theta_u}\int_{-\infty}^{\varepsilon}\exp\{a_u\}\Phi(b_u)ds\right.$$

$$-\frac{\theta_u}{2\theta_w+\theta_u}\int_{-\infty}^{\varepsilon}\exp\left\{2a_u+(\sigma_v/\theta_u)^2\right\}\Phi(b_u-\sigma_v/\theta_u)ds$$

$$+\frac{2\theta_w}{2\theta_w+\theta_u}\int_{-\infty}^{\varepsilon}\exp\{a_w\}\Phi(b_w)ds$$

$$\left.-\frac{\theta_w}{\theta_w+2\theta_u}\int_{-\infty}^{\varepsilon}\exp\left\{2a_w+(\sigma_v/\theta_w)^2\right\}\Phi(b_w-\sigma_v/\theta_w)ds\right]$$

and decomposing the shortcuts

$$F_{\varepsilon}(\varepsilon)=\int_{-\infty}^{\varepsilon} f_{\varepsilon}(s)ds=\frac{2}{\theta_w+\theta_u}\left[\frac{2\theta_u}{\theta_w+2\theta_u}\int_{-\infty}^{\varepsilon}\exp\left\{\frac{s}{\theta_u}+\frac{\sigma_v^2}{2\theta_u^2}\right\}\Phi\left(-\frac{s}{\sigma_v}-\frac{\sigma_v}{\theta_u}\right)ds\right.$$

$$-\quad\frac{\theta_u}{2\theta_w+\theta_u}\int_{-\infty}^{\varepsilon}\exp\left\{\frac{2s}{\theta_u}+\frac{2\sigma_v^2}{\theta_u^2}\right\}\Phi\left(-\frac{s}{\sigma_v}-\frac{2\sigma_v}{\theta_u}\right)ds$$

$$+\quad\frac{2\theta_w}{2\theta_w+\theta_u}\int_{-\infty}^{\varepsilon}\exp\left\{\frac{\sigma_v^2}{2\theta_w^2}-\frac{s}{\theta_w}\right\}\Phi\left(\frac{s}{\sigma_v}-\frac{\sigma_v}{\theta_w}\right)ds$$

$$\left.-\frac{\theta_w}{\theta_w+2\theta_u}\int_{-\infty}^{\varepsilon}\exp\left\{\frac{2\sigma_v^2}{\theta_w^2}-\frac{2s}{\theta_w}\right\}\Phi\left(\frac{s}{\sigma_v}-\frac{2\sigma_v}{\theta_w}\right)ds\right]$$

$$F_{\varepsilon}(\varepsilon)=\int_{-\infty}^{\varepsilon} f_{\varepsilon}(s)ds=\frac{2}{\theta_w+\theta_u}\left[\frac{2\theta_u\exp\left\{\frac{\sigma_v^2}{2\theta_u^2}\right\}}{\theta_w+2\theta_u}\int_{-\infty}^{\varepsilon}\exp\left\{\frac{s}{\theta_u}\right\}\Phi\left(-\frac{s}{\sigma_v}-\frac{\sigma_v}{\theta_u}\right)ds\right.$$

$$-\quad\frac{\theta_u\exp\left\{\frac{2\sigma_v^2}{\theta_u^2}\right\}}{2\theta_w+\theta_u}\int_{-\infty}^{\varepsilon}\exp\left\{\frac{2s}{\theta_u}\right\}\Phi\left(-\frac{s}{\sigma_v}-\frac{2\sigma_v}{\theta_u}\right)ds$$

$$+\quad\frac{2\theta_w\exp\left\{\frac{\sigma_v^2}{2\theta_w^2}\right\}}{2\theta_w+\theta_u}\int_{-\infty}^{\varepsilon}\exp\left\{-\frac{s}{\theta_w}\right\}\Phi\left(\frac{s}{\sigma_v}-\frac{\sigma_v}{\theta_w}\right)ds$$

$$\left.-\frac{\theta_w\exp\left\{\frac{2\sigma_v^2}{\theta_w^2}\right\}}{\theta_w+2\theta_u}\int_{-\infty}^{\varepsilon}\exp\left\{-\frac{2s}{\theta_w}\right\}\Phi\left(\frac{s}{\sigma_v}-\frac{2\sigma_v}{\theta_w}\right)ds\right]$$

We consider each integral in turn. We will use Owen (1980), p. 409, (eq. 101,000)

$$\int\exp\{\gamma x\}\Phi(\delta x)dx=\frac{1}{\gamma}\exp\{\gamma x\}\Phi(\delta x)-\frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}\Phi\left(\delta x-\frac{\gamma}{\delta}\right)$$

To calculate the distribution function we want the definite integral

$$\int_{-\infty}^{x} \exp\{\gamma s\}\Phi(\delta s)ds = \frac{1}{\gamma}\exp\{\gamma x\}\Phi(\delta x) - \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}\Phi\left(\delta x - \frac{\gamma}{\delta}\right)$$

$$-\frac{1}{\gamma}\lim_{s\to-\infty}\exp\{\gamma s\}\Phi(\delta s) + \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}\lim_{s\to-\infty}\Phi\left(\delta s - \frac{\gamma}{\delta}\right)$$

In order to determine the limits, we have to consider in each of the four cases the signs for γ, δ .

1st and 2nd Integral.

For the first two integrals, the signs of the matched coefficients are $\gamma > 0,\ \delta < 0$.

So in this case, $\lim_{s\to-\infty}\exp\{\gamma s\}\Phi(\delta s) = 0\cdot 1 = 0$, $\lim_{s\to-\infty}\Phi\left(\delta s - \frac{\gamma}{\delta}\right) = \Phi(\infty) = 1$. So the general formula to be used, after eliminating the zero-terms, is

$$\int_{-\infty}^{x} \exp\{\gamma s\}\Phi(\delta s)ds = \frac{1}{\gamma}\exp\{\gamma x\}\Phi(\delta x) - \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}\Phi\left(\delta x - \frac{\gamma}{\delta}\right) + \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}$$

$$= \frac{1}{\gamma}\exp\{\gamma x\}\Phi(\delta x) + \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}\Phi\left(-\delta x + \frac{\gamma}{\delta}\right)$$

where we have used the reflective symmetry of $\Phi(\cdot)$.

1st integral.

$$\int_{-\infty}^{\varepsilon} \exp\left\{\frac{s}{\theta_u}\right\}\Phi\left(-\frac{s}{\sigma_v} - \frac{\sigma_v}{\theta_u}\right)ds = \int_{-\infty}^{\varepsilon} \exp\left\{\frac{1}{\theta_u}s\right\}\Phi\left(-\frac{1}{\sigma_v}\left(s + \sigma_v^2/\theta_u\right)\right)ds$$

$$= \int_{-\infty}^{\varepsilon+\sigma_v^2/\theta_u} \exp\left\{s/\theta_u - \sigma_v^2/\theta_u^2\right\}\Phi\left(-\frac{s}{\sigma_v}\right)ds = \exp\left\{-\sigma_v^2/\theta_u^2\right\}\int_{-\infty}^{\varepsilon+\sigma_v^2/\theta_u} \exp\left\{s/\theta_u\right\}\Phi\left(-\frac{s}{\sigma_v}\right)ds$$

$$= \exp\left\{-\sigma_v^2/\theta_u^2\right\}\theta_u \exp\left\{\frac{\varepsilon}{\theta_u}+\sigma_v^2/\theta_u^2\right\}\Phi\left(-\frac{1}{\sigma_v}\left(\varepsilon+\sigma_v^2/\theta_u\right)\right)$$

$$+\exp\left\{-\sigma_v^2/\theta_u^2\right\}\theta_u \exp\left\{\frac{\sigma_v^2}{2\theta_u^2}\right\}\Phi\left(\frac{1}{\sigma_v}\left(\varepsilon+\sigma_v^2/\theta_u\right)-\frac{\sigma_v}{\theta_u}\right)$$

$$= \exp\left\{-\sigma_v^2/\theta_u^2\right\}\theta_u \exp\left\{\frac{\varepsilon}{\theta_u}+\sigma_v^2/\theta_u^2\right\}\Phi\left(-\frac{1}{\sigma_v}\left(\varepsilon+\sigma_v^2/\theta_u\right)\right)$$

$$+\exp\left\{-\sigma_v^2/\theta_u^2\right\}\theta_u \exp\left\{\frac{\sigma_v^2}{2\theta_u^2}\right\}\Phi\left(\frac{1}{\sigma_v}\left(\varepsilon+\sigma_v^2/\theta_u\right)-\frac{\sigma_v}{\theta_u}\right)$$

and so **the 1st component of the distribution function is**

$$\frac{2\theta_u \exp\left\{\frac{\sigma_v^2}{2\theta_u^2}\right\}}{\theta_w+2\theta_u}\left[\begin{array}{r}\exp\left\{-\sigma_v^2/\theta_u^2\right\}\theta_u \exp\left\{\frac{\varepsilon}{\theta_u}+\sigma_v^2/\theta_u^2\right\}\Phi\left(-\frac{1}{\sigma_v}\left(\varepsilon+\sigma_v^2/\theta_u\right)\right)\\ +\exp\left\{-\sigma_v^2/\theta_u^2\right\}\theta_u \exp\left\{\frac{\sigma_v^2}{2\theta_u^2}\right\}\Phi\left(\frac{1}{\sigma_v}\left(\varepsilon+\sigma_v^2/\theta_u\right)-\frac{\sigma_v}{\theta_u}\right)\end{array}\right]$$

$$= \frac{2\theta_u^2}{\theta_w+2\theta_u}\left[\exp\left\{\frac{\varepsilon}{\theta_u}+\frac{\sigma_v^2}{2\theta_u^2}\right\}\Phi\left(-\frac{\varepsilon}{\sigma_v}-\frac{\sigma_v}{\theta_u}\right)+\Phi\left(\frac{\varepsilon}{\sigma_v}\right)\right]$$

$$= \frac{2\theta_u^2}{\theta_w+2\theta_u}\left[\Phi\left(\frac{\varepsilon}{\sigma_v}\right)+\exp\left\{a_u\right\}\Phi\left(b_u\right)\right]$$

2nd integral.

$$\int_{-\infty}^{\varepsilon}\exp\left\{\frac{2s}{\theta_u}\right\}\Phi\left(-\frac{s}{\sigma_v}-\frac{2\sigma_v}{\theta_u}\right)ds = \int_{-\infty}^{\varepsilon}\exp\left\{\frac{2}{\theta_u}(s)\right\}\Phi\left(-\frac{1}{\sigma_v}\left(s+\frac{2\sigma_v^2}{\theta_u}\right)\right)ds$$

$$= \int_{-\infty}^{\varepsilon+2\sigma_v^2/\theta_u} \exp\left\{\frac{2}{\theta_u}\left(s-2\sigma_v^2/\theta_u\right)\right\}\Phi\left(-\frac{s}{\sigma_v}\right)ds$$

$$= \exp\left\{-\frac{4\sigma_v^2}{\theta_u^2}\right\}\int_{-\infty}^{\varepsilon+2\sigma_v^2/\theta_u} \exp\left\{\frac{2}{\theta_u}s\right\}\Phi\left(-\frac{s}{\sigma_v}\right)ds$$

$$= \exp\left\{-\frac{4\sigma_v^2}{\theta_u^2}\right\}\frac{\theta_u}{2}\exp\left\{\frac{2}{\theta_u}\left(\varepsilon+2\sigma_v^2/\theta_u\right)\right\}\Phi\left(-\frac{1}{\sigma_v}\left(\varepsilon+2\sigma_v^2/\theta_u\right)\right)$$

$$+\exp\left\{-\frac{4\sigma_v^2}{\theta_u^2}\right\}\frac{\theta_u}{2}\exp\left\{\frac{2\sigma_v^2}{\theta_u^2}\right\}\Phi\left(\frac{1}{\sigma_v}\left(\varepsilon+2\sigma_v^2/\theta_u\right)-\frac{2\sigma_v}{\theta_u}\right)$$

and so **the 2nd component of the distribution function is**

$$-\frac{\theta_u \exp\left\{\frac{2\sigma_v^2}{\theta_u^2}\right\}}{2\theta_w+\theta_u}\int_{-\infty}^{\varepsilon}\exp\left\{\frac{2s}{\theta_u}\right\}\Phi\left(-\frac{s}{\sigma_v}-\frac{2\sigma_v}{\theta_u}\right)ds$$

$$= -\frac{\theta_u \exp\left\{\frac{2\sigma_v^2}{\theta_u^2}\right\}}{2\theta_w+\theta_u}\left[\begin{array}{r}\exp\left\{-\frac{4\sigma_v^2}{\theta_u^2}\right\}\frac{\theta_u}{2}\exp\left\{\frac{2}{\theta_u}\left(\varepsilon+2\sigma_v^2/\theta_u\right)\right\}\Phi\left(-\frac{1}{\sigma_v}\left(\varepsilon+2\sigma_v^2/\theta_u\right)\right)\\ +\exp\left\{-\frac{4\sigma_v^2}{\theta_u^2}\right\}\frac{\theta_u}{2}\exp\left\{\frac{2\sigma_v^2}{\theta_u^2}\right\}\Phi\left(\frac{1}{\sigma_v}\left(\varepsilon+2\sigma_v^2/\theta_u\right)-\frac{2\sigma_v}{\theta_u}\right)\end{array}\right]$$

$$= -\frac{\theta_u^2}{2\left(2\theta_w+\theta_u\right)}\left[\exp\left\{\frac{2\varepsilon}{\theta_u}+\frac{2\sigma_v^2}{\theta_u^2}\right\}\Phi\left(-\frac{\varepsilon}{\sigma_v}-\frac{2\sigma_v}{\theta_u}\right)+\Phi\left(\frac{\varepsilon}{\sigma_v}\right)\right]$$

$$= -\frac{\theta_u^2}{2\left(2\theta_w+\theta_u\right)}\left[\Phi\left(\frac{\varepsilon}{\sigma_v}\right) \quad + \quad \exp\left\{2a_u+\frac{\sigma_v^2}{\theta_u^2}\right\}\Phi\left(b_u-\frac{\sigma_v}{\theta_u}\right)\right]$$

3d & 4th Integral.

For the 3d and 4th integral we have $\gamma < 0,\ \delta > 0$

So here, $\lim\limits_{s\to-\infty} \exp\{\gamma s\}\Phi(\delta s) = \infty \cdot 0 = 0, \quad \lim\limits_{s\to-\infty} \Phi\left(\delta s - \dfrac{\gamma}{\delta}\right) = 0$

So the general formula to be used, after eliminating the zero-terms, is

$$\int_{-\infty}^{x} \exp\{\gamma s\}\Phi(\delta s)\,ds = \frac{1}{\gamma}\exp\{\gamma x\}\Phi(\delta x) \;-\; \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}\Phi\left(\delta x - \frac{\gamma}{\delta}\right)$$

3d Integral.

$$\int_{-\infty}^{\varepsilon} \exp\left\{-\frac{s}{\theta_w}\right\}\Phi\left(\frac{s}{\sigma_v} - \frac{\sigma_v}{\theta_w}\right)ds = \int_{-\infty}^{\varepsilon} \exp\left\{-\frac{s}{\theta_w}\right\}\Phi\left(\frac{1}{\sigma_v}\left(s - \sigma_v^2/\theta_w\right)\right)ds$$

$$= \int_{-\infty}^{\varepsilon-\sigma_v^2/\theta_w} \exp\left\{-\frac{s+\sigma_v^2/\theta_w}{\theta_w}\right\}\Phi\left(\frac{1}{\sigma_v}s\right)ds$$

$$= \exp\left\{-\sigma_v^2/\theta_w^2\right\}\int_{-\infty}^{\varepsilon-\sigma_v^2/\theta_w} \exp\left\{-\frac{s}{\theta_w}\right\}\Phi\left(\frac{1}{\sigma_v}s\right)ds$$

$$= \exp\left\{-\sigma_v^2/\theta_w^2\right\}\left[\begin{array}{r} -\theta_w \exp\left\{-\dfrac{1}{\theta_w}\left(\varepsilon - \sigma_v^2/\theta_w\right)\right\}\Phi\left(\dfrac{1}{\sigma_v}\left(\varepsilon - \sigma_v^2/\theta_w\right)\right) \\ +\;\theta_w \exp\left\{\dfrac{\sigma_v^2}{2\theta_w^2}\right\}\Phi\left(\dfrac{1}{\sigma_v}\left(\varepsilon - \sigma_v^2/\theta_w\right) + \dfrac{\sigma_v}{\theta_w}\right) \end{array}\right]$$

So the 3d component of the distribution function is

$$\frac{2\theta_w \exp\left\{\dfrac{\sigma_v^2}{2\theta_w^2}\right\}}{2\theta_w+\theta_u}\int_{-\infty}^{\varepsilon}\exp\left\{-\frac{s}{\theta_w}\right\}\Phi\left(\frac{s}{\sigma_v}-\frac{\sigma_v}{\theta_w}\right)ds$$

$$=\frac{2\theta_w \exp\left\{\dfrac{\sigma_v^2}{2\theta_w^2}\right\}}{2\theta_w+\theta_u}\exp\left\{-\sigma_v^2/\theta_w^2\right\}\left[\begin{array}{r}-\theta_w\exp\left\{-\dfrac{1}{\theta_w}\left(\varepsilon-\sigma_v^2/\theta_w\right)\right\}\Phi\left(\dfrac{1}{\sigma_v}\left(\varepsilon-\sigma_v^2/\theta_w\right)\right)\\ +\,\theta_w\exp\left\{\dfrac{\sigma_v^2}{2\theta_w^2}\right\}\Phi\left(\dfrac{1}{\sigma_v}\left(\varepsilon-\sigma_v^2/\theta_w\right)+\dfrac{\sigma_v}{\theta_w}\right)\end{array}\right]$$

$$=\frac{2\theta_w^2}{2\theta_w+\theta_u}\left[-\exp\left\{-\frac{\varepsilon}{\theta_w}+\frac{\sigma_v^2}{2\theta_w^2}\right\}\Phi\left(\frac{\varepsilon}{\sigma_v}-\frac{\sigma_v}{\theta_w}\right)+\Phi\left(\frac{\varepsilon}{\sigma_v}\right)\right]$$

$$=\frac{2\theta_w^2}{2\theta_w+\theta_u}\left[\Phi\left(\frac{\varepsilon}{\sigma_v}\right)-\exp\left\{a_w\right\}\Phi\left(b_w\right)\right]$$

4th integral.

$$\int_{-\infty}^{\varepsilon}\exp\left\{-\frac{2s}{\theta_w}\right\}\Phi\left(\frac{s}{\sigma_v}-\frac{2\sigma_v}{\theta_w}\right)ds=\int_{-\infty}^{\varepsilon}\exp\left\{-\frac{2s}{\theta_w}\right\}\Phi\left(\frac{1}{\sigma_v}\left(s-\frac{2\sigma_v^2}{\theta_w}\right)\right)ds$$

$$=\int_{-\infty}^{\varepsilon-2\sigma_v^2/\theta_w}\exp\left\{-\frac{2}{\theta_w}\left(s+2\sigma_v^2/\theta_w\right)\right\}\Phi\left(\frac{s}{\sigma_v}\right)ds=\exp\left\{-2\sigma_v^2/\theta_w^2\right\}\int_{-\infty}^{\varepsilon-2\sigma_v^2/\theta_w}\exp\left\{-\frac{2s}{\theta_w}\right\}\Phi\left(\frac{s}{\sigma_v}\right)ds$$

$$=\exp\left\{-4\sigma_v^2/\theta_w^2\right\}\left[\begin{array}{r}-\dfrac{\theta_w}{2}\exp\left\{-\dfrac{2}{\theta_w}\left(\varepsilon-2\sigma_v^2/\theta_w\right)\right\}\Phi\left(\dfrac{1}{\sigma_v}\left(\varepsilon-2\sigma_v^2/\theta_w\right)\right)\\ +\dfrac{\theta_w}{2}\exp\left\{\dfrac{2\sigma_v^2}{\theta_w^2}\right\}\Phi\left(\dfrac{1}{\sigma_v}\left(\varepsilon-2\sigma_v^2/\theta_w\right)+\dfrac{2\sigma_v}{\theta_w}\right)\end{array}\right]$$

So the 4th component of the distribution function is

$$-\frac{\theta_w \exp\left\{\dfrac{2\sigma_v^2}{\theta_w^2}\right\}}{\theta_w + 2\theta_u} \int_{-\infty}^{\varepsilon} \exp\left\{-\frac{2s}{\theta_w}\right\} \Phi\left(\frac{s}{\sigma_v} - \frac{2\sigma_v}{\theta_w}\right) ds$$

$$= -\frac{\theta_w^2 \exp\left\{\dfrac{2\sigma_v^2}{\theta_w^2}\right\}}{2(\theta_w + 2\theta_u)} \exp\left\{-4\sigma_v^2/\theta_w^2\right\} \left[\begin{array}{r} -\dfrac{\theta_w}{2}\exp\left\{-\dfrac{2}{\theta_w}\left(\varepsilon - 2\sigma_v^2/\theta_w\right)\right\}\Phi\left(\dfrac{1}{\sigma_v}\left(\varepsilon - 2\sigma_v^2/\theta_w\right)\right) \\ +\dfrac{\theta_w}{2}\exp\left\{\dfrac{2\sigma_v^2}{\theta_w^2}\right\}\Phi\left(\dfrac{1}{\sigma_v}\left(\varepsilon - 2\sigma_v^2/\theta_w\right) + \dfrac{2\sigma_v}{\theta_w}\right) \end{array} \right]$$

$$= -\frac{\theta_w^2}{2(\theta_w + 2\theta_u)} \left[-\exp\left\{-\frac{2\varepsilon}{\theta_w} + \frac{2\sigma_v^2}{\theta_w^2}\right\} \Phi\left(\frac{\varepsilon}{\sigma_v} - \frac{2\sigma_v}{\theta_w}\right) + \Phi\left(\frac{\varepsilon}{\sigma_v}\right) \right]$$

$$= -\frac{\theta_w^2}{2(\theta_w + 2\theta_u)} \left[\Phi\left(\frac{\varepsilon}{\sigma_v}\right) - \exp\left\{2a_w + \frac{\sigma_v^2}{\theta_w^2}\right\} \Phi\left(b_w - \frac{\sigma_v}{\theta_w}\right) \right]$$

Collecting results,

$$\begin{aligned} F_\varepsilon(\varepsilon) = \int_{-\infty}^{\varepsilon} f_\varepsilon(s) ds = \frac{2}{\theta_w + \theta_u} \Bigg[& \frac{2\theta_u^2}{\theta_w + 2\theta_u} \left[\Phi\left(\frac{\varepsilon}{\sigma_v}\right) + \exp\{a_u\} \Phi(b_u) \right] \\ & - \frac{\theta_u^2}{2(2\theta_w + \theta_u)} \left[\Phi\left(\frac{\varepsilon}{\sigma_v}\right) + \exp\left\{2a_u + \frac{\sigma_v^2}{\theta_u^2}\right\} \Phi\left(b_u - \frac{\sigma_v}{\theta_u}\right) \right] \\ & + \frac{2\theta_w^2}{2\theta_w + \theta_u} \left[\Phi\left(\frac{\varepsilon}{\sigma_v}\right) - \exp\{a_w\} \Phi(b_w) \right] \\ & - \frac{\theta_w^2}{2(\theta_w + 2\theta_u)} \left[\Phi\left(\frac{\varepsilon}{\sigma_v}\right) - \exp\left\{2a_w + \frac{\sigma_v^2}{\theta_w^2}\right\} \Phi\left(b_w - \frac{\sigma_v}{\theta_w}\right) \right] \Bigg] \end{aligned}$$

Compacting a bit,

$$F_{\varepsilon}\left(\varepsilon\right)=\frac{2}{\theta_w+\theta_u}\left[\Phi\left(\frac{\varepsilon}{\sigma_v}\right)\left(\frac{2\theta_u^2}{\theta_w+2\theta_u}-\frac{\theta_u^2}{2\left(2\theta_w+\theta_u\right)}+\frac{2\theta_w^2}{2\theta_w+\theta_u}-\frac{\theta_w^2}{2\left(\theta_w+2\theta_u\right)}\right)\right.$$

$$+\frac{2\theta_u^2}{\theta_w+2\theta_u}\exp\left\{a_u\right\}\Phi\left(b_u\right)$$
$$-\frac{\theta_u^2}{2\left(2\theta_w+\theta_u\right)}\exp\left\{2a_u+\frac{\sigma_v^2}{\theta_u^2}\right\}\Phi\left(b_u-\frac{\sigma_v}{\theta_u}\right)$$

$$-\frac{2\theta_w^2}{2\theta_w+\theta_u}\exp\left\{a_w\right\}\Phi\left(b_w\right)$$

$$\left.+\frac{\theta_w^2}{2\left(\theta_w+2\theta_u\right)}\exp\left\{2a_w+\frac{\sigma_v^2}{\theta_w^2}\right\}\Phi\left(b_w-\frac{\sigma_v}{\theta_w}\right)\right]$$

We have,

$$\frac{2\theta_u^2}{\theta_w+2\theta_u}-\frac{\theta_u^2}{2\left(2\theta_w+\theta_u\right)}+\frac{2\theta_w^2}{2\theta_w+\theta_u}-\frac{\theta_w^2}{2\left(\theta_w+2\theta_u\right)}$$

$$=\frac{2\theta_u^2 2\left(2\theta_w+\theta_u\right)-\theta_u^2\left(\theta_w+2\theta_u\right)+2\theta_w^2 2\left(\theta_w+2\theta_u\right)-\theta_w^2\left(2\theta_w+\theta_u\right)}{2\left(\theta_w+2\theta_u\right)\left(2\theta_w+\theta_u\right)}$$

$$=\frac{8\theta_u^2\theta_w+4\theta_u^3-\theta_u^2\theta_w-2\theta_u^3+4\theta_w^3+8\theta_w^2\theta_u-2\theta_w^3-\theta_w^2\theta_u}{2\left(\theta_w+2\theta_u\right)\left(2\theta_w+\theta_u\right)}$$

$$=\frac{7\theta_w\theta_u^2+2\theta_u^3+2\theta_w^3+7\theta_w^2\theta_u}{2\left(\theta_w+2\theta_u\right)\left(2\theta_w+\theta_u\right)}=\frac{2\left(\theta_w+\theta_u\right)^3+\theta_w\theta_u\left(\theta_w+\theta_u\right)}{2\left(\theta_w+2\theta_u\right)\left(2\theta_w+\theta_u\right)}$$

So,

$$[3.124]:\ F_\varepsilon(\varepsilon)=\frac{2}{\theta_w+\theta_u}\left[\frac{2(\theta_w+\theta_u)^3+\theta_w\theta_u(\theta_w+\theta_u)}{2(\theta_w+2\theta_u)(2\theta_w+\theta_u)}\Phi\left(\frac{\varepsilon}{\sigma_v}\right)\right.$$

$$+\frac{2\theta_u^2}{\theta_w+2\theta_u}\exp\{a_u\}\Phi(b_u)-\frac{\theta_u^2}{2(2\theta_w+\theta_u)}\exp\left\{2a_u+\frac{\sigma_v^2}{\theta_u^2}\right\}\Phi\left(b_u-\frac{\sigma_v}{\theta_u}\right)$$

$$\left.-\frac{2\theta_w^2}{2\theta_w+\theta_u}\exp\{a_w\}\Phi(b_w)+\frac{\theta_w^2}{2(\theta_w+2\theta_u)}\exp\left\{2a_w+\frac{\sigma_v^2}{\theta_w^2}\right\}\Phi\left(b_w-\frac{\sigma_v}{\theta_w}\right)\right]$$

$$a_u=\frac{\varepsilon}{\theta_u}+\frac{\sigma_v^2}{2\theta_u^2}\ \Rightarrow\frac{2\varepsilon}{\theta_u}+\frac{2\sigma_v^2}{\theta_u^2}=2a_u+\frac{\sigma_v^2}{\theta_u^2},\quad b_u=-\left(\frac{\varepsilon}{\sigma_v}+\frac{\sigma_v}{\theta_u}\right)\Rightarrow-\left(\frac{\varepsilon}{\sigma_v}+\frac{2\sigma_v}{\theta_u}\right)=b_u-\frac{\sigma_v}{\theta_u}$$

$$a_w=\frac{\sigma_v^2}{2\theta_w^2}-\frac{\varepsilon}{\theta_w}\ \Rightarrow\frac{2\sigma_v^2}{\theta_w^2}-\frac{2\varepsilon}{\theta_w}=2a_w+\frac{\sigma_v^2}{\theta_w^2},\quad b_w=\frac{\varepsilon}{\sigma_v}-\frac{\sigma_v}{\theta_w}\Rightarrow\frac{\varepsilon}{\sigma_v}-\frac{2\sigma_v}{\theta_w}=b_w-\frac{\sigma_v}{\theta_w}$$

D. Individual Measures.

D.1. Conditional densities related to the w variable.

D.1.1. $f_{w|\varepsilon}(w|\varepsilon)$.

We need to calculate the conditional density

$$f_{w|\varepsilon}(w|\varepsilon)=\frac{f_{\varepsilon,w}(\varepsilon,w)}{f_\varepsilon(\varepsilon)}$$

We have $\varepsilon=v+w-u$. Set $\xi=v-u=\varepsilon-w$. Due to independence, we have

$$f_{\xi,w}(\xi,w)=f_\xi(\xi)\,f_w(w)=f_\xi(\varepsilon-w)\,f_w(w)=f_{\varepsilon,w}(\varepsilon,w)$$

the later because the the Jacobian determinant of the transformation is equal to unity.

So

$$f_{w|\varepsilon}\left(w|\varepsilon\right)=\frac{f_{\xi}\left(\varepsilon-w\right)f_{w}\left(w\right)}{f_{\varepsilon}\left(\varepsilon\right)}$$

We need to determine the density $f_{\xi}\left(\xi\right)$. We have $\xi=v-u\Rightarrow v=\xi+u$

$$f_{\xi}\left(\xi\right)=\int_{0}^{\infty}\frac{1}{\sigma_{v}}f_{v}\left(\xi+u\right)f_{u}\left(u\right)du=\int_{0}^{\infty}\frac{1}{\sigma_{v}}\phi\left(\frac{\xi+u}{\sigma_{v}}\right)\left[2f_{E}\left(u;\theta_{u}\right)-f_{E}\left(u;\theta_{u}/2\right)\right]du$$

$$=2\int_{0}^{\infty}\frac{1}{\sigma_{v}}\phi\left(\frac{\xi+u}{\sigma_{v}}\right)f_{E}\left(u;\theta_{u}\right)du-\int_{0}^{\infty}\frac{1}{\sigma_{v}}\phi\left(\frac{\xi+u}{\sigma_{v}}\right)f_{E}\left(u;\theta_{u}/2\right)du$$

$$=\frac{2}{\theta_{u}\sigma_{v}}\int_{0}^{\infty}\exp\left\{-u/\theta_{u}\right\}\phi\left(\frac{\xi+u}{\sigma_{v}}\right)du-\frac{2}{\theta_{u}\sigma_{v}}\int_{0}^{\infty}\exp\left\{-2u/\theta_{u}\right\}\phi\left(\frac{\xi+u}{\sigma_{v}}\right)du$$

Apply the transformation $y=\dfrac{\xi+u}{\sigma_{v}}\Rightarrow\begin{cases}u=\sigma_{v}y-\xi\\ du=\sigma_{v}dy\\ u=0\Rightarrow y=\xi/\sigma_{v}\\ w=\infty\Rightarrow y=\infty\end{cases}$ So

$$f_{\xi}\left(\xi\right)=\frac{2}{\theta_{u}}\int_{\xi/\sigma_{v}}^{\infty}\exp\left\{-\left(\sigma_{v}y-\xi\right)/\theta_{u}\right\}\phi\left(y\right)dy-\frac{2}{\theta_{u}}\int_{\xi/\sigma_{v}}^{\infty}\exp\left\{-2\left(\sigma_{v}y-\xi\right)/\theta_{u}\right\}\phi\left(y\right)dy$$

$$=\frac{2}{\theta_{u}}\exp\left\{\xi/\theta_{u}\right\}\int_{\xi/\sigma_{v}}^{\infty}\exp\left\{-y\sigma_{v}/\theta_{u}\right\}\phi\left(y\right)dy-\frac{2}{\theta_{u}}\exp\left\{2\xi/\theta_{u}\right\}\int_{\xi/\sigma_{v}}^{\infty}\exp\left\{-2y\sigma_{v}/\theta_{u}\right\}\phi\left(y\right)dy$$

Using Owen (1980) formula 100,010, p. 409

$$f_{\xi}(\xi)=\frac{2}{\theta_u}\exp\{\xi/\theta_u\}\left[\exp\left\{\frac{\sigma_v^2}{2\theta_u^2}\right\}\cdot 1-\exp\left\{\frac{\sigma_v^2}{2\theta_u^2}\right\}\Phi\left(\frac{\xi}{\sigma_v}+\frac{\sigma_v}{\theta_u}\right)\right]$$

$$-\frac{2}{\theta_u}\exp\{2\xi/\theta_u\}\left[\exp\left\{\frac{2\sigma_v^2}{\theta_u^2}\right\}\cdot 1-\exp\left\{\frac{2\sigma_v^2}{\theta_u^2}\right\}\Phi\left(\frac{\xi}{\sigma_v}+\frac{2\sigma_v}{\theta_u}\right)\right]$$

$$=\frac{2}{\theta_u}\exp\{\xi/\theta_u\}\exp\left\{\frac{\sigma_v^2}{2\theta_u^2}\right\}\Phi\left(-\frac{\xi}{\sigma_v}-\frac{\sigma_v}{\theta_u}\right)-\frac{2}{\theta_u}\exp\{2\xi/\theta_u\}\exp\left\{\frac{2\sigma_v^2}{\theta_u^2}\right\}\Phi\left(-\frac{\xi}{\sigma_v}-\frac{2\sigma_v}{\theta_u}\right)$$

$$\Rightarrow f_{\xi}(\xi)=\frac{2}{\theta_u}\exp\left\{\frac{\xi}{\theta_u}+\frac{\sigma_v^2}{2\theta_u^2}\right\}\Phi\left(-\frac{\xi}{\sigma_v}-\frac{\sigma_v}{\theta_u}\right)-\frac{2}{\theta_u}\exp\left\{\frac{2\xi}{\theta_u}+\frac{2\sigma_v^2}{\theta_u^2}\right\}\Phi\left(-\frac{\xi}{\sigma_v}-\frac{2\sigma_v}{\theta_u}\right)$$

So

$$f_{\xi}(\varepsilon-w)=\frac{2}{\theta_u}\exp\left\{\frac{\varepsilon-w}{\theta_u}+\frac{\sigma_v^2}{2\theta_u^2}\right\}\Phi\left(-\frac{\varepsilon-w}{\sigma_v}-\frac{\sigma_v}{\theta_u}\right)-\frac{2}{\theta_u}\exp\left\{\frac{2(\varepsilon-w)}{\theta_u}+\frac{2\sigma_v^2}{\theta_u^2}\right\}\Phi\left(-\frac{\varepsilon-w}{\sigma_v}-\frac{2\sigma_v}{\theta_u}\right)$$

$$=\frac{2}{\theta_u}\exp\left\{\frac{\varepsilon}{\theta_u}+\frac{\sigma_v^2}{2\theta_u^2}\right\}\exp\left\{-\frac{w}{\theta_u}\right\}\Phi\left(\frac{w}{\sigma_v}-\frac{\varepsilon}{\sigma_v}-\frac{\sigma_v}{\theta_u}\right)$$

$$-\frac{2}{\theta_u}\exp\left\{\frac{2\varepsilon}{\theta_u}+\frac{2\sigma_v^2}{\theta_u^2}\right\}\exp\left\{-\frac{2w}{\theta_u}\right\}\Phi\left(\frac{w}{\sigma_v}-\frac{\varepsilon}{\sigma_v}-\frac{2\sigma_v}{\theta_u}\right)$$

$$=\frac{2}{\theta_u}\exp\left\{-\frac{w}{\theta_u}\right\}\exp\{a_u\}\Phi\left(\frac{w}{\sigma_v}+b_u\right)-\frac{2}{\theta_u}\exp\left\{-\frac{2w}{\theta_u}\right\}\exp\left\{2a_u+\frac{\sigma_v^2}{\theta_u^2}\right\}\Phi\left(\frac{w}{\sigma_v}+b_u-\frac{\sigma_v}{\theta_u}\right)$$

So,

$$[3.129]:\ f_{w|\varepsilon}\left(w|\varepsilon\right)=\frac{f_w\left(w\right)}{f_\varepsilon\left(\varepsilon\right)}\frac{2}{\theta_u}\left[\exp\left\{-\frac{w}{\theta_u}\right\}\exp\left\{a_u\right\}\Phi\left(\frac{w}{\sigma_v}+b_u\right)\right.$$

$$\left.-\exp\left\{-\frac{2w}{\theta_u}\right\}\exp\left\{2a_u+\frac{\sigma_v^2}{\theta_u^2}\right\}\Phi\left(\frac{w}{\sigma_v}+b_u-\frac{\sigma_v}{\theta_u}\right)\right]$$

D.1.2. Related to $f_{q|\varepsilon}\left(q|\varepsilon\right),\ q=\exp\left\{w\right\}$.

$f_{w|\varepsilon}\left(w|\varepsilon\right)\equiv h\left(w,\varepsilon\right)$, and we want to apply on the bivariate argument of the function the transformation

$$T\left(w,\varepsilon\right)=\left(q\left(w,\varepsilon\right),k\left(w,\varepsilon\right)\right)$$

$$q\left(w,\varepsilon\right)=\exp\left\{w\right\}\Rightarrow w=\ln q,\ \ q\geq 1\,,\quad k\left(w,\varepsilon\right)=\varepsilon$$

Consider the determinant of the Jacobian determinant

$$J\left(q,k\right)=\frac{\partial w}{\partial q}\frac{\partial \varepsilon}{\partial k}-\frac{\partial w}{\partial k}\frac{\partial \varepsilon}{\partial q}=\frac{\partial \ln q}{\partial q}\frac{\partial \varepsilon}{\partial \varepsilon}-\frac{\partial \ln q}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial q}=\frac{1}{q}\cdot 1-0=\frac{1}{q}$$

So (see eg. Stirzaker 2003, p. 343), we have

$q=\exp\left\{w\right\},\quad f_{q|\varepsilon}\left(q|\varepsilon\right)=\left|J\left(q,k\right)\right|f_{w|\varepsilon}\left(\ln q|\varepsilon\right)$ and so

$$f_{q|\varepsilon}\left(q|\varepsilon\right)=\frac{1}{q}\frac{f_w\left(\ln q\right)}{f_\varepsilon\left(\varepsilon\right)}\frac{2}{\theta_u}\left[\exp\left\{-\frac{\ln q}{\theta_u}\right\}\exp\left\{a_u\right\}\Phi\left(\frac{\ln q}{\sigma_v}+b_u\right)\right.$$

$$\left.-\exp\left\{-\frac{2\ln q}{\theta_u}\right\}\exp\left\{2a_u+\frac{\sigma_v^2}{\theta_u^2}\right\}\Phi\left(\frac{\ln q}{\sigma_v}+b_u-\frac{\sigma_v}{\theta_u}\right)\right]$$

$$f_{q|\varepsilon}\left(q|\varepsilon\right)=\frac{1}{f_{\varepsilon}\left(\varepsilon\right)}\frac{2}{\theta_u}\left[\frac{2}{\theta_w}q^{-1/\theta_w}-\frac{2}{\theta_w}q^{-2/\theta_w}\right]\left[q^{-1/\theta_u-1}\exp\left\{a_u\right\}\Phi\left(\frac{\ln q}{\sigma_v}+b_u\right)\right.$$

$$\left.-q^{-2/\theta_u-1}\exp\left\{2a_u+\frac{\sigma_v^2}{\theta_u^2}\right\}\Phi\left(\frac{\ln q}{\sigma_v}+b_u-\frac{\sigma_v}{\theta_u}\right)\right]$$

[3.136]: $q=\exp\left\{w\right\}$,

$$f_{q|\varepsilon}\left(q|\varepsilon\right)=\frac{4}{\theta_w\theta_u}\frac{\left(q^{-1/\theta_w}-q^{-2/\theta_w}\right)}{f_{\varepsilon}\left(\varepsilon\right)}\left[q^{-1/\theta_u-1}\exp\left\{a_u\right\}\Phi\left(\frac{\ln q}{\sigma_v}+b_u\right)\right.$$

$$\left.-q^{-2/\theta_u-1}\exp\left\{2a_u+\frac{\sigma_v^2}{\theta_u^2}\right\}\Phi\left(\frac{\ln q}{\sigma_v}+b_u-\frac{\sigma_v}{\theta_u}\right)\right]$$

D.1.3. Related to $f_{q|\varepsilon}\left(q|\varepsilon\right)$, $q=\exp\left\{-w\right\}$.

As before,

$f_{w|\varepsilon}\left(w|\varepsilon\right)\equiv h\left(w,\varepsilon\right)$, and we want to apply on the bivariate argument of the function the transformation

$$T\left(w,\varepsilon\right)=\left(q\left(w,\varepsilon\right),k\left(w,\varepsilon\right)\right)$$

$$q\left(w,\varepsilon\right)=\exp\left\{-w\right\}\Rightarrow w=-\ln q,\ 0<q\leq 1,\quad k\left(w,\varepsilon\right)=\varepsilon$$

The Jacobian determinant is

$$J\left(q,k\right)=\frac{\partial w}{\partial q}\frac{\partial\varepsilon}{\partial k}-\frac{\partial w}{\partial k}\frac{\partial\varepsilon}{\partial q}=\frac{\partial\left(-\ln q\right)}{\partial q}\frac{\partial\varepsilon}{\partial\varepsilon}-\frac{\partial\ln q}{\partial\varepsilon}\frac{\partial\varepsilon}{\partial q}=-\frac{1}{q}\cdot 1-0=-\frac{1}{q}$$

So $q=\exp\left\{-w\right\}$, $\quad f_{q|\varepsilon}\left(q|\varepsilon\right)=\left|J\left(q,k\right)\right|f_{w|\varepsilon}\left(-\ln q|\varepsilon\right)$

$$f_{q|\varepsilon}\left(q|\varepsilon\right)=\frac{1}{q}\frac{f_w\left(-\ln q\right)}{f_\varepsilon\left(\varepsilon\right)}\frac{2}{\theta_u}\left[\exp\left\{-\frac{-\ln q}{\theta_u}\right\}\exp\left\{a_u\right\}\Phi\left(\frac{-\ln q}{\sigma_v}+b_u\right)\right.$$

$$\left.-\exp\left\{-\frac{2\left(-\ln q\right)}{\theta_u}\right\}\exp\left\{2a_u+\frac{\sigma_v^2}{\theta_u^2}\right\}\Phi\left(\frac{-\ln q}{\sigma_v}+b_u-\frac{\sigma_v}{\theta_u}\right)\right]$$

$$f_{q|\varepsilon}\left(q|\varepsilon\right)=\frac{1}{f_\varepsilon\left(\varepsilon\right)}\frac{2}{\theta_u}\left[\frac{2}{\theta_w}q^{1/\theta_w-1}-\frac{2}{\theta_w}q^{2/\theta_w-1}\right]\left[q^{1/\theta_u}\exp\left\{a_u\right\}\Phi\left(\frac{-\ln q}{\sigma_v}+b_u\right)\right.$$

$$\left.-q^{2/\theta_u}\exp\left\{2a_u+\frac{\sigma_v^2}{\theta_u^2}\right\}\Phi\left(\frac{-\ln q}{\sigma_v}+b_u-\frac{\sigma_v}{\theta_u}\right)\right]$$

$[3.137]:\ q=\exp\left\{-w\right\},$

$$f_{q|\varepsilon}\left(q|\varepsilon\right)=\frac{4}{\theta_w\theta_u}\frac{\left(q^{1/\theta_w-1}-q^{2/\theta_w-1}\right)}{f_\varepsilon\left(\varepsilon\right)}\left[q^{1/\theta_u}\exp\left\{a_u\right\}\Phi\left(\frac{-\ln q}{\sigma_v}+b_u\right)\right.$$

$$\left.-q^{2/\theta_u}\exp\left\{2a_u+\frac{\sigma_v^2}{\theta_u^2}\right\}\Phi\left(\frac{-\ln q}{\sigma_v}+b_u-\frac{\sigma_v}{\theta_u}\right)\right]$$

D.2. Conditional densities related to the u variable.

D.2.1. $f_{u|\varepsilon}\left(u|\varepsilon\right)$.

We need first to calculate the conditional density

$$f_{u|\varepsilon}\left(u|\varepsilon\right)=\frac{f_{\varepsilon,u}\left(\varepsilon,u\right)}{f_\varepsilon\left(\varepsilon\right)}$$

We have $\varepsilon=v+w-u$. Set here $\xi=v+w=\varepsilon+u$. Due to independence, we have

$$f_{\xi,u}(\xi,u) = f_{\xi}(\xi)\, f_u(u) = f_{\xi}(\varepsilon+u)\, f_u(u) = f_{\varepsilon,u}(\varepsilon,u)$$

the later because the the Jacobian determinant of the transformation is equal to unity. So

$$f_{u|\varepsilon}(u|\varepsilon) = \frac{f_{\xi}(\varepsilon+u)\, f_u(u)}{f_{\varepsilon}(\varepsilon)}$$

We need to determine the density $f_{\xi}(\xi)$. We have $\xi = v + w \Rightarrow v = \xi - w$

$$f_{\xi}(\xi) = \int_0^{\infty} \frac{1}{\sigma_v} f_v(\xi - w)\, f_w(w)\, dw = \int_0^{\infty} \frac{1}{\sigma_v} \phi\left(\frac{\xi - w}{\sigma_v}\right)\left[2 f_E(w;\theta_w) - f_E(w;\theta_w/2)\right] dw$$

$$= 2\int_0^{\infty} \frac{1}{\sigma_v} \phi\left(\frac{\xi - w}{\sigma_v}\right) f_E(w;\theta_w)\, dw \;-\; \int_0^{\infty} \frac{1}{\sigma_v} \phi\left(\frac{\xi - w}{\sigma_v}\right) f_E(w;\theta_w/2)\, dw$$

$$= \frac{2}{\theta_w \sigma_v} \int_0^{\infty} \exp\{-w/\theta_w\}\, \phi\left(\frac{\xi - w}{\sigma_v}\right) dw \;-\; \frac{2}{\theta_w \sigma_v} \int_0^{\infty} \exp\{-2w/\theta_w\}\, \phi\left(\frac{\xi - w}{\sigma_v}\right) dw$$

Apply the transformation $\quad y = \dfrac{\xi - w}{\sigma_v} \Rightarrow \begin{cases} w = -\sigma_v y + \xi \\ dw = -\sigma_v dy \\ w = 0 \Rightarrow y = \xi/\sigma_v \\ w = \infty \Rightarrow y = -\infty \end{cases}$ So

$$f_{\xi}(\xi) = -\frac{2}{\theta_w} \int_{\xi/\sigma_v}^{-\infty} \exp\{-(-\sigma_v y + \xi)/\theta_w\}\, \phi(y)\, dy \;+\; \frac{2}{\theta_w} \int_{\xi/\sigma_v}^{-\infty} \exp\{-2(-\sigma_v y + \xi)/\theta_w\}\, \phi(y)\, dy$$

Swapping the limits of integration and re-arrange

$$\ldots = \frac{2}{\theta_w} \int_{-\infty}^{\xi/\sigma_v} \exp\left\{-\left(-\sigma_v y + \xi\right)/\theta_w\right\} \phi\left(y\right) dy \;\; - \frac{2}{\theta_w} \int_{-\infty}^{\xi/\sigma_v} \exp\left\{-2\left(-\sigma_v y + \xi\right)/\theta_w\right\} \phi\left(y\right) dy$$

$$\ldots = \frac{2}{\theta_w} \exp\left\{-\xi/\theta_w\right\} \int_{-\infty}^{\xi/\sigma_v} \exp\left\{y\sigma_v/\theta_w\right\} \phi\left(y\right) dy - \frac{2}{\theta_w} \exp\left\{-2\xi/\theta_w\right\} \int_{-\infty}^{\xi/\sigma_v} \exp\left\{2y\sigma_v/\theta_w\right\} \phi\left(y\right) dy$$

Using Owen (1980) formula 100,010, p. 409,

$$f_\xi\left(\xi\right) = \frac{2}{\theta_w} \exp\left\{-\xi/\theta_w\right\} \left[\exp\left\{\frac{\sigma_v^2}{2\theta_w^2}\right\} \cdot \Phi\left(\frac{\xi}{\sigma_v} - \frac{\sigma_v}{\theta_w}\right) - 0\right]$$

$$- \frac{2}{\theta_w} \exp\left\{-2\xi/\theta_w\right\} \left[\exp\left\{\frac{2\sigma_v^2}{\theta_w^2}\right\} \cdot \Phi\left(\frac{\xi}{\sigma_v} - \frac{2\sigma_v}{\theta_w}\right) - 0\right]$$

$$= \frac{2}{\theta_w} \exp\left\{-\frac{\xi}{\theta_w} + \frac{\sigma_v^2}{2\theta_w^2}\right\} \cdot \Phi\left(\frac{\xi}{\sigma_v} - \frac{\sigma_v}{\theta_w}\right) \; - \frac{2}{\theta_w} \exp\left\{-\frac{2\xi}{\theta_w} + \frac{2\sigma_v^2}{\theta_w^2}\right\} \cdot \Phi\left(\frac{\xi}{\sigma_v} - \frac{2\sigma_v}{\theta_w}\right)$$

So

$$f_\xi\left(\varepsilon + u\right) = \frac{2}{\theta_w} \exp\left\{-\frac{\varepsilon + u}{\theta_w} + \frac{\sigma_v^2}{2\theta_w^2}\right\} \cdot \Phi\left(\frac{\varepsilon + u}{\sigma_v} - \frac{\sigma_v}{\theta_w}\right) - \frac{2}{\theta_w} \exp\left\{-\frac{2\left(\varepsilon + u\right)}{\theta_w} + \frac{2\sigma_v^2}{\theta_w^2}\right\} \cdot \Phi\left(\frac{\varepsilon + u}{\sigma_v} - \frac{2\sigma_v}{\theta_w}\right)$$

$$= \frac{2}{\theta_w} \exp\left\{-\frac{\varepsilon}{\theta_w} + \frac{\sigma_v^2}{2\theta_w^2}\right\} \exp\left\{-\frac{u}{\theta_w}\right\} \Phi\left(\frac{u}{\sigma_v} + \frac{\varepsilon}{\sigma_v} - \frac{\sigma_v}{\theta_w}\right)$$

$$- \frac{2}{\theta_w} \exp\left\{-\frac{2\varepsilon}{\theta_w} + \frac{2\sigma_v^2}{\theta_w^2}\right\} \exp\left\{-\frac{2u}{\theta_w}\right\} \Phi\left(\frac{u}{\sigma_v} + \frac{\varepsilon}{\sigma_v} - \frac{2\sigma_v}{\theta_w}\right)$$

$$= \frac{2}{\theta_w} \exp\left\{a_w\right\} \exp\left\{-\frac{u}{\theta_w}\right\} \Phi\left(\frac{u}{\sigma_v} + b_w\right)$$

$$- \frac{2}{\theta_w} \exp\left\{2a_w + \frac{\sigma_v^2}{\theta_w^2}\right\} \exp\left\{-\frac{2u}{\theta_w}\right\} \Phi\left(\frac{u}{\sigma_v} + b_w - \frac{\sigma_v}{\theta_w}\right)$$

So

$$[3.130]:\ f_{u|\varepsilon}\left(u|\varepsilon\right)=\frac{f_u\left(u\right)}{f_\varepsilon\left(\varepsilon\right)}\frac{2}{\theta_w}\left[\exp\left\{-\frac{u}{\theta_w}\right\}\exp\left\{a_w\right\}\Phi\left(\frac{u}{\sigma_v}+b_w\right)\right.$$
$$\left.-\exp\left\{-\frac{2u}{\theta_w}\right\}\exp\left\{2a_w+\frac{\sigma_v^2}{\theta_w^2}\right\}\Phi\left(\frac{u}{\sigma_v}+b_w-\frac{\sigma_v}{\theta_w}\right)\right]$$

D.2.2. Related to $f_{q|\varepsilon}\left(q|\varepsilon\right),\ q=\exp\{-u\}$.

As before,

$f_{u|\varepsilon}\left(u|\varepsilon\right)\equiv h\left(u,\varepsilon\right)$, and we want to apply on the bivariate argument of the function the transformation

$$T\left(u,\varepsilon\right)=\left(q\left(u,\varepsilon\right),k\left(u,\varepsilon\right)\right)$$

$$q\left(u,\varepsilon\right)=\exp\{-u\}\Rightarrow u=-\ln q,\ \ 0<q\le 1\,,\quad k\left(u,\varepsilon\right)=\varepsilon$$

The Jacobian determinant is

$$J\left(q,k\right)=\frac{\partial u}{\partial q}\frac{\partial \varepsilon}{\partial k}-\frac{\partial u}{\partial k}\frac{\partial \varepsilon}{\partial q}=\frac{\partial\left(-\ln q\right)}{\partial q}\frac{\partial \varepsilon}{\partial \varepsilon}-\frac{\partial \ln q}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial q}=-\frac{1}{q}\cdot 1-0=-\frac{1}{q}$$

So $q=\exp\{-u\}$, $\quad f_{q|\varepsilon}\left(q|\varepsilon\right)=\left|J\left(q,k\right)\right|f_{u|\varepsilon}\left(-\ln q|\varepsilon\right)$

$$f_{q|\varepsilon}\left(q|\varepsilon\right)=\frac{1}{q}\frac{f_u\left(-\ln q\right)}{f_\varepsilon\left(\varepsilon\right)}\frac{2}{\theta_w}\left[q^{1/\theta_w}\exp\left\{a_w\right\}\Phi\left(\frac{-\ln q}{\sigma_v}+b_w\right)\right.$$
$$\left.-q^{2/\theta_w}\exp\left\{2a_w+\frac{\sigma_v^2}{\theta_w^2}\right\}\Phi\left(\frac{-\ln q}{\sigma_v}+b_w-\frac{\sigma_v}{\theta_w}\right)\right]$$

$$f_{q|\varepsilon}\left(q|\varepsilon\right)=\frac{1}{f_{\varepsilon}\left(\varepsilon\right)}\frac{2}{\theta_w}\left[\frac{2}{\theta_u}q^{1/\theta_u-1}-\frac{2}{\theta_u}q^{2/\theta_u-1}\right]\left[q^{1/\theta_w}\exp\left\{a_w\right\}\Phi\left(\frac{-\ln q}{\sigma_v}+b_w\right)\right.$$

$$\left.-q^{2/\theta_w}\exp\left\{2a_w+\frac{\sigma_v^2}{\theta_w^2}\right\}\Phi\left(\frac{-\ln q}{\sigma_v}+b_w-\frac{\sigma_v}{\theta_w}\right)\right]$$

[3.138]: $q=\exp\left\{-u\right\}$,

$$f_{q|\varepsilon}\left(q|\varepsilon\right)=\frac{1}{f_{\varepsilon}\left(\varepsilon\right)}\frac{4}{\theta_w\theta_u}\left(q^{1/\theta_u-1}-q^{2/\theta_u-1}\right)\left[q^{1/\theta_w}\exp\left\{a_w\right\}\Phi\left(\frac{-\ln q}{\sigma_v}+b_w\right)\right.$$

$$\left.-q^{2/\theta_w}\exp\left\{2a_w+\frac{\sigma_v^2}{\theta_w^2}\right\}\Phi\left(\frac{-\ln q}{\sigma_v}+b_w-\frac{\sigma_v}{\theta_w}\right)\right]$$

D.3. Related to the $z=w-u$ variable.

D.3.1. $f_{z|\varepsilon}\left(z|\varepsilon\right)$.

We need to calculate the conditional density

$$f_{z|\varepsilon}\left(z|\varepsilon\right)=\frac{f_{\varepsilon,z}\left(\varepsilon,z\right)}{f_{\varepsilon}\left(\varepsilon\right)}$$

We have $\varepsilon=v+z\Rightarrow v=\varepsilon-z$. Due to independence, we have

$$f_{v,z}\left(v,z\right)=f_v\left(v\right)f_z\left(z\right)=f_v\left(\varepsilon-z\right)f_z\left(z\right)=f_{\varepsilon,z}\left(\varepsilon,z\right)$$

So

$$f_{z|\varepsilon}\left(z|\varepsilon\right)=\frac{f_v\left(\varepsilon-z\right)f_z\left(z\right)}{f_{\varepsilon}\left(\varepsilon\right)}$$

The density $f_z\left(z\right)$ has branches. So we have

$$[3.131]: f_{z|\varepsilon}(z|\varepsilon) = \begin{cases} \dfrac{1}{\sigma_v f_\varepsilon(\varepsilon)}\phi\left(\dfrac{\varepsilon - z}{\sigma_v}\right)\dfrac{2\theta_u}{\theta_w+\theta_u}\left[\dfrac{2\exp\{z/\theta_u\}}{\theta_w+2\theta_u} - \dfrac{\exp\{2z/\theta_u\}}{2\theta_w+\theta_u}\right] & z\leq 0 \\ \dfrac{1}{\sigma_v f_\varepsilon(\varepsilon)}\phi\left(\dfrac{\varepsilon - z}{\sigma_v}\right)\dfrac{2\theta_w}{\theta_w+\theta_u}\left[\dfrac{2\exp\{-z/\theta_w\}}{2\theta_w+\theta_u} - \dfrac{\exp\{-2z/\theta_w\}}{\theta_w+2\theta_u}\right] & z>0 \end{cases}$$

D.3.2. Related to $f_{q|\varepsilon}(q|\varepsilon),\ q=\exp\{z\}$.

For the logarithmic specification case we need $f_{z|\varepsilon}(z|\varepsilon)\equiv h(z,\varepsilon)$, and we want to apply on the bivariate argument of the function the transformation

$$T(z,\varepsilon)=\left(q(z,\varepsilon),k(z,\varepsilon)\right), \qquad q(z,\varepsilon)=\exp\{z\}\Rightarrow z=\ln q, \quad k(z,\varepsilon)=\varepsilon$$

Consider the determinant of the Jacobian determinant

$$J(q,k)=\frac{\partial z}{\partial q}\frac{\partial \varepsilon}{\partial k}-\frac{\partial z}{\partial k}\frac{\partial \varepsilon}{\partial q}=\frac{\partial \ln q}{\partial q}\frac{\partial \varepsilon}{\partial \varepsilon}-\frac{\partial \ln q}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial q}=\frac{1}{q}\cdot 1-0=\frac{1}{q}$$

So we have
$$q=\exp\{z\}, \quad f_{q|\varepsilon}(q|\varepsilon)=\left|J(q,k)\right| f_{z|\varepsilon}(\ln q|\varepsilon),$$
and so
$$z\leq 0\Rightarrow 0<q\leq 1, \quad z>0\Rightarrow q>1$$

$$f_{q|\varepsilon}(q|\varepsilon) = \begin{cases} \dfrac{1}{q}\dfrac{1}{\sigma_v f_\varepsilon(\varepsilon)}\phi\left(\dfrac{\varepsilon - \ln q}{\sigma_v}\right)\dfrac{2\theta_u}{\theta_w+\theta_u}\left[\dfrac{2}{\theta_w+2\theta_u}q^{1/\theta_u} - \dfrac{1}{2\theta_w+\theta_u}q^{2/\theta_u}\right] & 0<q\leq 1 \\ \dfrac{1}{q}\dfrac{1}{\sigma_v f_\varepsilon(\varepsilon)}\phi\left(\dfrac{\varepsilon - \ln q}{\sigma_v}\right)\dfrac{2\theta_w}{\theta_w+\theta_u}\left[\dfrac{2}{2\theta_w+\theta_u}q^{-1/\theta_w} - \dfrac{1}{\theta_w+2\theta_u}q^{-2/\theta_w}\right] & q>1 \end{cases}$$

$$
=\begin{cases}
\dfrac{1}{\sigma_v f_\varepsilon(\varepsilon)}\phi\left(\dfrac{\varepsilon-\ln q}{\sigma_v}\right)\dfrac{2\theta_u}{\theta_w+\theta_u}\left[\dfrac{2q^{1/\theta_u-1}}{\theta_w+2\theta_u} - \dfrac{q^{2/\theta_u-1}}{2\theta_w+\theta_u}\right] & 0<q\le 1 \\[2ex]
\dfrac{1}{\sigma_v f_\varepsilon(\varepsilon)}\phi\left(\dfrac{\varepsilon-\ln q}{\sigma_v}\right)\dfrac{2\theta_w}{\theta_w+\theta_u}\left[\dfrac{2q^{-1/\theta_w-1}}{2\theta_w+\theta_u} - \dfrac{q^{-2/\theta_w-1}}{\theta_w+2\theta_u}\right] & q>1
\end{cases}
$$

So finally,

$$[3.139]:\quad q=\exp\{w-u\}=\exp\{z\}$$

$$
f_{q|\varepsilon}(q|\varepsilon)=\begin{cases}
\dfrac{2\theta_u}{\sigma_v(\theta_w+\theta_u)}\dfrac{\phi\left((\varepsilon-\ln q)/\sigma_v\right)}{f_\varepsilon(\varepsilon)}\left[\dfrac{2q^{1/\theta_u-1}}{\theta_w+2\theta_u} - \dfrac{q^{2/\theta_u-1}}{2\theta_w+\theta_u}\right] & 0<q\le 1 \\[2ex]
\dfrac{2\theta_w}{\sigma_v(\theta_w+\theta_u)}\dfrac{\phi\left((\varepsilon-\ln q)/\sigma_v\right)}{f_\varepsilon(\varepsilon)}\left[\dfrac{2q^{-1/\theta_w-1}}{2\theta_w+\theta_u} - \dfrac{q^{-2/\theta_w-1}}{\theta_w+2\theta_u}\right] & q>1
\end{cases}
$$

E. Distributional connections.

E.1. $q=\exp\{w\},\quad q\in[1,\infty)$.

We have $\Pr(\exp\{w\}\le q)=\Pr(w\le \ln q)=F_w(\ln q)$

Let $w\sim GE(2,\theta_w,0)$. Then

$$F_w(w)=\left(1-\exp\{-w/\theta_w\}\right)^2 \;\Rightarrow F_w(\ln q)=\left(1-\exp\{-\ln q/\theta_w\}\right)^2=\left(1-q^{-1/\theta_w}\right)^2$$

So $F_q(q)=\left(1-\dfrac{1}{q^{1/\theta_w}}\right)^2$

But $1-\frac{1}{q^{1/\theta_w}}$ is the distribution function of a Pareto random variable with minimum value equal to unity, and shape parameter $1/\theta_w$. Therefore, its square is the distribution function of the maximum of two i.i.d Pareto random variables with these parameters.

E.2. $q=1-\exp\{-w\},\quad q\in[0,1)$

$$\Pr(1-\exp\{-w\}\le q)=\Pr(1-q\le\exp\{-w\})=\Pr(\ln(1-q)\le -w)=\Pr(w\le-\ln(1-q))$$

So $F_q(q)=F_w(-\ln(1-q))=\left(1-\exp\{\ln(1-q)/\theta_w\}\right)^2=\left(1-(1-q)^{1/\theta_w}\right)^2$

The function $1-(1-q)^{1/\theta_w}$ is the distribution function of the Kumaraswamy distribution, which has general form $F_k(k)=1-\left(1-k^{\alpha}\right)^{\beta}$, for parameter values $\alpha=1,\ \beta=1/\theta_w$. Therefore its square is the distribution function of two i.i.d. Kumaraswamy random variables with these parameter values.

E.3. $q=\exp\{-u\},\quad q\in(0,1]$

$$F_q(q)=\Pr(\exp\{-u\}\le q)=\Pr(-u\le\ln q)=\Pr(-\ln q\le u)=1-\Pr(u\le-\ln q)=1-F_u(-\ln q)$$

If $u\sim GE(2,\theta_u,0)$, then $F_q(q)=1-F_u(-\ln q)=1-\left(1-\exp\{\ln q/\theta_u\}\right)^2=1-\left(1-q^{1/\theta_u}\right)^2$

Note that q^{1/θ_u} is the distribution function of a Beta random variable with parameters $\alpha=1/\theta_w,\ \beta=1$. Then the expression $1-\left(1-q^{1/\theta_u}\right)^2$ is the distribution function of the minimum of two i.i.d Beta random variables with these parameter values.

--

Chapter 4
Dependence and Endogeneity.

TECHNICAL APPENDIX

Section I.

A. Range of correlation in Freund's bivariate Exponential Extension.

Pearson's correlation coefficient is

$$\rho = \frac{a'b' - ab}{\sqrt{\left(a'^2 + 2ab + b^2\right)\left(b'^2 + 2ab + a^2\right)}}$$

$$= \frac{a'b'}{\sqrt{\left(a'^2 + 2ab + b^2\right)\left(b'^2 + 2ab + a^2\right)}} - \frac{ab}{\sqrt{\left(a'^2 + 2ab + b^2\right)\left(b'^2 + 2ab + a^2\right)}}$$

1) Positive Correlation.

The correlation coefficient can attain the value unity, if $a' \to \infty, b' \to \infty$. In such a case the negative term goes to zero while in the positive component, the leading term in the denominator is $\sqrt{a'^2 b'^2} = a'\ b'$ which cancels out with the numerator resulting in the value of unity.

2) Negative Correlation.

We obtain maximum negative correlation if the first component goes to zero and the second is maximized (without the negative sign). For the first component to go to zero, we examine the case where *both* $a', b' \to 0$. This also contributes to increase the value of the second component. Under this we want to maximize

$$\max_{a,b} \frac{ab}{\sqrt{\left(2ab + b^2\right)\left(2ab + a^2\right)}} = \max_{a,b} \frac{\sqrt{ab}}{\sqrt{(2a+b)(2b+a)}}$$

The argmax will be the same for the square, $\frac{ab}{(2a+b)(2b+a)} = \frac{ab}{5ab+2a^2+2b^2}$

We have

$$\frac{\partial}{\partial a}\left(\frac{ab}{5ab+2a^2+2b^2}\right) = 0 \Rightarrow b\left(5ab+2a^2+2b^2\right) - ab\left(5b+4a\right) = 0$$

$$\Rightarrow b\left(5ab+2a^2+2b^2-5ab-4a^2\right) = 0 \Rightarrow 2b^2-2a^2 = 0$$

$$\Rightarrow a = b$$

Also,

$$\frac{\partial}{\partial b}\left(\frac{ab}{5ab+2a^2+2b^2}\right) = 0 \Rightarrow a\left(5ab+2a^2+2b^2\right) - ab\left(5a+4b\right) = 0$$

$$\Rightarrow a\left(5ab+2a^2+2b^2-5ab-4b^2\right) = 0 \Rightarrow 2a^2-2b^2 = 0$$

$$\Rightarrow a = b$$

So the condition to maximize the second term (and so minimize the correlation coefficient) is $a = b$. Plugging this together with $a', b' \to 0$ in the expression for ρ we arrive at

$$\rho_{\min} = -\frac{a^2}{\sqrt{3a^2 \cdot 3a^2}} = -\frac{1}{3} \ .$$

B. The distribution of the Correlated Exponential 2TSF error term.

B1. The density and distribution function of $w-u$.

Set $z=w-u \Rightarrow u=z+w$. Then

$$f_{wu}(w,u)=\begin{cases} ab'e^{-b'u}e^{-(a+b-b')(z+u)} & z<0 \\ \\ a'be^{-a'(z+u)}e^{-(a+b-a')u} & z\geq 0 \end{cases}$$

Limits of Integration. We have $P(Z\leq z)=P(W-U\leq z)$. So $w \leq z+u$ and since $0\leq w$ it follows that we must have $-z \leq u$. This always holds when z is positive, but it restricts the integration interval when z is negative. So we have

For $z<0$

$$f_Z(z)=\int_{-z}^{\infty}ab'e^{-b'u}e^{-(a+b-b')(u+z)}du=ab'e^{-(a+b-b')z}\int_{-z}^{\infty}e^{-(a+b)u}du$$

$$=\frac{ab'}{a+b}e^{-(a+b-b')z}e^{(a+b)z} \;=\; \frac{ab'}{a+b}e^{b'z}$$

For $z\geq 0$

$$f_Z(z)=\int_{0}^{\infty}a'be^{-a'(z+u)}e^{-(a+b-a')u}du=a'be^{-a'z}\int_{0}^{\infty}e^{-(a+b)u}du=\;\frac{ba'}{a+b}e^{-a'z}$$

Therefore the density of $z=w-u$ is

$$f_Z(z)=\begin{cases} \dfrac{ab'}{a+b}e^{b'z} & z<0 \\ \\ \dfrac{ba'}{a+b}e^{-a'z} & z\geq 0 \end{cases}$$

Defining $m\equiv \dfrac{a}{a+b}$ we can write

$$[4.18]: \quad f_Z(z) = \begin{cases} mb' \exp\{b'z\} e^{b'z} & z < 0 \\ (1-m)a' \exp\{-a'z\} e^{-a'z} & z \geq 0 \end{cases}$$

The distribution function is

$$F_z(z) = \begin{cases} \int_{-\infty}^{z} \frac{ab'}{a+b} \exp\{b's\} ds & z < 0 \\ F_z(0) + \int_0^z \frac{ba'}{a+b} \exp\{-a's\} ds & z \geq 0 \end{cases} = \begin{cases} \frac{ab'}{a+b} \frac{1}{b'} \exp\{b's\}\Big|_{-\infty}^{z} & z < 0 \\ F_z(0) - \frac{ba'}{a+b} \frac{1}{a'} \exp\{-a's\}\Big|_{0}^{z} & z \geq 0 \end{cases}$$

$$[4.20]: \quad F_z(z) = \begin{cases} m \exp\{b'z\} & z < 0 \\ 1 - (1-m)\exp\{-a'z\} & z \geq 0 \end{cases}$$

B.2 The 2TSF Correlated Exponential three-component error density.

Let $v \sim N(0, \sigma_v^2)$ independent of z. Consider $\varepsilon = v + z \Rightarrow v = \varepsilon - z$. We have

$$f_\varepsilon(\varepsilon) = \int_{-\infty}^{0} \frac{1}{\sigma_v \sqrt{2\pi}} \frac{ab'}{a+b} \exp\{b'z\} \exp\left\{-\tfrac{1}{2\sigma_v^2}(\varepsilon - z)^2\right\} dz$$

$$+ \int_0^{\infty} \frac{1}{\sigma_v \sqrt{2\pi}} \frac{ba'}{a+b} \exp\{-a'z\} \exp\left\{-\tfrac{1}{2\sigma_v^2}(\varepsilon - z)^2\right\} dz$$

$$= \frac{1}{\sigma_v\sqrt{2\pi}} \frac{ab'}{a+b} \exp\left\{-\tfrac{1}{2\sigma_v^2}\varepsilon^2\right\} \int_{-\infty}^{0} \exp\left\{-\tfrac{1}{2\sigma_v^2}z^2\right\} \exp\left\{\left(b' + \frac{\varepsilon}{\sigma_v^2}\right)z\right\} dz$$

$$+ \frac{1}{\sigma_v\sqrt{2\pi}} \frac{ba'}{a+b} \exp\left\{-\tfrac{1}{2\sigma_v^2}\varepsilon^2\right\} \int_{0}^{\infty} \exp\left\{-\tfrac{1}{2\sigma_v^2}z^2\right\} \exp\left\{\left(\frac{\varepsilon}{\sigma_v^2} - a'\right)z\right\} dz$$

Swap the limits of integration on the first Integral

$$= \frac{1}{\sigma_v\sqrt{2\pi}} \frac{ab'}{a+b} \exp\left\{-\tfrac{1}{2\sigma_v^2}\varepsilon^2\right\} \int_{0}^{\infty} \exp\left\{-\tfrac{1}{2\sigma_v^2}z^2\right\} \exp\left\{-\left(b' + \frac{\varepsilon}{\sigma_v^2}\right)z\right\} dz$$

$$+ \frac{1}{\sigma_v\sqrt{2\pi}} \frac{ba'}{a+b} \exp\left\{-\tfrac{1}{2\sigma_v^2}\varepsilon^2\right\} \int_{0}^{\infty} \exp\left\{-\tfrac{1}{2\sigma_v^2}z^2\right\} \exp\left\{\left(\frac{\varepsilon}{\sigma_v^2} - a'\right)z\right\} dz$$

The integrals evaluate to

$$= \frac{1}{\sigma_v\sqrt{2\pi}} \frac{ab'}{a+b} \exp\left\{-\tfrac{1}{2\sigma_v^2}\varepsilon^2\right\} \frac{\sigma_v\sqrt{\pi}}{\sqrt{2}} \exp\left\{\tfrac{1}{2\sigma_v^2}\varepsilon^2\right\} \exp\left\{b'\varepsilon + \tfrac{1}{2}\sigma_v^2 b'^2\right\} \left(2 - 2\Phi\left(\frac{\varepsilon}{\sigma_v} + \sigma_v b'\right)\right)$$

$$+ \frac{1}{\sigma_v\sqrt{2\pi}} \frac{ba'}{a+b} \exp\left\{-\tfrac{1}{2\sigma_v^2}\varepsilon^2\right\} \frac{\sigma_v\sqrt{\pi}}{\sqrt{2}} \exp\left\{\tfrac{1}{2\sigma_v^2}\varepsilon^2\right\} \exp\left\{-a'\varepsilon + \tfrac{1}{2}\sigma_v^2 a'^2\right\} \left(2 - 2\Phi\left(-\frac{\varepsilon}{\sigma_v} + \sigma_v a'\right)\right)$$

which finally gives

$$[4.8]: \quad f_\varepsilon(\varepsilon) = \frac{a}{a+b} b' \exp\left\{b'\varepsilon + \tfrac{1}{2}\sigma_v^2 b'^2\right\} \Phi\left(-\frac{\varepsilon}{\sigma_v} - \sigma_v b'\right)$$

$$+ \frac{b}{a+b} a' \exp\left\{-a'\varepsilon + \tfrac{1}{2}\sigma_v^2 a'^2\right\} \Phi\left(\frac{\varepsilon}{\sigma_v} - \sigma_v a'\right)$$

Defining the shortcuts

$$\omega_2 \equiv \frac{\varepsilon}{\sigma_v} + b'\sigma_v, \quad \omega_3 \equiv \frac{\varepsilon}{\sigma_v} - a'\sigma_v,$$

we have that

$$\tfrac{1}{2}\omega_2^2 = \tfrac{1}{2}\left(\varepsilon/\sigma_v\right)^2 + b'\varepsilon + \tfrac{1}{2}\sigma_v^2 b'^2 \Rightarrow b'\varepsilon + \tfrac{1}{2}\sigma_v^2 b'^2 = \tfrac{1}{2}\omega_2^2 - \tfrac{1}{2}\left(\varepsilon/\sigma_v\right)^2$$

$$\tfrac{1}{2}\omega_3^2 = \tfrac{1}{2}\left(\varepsilon/\sigma_v\right)^2 - a'\varepsilon + \tfrac{1}{2}\sigma_v^2 a'^2 \Rightarrow -a'\varepsilon + \tfrac{1}{2}\sigma_v^2 a'^2 = \tfrac{1}{2}\omega_3^2 - \tfrac{1}{2}\left(\varepsilon/\sigma_v\right)^2$$

So we can also write

$$f_\varepsilon(\varepsilon) = \frac{\exp\left\{-\tfrac{1}{2}\left(\varepsilon/\sigma_v\right)^2\right\}}{a+b}\left[ab'\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\Phi\left(-\omega_2\right) + a'b\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\Phi\left(\omega_3\right)\right]$$

Using $m \equiv \dfrac{a}{a+b}$ and $\exp\left\{-\tfrac{1}{2}\left(\varepsilon/\sigma_v\right)^2\right\} = \sqrt{2\pi}\cdot\phi\left(\varepsilon/\sigma_v\right)$ we can write

$$\boxed{[4.9]: \quad f_\varepsilon(\varepsilon) = \sqrt{2\pi}\phi\left(\varepsilon/\sigma_v\right)\left[mb'\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\Phi\left(-\omega_2\right) + (1-m)a'\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\Phi\left(\omega_3\right)\right] \;.}$$

B.3 The distribution function of the 2TSF Correlated Exponential composite error.

We have obtained the three-component error density

$$f_\varepsilon(\varepsilon) = \frac{a}{a+b}b'\exp\left\{b'\varepsilon + \tfrac{1}{2}\sigma_v^2 b'^2\right\}\Phi\left(-\frac{\varepsilon}{\sigma_v} - \sigma_v b'\right)$$

$$+ \frac{b}{a+b}a'\exp\left\{-a'\varepsilon + \tfrac{1}{2}\sigma_v^2 a'^2\right\}\Phi\left(\frac{\varepsilon}{\sigma_v} - \sigma_v a'\right)$$

or

$$f_\varepsilon(\varepsilon) = \psi_1 I_1 \; + \; \psi_2 I_2$$

$$\psi_1 = \frac{ab'}{a+b}\exp\left\{\tfrac{1}{2}\sigma_v^2 b'^2\right\}, \;\; I_1 = \exp\{b'\varepsilon\}\,\Phi\left(-\frac{1}{\sigma_v}\left(\varepsilon + \sigma_v^2 b'\right)\right)$$

$$\psi_2 = \frac{ba'}{a+b}\exp\left\{\tfrac{1}{2}\sigma_v^2 a'^2\right\}, \;\; I_2 = \exp\{-a'\varepsilon\}\,\Phi\left(\frac{1}{\sigma_v}\left(\varepsilon - \sigma_v^2 a'\right)\right)$$

In Owen (1980), p. 409, (eq. 101,000), we find the indefinite integral

$$\int \exp\{\gamma x\}\,\Phi(\delta x)\,dx = \frac{1}{\gamma}\exp\{\gamma x\}\,\Phi(\delta x) \; - \; \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}\Phi\left(\delta x - \frac{\gamma}{\delta}\right)$$

To calculate the distribution function we want the definite integral

$$\int_{-\infty}^{x} \exp\{\gamma s\}\,\Phi(\delta s)\,ds = \frac{1}{\gamma}\exp\{\gamma x\}\,\Phi(\delta x) \; - \; \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}\Phi\left(\delta x - \frac{\gamma}{\delta}\right)$$

$$-\frac{1}{\gamma}\lim_{s\to-\infty}\exp\{\gamma s\}\,\Phi(\delta s) \; + \; \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}\lim_{s\to-\infty}\Phi\left(\delta s - \frac{\gamma}{\delta}\right)$$

Looking at our integrands I_1, I_2, in order to determine the limits, we have to consider alternating pairs of signs for γ, δ.

A) For integrand I_1 the corresponding signs are $\gamma = b' > 0, \;\; \delta = -\dfrac{1}{\sigma_v} < 0$

In this case, $\lim_{s\to-\infty}\exp\{\gamma s\}\,\Phi(\delta s) = 0\cdot 1 = 0,$ $\quad \lim_{s\to-\infty}\Phi\left(\delta s - \dfrac{\gamma}{\delta}\right) = \Phi(\infty) = 1$. So the general formula to be used, after eliminating the zero-terms, is

$$\int_{-\infty}^{x} \exp\{\gamma s\}\,\Phi(\delta s)\,ds = \frac{1}{\gamma}\exp\{\gamma x\}\,\Phi(\delta x) \;-\; \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}\Phi\left(\delta x - \frac{\gamma}{\delta}\right) \;+\; \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}$$

$$= \frac{1}{\gamma}\exp\{\gamma x\}\,\Phi(\delta x) \;+\; \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}\Phi\left(-\delta x + \frac{\gamma}{\delta}\right)$$

where we have used the reflective symmetry of $\Phi(\cdot)$.

We have

$$\int_{-\infty}^{\varepsilon} I_1 ds = \int_{-\infty}^{\varepsilon} \exp\{b's\}\,\Phi\left(-\frac{1}{\sigma_v}\left(s + \sigma_v^2 b'\right)\right)ds = \int_{-\infty}^{\varepsilon+\sigma_v^2 b'} \exp\left\{b's - \sigma_v^2 b'^2\right\}\Phi\left(-\frac{s}{\sigma_v}\right)ds$$

$$= \exp\left\{-\sigma_v^2 b'^2\right\}\int_{-\infty}^{\varepsilon+\sigma_v^2 b'} \exp\{b's\}\,\Phi\left(-\frac{s}{\sigma_v}\right)ds$$

which is now in the appropriate form to use Owen's formula. The correspondance of coeffcients is $\gamma = b' > 0,\; \delta = -\dfrac{1}{\sigma_v} < 0$. We get

$$\int_{-\infty}^{\varepsilon} I_1 ds = \exp\left\{-\sigma_v^2 b'^2\right\}\left[\frac{1}{b'}\exp\left\{b'\varepsilon + \sigma_v^2 b'^2\right\}\Phi\left(-\frac{\varepsilon}{\sigma_v} - \sigma_v b'\right) \;+\; \frac{1}{b'}\exp\left\{\tfrac{1}{2}\sigma_v^2 b'^2\right\}\Phi\left(\frac{\varepsilon + \sigma_v^2 b'}{\sigma_v} - \sigma_v b'\right)\right]$$

$$\Rightarrow \int_{-\infty}^{\varepsilon} I_1 ds = \frac{1}{b'}\left[\exp\{b'\varepsilon\}\,\Phi\left(-\frac{\varepsilon}{\sigma_v} - \sigma_v b'\right) \;+\; \exp\left\{-\tfrac{1}{2}\sigma_v^2 b'^2\right\}\Phi\left(\frac{\varepsilon}{\sigma_v}\right)\right]$$

and

$$\psi_1\int_{-\infty}^{\varepsilon} I_1 ds = \frac{ab'}{a+b}\exp\left\{\tfrac{1}{2}\sigma_v^2 b'^2\right\}\frac{1}{b'}\left[\exp\{b'\varepsilon\}\,\Phi\left(-\frac{\varepsilon}{\sigma_v} - \sigma_v b'\right) \;+\; \exp\left\{-\tfrac{1}{2}\sigma_v^2 b'^2\right\}\Phi\left(\frac{\varepsilon}{\sigma_v}\right)\right]$$

$$= \frac{a}{a+b}\left[\exp\left\{\tfrac{1}{2}\sigma_v^2 b'^2\right\}\exp\{b'\varepsilon\}\,\Phi\left(-\frac{\varepsilon}{\sigma_v} - \sigma_v b'\right) + \Phi\left(\frac{\varepsilon}{\sigma_v}\right)\right]$$

We turn now to the second integrand.

B) For the integrand I_2 the corresponding signs are $\gamma = -a' < 0,\ \delta = \dfrac{1}{\sigma_v} > 0$

In this case, $\lim\limits_{s \to -\infty} \exp\{\gamma s\}\Phi(\delta s) = 0,\qquad \lim\limits_{s \to -\infty} \Phi\left(\delta s - \dfrac{\gamma}{\delta}\right) = \Phi(-\infty) = 0$. So the general formula to be used for I_2 is

$$\int_{-\infty}^{x} \exp\{\gamma s\}\Phi(\delta s)\,ds = \frac{1}{\gamma}\exp\{\gamma x\}\Phi(\delta x) \ - \ \frac{1}{\gamma}\exp\left\{\frac{\gamma^2}{2\delta^2}\right\}\Phi\left(\delta x - \frac{\gamma}{\delta}\right)$$

We have

$$\int_{-\infty}^{\varepsilon} I_2 ds = \int_{-\infty}^{\varepsilon} \exp\{-a's\}\,\Phi\left(\frac{1}{\sigma_v}\left(s - \sigma_v^2 a'\right)\right)ds = \int_{-\infty}^{\varepsilon - \sigma_v^2 a'} \exp\{-a's - \sigma_v^2 a'^2\}\,\Phi\left(\frac{s}{\sigma_v}\right)ds$$

$$= \exp\{-\sigma_v^2 a'^2\}\int_{-\infty}^{\varepsilon - \sigma_v^2 a'} \exp\{-a's\}\,\Phi\left(\frac{s}{\sigma_v}\right)ds$$

which is now in the appropriate form. Matching coefficients, we have

$$\int_{-\infty}^{\varepsilon} I_2 ds = \exp\{-\sigma_v^2 a'^2\}\left[-\frac{1}{a'}\exp\{-a'\varepsilon + \sigma_v^2 a'\}\,\Phi\left(\frac{\varepsilon - \sigma_v^2 a'}{\sigma_v}\right) \ + \ \frac{1}{a'}\exp\{\tfrac{1}{2}a'^2\sigma_v^2\}\,\Phi\left(\frac{\varepsilon - \sigma_v^2 a'}{\sigma_v} + \sigma_v a'\right)\right]$$

$$= \frac{1}{a'}\left[\exp\{-\tfrac{1}{2}a'^2\sigma_v^2\}\,\Phi\left(\frac{\varepsilon}{\sigma_v}\right) \ - \ \exp\{-a'\varepsilon\}\,\Phi\left(\frac{\varepsilon}{\sigma_v} - \sigma_v a'\right)\ \right]$$

and

$$\psi_2\int_{-\infty}^{\varepsilon} I_2 ds = \frac{ba'}{a+b}\exp\left\{\tfrac{1}{2}\sigma_v^2 a'^2\right\}\frac{1}{a'}\left[\exp\left\{-\tfrac{1}{2}a'^2\sigma_v^2\right\}\Phi\left(\frac{\varepsilon}{\sigma_v}\right) - \exp\left\{-a'\varepsilon\right\}\Phi\left(\frac{\varepsilon}{\sigma_v}-\sigma_v a'\right)\right]$$

$$= \frac{b}{a+b}\left[\Phi\left(\frac{\varepsilon}{\sigma_v}\right) - \exp\left\{\tfrac{1}{2}\sigma_v^2 a'^2\right\}\exp\left\{-a'\varepsilon\right\}\Phi\left(\frac{\varepsilon}{\sigma_v}-\sigma_v a'\right)\right]$$

So the cumulative distribution function of the composite error term is

$$F_\varepsilon(\varepsilon) = \int_{-\infty}^{\varepsilon} f_\varepsilon(s)\,ds = \psi_1\int_{-\infty}^{\varepsilon} I_1 ds + \psi_2\int_{-\infty}^{\varepsilon} I_2 ds$$

$$F_\varepsilon(\varepsilon) = \frac{a}{a+b}\left[\exp\left\{\tfrac{1}{2}\sigma_v^2 b'^2\right\}\exp\left\{b'\varepsilon\right\}\Phi\left(-\frac{\varepsilon}{\sigma_v}-\sigma_v b'\right) + \Phi\left(\frac{\varepsilon}{\sigma_v}\right)\right]$$

$$+ \frac{b}{a+b}\left[\Phi\left(\frac{\varepsilon}{\sigma_v}\right) - \exp\left\{\tfrac{1}{2}\sigma_v^2 a'^2\right\}\exp\left\{-a'\varepsilon\right\}\Phi\left(\frac{\varepsilon}{\sigma_v}-\sigma_v a'\right)\right]$$

and finally,

$$[4.11]:\quad F_\varepsilon(\varepsilon) = \Phi\left(\frac{\varepsilon}{\sigma_v}\right) + \frac{a}{a+b}\exp\left\{\tfrac{1}{2}\sigma_v^2 b'^2\right\}\exp\left\{b'\varepsilon\right\}\Phi\left(-\frac{\varepsilon}{\sigma_v}-\sigma_v b'\right)$$

$$- \frac{b}{a+b}\exp\left\{\tfrac{1}{2}\sigma_v^2 a'^2\right\}\exp\left\{-a'\varepsilon\right\}\Phi\left(\frac{\varepsilon}{\sigma_v}-\sigma_v a'\right)$$

Using the shortcuts defined before $\omega_2 \equiv \dfrac{\varepsilon}{\sigma_v} + b'\sigma_v,\quad \omega_3 \equiv \dfrac{\varepsilon}{\sigma_v} - a'\sigma_v$

and $m \equiv \dfrac{a}{a+b}$, $\exp\left\{-\tfrac{1}{2}(\varepsilon/\sigma_v)^2\right\} = \sqrt{2\pi}\cdot\phi(\varepsilon/\sigma_v)$ we can write

$$[4.12]:\quad F_\varepsilon(\varepsilon) = \Phi(\varepsilon/\sigma_v) + \sqrt{2\pi}\cdot\phi(\varepsilon/\sigma_v)\left[m\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\Phi(-\omega_2) - (1-m)\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\Phi(\omega_3)\right] .$$

C. Moment generating functions and skewness of the composite error term.

C.1. The moment generating function of $z = w - u$.

The moment generating function of $z = w - u$ is

$$M_z(s) = \int_{-\infty}^{\infty} f_z(z) e^{sz} dz = \frac{ab'}{a+b}\int_{-\infty}^{0} e^{b'z} e^{sz} dz + \frac{ba'}{a+b}\int_{0}^{\infty} e^{-a'z} e^{sz} dz$$

$$= \frac{ab'}{a+b}\int_{0}^{\infty} e^{-(b'+s)z} dz + \frac{ba'}{a+b}\int_{0}^{\infty} e^{-(a'-s)z} dz$$

$$\Rightarrow M_z(s) = \frac{a}{a+b}\frac{b'}{(b'+s)} + \frac{b}{a+b}\frac{a'}{(a'-s)}$$

1st moment.

$$\frac{\partial M_z(s)}{\partial s} = -\frac{a}{a+b}\frac{b'}{(b'+s)^2} + \frac{b}{a+b}\frac{a'}{(a'-s)^2} \Rightarrow \frac{\partial M_z(0)}{\partial s} = \frac{b}{a'(a+b)} - \frac{a}{b'(a+b)}$$

$$\Rightarrow E(z) = \frac{bb' - aa'}{a'b'(a+b)} = \frac{(1-m)b' - ma'}{a'b'}$$

2nd moment.

$$\frac{\partial^2 M_z(s)}{\partial s^2} = \frac{a}{a+b}\frac{2b'}{(b'+s)^3} + \frac{b}{a+b}\frac{2a'}{(a'-s)^3} \Rightarrow \frac{\partial^2 M_z(0)}{\partial s^2} = \frac{2a}{b'^2(a+b)} + \frac{2b}{a'^2(a+b)}$$

$$\Rightarrow E(z^2) = 2\frac{aa'^2 + bb'^2}{a'^2 b'^2 (a+b)} = 2\frac{ma'^2 + (1-m)b'^2}{a'^2 b'^2}$$

3d moment.

$$\frac{\partial^3 M_z(s)}{\partial s^3} = -\frac{a}{a+b}\frac{6b'}{(b'+s)^4} + \frac{b}{a+b}\frac{6a'}{(a'-s)^4} \Rightarrow \frac{\partial^3 M_z(0)}{\partial s^3} = -\frac{6a}{b'^3(a+b)} + \frac{6b}{a'^3(a+b)}$$

$$\Rightarrow E(z^3) = 6\frac{-aa'^3+bb'^3}{a'^3b'^3(a+b)} = 6\frac{(1-m)b'^3-ma'^3}{a'^3b'^3}$$

4th moment.

$$\frac{\partial^4 M_z(s)}{\partial s^4} = \frac{a}{a+b}\frac{24b'}{(b'+s)^5} + \frac{b}{a+b}\frac{24a'}{(a'-s)^5} \Rightarrow \frac{\partial^4 M_z(0)}{\partial s^4} = \frac{24a}{b'^4(a+b)} + \frac{24b}{a'^4(a+b)}$$

$$\Rightarrow E(z^4) = 24\frac{aa'^4+bb'^4}{a'^4b'^4(a+b)}$$

C.2 The moment generating function of $\varepsilon = v+z$.

Since the zero-mean normal random variable ε is independent from z we have that $M_\varepsilon(s) = M_z(s)M_v(s)$. The first four derivatives and moments are

1st moment.

$$\frac{\partial M_\varepsilon(s)}{\partial s} = \frac{\partial M_z(s)}{\partial s}M_v(s) + M_z(s)\frac{\partial M_v(s)}{\partial s} \Rightarrow \frac{\partial M_\varepsilon(0)}{\partial s} = \frac{\partial M_z(0)}{\partial s}$$

$$\Rightarrow E(\varepsilon) = E(z) = \frac{bb'-aa'}{a'b'(a+b)} = \frac{1-m}{a'} - \frac{m}{b'}$$

2nd moment.

$$\frac{\partial^2 M_\varepsilon(s)}{\partial s^2} = \frac{\partial^2 M_z(s)}{\partial s^2}M_v(s) + 2\frac{\partial M_z(s)}{\partial s}\frac{\partial M_v(s)}{\partial s} + M_z(s)\frac{\partial^2 M_v(s)}{\partial s^2}$$

$$\Rightarrow \frac{\partial M_\varepsilon(0)}{\partial s} = \frac{\partial^2 M_z(0)}{\partial s^2} + \frac{\partial^2 M_v(s)}{\partial s^2}$$

$$\Rightarrow E\left(\varepsilon^2\right)=E\left(z^2\right)+E\left(v^2\right)=2\frac{aa'^2+bb'^2}{a'^2b'^2\left(a+b\right)}+\sigma_v^2=2\frac{ma'^2+\left(1-m\right)b'^2}{a'^2b'^2}+\sigma_v^2$$

3d moment.

$$\frac{\partial^3 M_\varepsilon\left(s\right)}{\partial s^3}=\frac{\partial^3 M_z\left(s\right)}{\partial s^3}M_v\left(s\right)+\frac{\partial^2 M_z\left(s\right)}{\partial s^2}\frac{\partial\ M_v\left(s\right)}{\partial s}$$

$$+2\frac{\partial^2 M_z\left(s\right)}{\partial s^2}\frac{\partial M_v\left(s\right)}{\partial s}+2\frac{\partial M_z\left(s\right)}{\partial s}\frac{\partial^2 M_v\left(s\right)}{\partial s^2}$$

$$+\frac{\partial M_z\left(s\right)}{\partial s}\frac{\partial^2 M_v\left(s\right)}{\partial s^2}+M_z\left(s\right)\frac{\partial^3 M_v\left(s\right)}{\partial s^3}$$

$$\frac{\partial^3 M_\varepsilon\left(0\right)}{\partial s^3}=\frac{\partial^3 M_z\left(0\right)}{\partial s^3}+3\frac{\partial M_z\left(0\right)}{\partial s}\frac{\partial^2 M_v\left(0\right)}{\partial s^2}$$

So

$$E\left(\varepsilon^3\right)=E\left(z^3\right)+3E\left(z\right)E\left(v^2\right)=6\frac{-aa'^3+bb'^3}{a'^3b'^3\left(a+b\right)}+3\frac{bb'-aa'}{a'b'\left(a+b\right)}\sigma_v^2$$

$$\Rightarrow E\left(\varepsilon^3\right)=E\left(z^3\right)+3E\left(z\right)E\left(v^2\right)=6\frac{-ma'^3+\left(1-m\right)b'^3}{a'^3b'^3}+3\frac{\left(1-m\right)b'-ma'}{a'b'}\sigma_v^2$$

C.3 Sign of skewness of ε.

The direction of skewness will depend on the sign of the third central moment. Using the relation between central and raw moments we have

$$E\left(\varepsilon-E(\varepsilon)\right)^3=E\left(\varepsilon^3\right)-3E\left(\varepsilon\right)E\left(\varepsilon^2\right)+2\left[E\left(\varepsilon\right)\right]^3$$

Using the previous results, in our case we have

$$E\left(\varepsilon - E(\varepsilon)\right)^3 = E\left(z^3\right) + 3E(z)E\left(v^2\right) - 3E(z)\left[E\left(z^2\right) + E\left(v^2\right)\right] + 2\left[E(z)\right]^3$$

$$= E\left(z^3\right) - 3E(z)E\left(z^2\right) + 2\left[E(z)\right]^3 = E\left(z - E(z)\right)^3$$

$$= E\left(z^3\right) - E(z)\left[3E\left(z^2\right) - 2\left[E(z)\right]^2\right] = E\left(z^3\right) - E(z)\left[E\left(z^2\right) + 2\operatorname{Var}(z)\right]$$

Since $\left[E\left(z^2\right) + 2\operatorname{Var}(z)\right] > 0$ we see that if $E\left(z^3\right) < 0,\ E(z) > 0$ we will obtain certainly negative skewness, while if $E\left(z^3\right) > 0,\ E(z) < 0$ we will certainly obtain positive skewness.

From the raw moments obtained earlier, we have

$$E(z) = \frac{(1-m)b' - ma'}{a'b'},\quad E\left(z^3\right) = 6\frac{(1-m)b'^3 - ma'^3}{a'^3 b'^3}$$ So we will have

$E\left(z^3\right) < 0,\ E(z) > 0$ if

$$(1-m)b'^3 - ma'^3 < 0 \Rightarrow \frac{b'}{a'} < \left(\frac{m}{1-m}\right)^{1/3} \text{ and } (1-m)b' - ma' < 0 \Rightarrow \frac{b'}{a'} > \frac{m}{1-m}$$

or $\dfrac{m}{1-m} < \dfrac{b'}{a'} < \left(\dfrac{m}{1-m}\right)^{1/3}$.

This will be feasible if $m < 1/2$, and then it will hold if b'/a' falls in the interval. Analogous results obtain for the case $E\left(z^3\right) > 0,\ E(z) < 0$.

D. Testing for the existence of dependence.

Freund's bivariate Exponential Extension nests the independence case, when $a = a'$ and $b = b'$. This implies that *if* the two one-sided components are independent we will have

$$m \equiv \frac{a}{a+b} = \frac{a'}{a'+b'} \Rightarrow (1-m)a' = mb'$$

The resulting equation is a *necessary* condition for independence, and it is formed by identifiable parameters. So we can formulate an asymptotically valid statistical test with null hypothesis $\mathrm{H}_0 : (1-m)a' = mb'$, which in practice will be based on the difference $T = (1-\hat{m})\hat{a}' - \hat{m}\hat{b}'$.

If the null hypothesis is rejected we have statistical support for dependence. But if the null hypothesis is not rejected, there is still the possibility that dependence exists of a restricted nature, and the test is inconclusive.

We note that the test shares the same philosophy and some common characteristics as the well known Hausman (1978) tests (or "vector of contrasts" tests), since it too essentially tests whether two estimators have the same probability limit. We will discuss similarities and differences as we develop the test.

In the context of maximum likelihood estimation of the model, let the full parameter vector be $\mathbf{q} = (\theta_1\ \theta_2)'$, $\theta_1 = (a', b', m)'$ and θ_2 containing all other parameters of the model. Let the log-likelihood of the model be $L(\mathbf{q}) \equiv \sum_{i=1}^{n} \ln \ell_i(\mathbf{q})$. We will denote with a zero-subscript the true parameter values.

The first-order condition for a maximum is $\dfrac{\partial L(\hat{\mathbf{q}})}{\partial \mathbf{q}} = \mathbf{0}$ and applying (per row) a mean-value expansion we have

$$\frac{\partial L(\hat{\mathbf{q}})}{\partial \mathbf{q}} = \mathbf{0} \Rightarrow \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} + \frac{\partial^2 L(\bar{\mathbf{q}})}{\partial \mathbf{q} \partial \mathbf{q}'}(\hat{\mathbf{q}} - \mathbf{q}_0) = \mathbf{0} \Rightarrow \hat{\mathbf{q}} = \mathbf{q}_0 - \left[\frac{1}{n}\frac{\partial^2 L(\bar{\mathbf{q}})}{\partial \mathbf{q} \partial \mathbf{q}'}\right]^{-1}\left(\frac{1}{n}\frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}}\right)$$

We partition the system of equations as follows

$$\begin{bmatrix} \hat{a}' \\ \hat{b}' \\ \hat{m} \\ \hat{\theta}_2 \end{bmatrix} = \begin{bmatrix} a' \\ b' \\ m \\ \theta_2 \end{bmatrix} - \begin{bmatrix} \mathbf{h}'_a \\ \mathbf{h}'_b \\ \mathbf{h}'_m \\ \mathrm{H}_\theta \end{bmatrix} \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right) \quad \text{which gives us}$$

$$\hat{a}' = a' - \mathbf{h}'_a \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right), \qquad \hat{b}' = b' - \mathbf{h}'_b \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right), \qquad \hat{m} = m - \mathbf{h}'_m \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right)$$

We turn to calculate the statistic.

$$T = (1-\hat{m})\hat{a}' - \hat{m}\hat{b}' = \left[1 - m + \mathbf{h}'_m \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right) \right] \left[a' - \mathbf{h}'_a \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right) \right]$$

$$- \left[m - \mathbf{h}'_m \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right) \right] \left[b' - \mathbf{h}'_b \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right) \right]$$

$$= (1-m)a' - (1-m)\mathbf{h}'_a \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right) + a'\mathbf{h}'_m \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right) - \mathbf{h}'_m \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right) \mathbf{h}'_a \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right)$$

$$-mb' + m\mathbf{h}'_b \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right) + b'\mathbf{h}'_m \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right) - \mathbf{h}'_m \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right) \mathbf{h}'_b \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right)$$

$$= \left[(1-m)a' - mb' \right] - (1-m)\mathbf{h}'_a \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right) + (a'+b')\mathbf{h}'_m \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right) + m\mathbf{h}'_b \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right)$$

$$- \mathbf{h}'_m \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right) \mathbf{h}'_a \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right) - \mathbf{h}'_m \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right) \mathbf{h}'_b \left(\frac{1}{n} \frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \right)$$

$$= \left[(1-m)a' - mb'\right] + (1-m)(\hat{a}' - a') - (a'+b')(\hat{m} - m) - m\left(\hat{b}' - b'\right)$$

$$- \mathbf{h}'_m\left(\frac{1}{n}\frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}}\right)\mathbf{h}'_a\left(\frac{1}{n}\frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}}\right) - \mathbf{h}'_m\left(\frac{1}{n}\frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}}\right)\mathbf{h}'_b\left(\frac{1}{n}\frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}}\right)$$

Multiply throughout by $\sqrt{n}$

$$\sqrt{n}T = \sqrt{n}\left[(1-m)a' - mb'\right] + (1-m)\left[\sqrt{n}(\hat{a}' - a')\right] - (a'+b')\left[\sqrt{n}(\hat{m} - m)\right] - m\left[\sqrt{n}\left(\hat{b}' - b'\right)\right]$$

$$- \mathbf{h}'_m\left(\frac{1}{n}\frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}}\right)\mathbf{h}'_a\left(\frac{1}{\sqrt{n}}\frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}}\right) - \mathbf{h}'_m\left(\frac{1}{n}\frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}}\right)\mathbf{h}'_b\left(\frac{1}{\sqrt{n}}\frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}}\right)$$

First, note that if the null hypothesis $\mathrm{H}_0 : (1-m)a' = mb'$ is not correct, the value of the statistic goes to infinity due to its first term, and so the test is consistent. On the other hand, due to this fact, power studies can only be of a local nature, as in Hausman (1978).

Moreover, with an ergodic stationary sample and under correct specification, we have

$$\frac{1}{n}\frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}} \xrightarrow{p} E\left[\ell_i(\mathbf{q}_0)\right] = 0$$

So the last two terms vanish asymptotically, since the components of the Hessian inverse converge by assumption, and $\frac{1}{\sqrt{n}}\frac{\partial L(\mathbf{q}_0)}{\partial \mathbf{q}}$ is a well-defined random variable at the limit. Therefore the statistic under the null hypothesis is

$$\boxed{[4.15]: \ \sqrt{n}T\Big|_{\mathrm{H}_0} \longrightarrow (1-m)\left[\sqrt{n}(\hat{a}' - a')\right] - (a'+b')\left[\sqrt{n}(\hat{m} - m)\right] - m\left[\sqrt{n}\left(\hat{b}' - b'\right)\right]}$$

Given the asymptotic properties of the maximum likelihood estimator, at the limit this is a linear combination of three dependent zero-mean normal random variables and so a normal random variable itself. Regarding the limiting variance of the statistic, it does not have a simple form as in the Hausman tests case, because the simplification there rested on an assumption that one of the two contrasted estimators is efficient while the other was not, under the null. No such properties exist in our case, so we cannot avoid the "messy calculations" to obtain the variance. But in practice, all necessary magnitudes will be provided by the variance-covariance matrix of the MLE estimator. Write temporarily for compactness

$$\sqrt{n}T\Big|_{\mathrm{H}_0} = A - M - B \Rightarrow$$

$$\mathrm{Var}\left(\sqrt{n}T\right) = \mathrm{Var}(A) + \mathrm{Var}(M) + \mathrm{Var}(B) - 2\,\mathrm{Cov}(A,M) - 2\,\mathrm{Cov}(A-M,B)$$

$$= \mathrm{Var}(A) + \mathrm{Var}(M) + \mathrm{Var}(B) - 2\,\mathrm{Cov}(A,M) - 2\left[E(AB-MB) - (E(A)-E(M))E(B)\right]$$

$$\Rightarrow \mathrm{Var}\left(\sqrt{n}T\right) = \mathrm{Var}(A) + \mathrm{Var}(M) + \mathrm{Var}(B) - 2\,\mathrm{Cov}(A,M) - 2\,\mathrm{Cov}(A,B) + 2\,\mathrm{Cov}(M,B)$$

We have

$$\mathrm{Var}(A) = n(1-m)^2\,\mathrm{Var}(\hat{a}'), \quad \mathrm{Var}(M) = n(a'+b')^2\,\mathrm{Var}(\hat{m}), \quad \mathrm{Var}(B) = nm^2\,\mathrm{Var}\left(\hat{b}'\right)$$

$$\mathrm{Cov}(A,M) = n(1-m)(a'+b')\mathrm{Cov}(\hat{a}',\hat{m}), \qquad \mathrm{Cov}(A,B) = n(1-m)m\,\mathrm{Cov}\left(\hat{a}',\hat{b}'\right)$$

$$\mathrm{Cov}(M,B) = n(a'+b')m\,\mathrm{Cov}\left(\hat{m},\hat{b}'\right)$$

So

$$\boxed{\begin{aligned}[4.16]:\ \mathrm{Var}(T) = {} & (1-m)^2\,\mathrm{Var}(\hat{a}') + (a'+b')^2\,\mathrm{Var}(\hat{m}) + m^2\,\mathrm{Var}\left(\hat{b}'\right) \\ & - 2(1-m)(a'+b')\mathrm{Cov}(\hat{a}',\hat{m}) - 2(1-m)m\,\mathrm{Cov}\left(\hat{a}',\hat{b}'\right) + 2(a'+b')m\,\mathrm{Cov}\left(\hat{m},\hat{b}'\right)\end{aligned}}$$

All the variance and covariance terms are consistently estimated by the VCV matrix of the MLE, and for the scaling constants we can use their consistent estimators. It follows that

$$\hat{q} = \frac{T^2}{\hat{\mathrm{V}}\mathrm{ar}(T)} = \frac{\left((1-\hat{m})\hat{a}' - \hat{m}\hat{b}'\right)^2}{\hat{\mathrm{V}}\mathrm{ar}(T)} \xrightarrow[\mathrm{H}_0]{d} \chi_1^2 .$$

D.1. Simulating the distribution of the statistic under the null hypothesis.

We performed the following small Monte Carlo study:

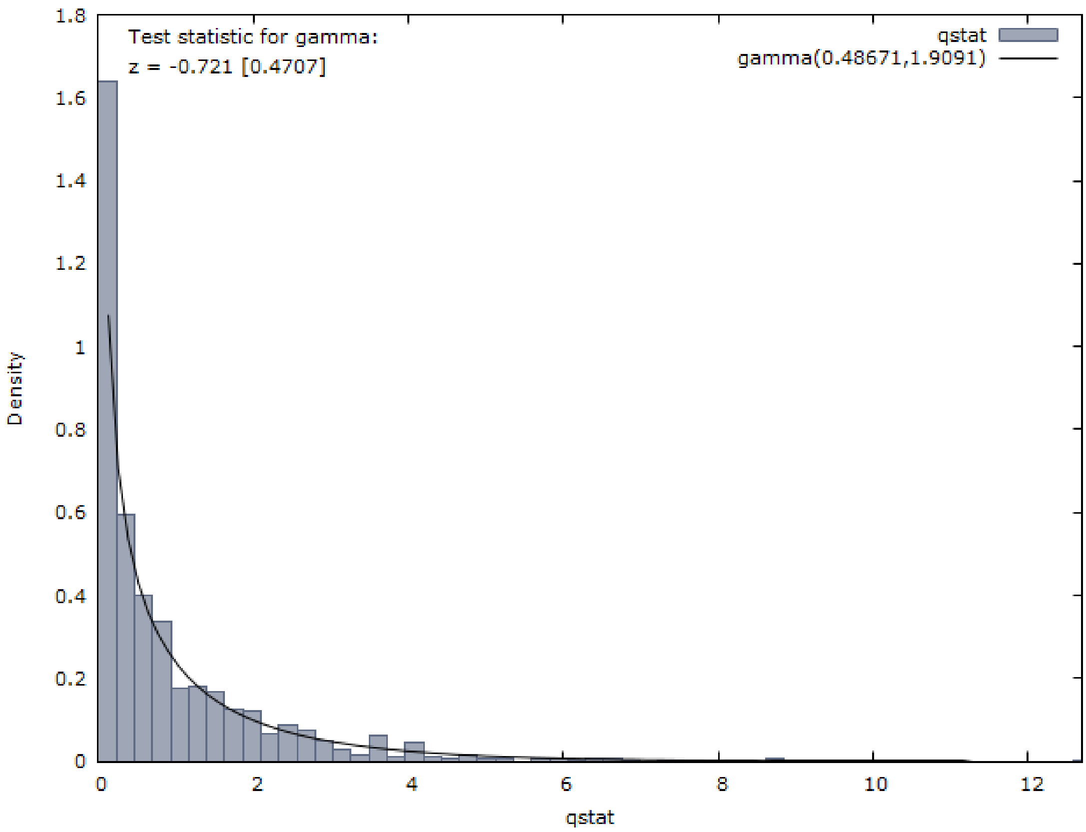
We considered 1000 runs, each on a sample of size *n*=5,000.

In each case we generated the following independent random variables

$$v \sim N\left(0,\sigma_v^2\right),\ \sigma_v = 0.3, \quad w \sim \mathrm{Exp}\left(\sigma_w\right),\ \sigma_w = 1/a',\ a' = 2 \Rightarrow \sigma_w = 0.5,$$

$$u \sim \mathrm{Exp}\left(\sigma_u\right),\ \sigma_w = 1/b',\ b' = 2.5 \Rightarrow \sigma_u = 0.4, \quad \Rightarrow m = \frac{a'}{a'+b'} = 0.444$$

We then estimated by maximum likelihood the parameters of $\varepsilon = v + w - u$ using the 2TSF Correlated exponential density, and used the estimates to calculate the statistic $\hat{q}$. Its empirical frequency distribution was



We see that the distribution is very close to a chi-square distribution with one degree of freedom (which is a Gamma distribution with shape parameter 0.5 and scale parameter 2).

E. Conditional expected values.

E.1. The joint density $f_{\varepsilon,w}(\varepsilon,w)$.

To obtain conditional expected values we need the conditional density, $f_{w|\varepsilon}=\dfrac{f_{\varepsilon,w}(\varepsilon,w)}{f_{\varepsilon}(\varepsilon)}$.

We have already derived the denominator, so what we need is the joint density.

We have $\varepsilon=v+w-u$. Define $q\equiv v-u=\varepsilon-w,\;\; v=q+u$. Then

$$f_{q,w}(q,w)=\int_0^\infty f_{v,w,u}(q+u,w,u)\,du=\int_0^\infty f_v(q+u)\,f_{w,u}(w,u)\,du\,.$$

Then, since the determinant of the Jacobian of the transformation $(q,w)\to(\varepsilon-w,w)$ is unity, we get

$$f_{\varepsilon,w}(\varepsilon,w)=f_{q,w}(\varepsilon-w,w)$$

We have

$$f_v(q+u)=\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma_v^2}(q+u)^2\right\}=\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\frac{q^2}{2\sigma_v^2}\right\}\exp\left\{-\frac{u^2}{2\sigma_v^2}-\frac{qu}{\sigma_v^2}\right\}$$

$$=A(q)\exp\left\{-\frac{u^2}{2\sigma_v^2}-\frac{qu}{\sigma_v^2}\right\},\quad A(q)=\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\frac{q^2}{2\sigma_v^2}\right\}$$

We have also to take into account the fact that the joint density $f_{w,u}(w,u)$ has branches. This leads to

$$\int_0^\infty f_v(q+u)\,f_{w,u}(w,u)\,du=\int_0^w f_v(q+u)\,f_{w>u}(w,u)\,du+\int_w^\infty f_v(q+u)\,f_{w<u}(w,u)\,du$$

We consider each integral separately

$$I_1=\int_0^w f_v\left(q+u\right)f_{w>u}\left(w,u\right)du=A\left(q\right)\int_0^w \exp\left\{-\frac{u^2}{2\sigma_v^2}-\frac{qu}{\sigma_v^2}\right\}a'b\exp\left\{-a'w-\left(a+b-a'\right)u\right\}du$$

$$=A\left(q\right)a'b\exp\left\{-a'w\right\}\int_0^w \exp\left\{-\frac{u^2}{2\sigma_v^2}-\frac{qu}{\sigma_v^2}\right\}\exp\left\{-\left(a+b-a'\right)u\right\}du$$

$$=A\left(q\right)a'b\exp\left\{-a'w\right\}\int_0^w \exp\left\{-\frac{u^2}{2\sigma_v^2}-\left(\frac{q}{\sigma_v^2}+\left(a+b-a'\right)\right)u\right\}du$$

$$=A\left(q\right)a'b\exp\left\{-a'w\right\}\times\left[\int_0^\infty \exp\left\{-\frac{u^2}{2\sigma_v^2}-\left(\frac{q}{\sigma_v^2}+\left(a+b-a'\right)\right)u\right\}du\right.$$
$$\left.-\int_w^\infty \exp\left\{-\frac{u^2}{2\sigma_v^2}-\left(\frac{q}{\sigma_v^2}+\left(a+b-a'\right)\right)u\right\}du\right]$$

$$=A\left(q\right)a'b\exp\left\{-a'w\right\}\sqrt{\pi}\frac{\sigma_v}{\sqrt{2}}\exp\left\{\frac{\sigma_v^2}{2}\left(\frac{q}{\sigma_v^2}+\left(a+b-a'\right)\right)^2\right\}$$
$$\times\left[1-\mathrm{erf}\left(\left[\frac{q}{\sigma_v^2}+\left(a+b-a'\right)\right]\frac{\sigma_v}{\sqrt{2}}\right)-1+\mathrm{erf}\left(\left[\frac{q}{\sigma_v^2}+\left(a+b-a'\right)\right]\frac{\sigma_v}{\sqrt{2}}+\frac{w}{\sigma_v\sqrt{2}}\right)\right]$$

$$=A\left(q\right)a'b\exp\left\{-a'w\right\}\sqrt{\pi}\frac{\sigma_v}{\sqrt{2}}\exp\left\{\frac{\sigma_v^2}{2}\left(\frac{q}{\sigma_v^2}+\left(a+b-a'\right)\right)^2\right\}$$
$$\times\left[2\Phi\left(\left[\frac{q}{\sigma_v}+\left(a+b-a'\right)\sigma_v\right]+\frac{w}{\sigma_v}\right)-2\Phi\left(\frac{q}{\sigma_v}+\left(a+b-a'\right)\sigma_v\right)\right]$$

$$=a'b\exp\left\{-a'w\right\}\exp\left\{-\frac{q^2}{2\sigma_v^2}\right\}\exp\left\{\frac{1}{2}\left(\frac{q}{\sigma_v}+\left(a+b-a'\right)\sigma_v\right)^2\right\}$$
$$\times\left[\Phi\left(\left[\frac{q}{\sigma_v}+\left(a+b-a'\right)\sigma_v\right]+\frac{w}{\sigma_v}\right)-\Phi\left(\frac{q}{\sigma_v}+\left(a+b-a'\right)\sigma_v\right)\right]$$

$$\Rightarrow I_1 = a'b\exp\left\{\frac{1}{2}(a+b-a')^2\sigma_v^2\right\}\exp\{-a'w\}\exp\{(a+b-a')q\}$$

$$\times\left[\Phi\left(\left[\frac{q}{\sigma_v}+(a+b-a')\sigma_v\right]+\frac{w}{\sigma_v}\right)-\Phi\left(\frac{q}{\sigma_v}+(a+b-a')\sigma_v\right)\right]$$

For the second integral of $\int_0^{\infty} f_v(q+u)\, f_{w,u}(w,u)\,du$ we have

$$I_2=\int_w^{\infty} f_v(q+u)\, f_{w<u}(w,u)\,du = A(q)\int_w^{\infty}\exp\left\{-\frac{u^2}{2\sigma_v^2}-\frac{qu}{\sigma_v^2}\right\}ab'\exp\{-b'u-(a+b-b')w\}\,du$$

$$=A(q)\,ab'\exp\{-(a+b-b')w\}\int_w^{\infty}\exp\left\{-\frac{u^2}{2\sigma_v^2}-\frac{qu}{\sigma_v^2}\right\}ab'\exp\{-b'u\}\,du$$

$$=A(q)\,ab'\exp\{-(a+b-b')w\}\int_w^{\infty}\exp\left\{-\frac{u^2}{2\sigma_v^2}-\left(\frac{q}{\sigma_v^2}+b'\right)u\right\}du$$

$$=A(q)\,ab'\exp\{-(a+b-b')w\}\int_w^{\infty}\exp\left\{-\frac{u^2}{2\sigma_v^2}-\left(\frac{q}{\sigma_v^2}+b'\right)u\right\}du$$

$$=A(q)\,ab'\exp\{-(a+b-b')w\}\sqrt{\pi}\,\frac{\sigma_v}{\sqrt{2}}\exp\left\{\frac{\sigma_v^2}{2}\left(\frac{q}{\sigma_v^2}+b'\right)^2\right\}$$

$$\times\left[1-\operatorname{erf}\left(\left(\frac{q}{\sigma_v^2}+b'\right)\frac{\sigma_v}{\sqrt{2}}+\frac{w}{\sigma_v\sqrt{2}}\right)\right]$$

$$=\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\frac{q^2}{2\sigma_v^2}\right\}ab'\exp\{-(a+b-b')w\}\sqrt{\pi}\,\frac{\sigma_v}{\sqrt{2}}\exp\left\{\frac{1}{2}\left(\frac{q}{\sigma_v}+b'\sigma_v\right)^2\right\}$$

$$\times\left[2-2\Phi\left(\left(\frac{q}{\sigma_v}+b'\sigma_v\right)+\frac{w}{\sigma_v}\right)\right]$$

$$= ab' \exp\left\{-\left(a+b-b'\right)w\right\} \exp\left\{-\frac{q^2}{2\sigma_v^2}\right\} \exp\left\{\frac{1}{2}\left(\frac{q}{\sigma_v}+b'\sigma_v\right)^2\right\}$$
$$\times\left[\Phi\left(-\left(\frac{q}{\sigma_v}+b'\sigma_v\right)-\frac{w}{\sigma_v}\right)\right]$$

$$\Rightarrow I_2 = ab' \exp\left\{\frac{1}{2}\left(b'\sigma_v\right)^2\right\} \exp\left\{-\left(a+b-b'\right)w\right\} \exp\left\{b'q\right\} \cdot \left[\Phi\left(-\left(\frac{q}{\sigma_v}+b'\sigma_v\right)-\frac{w}{\sigma_v}\right)\right]$$

So

$$f_{q,w}\left(q,w\right) = a'b \exp\left\{\frac{1}{2}\left(a+b-a'\right)^2 \sigma_v^2\right\} \exp\left\{-a'w\right\} \exp\left\{\left(a+b-a'\right)q\right\}$$
$$\times\left[\Phi\left(\left[\frac{q}{\sigma_v}+\left(a+b-a'\right)\sigma_v\right]+\frac{w}{\sigma_v}\right)-\Phi\left(\left[\frac{q}{\sigma_v}+\left(a+b-a'\right)\sigma_v\right]\right)\right]$$

$$+\ ab' \exp\left\{\frac{1}{2}\left(b'\sigma_v\right)^2\right\} \exp\left\{-\left(a+b-b'\right)w\right\} \exp\left\{b'q\right\} \cdot \left[\Phi\left(-\left(\frac{q}{\sigma_v}+b'\sigma_v\right)-\frac{w}{\sigma_v}\right)\right]$$

Inserting $q = \varepsilon - w$ we have

$$f_{\varepsilon,w}\left(\varepsilon,w\right) = a'b \exp\left\{\frac{1}{2}\left(a+b-a'\right)^2 \sigma_v^2\right\} \exp\left\{-a'w\right\} \exp\left\{\left(a+b-a'\right)\left(\varepsilon-w\right)\right\}$$
$$\times\left[\Phi\left(\left[\frac{\varepsilon-w}{\sigma_v}+\left(a+b-a'\right)\sigma_v\right]+\frac{w}{\sigma_v}\right)-\Phi\left(\left[\frac{\varepsilon-w}{\sigma_v}+\left(a+b-a'\right)\sigma_v\right]\right)\right]$$

$$+\ ab' \exp\left\{\frac{1}{2}\left(b'\sigma_v\right)^2\right\} \exp\left\{-\left(a+b-b'\right)w\right\} \exp\left\{b'\left(\varepsilon-w\right)\right\} \cdot \left[\Phi\left(-\left(\frac{\varepsilon-w}{\sigma_v}+b'\sigma_v\right)-\frac{w}{\sigma_v}\right)\right]$$

and simplifying,

$$f_{\varepsilon,w}(\varepsilon,w) = a'b\exp\left\{\frac{1}{2}(a+b-a')^2\sigma_v^2\right\}\exp\{(a+b-a')\varepsilon\}\exp\{-(a+b)w\}$$
$$\times\left[\Phi\left(\frac{\varepsilon}{\sigma_v}+(a+b-a')\sigma_v\right)-\Phi\left(\frac{\varepsilon}{\sigma_v}+(a+b-a')\sigma_v-\frac{w}{\sigma_v}\right)\right]$$

$$+\,ab'\exp\left\{\frac{1}{2}(b'\sigma_v)^2\right\}\exp\{b'\varepsilon\}\exp\{-(a+b)w\}\Phi\left(-\frac{\varepsilon}{\sigma_v}-b'\sigma_v\right)$$

We define now one more shortcut expression

Set $$\omega_1 = \frac{\varepsilon}{\sigma_v}+(a+b-a')\sigma_v$$

and note that

$$\frac{1}{2}\omega_1^2 = \frac{\varepsilon^2}{2\sigma_v^2}+(a+b-a')\sigma_v+\frac{1}{2}(a+b-a')^2\sigma_v^2 \Rightarrow \frac{1}{2}\omega_1^2-\frac{\varepsilon^2}{2\sigma_v^2} = (a+b-a')\sigma_v+\frac{1}{2}(a+b-a')^2\sigma_v^2$$

Using also $$\omega_2 = \frac{\varepsilon}{\sigma_v}+b'\sigma_v$$

and remembering that $$\frac{1}{2}\omega_2^2 = \frac{\varepsilon^2}{2\sigma_v^2}+b'\varepsilon+\frac{1}{2}(b'\sigma_v)^2 \Rightarrow \frac{1}{2}\omega_2^2-\frac{\varepsilon^2}{2\sigma_v^2} = b'\varepsilon+\frac{1}{2}(b'\sigma_v)^2$$

we can re-write the joint density as

$$f_{\varepsilon,w}(\varepsilon,w) = a'b\exp\left\{\tfrac{1}{2}\omega_1^2-\tfrac{1}{2}(\varepsilon/\sigma_v)^2\right\}\exp\{-(a+b)w\}\cdot\left[\Phi(\omega_1)-\Phi\left(\omega_1-\frac{w}{\sigma_v}\right)\right]$$

$$+\,ab'\exp\left\{\tfrac{1}{2}\omega_2^2-\tfrac{1}{2}(\varepsilon/\sigma_v)^2\right\}\exp\{-(a+b)w\}\Phi(-\omega_2)$$

$$\Rightarrow f_{\varepsilon,w}(\varepsilon,w)=\exp\{-(a+b)w\}\begin{bmatrix} a'b\exp\left\{\frac{1}{2}\omega_1^2-\frac{\varepsilon^2}{2\sigma_v^2}\right\}\Phi(\omega_1) \\ +ab'\exp\left\{\frac{1}{2}\omega_2^2-\frac{\varepsilon^2}{2\sigma_v^2}\right\}\Phi(-\omega_2)\end{bmatrix}$$

$$-a'b\exp\left\{\frac{1}{2}\omega_1^2-\frac{\varepsilon^2}{2\sigma_v^2}\right\}\exp\{-(a+b)w\}\Phi\left(\omega_1-\frac{w}{\sigma_v}\right)$$

$$\boxed{\begin{array}{l} f_{\varepsilon,w}(\varepsilon,w)=\exp\{-(a+b)w\}\left[\Omega_1\Phi(\omega_1)+\Omega_2\Phi(-\omega_2)\right]-\Omega_1\cdot\exp\{-(a+b)w\}\Phi\left(\omega_1-\frac{w}{\sigma_v}\right) \\ \text{with} \\ \Omega_1\equiv a'b\exp\left\{\frac{1}{2}\omega_1^2-\frac{\varepsilon^2}{2\sigma_v^2}\right\},\quad \Omega_2\equiv ab'\exp\left\{\frac{1}{2}\omega_2^2-\frac{\varepsilon^2}{2\sigma_v^2}\right\} \\ \omega_1=\frac{\varepsilon}{\sigma_v}+(a+b-a')\sigma_v,\qquad \omega_2=\frac{\varepsilon}{\sigma_v}+b'\sigma_v \end{array}}$$

E.2. The conditional expected value $E(w|\varepsilon)$.

$$E(w|\varepsilon)=\frac{1}{f_\varepsilon(\varepsilon)}\int_0^\infty wf_{\varepsilon,w}(\varepsilon,w)dw=\frac{1}{f_\varepsilon(\varepsilon)}\int_0^\infty w\exp\{-(a+b)w\}\cdot\left[\Omega_1\cdot\Phi(\omega_1)+\Omega_2\cdot\Phi(-\omega_2)\right]dw$$

$$-\frac{1}{f_\varepsilon(\varepsilon)}\int_0^\infty w\cdot\Omega_1\cdot\exp\{-(a+b)w\}\Phi\left(\omega_1-\frac{w}{\sigma_v}\right)dw$$

$$=\frac{1}{f_\varepsilon(\varepsilon)}\frac{\Omega_1\cdot\Phi(\omega_1)+\Omega_2\cdot\Phi(-\omega_2)}{(a+b)^2}-\frac{\Omega_1}{f_\varepsilon(\varepsilon)}\int_0^\infty w\cdot\exp\{-(a+b)w\}\Phi\left(\omega_1-\frac{w}{\sigma_v}\right)dw$$

For the remaining integral, we apply the transformation

$$y=\omega_1-\frac{w}{\sigma_v}\Rightarrow\begin{cases} w=\sigma_v\omega_1-\sigma_v y \\ dw=-\sigma_v dy \\ w=0\Rightarrow y=\omega_1 \\ w=\infty\Rightarrow y=-\infty \end{cases}\qquad \text{So,}$$

$$\int_0^{\infty} w\cdot\exp\{-(a+b)w\}\Phi\left(\omega_1-\frac{w}{\sigma_v}\right)dw=-\int_{\omega_1}^{-\infty}(\sigma_v\omega_1-\sigma_v y)\cdot\exp\{-(a+b)(\sigma_v\omega_1-\sigma_v y)\}\Phi(y)\sigma_v dy$$

$$=\sigma_v\int_{-\infty}^{\omega_1}(\sigma_v\omega_1-\sigma_v y)\cdot\exp\{-(a+b)(\sigma_v\omega_1-\sigma_v y)\}\Phi(y)dy$$

$$=\sigma_v^2\omega_1\exp\{-(a+b)\sigma_v\omega_1\}\int_{-\infty}^{\omega_1}\exp\{(a+b)\sigma_v y\}\Phi(y)dy$$

$$-\sigma_v^2\exp\{-(a+b)\sigma_v\omega_1\}\int_{-\infty}^{\omega_1}y\exp\{(a+b)\sigma_v y\}\Phi(y)dy$$

Applying expressions 101,000 and 101,001 from Owen (1980), p. 409, for the 1st and 2nd integral respectively we have, for the 1st integral

$$\int_{-\infty}^{\omega_1}\exp\{(a+b)\sigma_v y\}\Phi(y)dy=\frac{1}{(a+b)\sigma_v}\exp\{(a+b)\sigma_v y\}\Phi(y)\Big|_{-\infty}^{\omega_1}$$
$$-\frac{\exp\left\{\frac{1}{2}(a+b)^2\sigma_v^2\right\}}{(a+b)\sigma_v}\Phi\left(y-(a+b)\sigma_v\right)\Big|_{-\infty}^{\omega_1}$$

$$=\frac{\exp\{(a+b)\sigma_v\omega_1\}\Phi(\omega_1)\ -\ \exp\left\{\frac{1}{2}(a+b)^2\sigma_v^2\right\}\Phi\left(\omega_1-(a+b)\sigma_v\right)}{(a+b)\sigma_v}$$

and for the 2nd integral

$$\int_{-\infty}^{\omega_1} y\exp\left\{(a+b)\sigma_v y\right\}\Phi(y)\,dy = \frac{(a+b)\sigma_v y-1}{(a+b)^2\sigma_v^2}\Phi(y)\exp\left\{(a+b)\sigma_v y\right\}\Big|_{-\infty}^{\omega_1}$$

$$-\ \frac{(a+b)^2\sigma_v^2-1}{(a+b)^2\sigma_v^2}\exp\left\{\tfrac{1}{2}(a+b)^2\sigma_v^2\right\}\Phi\left(y-(a+b)\sigma_v\right)\Big|_{-\infty}^{\omega_1}$$

$$+\ \frac{1}{(a+b)\sigma_v}\exp\left\{\tfrac{1}{2}(a+b)^2\sigma_v^2\right\}\phi\left(y-(a+b)\sigma_v\right)\Big|_{-\infty}^{\omega_1}$$

$$=\frac{(a+b)\sigma_v\omega_1-1}{(a+b)^2\sigma_v^2}\Phi(\omega_1)\exp\left\{(a+b)\sigma_v\omega_1\right\}\ -\ \frac{(a+b)^2\sigma_v^2-1}{(a+b)^2\sigma_v^2}\exp\left\{\tfrac{1}{2}(a+b)^2\sigma_v^2\right\}\Phi\left(\omega_1-(a+b)\sigma_v\right)$$

$$+\ \frac{1}{(a+b)\sigma_v}\exp\left\{\tfrac{1}{2}(a+b)^2\sigma_v^2\right\}\phi\left(\omega_1-(a+b)\sigma_v\right)$$

Then,

$$\int_0^{\infty} w\cdot\exp\left\{-(a+b)w\right\}\Phi\left(\omega_1-\frac{w}{\sigma_v}\right)dw=$$

$$=\sigma_v^2\omega_1\exp\left\{-(a+b)\sigma_v\omega_1\right\}\frac{\exp\left\{(a+b)\sigma_v\omega_1\right\}\Phi(\omega_1)\ -\ \exp\left\{\tfrac{1}{2}(a+b)^2\sigma_v^2\right\}\Phi\left(\omega_1-(a+b)\sigma_v\right)}{(a+b)\sigma_v}$$

$$-\sigma_v^2\exp\left\{-(a+b)\sigma_v\omega_1\right\}\frac{(a+b)\sigma_v\omega_1-1}{(a+b)^2\sigma_v^2}\Phi(\omega_1)\exp\left\{(a+b)\sigma_v\omega_1\right\}$$

$$+\ \sigma_v^2\exp\left\{-(a+b)\sigma_v\omega_1\right\}\frac{(a+b)^2\sigma_v^2-1}{(a+b)^2\sigma_v^2}\exp\left\{\tfrac{1}{2}(a+b)^2\sigma_v^2\right\}\Phi\left(\omega_1-(a+b)\sigma_v\right)$$

$$-\sigma_v^2\exp\left\{-(a+b)\sigma_v\omega_1\right\}\frac{1}{(a+b)\sigma_v}\exp\left\{\tfrac{1}{2}(a+b)^2\sigma_v^2\right\}\phi\left(\omega_1-(a+b)\sigma_v\right)$$

Applying the shortcut expressions, note that

$$\frac{1}{2}\omega_3^2 = \frac{1}{2}\left(\omega_1 - (a+b)\sigma_v\right)^2 = \frac{1}{2}\omega_1^2 - (a+b)\sigma_v\omega_1 + \frac{1}{2}(a+b)^2\sigma_v^2$$

$$\Rightarrow \frac{1}{2}\left(\omega_3^2 - \omega_1^2\right) = -(a+b)\sigma_v\omega_1 + \frac{1}{2}(a+b)^2\sigma_v^2$$

Simplifying and substituting,

$$\int_0^\infty w \cdot \exp\left\{-(a+b)w\right\}\Phi\left(\omega_1 - \frac{w}{\sigma_v}\right)dw =$$

$$= \frac{\sigma_v\omega_1\Phi(\omega_1) - \sigma_v\omega_1\exp\left\{\tfrac{1}{2}\left(\omega_3^2-\omega_1^2\right)\right\}\Phi(\omega_3)}{(a+b)} - \frac{(a+b)\sigma_v\omega_1 - 1}{(a+b)^2}\Phi(\omega_1)$$

$$+ \frac{(a+b)^2\sigma_v^2 - 1}{(a+b)^2}\exp\left\{\tfrac{1}{2}\left(\omega_3^2-\omega_1^2\right)\right\}\Phi(\omega_3) - \frac{\sigma_v}{(a+b)}\exp\left\{\tfrac{1}{2}\left(\omega_3^2-\omega_1^2\right)\right\}\phi(\omega_3)$$

$$= \frac{(a+b)\sigma_v\omega_1 - (a+b)\sigma_v\omega_1 + 1}{(a+b)^2}\Phi(\omega_1) + \left[\frac{(a+b)^2\sigma_v^2 - 1}{(a+b)^2} - \frac{\sigma_v\omega_1}{(a+b)}\right]\exp\left\{\tfrac{1}{2}\left(\omega_3^2-\omega_1^2\right)\right\}\Phi(\omega_3)$$

$$- \frac{\sigma_v}{(a+b)}\exp\left\{\tfrac{1}{2}\left(\omega_3^2-\omega_1^2\right)\right\}\phi(\omega_3)$$

$$= \frac{\Phi(\omega_1)}{(a+b)^2} + \left[\frac{(a+b)^2\sigma_v^2 - 1 - (a+b)\sigma_v\omega_1}{(a+b)^2}\right]\exp\left\{\tfrac{1}{2}\left(\omega_3^2-\omega_1^2\right)\right\}\Phi(\omega_3)$$

$$- \frac{\sigma_v}{(a+b)}\exp\left\{\tfrac{1}{2}\left(\omega_3^2-\omega_1^2\right)\right\}\phi(\omega_3)$$

$$= \frac{\Phi(\omega_1)}{(a+b)^2} \quad + \frac{(a+b)\sigma_v\left[(a+b)\sigma_v - \omega_1\right]-1}{(a+b)^2}\exp\left\{\tfrac{1}{2}\left(\omega_3^2-\omega_1^2\right)\right\}\Phi(\omega_3)$$

$$-\frac{\sigma_v}{(a+b)}\exp\left\{\tfrac{1}{2}\left(\omega_3^2-\omega_1^2\right)\right\}\phi(\omega_3)$$

So

$$E(w|\varepsilon) = \frac{1}{f_\varepsilon(\varepsilon)}\frac{\Omega_1\cdot\Phi(\omega_1)+\Omega_2\cdot\Phi(-\omega_2)}{(a+b)^2} - \frac{\Omega_1}{f_\varepsilon(\varepsilon)}\frac{\Phi(\omega_1)}{(a+b)^2}$$

$$-\frac{\Omega_1}{f_\varepsilon(\varepsilon)}\frac{(a+b)\sigma_v\left[(a+b)\sigma_v-\omega_1\right]-1}{(a+b)^2}\exp\left\{\tfrac{1}{2}\left(\omega_3^2-\omega_1^2\right)\right\}\Phi(\omega_3)$$

$$+\frac{\Omega_1}{f_\varepsilon(\varepsilon)}\frac{\sigma_v}{(a+b)}\exp\left\{\tfrac{1}{2}\left(\omega_3^2-\omega_1^2\right)\right\}\phi(\omega_3)$$

But

$$(a+b)\sigma_v-\omega_1 = (a+b)\sigma_v - \frac{\varepsilon}{\sigma_v}-(a+b-a')\sigma_v \;=\; -\frac{\varepsilon}{\sigma_v}+a'\sigma_v = -\omega_3$$

while $(a+b)\sigma_v = \omega_1 - \omega_3$

Cancelling out and compacting

$$E(w|\varepsilon) = \frac{1}{f_\varepsilon(\varepsilon)}\frac{\Omega_2\cdot\Phi(-\omega_2)}{(a+b)^2}$$

$$-\frac{\Omega_1}{f_\varepsilon(\varepsilon)}\exp\left\{\tfrac{1}{2}\left(\omega_3^2-\omega_1^2\right)\right\}\left[\frac{-(\omega_1-\omega_3)\omega_3-1}{(a+b)^2}\Phi(\omega_3)-\frac{\sigma_v}{(a+b)}\phi(\omega_3)\right]$$

$$E\left(w|\varepsilon\right)=\frac{1}{f_{\varepsilon}\left(\varepsilon\right)}\frac{\Omega_2\cdot\Phi\left(-\omega_2\right)}{\left(a+b\right)^2}$$

$$+\frac{\Omega_1}{f_{\varepsilon}\left(\varepsilon\right)}\exp\left\{\tfrac{1}{2}\left(\omega_3^2-\omega_1^2\right)\right\}\left[\frac{\omega_1\omega_3-\omega_3^2+1}{\left(a+b\right)^2}\Phi\left(\omega_3\right)+\frac{\sigma_v}{\left(a+b\right)}\phi\left(\omega_3\right)\right]$$

This can be simplified further. We have

$$\Omega_1\equiv a'b\exp\left\{\tfrac{1}{2}\omega_1^2-\tfrac{1}{2}\left(\varepsilon/\sigma_v\right)^2\right\},\quad \Omega_2\equiv ab'\exp\left\{\tfrac{1}{2}\omega_2^2-\tfrac{1}{2}\left(\varepsilon/\sigma_v\right)^2\right\}$$

$$\omega_1=\frac{\varepsilon}{\sigma_v}+\left(a+b-a'\right)\sigma_v,\quad \omega_2=\frac{\varepsilon}{\sigma_v}+b'\sigma_v,\quad \omega_3=\frac{\varepsilon}{\sigma_v}-a'\sigma_v$$

$$f_{\varepsilon}\left(\varepsilon\right)=\frac{\exp\left\{-\tfrac{1}{2}\left(\varepsilon/\sigma_v\right)^2\right\}}{a+b}\left[ab'\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\Phi\left(-\omega_2\right)+a'b\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\Phi\left(\omega_3\right)\right]$$

Using these,

$$E\left(w|\varepsilon\right)=\frac{ab'\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\Phi\left(-\omega_2\right)+a'b\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\left[\left(\omega_1\omega_3-\omega_3^2+1\right)\Phi\left(\omega_3\right)+\left(\omega_1-\omega_3\right)\phi\left(\omega_3\right)\right]}{\left[ab'\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\Phi\left(-\omega_2\right)+a'b\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\Phi\left(\omega_3\right)\right]\left(a+b\right)}$$

$$=\frac{1}{\left(a+b\right)}+\frac{\left(\omega_1-\omega_3\right)a'b\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\left[\omega_3\Phi\left(\omega_3\right)+\phi\left(\omega_3\right)\right]}{\left[ab'\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\Phi\left(-\omega_2\right)+a'b\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\Phi\left(\omega_3\right)\right]\left(a+b\right)}$$

[4.25]: $$E\left(w|\varepsilon\right)=\frac{1}{\left(a+b\right)}+\frac{\sigma_v a'b\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\left[\omega_3\Phi\left(\omega_3\right)+\phi\left(\omega_3\right)\right]}{ab'\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\Phi\left(-\omega_2\right)+a'b\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\Phi\left(\omega_3\right)}.$$

E.3.The conditional expected value $E\left(\exp\{\pm w\}|\varepsilon\right)$.

$$E\left(\exp\{\pm w\}|\varepsilon\right)=\frac{1}{f_{\varepsilon}(\varepsilon)}\int_{0}^{\infty}\exp\{\pm w\}f_{\varepsilon,w}(\varepsilon,w)\,dw$$

$$=\frac{1}{f_{\varepsilon}(\varepsilon)}\int_{0}^{\infty}\exp\{\pm w\}\exp\{-(a+b)w\}\cdot\left[\Omega_1\cdot\Phi(\omega_1)+\Omega_2\cdot\Phi(-\omega_2)\right]dw$$

$$-\frac{1}{f_{\varepsilon}(\varepsilon)}\int_{0}^{\infty}\exp\{\pm w\}\cdot\Omega_1\cdot\exp\{-(a+b)w\}\Phi\left(\omega_1-\frac{w}{\sigma_v}\right)dw$$

$$=\frac{1}{f_{\varepsilon}(\varepsilon)}\int_{0}^{\infty}\exp\left\{\left[\pm1-(a+b)\right]w\right\}\cdot\left[\Omega_1\cdot\Phi(\omega_1)+\Omega_2\cdot\Phi(-\omega_2)\right]dw$$

$$-\frac{1}{f_{\varepsilon}(\varepsilon)}\int_{0}^{\infty}\exp\left\{\left[\pm1-(a+b)\right]w\right\}\cdot\Omega_1\cdot\Phi\left(\omega_1-\frac{w}{\sigma_v}\right)dw$$

$$E\left(\exp\{\pm w\}|\varepsilon\right)=\frac{\left[\Omega_1\cdot\Phi(\omega_1)+\Omega_2\cdot\Phi(-\omega_2)\right]}{f_{\varepsilon}(\varepsilon)}\int_{0}^{\infty}\exp\left\{\left[\pm1-(a+b)\right]w\right\}dw$$

$$-\frac{\Omega_1}{f_{\varepsilon}(\varepsilon)}\int_{0}^{\infty}\exp\left\{\left[\pm1-(a+b)\right]w\right\}\cdot\Phi\left(\omega_1-\frac{w}{\sigma_v}\right)dw$$

For $E\left(\exp\{+w\}|\varepsilon\right)$ to converge we require $(a+b)>1$(which is what we have observed in empirical applications). Given this,

$$E\left(\exp\{\pm w\}|\varepsilon\right)=\frac{\left[\Omega_1\cdot\Phi(\omega_1)+\Omega_2\cdot\Phi(-\omega_2)\right]}{f_{\varepsilon}(\varepsilon)}\frac{1}{(a+b)\mp1}$$

$$-\frac{\Omega_1}{f_{\varepsilon}(\varepsilon)}\int_{0}^{\infty}\exp\left\{\left[\pm1-(a+b)\right]w\right\}\cdot\Phi\left(\omega_1-\frac{w}{\sigma_v}\right)dw$$

For the remaining integral, denote it I_1 we apply again the transformation

$$y=\omega_1-\frac{w}{\sigma_v}\Rightarrow\begin{cases} w=\sigma_v\omega_1-\sigma_v y \\ dw=-\sigma_v dy \\ w=0\Rightarrow y=\omega_1 \\ w=\infty\Rightarrow y=-\infty \end{cases}$$ So,

$$I_1=\int_0^{\infty}\exp\left\{\left[\pm 1-(a+b)\right]w\right\}\cdot\Phi\left(\omega_1-\frac{w}{\sigma_v}\right)dw=-\int_{\omega_1}^{-\infty}\exp\left\{\left[\pm 1-(a+b)\right]\left(\sigma_v\omega_1-\sigma_v y\right)\right\}\cdot\Phi(y)\sigma_v dy$$

$$=\sigma_v\exp\left\{\left[\pm 1-(a+b)\right]\sigma_v\omega_1\right\}\int_{-\infty}^{\omega_1}\exp\left\{\left[(a+b)\mp 1\right]\sigma_v y\right\}\cdot\Phi(y)dy$$

Using the same formula 101,000 from Owen(1980) as before, we have

$$\int_{-\infty}^{\omega_1}\exp\left\{\left[(a+b)\mp 1\right]\sigma_v y\right\}\cdot\Phi(y)dy=\frac{\exp\left\{\left[(a+b)\mp 1\right]\sigma_v\omega_1\right\}}{\left[(a+b)\mp 1\right]\sigma_v}\Phi(\omega_1)$$

$$-\frac{\exp\left\{\frac{1}{2}\left[(a+b)\mp 1\right]^2\sigma_v^2\right\}\Phi\left(\omega_1-\left[(a+b)\mp 1\right]\sigma_v\right)}{\left[(a+b)\mp 1\right]\sigma_v}$$

(since the elements pertaining to the minus infinity limit go to zero irrespective of the value of $\left[(a+b)\mp 1\right]$. So

$$I_1=\sigma_v\exp\left\{\left[\pm 1-(a+b)\right]\sigma_v\omega_1\right\}\frac{\exp\left\{\left[(a+b)\mp 1\right]\sigma_v\omega_1\right\}}{\left[(a+b)\mp 1\right]\sigma_v}\Phi(\omega_1)$$

$$-\sigma_v\exp\left\{\left[\pm 1-(a+b)\right]\sigma_v\omega_1\right\}\frac{\exp\left\{\frac{1}{2}\left[(a+b)\mp 1\right]^2\sigma_v^2\right\}\Phi\left(\omega_1-\left[(a+b)\mp 1\right]\sigma_v\right)}{\left[(a+b)\mp 1\right]\sigma_v}$$

$$= \frac{\Phi(\omega_1) - \exp\left\{-\left[(a+b)\mp 1\right]\sigma_v \omega_1\right\} \exp\left\{\tfrac{1}{2}\left[(a+b)\mp 1\right]^2 \sigma_v^2\right\} \Phi\left(\omega_1 - \left[(a+b)\mp 1\right]\sigma_v\right)}{\left[(a+b)\mp 1\right]}$$

and

$$E\left(\exp\{\pm w\} \mid \varepsilon\right) = \frac{1}{f_\varepsilon(\varepsilon)} \frac{\left[\Omega_1 \cdot \Phi(\omega_1) + \Omega_2 \cdot \Phi(-\omega_2)\right]}{\left[(a+b)\mp 1\right]}$$

$$- \frac{\Omega_1}{f_\varepsilon(\varepsilon)} \frac{\Phi(\omega_1) - \exp\left\{-\left[(a+b)\mp 1\right]\sigma_v \omega_1\right\} \exp\left\{\tfrac{1}{2}\left[(a+b)\mp 1\right]^2 \sigma_v^2\right\} \Phi\left(\omega_1 - \left[(a+b)\mp 1\right]\sigma_v\right)}{\left[(a+b)\mp 1\right]}$$

$$= \frac{1}{f_\varepsilon(\varepsilon)} \frac{\Omega_2 \cdot \Phi(-\omega_2)}{\left[(a+b)\mp 1\right]}$$

$$+ \frac{\Omega_1}{f_\varepsilon(\varepsilon)} \frac{\exp\left\{-\left[(a+b)\mp 1\right]\sigma_v \omega_1\right\} \exp\left\{\tfrac{1}{2}\left[(a+b)\mp 1\right]^2 \sigma_v^2\right\} \Phi\left(\omega_1 - \left[(a+b)\mp 1\right]\sigma_v\right)}{(a+b\mp 1)}$$

Also note that we can write

$$\exp\left\{-\left[(a+b)\mp 1\right]\sigma_v \omega_1\right\} \exp\left\{\tfrac{1}{2}\left[(a+b)\mp 1\right]^2 \sigma_v^2\right\} =$$

$$= \exp\left\{\tfrac{1}{2}\left(\omega_1 - \left[(a+b)\mp 1\right]\sigma_v\right)^2\right\} \exp\left\{-\tfrac{1}{2}\omega_1^2\right\} = \exp\left\{\tfrac{1}{2}\left(\omega_3 \pm \sigma_v\right)^2 - \tfrac{1}{2}\omega_1^2\right\}$$

So, since also $\Omega_1 \equiv a'b \exp\left\{\tfrac{1}{2}\omega_1^2 - \tfrac{1}{2}(\varepsilon/\sigma_v)^2\right\}$, $\quad \Omega_2 \equiv ab' \exp\left\{\tfrac{1}{2}\omega_2^2 - \tfrac{1}{2}(\varepsilon/\sigma_v)^2\right\}$

we get

$$E\left(\exp\{\pm w\}|\varepsilon\right)=\sigma_v \exp\left\{-\frac{\varepsilon^2}{2\sigma_v^2}\right\}\frac{ab'\exp\left\{\frac{1}{2}\omega_2^2\right\}\Phi(-\omega_2) + a'b\exp\left\{\frac{1}{2}\left(\omega_3\pm\sigma_v\right)^2\right\}\Phi\left(\omega_3\pm\sigma_v\right)}{f_\varepsilon(\varepsilon)(a+b\mp 1)}$$

and since

$$f_\varepsilon(\varepsilon)=\frac{\exp\left\{-\frac{1}{2}\left(\varepsilon/\sigma_v\right)^2\right\}}{a+b}\left[ab'\exp\left\{\frac{1}{2}\omega_2^2\right\}\Phi(-\omega_2)+a'b\exp\left\{\frac{1}{2}\omega_3^2\right\}\Phi(\omega_3)\right]$$

we get

$$[4.28]:\ E\left(\exp\{\pm w\}|\varepsilon\right)=$$
$$=\frac{a+b}{a+b\mp 1}\frac{ab'\exp\left\{\frac{1}{2}\omega_2^2\right\}\Phi(-\omega_2) + a'b\exp\left\{\frac{1}{2}\left(\omega_3\pm\sigma_v\right)^2\right\}\Phi\left(\omega_3\pm\sigma_v\right)}{\left[ab'\exp\left\{\frac{1}{2}\omega_2^2\right\}\Phi(-\omega_2)+a'b\exp\left\{\frac{1}{2}\omega_3^2\right\}\Phi(\omega_3)\right]}\ .$$

E.4. The joint density $f_{\varepsilon,u}(\varepsilon,u)$.

To obtain conditional expected values related to the component u we need the conditional density, $f_{u|\varepsilon}=\dfrac{f_{\varepsilon,u}(\varepsilon,u)}{f_\varepsilon(\varepsilon)}$.

We have already derived the denominator, so what we need is the joint density.

We have $\varepsilon=v+w-u$. Define here the auxiliary variable $q\equiv v+w=\varepsilon+u,\ \ v=q-w$. Then

$$f_{q,u}(q,u)=\int_0^\infty f_{v,w,u}(q-w,w,u)dw=\int_0^\infty f_v(q-w)\,f_{w,u}(w,u)dw.$$

Then, since the determinant of the Jacobian of the transformation $(q,u)\to(\varepsilon+u,u)$ is unity, we get

$$f_{\varepsilon,u}(\varepsilon,u)=f_{q,u}(\varepsilon+u,u)$$

We have

$$f_v(q-w)=\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma_v^2}(q-w)^2\right\}=\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\frac{q^2}{2\sigma_v^2}\right\}\exp\left\{-\frac{w^2}{2\sigma_v^2}+\frac{qw}{\sigma_v^2}\right\}$$

$$=A(q)\exp\left\{-\frac{w^2}{2\sigma_v^2}+\frac{qw}{\sigma_v^2}\right\},\quad A(q)=\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{-\frac{q^2}{2\sigma_v^2}\right\}$$

We have also to take into account the fact that the joint density $f_{w,u}(w,u)$ has branches. This leads to

$$\int_0^\infty f_v(q-w)\,f_{w,u}(w,u)\,dw=\int_u^\infty f_v(q-w)\,f_{w>u}(w,u)\,dw+\int_0^u f_v(q-w)\,f_{w<u}(w,u)\,dw$$

which leads to two integrals to be evaluated,

$$I_1=\int_u^\infty f_v(q-w)\,f_{w>u}(w,u)\,dw=A(q)\int_u^\infty\exp\left\{-\frac{w^2}{2\sigma_v^2}+\frac{qw}{\sigma_v^2}\right\}a'b\exp\left\{-a'w-(a+b-a')u\right\}dw$$

$$I_2=\int_0^u f_v(q-w)\,f_{w<u}(w,u)\,dw=A(q)\int_0^u\exp\left\{-\frac{w^2}{2\sigma_v^2}+\frac{qw}{\sigma_v^2}\right\}ab'\exp\left\{-b'u-(a+b-b')w\right\}dw$$

For the **first integral, we have**

$$I_1=A(q)a'b\exp\left\{-(a+b-a')u\right\}\int_u^\infty\exp\left\{-\frac{w^2}{2\sigma_v^2}+\frac{qw}{\sigma_v^2}\right\}\exp\left\{-a'w\right\}dw$$

$$=A(q)a'b\exp\left\{-(a+b-a')u\right\}\int_u^\infty\exp\left\{-\frac{w^2}{2\sigma_v^2}-\left(a'-\frac{q}{\sigma_v^2}\right)w\right\}dw$$

$$= A(q)\, a'b \exp\{-(a+b-a')u\}$$
$$\times\ \sqrt{\pi}\, \frac{\sigma_v}{\sqrt{2}} \exp\left\{\frac{\sigma_v^2}{2}\left(a' - \frac{q}{\sigma_v^2}\right)^2\right\}\left[1 - \mathrm{erf}\left(\left(a' - \frac{q}{\sigma_v^2}\right)\frac{\sigma_v}{\sqrt{2}} + \frac{u}{\sigma_v \sqrt{2}}\right)\right]$$

$$= \frac{1}{\sigma_v \sqrt{2\pi}} \exp\left\{-\frac{q^2}{2\sigma_v^2}\right\} a'b \exp\{-(a+b-a')u\}$$
$$\times\ \sqrt{\pi}\, \frac{\sigma_v}{\sqrt{2}} \exp\left\{\frac{\sigma_v^2}{2}\left(a' - \frac{q}{\sigma_v^2}\right)^2\right\}\left[2 - 2\Phi\left(a'\sigma_v - \frac{q}{\sigma_v} + \frac{u}{\sigma_v}\right)\right]$$

$$= \exp\left\{-\frac{q^2}{2\sigma_v^2}\right\} a'b \exp\{-(a+b-a')u\} \exp\left\{\frac{1}{2}\left(a'\sigma_v - \frac{q}{\sigma_v}\right)^2\right\}\Phi\left(-a'\sigma_v - \frac{u}{\sigma_v} + \frac{q}{\sigma_v}\right)$$

$$= a'b \exp\{-(a+b-a')u\} \exp\left\{-a'q + \tfrac{1}{2} a'^2 \sigma_v^2\right\}\Phi\left(-a'\sigma_v - \frac{u}{\sigma_v} + \frac{q}{\sigma_v}\right)$$

Note that here $q \equiv \varepsilon + u$. Substituting, we obtain

$$I_1 = \int_u^\infty f_v(q-w)\, f_{w>u}(w,u)\, dw$$
$$= a'b \exp\{-(a+b-a')u\} \exp\left\{-a'(\varepsilon+u) + \tfrac{1}{2} a'^2 \sigma_v^2\right\}\Phi\left(-a'\sigma_v - \frac{u}{\sigma_v} + \frac{\varepsilon+u}{\sigma_v}\right)$$

and canceling off

$$I_1 = \int_u^\infty f_v(q-w)\, f_{w>u}(w,u)\, dw$$
$$= a'b \exp\{-(a+b)u\} \exp\left\{-a'\varepsilon + \tfrac{1}{2} a'^2 \sigma_v^2\right\}\Phi\left(-a'\sigma_v + \frac{\varepsilon}{\sigma_v}\right)$$

Finally, using one of the previous shorthands, $\omega_3 = \dfrac{\varepsilon}{\sigma_v} - a'\sigma_v$ we can write

$$I_1 = \int_u^\infty f_v(q-w)\, f_{w>u}(w,u)\, dw$$
$$= a'b \exp\left\{-\tfrac{1}{2}(\varepsilon/\sigma_v)^2\right\} \exp\left\{\tfrac{1}{2}\omega_3^2\right\}\Phi(\omega_3) \exp\{-(a+b)u\}$$

For the **2nd integral** we have

$$I_2 = \int_0^u f_v(q-w)\, f_{w<u}(w,u)\, dw = A(q)\int_0^u \exp\left\{-\frac{w^2}{2\sigma_v^2}+\frac{qw}{\sigma_v^2}\right\} ab' \exp\left\{-b'u-(a+b-b')w\right\} dw$$

$$= A(q)\, ab' \exp\{-b'u\} \int_0^u \exp\left\{-\frac{w^2}{2\sigma_v^2}-\left((a+b-b')-\frac{q}{\sigma_v^2}\right)w\right\} dw$$

$$= A(q)\, ab' \exp\{-b'u\}\left[\int_0^\infty \exp\left\{-\frac{w^2}{2\sigma_v^2}-\left((a+b-b')-\frac{q}{\sigma_v^2}\right)w\right\} dw\right.$$

$$\left. -\int_u^\infty \exp\left\{-\frac{w^2}{2\sigma_v^2}-\left((a+b-b')-\frac{q}{\sigma_v^2}\right)w\right\} dw\right]$$

$$= A(q)\, ab' \exp\{-b'u\} \sqrt{\pi}\,\frac{\sigma_v}{\sqrt{2}} \exp\left\{\frac{\sigma_v^2}{2}\left((a+b-b')-\frac{q}{\sigma_v^2}\right)^2\right\}$$

$$\times\left[1-\operatorname{erf}\left(\left((a+b-b')-\frac{q}{\sigma_v^2}\right)\frac{\sigma_v}{\sqrt{2}}\right)-1+\operatorname{erf}\left(\left((a+b-b')-\frac{q}{\sigma_v^2}\right)\frac{\sigma_v}{\sqrt{2}}+\frac{u}{\sqrt{2}\sigma_v}\right)\right]$$

$$= \frac{1}{\sigma_v\sqrt{2\pi}} \exp\left\{-\frac{q^2}{2\sigma_v^2}\right\} ab' \exp\{-b'u\} \sqrt{\pi}\,\frac{\sigma_v}{\sqrt{2}} \exp\left\{\frac{\sigma_v^2}{2}\left((a+b-b')-\frac{q}{\sigma_v^2}\right)^2\right\}$$

$$\times\left[2\Phi\left(\frac{q}{\sigma_v}-(a+b-b')\sigma_v\right)-2\Phi\left(\frac{q}{\sigma_v}-\frac{u}{\sigma_v}-(a+b-b')\sigma_v\right)\right]$$

$$= \exp\left\{-\frac{q^2}{2\sigma_v^2}\right\} ab' \exp\{-b'u\} \exp\left\{\frac{1}{2}\left((a+b-b')\sigma_v-\frac{q}{\sigma_v}\right)^2\right\}$$

$$\times\left[\Phi\left(\frac{q}{\sigma_v}-(a+b-b')\sigma_v\right)-\Phi\left(\frac{q}{\sigma_v}-\frac{u}{\sigma_v}-(a+b-b')\sigma_v\right)\right]$$

$$= ab'\exp\left\{-b'u\right\}\exp\left\{\frac{1}{2}\left(a+b-b'\right)^2\sigma_v^2-\left(a+b-b'\right)q\right\}$$
$$\times\left[\Phi\left(\frac{q}{\sigma_v}-\left(a+b-b'\right)\sigma_v\right)-\Phi\left(\frac{q}{\sigma_v}-\frac{u}{\sigma_v}-\left(a+b-b'\right)\sigma_v\right)\right]$$

Here we have defined $q\equiv v+w=\varepsilon+u$. Inserting

$$I_2=\int_0^u f_v\left(q-w\right)f_{w<u}\left(w,u\right)dw=ab'\exp\left\{-b'u\right\}\exp\left\{\frac{1}{2}\left(a+b-b'\right)^2\sigma_v^2-\left(a+b-b'\right)\left(\varepsilon+u\right)\right\}$$
$$\times\left[\Phi\left(\frac{\varepsilon+u}{\sigma_v}-\left(a+b-b'\right)\sigma_v\right)-\Phi\left(\frac{\varepsilon+u}{\sigma_v}-\frac{u}{\sigma_v}-\left(a+b-b'\right)\sigma_v\right)\right]$$

$$=ab'\exp\left\{-\left(a+b\right)u\right\}\exp\left\{\frac{1}{2}\left(a+b-b'\right)^2\sigma_v^2-\left(a+b-b'\right)\varepsilon\right\}$$
$$\times\left[\Phi\left(\frac{\varepsilon}{\sigma_v}-\left(a+b-b'\right)\sigma_v+\frac{u}{\sigma_v}\right)-\Phi\left(\frac{\varepsilon}{\sigma_v}-\left(a+b-b'\right)\sigma_v\right)\right]$$

Defining one more shorthand, $\omega_4=\dfrac{\varepsilon}{\sigma_v}-\left(a+b-b'\right)\sigma_v$ we compact into

$$I_2=\int_0^u f_v\left(q-w\right)f_{w<u}\left(w,u\right)dw=ab'\exp\left\{-\tfrac{1}{2}\left(\varepsilon/\sigma_v\right)^2\right\}\exp\left\{\frac{1}{2}\omega_4^2\right\}\exp\left\{-\left(a+b\right)u\right\}$$
$$\times\left[\Phi\left(\omega_4+\frac{u}{\sigma_v}\right)-\Phi\left(\omega_4\right)\right]$$

So

$$f_{\varepsilon,u}\left(\varepsilon,u\right)=a'b\exp\left\{-\tfrac{1}{2}\left(\varepsilon/\sigma_v\right)^2\right\}\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\Phi\left(\omega_3\right)\exp\left\{-\left(a+b\right)u\right\}$$
$$+\,ab'\exp\left\{-\tfrac{1}{2}\left(\varepsilon/\sigma_v\right)^2\right\}\exp\left\{\frac{1}{2}\omega_4^2\right\}\exp\left\{-\left(a+b\right)u\right\}\times\left[\Phi\left(\omega_4+\frac{u}{\sigma_v}\right)-\Phi\left(\omega_4\right)\right]$$

$$\omega_3=\frac{\varepsilon}{\sigma_v}-a'\sigma_v,\qquad \omega_4=\frac{\varepsilon}{\sigma_v}-\left(a+b-b'\right)\sigma_v$$

or

$$f_{\varepsilon,u}(\varepsilon,u)=\exp\left\{-(a+b)u\right\}\left(\Omega_4\left[\Phi\left(\omega_4+u/\sigma_v\right)-\Phi\left(\omega_4\right)\right]+\Omega_3\cdot\Phi\left(\omega_3\right)\right)$$

$$\Omega_3\equiv a'b\exp\left\{\frac{1}{2}\omega_3^2-\frac{\varepsilon^2}{2\sigma_v^2}\right\},\quad \Omega_4\equiv ab'\exp\left\{\frac{1}{2}\omega_4^2-\frac{\varepsilon^2}{2\sigma_v^2}\right\},\omega_4=\frac{\varepsilon}{\sigma_v}-\left(a+b-b'\right)\sigma_v$$

E.5.The conditional expected value $E\left(u|\varepsilon\right)$.

$$E\left(u|\varepsilon\right)=\frac{1}{f_\varepsilon(\varepsilon)}\int_0^\infty u f_{\varepsilon,u}(\varepsilon,u)\,du=\frac{1}{f_\varepsilon(\varepsilon)}\int_0^\infty u\exp\left\{-(a+b)u\right\}\cdot\left[\Omega_3\cdot\Phi\left(\omega_3\right)-\Omega_4\cdot\Phi\left(\omega_4\right)\right]du$$

$$+\frac{1}{f_\varepsilon(\varepsilon)}\int_0^\infty u\cdot\Omega_4\cdot\exp\left\{-(a+b)u\right\}\Phi\left(\omega_4+u/\sigma_v\right)du$$

$$=\frac{1}{f_\varepsilon(\varepsilon)}\frac{\Omega_3\cdot\Phi\left(\omega_3\right)-\Omega_4\cdot\Phi\left(\omega_4\right)}{(a+b)^2}+\frac{\Omega_4}{f_\varepsilon(\varepsilon)}\int_0^\infty w\cdot\exp\left\{-(a+b)w\right\}\Phi\left(\omega_4+u/\sigma_v\right)du$$

For the remaining integral, we apply the transformation

$$y=\omega_4+\frac{u}{\sigma_v}\Rightarrow\begin{cases}u=\sigma_v y-\sigma_v\omega_4\\ du=\sigma_v dy\\ u=0\Rightarrow y=\omega_4\\ u=\infty\Rightarrow y=\infty\end{cases}\qquad \text{So,}$$

$$\int_0^\infty u\cdot\exp\left\{-(a+b)u\right\}\Phi\left(\omega_4+u/\sigma_v\right)du=$$

$$=\sigma_v\int_{\omega_4}^\infty\left(\sigma_v y-\sigma_v\omega_4\right)\exp\left\{-(a+b)\left(\sigma_v y-\sigma_v\omega_4\right)\right\}\Phi(y)\,dy$$

$$= \sigma_v^2 \exp\left\{(a+b)\sigma_v \omega_4\right\} \times \left[\int_{\omega_4}^{\infty} y \exp\left\{-(a+b)\sigma_v y\right\} \Phi(y)\,dy \right.$$

$$\left. -\omega_4 \int_{\omega_4}^{\infty} \exp\left\{-(a+b)\sigma_v y\right\} \Phi(y)\,dy \right]$$

Using the same formulas as in the previous section we have

for the 1st integral

$$\begin{aligned}
\int_{\omega_4}^{\infty} y \exp\left\{-(a+b)\sigma_v y\right\} \Phi(y)\,dy = 0 \; & - \frac{(a+b)^2 \sigma_v^2 - 1}{(a+b)^2 \sigma_v^2} \exp\left\{\tfrac{1}{2}(a+b)^2 \sigma_v^2\right\} + 0 \\
& - \frac{-(a+b)\sigma_v \omega_4 - 1}{(a+b)^2 \sigma_v^2} \Phi(\omega_4) \exp\left\{-(a+b)\sigma_v \omega_4\right\} \\
& + \frac{(a+b)^2 \sigma_v^2 - 1}{(a+b)^2 \sigma_v^2} \exp\left\{\tfrac{1}{2}(a+b)^2 \sigma_v^2\right\} \Phi\left(\omega_4 + (a+b)\sigma_v\right) \\
& - \frac{1}{-(a+b)\sigma_v} \exp\left\{\tfrac{1}{2}(a+b)^2 \sigma_v^2\right\} \phi\left(\omega_4 + (a+b)\sigma_v\right)
\end{aligned}$$

$$\omega_4 + (a+b)\sigma_v = \frac{\varepsilon}{\sigma_v} - (a+b-b')\sigma_v + (a+b)\sigma_v = \frac{\varepsilon}{\sigma_v} + b'\sigma_v = \omega_2$$

So

$$\begin{aligned}
\int_{\omega_4}^{\infty} y \exp\left\{-(a+b)\sigma_v y\right\} \Phi(y)\,dy = & - \frac{(a+b)^2 \sigma_v^2 - 1}{(a+b)^2 \sigma_v^2} \exp\left\{\tfrac{1}{2}(a+b)^2 \sigma_v^2\right\} \Phi(-\omega_2) \\
& + \frac{(a+b)\sigma_v \omega_4 + 1}{(a+b)^2 \sigma_v^2} \Phi(\omega_4) \exp\left\{-(a+b)\sigma_v \omega_4\right\} \\
& + \frac{1}{(a+b)\sigma_v} \exp\left\{\tfrac{1}{2}(a+b)^2 \sigma_v^2\right\} \phi(\omega_2)
\end{aligned}$$

For the 2nd integral

$$\int_{\omega_4}^{\infty} \exp\left\{-(a+b)\sigma_v y\right\}\Phi(y)\,dy = 0 + \frac{1}{(a+b)\sigma_v}\exp\left\{\tfrac{1}{2}(a+b)^2\sigma_v^2\right\}\Phi(\infty)$$

$$-\frac{1}{-(a+b)\sigma_v}\exp\left\{-(a+b)\sigma_v\omega_4\right\}+\frac{1}{-(a+b)\sigma_v}\exp\left\{\tfrac{1}{2}(a+b)^2\sigma_v^2\right\}\Phi\left(\omega_4+(a+b)\sigma_v\right)$$

$$=\frac{1}{(a+b)\sigma_v}\exp\left\{\tfrac{1}{2}(a+b)^2\sigma_v^2\right\}\left[1-\Phi\left(\omega_4+(a+b)\sigma_v\right)\right]+\frac{1}{(a+b)\sigma_v}\exp\left\{-(a+b)\sigma_v\omega_4\right\}$$

Note that $\omega_4+(a+b)\sigma_v = \dfrac{\varepsilon}{\sigma_v}-(a+b-b')\sigma_v+(a+b)\sigma_v = \dfrac{\varepsilon}{\sigma_v}+b'\sigma_v = \omega_2$

So

$$\int_{\omega_4}^{\infty} \exp\left\{-(a+b)\sigma_v y\right\}\Phi(y)\,dy$$

$$=\frac{1}{(a+b)\sigma_v}\exp\left\{\tfrac{1}{2}(a+b)^2\sigma_v^2\right\}\Phi(-\omega_2)+\frac{1}{(a+b)\sigma_v}\exp\left\{-(a+b)\sigma_v\omega_4\right\}$$

and therefore,

$$\int_0^{\infty} u\cdot\exp\left\{-(a+b)u\right\}\Phi\left(\omega_4+u/\sigma_v\right)du=\sigma_v^2\exp\left\{(a+b)\sigma_v\omega_4\right\}\times\left[\int_{\omega_4}^{\infty} y\exp\left\{-(a+b)\sigma_v y\right\}\Phi(y)\,dy\right.$$

$$\left.-\omega_4\int_{\omega_4}^{\infty}\exp\left\{-(a+b)\sigma_v y\right\}\Phi(y)\,dy\right]$$

$$
\begin{aligned}
&= -\sigma_v^2 \exp\left\{(a+b)\sigma_v\omega_4\right\} \frac{(a+b)^2\sigma_v^2 - 1}{(a+b)^2\sigma_v^2} \exp\left\{\tfrac{1}{2}(a+b)^2\sigma_v^2\right\}\Phi(-\omega_2) \\
&\quad + \sigma_v^2 \exp\left\{(a+b)\sigma_v\omega_4\right\} \frac{(a+b)\sigma_v\omega_4 + 1}{(a+b)^2\sigma_v^2}\Phi(\omega_4)\exp\left\{-(a+b)\sigma_v\omega_4\right\} \\
&\quad + \sigma_v^2 \exp\left\{(a+b)\sigma_v\omega_4\right\} \frac{1}{(a+b)\sigma_v} \exp\left\{\tfrac{1}{2}(a+b)^2\sigma_v^2\right\}\phi(\omega_2) \\
&\quad - \sigma_v^2 \exp\left\{(a+b)\sigma_v\omega_4\right\} \frac{\omega_4}{(a+b)\sigma_v} \exp\left\{\tfrac{1}{2}(a+b)^2\sigma_v^2\right\}\Phi(-\omega_2) \\
&\quad - \sigma_v^2 \exp\left\{(a+b)\sigma_v\omega_4\right\} \frac{\omega_4}{(a+b)\sigma_v} \exp\left\{-(a+b)\sigma_v\omega_4\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{(a+b)^2\sigma_v^2 - 1}{(a+b)^2} \exp\left\{(a+b)\sigma_v\omega_4\right\} \exp\left\{\tfrac{1}{2}(a+b)^2\sigma_v^2\right\}\Phi(-\omega_2) \\
&\quad + \frac{(a+b)\sigma_v\omega_4 + 1}{(a+b)^2}\Phi(\omega_4) \\
&\quad + \sigma_v \frac{1}{(a+b)} \exp\left\{(a+b)\sigma_v\omega_4\right\} \exp\left\{\tfrac{1}{2}(a+b)^2\sigma_v^2\right\}\phi(\omega_2) \\
&\quad - \sigma_v \frac{\omega_4}{(a+b)} \exp\left\{(a+b)\sigma_v\omega_4\right\} \exp\left\{\tfrac{1}{2}(a+b)^2\sigma_v^2\right\}\Phi(-\omega_2) \\
&\quad - \sigma_v \frac{\omega_4}{(a+b)}
\end{aligned}
$$

Note that

$$
\exp\left\{(a+b)\sigma_v\omega_4\right\}\exp\left\{\tfrac{1}{2}(a+b)^2\sigma_v^2\right\} = \exp\left\{\tfrac{1}{2}\left(\omega_4 + (a+b)\sigma_v\right)^2 - \tfrac{1}{2}\omega_4^2\right\} = \exp\left\{\tfrac{1}{2}\omega_2^2 - \tfrac{1}{2}\omega_4^2\right\}
$$

So

$$\int_0^{\infty} u \cdot \exp\{-(a+b)u\}\Phi(\omega_4 + u/\sigma_v)\,du$$

$$= -\exp\left\{\tfrac{1}{2}\omega_2^2 - \tfrac{1}{2}\omega_4^2\right\}\Phi(-\omega_2)\left[\frac{(a+b)^2\sigma_v^2 - 1}{(a+b)^2} + \sigma_v \frac{\omega_4}{(a+b)}\right]$$

$$+ \frac{(a+b)\sigma_v\omega_4 + 1}{(a+b)^2}\Phi(\omega_4) \; - \; \sigma_v \frac{\omega_4}{(a+b)} \; + \; \sigma_v \frac{1}{(a+b)}\exp\left\{\tfrac{1}{2}\omega_2^2 - \tfrac{1}{2}\omega_4^2\right\}\phi(\omega_2)$$

$$\int_0^{\infty} u \cdot \exp\{-(a+b)u\}\Phi(\omega_4 + u/\sigma_v)\,du$$

$$= -\exp\left\{\tfrac{1}{2}\omega_2^2 - \tfrac{1}{2}\omega_4^2\right\}\Phi(-\omega_2)\left[\frac{(a+b)^2\sigma_v^2 - 1 + (a+b)\sigma_v\omega_4}{(a+b)^2}\right]$$

$$+ \frac{(a+b)\sigma_v\omega_4 + 1}{(a+b)^2}\Phi(\omega_4) \; - \; \frac{\sigma_v\omega_4}{(a+b)} \; + \; \frac{\sigma_v}{(a+b)}\exp\left\{\tfrac{1}{2}\omega_2^2 - \tfrac{1}{2}\omega_4^2\right\}\phi(\omega_2)$$

Also, since ,

$$\omega_4 = \frac{\varepsilon}{\sigma_v} - (a+b-b')\sigma_v = \omega_2 - (a+b)\sigma_v \;\Rightarrow \omega_4(a+b)\sigma_v = (a+b)\sigma_v\omega_2 - (a+b)^2\sigma_v^2$$

we have

$$\int_0^{\infty} u \cdot \exp\{-(a+b)u\}\Phi(\omega_4 + u/\sigma_v)\,du$$

$$= -\exp\left\{\tfrac{1}{2}\omega_2^2 - \tfrac{1}{2}\omega_4^2\right\}\Phi(-\omega_2)\left[\frac{(a+b)\sigma_v\omega_2 - 1}{(a+b)^2}\right]$$

$$+ \frac{(a+b)\sigma_v\omega_4 + 1}{(a+b)^2}\Phi(\omega_4) \; - \; \frac{\sigma_v\omega_4}{(a+b)} \; + \; \frac{\sigma_v}{(a+b)}\exp\left\{\tfrac{1}{2}\omega_2^2 - \tfrac{1}{2}\omega_4^2\right\}\phi(\omega_2)$$

Going back to the full expression,

$$E(u|\varepsilon)=\frac{1}{f_{\varepsilon}(\varepsilon)}\frac{\Omega_3\cdot\Phi(\omega_3)-\Omega_4\cdot\Phi(\omega_4)}{(a+b)^2}\;+\;\frac{\Omega_4}{f_{\varepsilon}(\varepsilon)}\int_0^{\infty} w\cdot\exp\{-(a+b)w\}\Phi(\omega_4+u/\sigma_v)du$$

$$=\frac{1}{f_{\varepsilon}(\varepsilon)}\frac{\Omega_3\cdot\Phi(\omega_3)}{(a+b)^2}\;-\;\frac{\Omega_4\cdot\Phi(\omega_4)}{f_{\varepsilon}(\varepsilon)(a+b)^2}$$

$$-\frac{\Omega_4}{f_{\varepsilon}(\varepsilon)}\exp\left\{\tfrac{1}{2}\omega_2^2-\tfrac{1}{2}\omega_4^2\right\}\Phi(-\omega_2)\left[\frac{(a+b)\sigma_v\omega_2-1}{(a+b)^2}\right]$$

$$+\frac{\Omega_4}{f_{\varepsilon}(\varepsilon)}\,\frac{(a+b)\sigma_v\omega_4}{(a+b)^2}\;+\;\frac{\Omega_4\cdot\Phi(\omega_4)}{f_{\varepsilon}(\varepsilon)(a+b)^2}$$

$$-\frac{\Omega_4}{f_{\varepsilon}(\varepsilon)}\,\frac{\sigma_v\omega_4}{(a+b)}\;+\;\frac{\Omega_4}{f_{\varepsilon}(\varepsilon)}\,\frac{\sigma_v}{(a+b)}\exp\left\{\tfrac{1}{2}\omega_2^2-\tfrac{1}{2}\omega_4^2\right\}\phi(\omega_2)$$

$$=\frac{1}{f_{\varepsilon}(\varepsilon)}\frac{\Omega_3\cdot\Phi(\omega_3)}{(a+b)^2}\;-\;\frac{\Omega_4}{f_{\varepsilon}(\varepsilon)}\exp\left\{\tfrac{1}{2}\omega_2^2-\tfrac{1}{2}\omega_4^2\right\}\Phi(-\omega_2)\left[\frac{(a+b)\sigma_v\omega_2-1}{(a+b)^2}\right]$$

$$+\frac{\Omega_4}{f_{\varepsilon}(\varepsilon)}\left[\frac{(a+b)\sigma_v\omega_4}{(a+b)^2}-\frac{\sigma_v\omega_4}{(a+b)}\right]\;+\;\frac{\Omega_4}{f_{\varepsilon}(\varepsilon)}\frac{\sigma_v}{(a+b)}\exp\left\{\tfrac{1}{2}\omega_2^2-\tfrac{1}{2}\omega_4^2\right\}\phi(\omega_2)$$

$$=\frac{1}{f_{\varepsilon}(\varepsilon)}\frac{\Omega_3\cdot\Phi(\omega_3)}{(a+b)^2}\;-\;\frac{\Omega_4}{f_{\varepsilon}(\varepsilon)(a+b)^2}\exp\left\{\tfrac{1}{2}\omega_2^2-\tfrac{1}{2}\omega_4^2\right\}\Phi(-\omega_2)\left[(a+b)\sigma_v\omega_2-1\right]$$

$$+\;\frac{\Omega_4}{f_{\varepsilon}(\varepsilon)}\frac{\sigma_v}{(a+b)}\exp\left\{\tfrac{1}{2}\omega_2^2-\tfrac{1}{2}\omega_4^2\right\}\phi(\omega_2)$$

We start the process of simplification.

We have

$$f_{\varepsilon}(\varepsilon)=\frac{\exp\left\{-\tfrac{1}{2}(\varepsilon/\sigma_v)^2\right\}}{a+b}\left[ab'\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\Phi(-\omega_2)\;+\;a'b\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\Phi(\omega_3)\right]$$

and

$$\Omega_3 \equiv a'b\exp\left\{\frac{1}{2}\omega_3^2 - \frac{\varepsilon^2}{2\sigma_v^2}\right\}, \quad \Omega_4 \equiv ab'\exp\left\{\frac{1}{2}\omega_4^2 - \frac{\varepsilon^2}{2\sigma_v^2}\right\}$$

So

$$\frac{1}{f_\varepsilon(\varepsilon)}\frac{\Omega_3}{(a+b)^2} = \frac{a'b\exp\left\{\frac{1}{2}\omega_3^2\right\}}{\left[ab'\exp\left\{\frac{1}{2}\omega_2^2\right\}\Phi(-\omega_2) + a'b\exp\left\{\frac{1}{2}\omega_3^2\right\}\Phi(\omega_3)\right](a+b)}$$

also

$$\frac{\Omega_4\exp\left\{\frac{1}{2}\omega_2^2 - \frac{1}{2}\omega_4^2\right\}}{f_\varepsilon(\varepsilon)(a+b)^2} = \frac{ab'\exp\left\{\frac{1}{2}\omega_4^2 - \frac{\varepsilon^2}{2\sigma_v^2}\right\}\exp\left\{\frac{1}{2}\omega_2^2 - \frac{1}{2}\omega_4^2\right\}}{\frac{\exp\left\{-\frac{1}{2}(\varepsilon/\sigma_v)^2\right\}}{a+b}\left[ab'\exp\left\{\frac{1}{2}\omega_2^2\right\}\Phi(-\omega_2) + a'b\exp\left\{\frac{1}{2}\omega_3^2\right\}\Phi(\omega_3)\right](a+b)^2}$$

$$\frac{\Omega_4\exp\left\{\frac{1}{2}\omega_2^2 - \frac{1}{2}\omega_4^2\right\}}{f_\varepsilon(\varepsilon)(a+b)^2} = \frac{ab'\exp\left\{\frac{1}{2}\omega_2^2\right\}}{\left[ab'\exp\left\{\frac{1}{2}\omega_2^2\right\}\Phi(-\omega_2) + a'b\exp\left\{\frac{1}{2}\omega_3^2\right\}\Phi(\omega_3)\right](a+b)}$$

and

$$\frac{\Omega_4}{f_\varepsilon(\varepsilon)}\frac{1}{(a+b)} = \frac{ab'\exp\left\{\frac{1}{2}\omega_4^2\right\}}{\left[ab'\exp\left\{\frac{1}{2}\omega_2^2\right\}\Phi(-\omega_2) + a'b\exp\left\{\frac{1}{2}\omega_3^2\right\}\Phi(\omega_3)\right]}$$

Therefore

$$E(u|\varepsilon) = \frac{a'b\exp\left\{\frac{1}{2}\omega_3^2\right\}\Phi(\omega_3)}{\left[ab'\exp\left\{\frac{1}{2}\omega_2^2\right\}\Phi(-\omega_2) + a'b\exp\left\{\frac{1}{2}\omega_3^2\right\}\Phi(\omega_3)\right](a+b)}$$

$$-\frac{ab'\exp\left\{\frac{1}{2}\omega_2^2\right\}\Phi(-\omega_2)\left[(a+b)\sigma_v\omega_2 - 1\right]}{\left[ab'\exp\left\{\frac{1}{2}\omega_2^2\right\}\Phi(-\omega_2) + a'b\exp\left\{\frac{1}{2}\omega_3^2\right\}\Phi(\omega_3)\right](a+b)}$$

$$+\ \frac{\sigma_v ab'\exp\left\{\frac{1}{2}\omega_2^2\right\}\phi(\omega_2)}{\left[ab'\exp\left\{\frac{1}{2}\omega_2^2\right\}\Phi(-\omega_2) + a'b\exp\left\{\frac{1}{2}\omega_3^2\right\}\Phi(\omega_3)\right]}$$

So

$$E\left(u|\varepsilon\right)=\frac{1}{a+b}-\frac{\sigma_v\omega_2 ab'\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\Phi\left(-\omega_2\right)}{\left[ab'\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\Phi\left(-\omega_2\right)+a'b\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\Phi\left(\omega_3\right)\right]}$$

$$+\frac{\sigma_v ab'\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\phi\left(\omega_2\right)}{\left[ab'\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\Phi\left(-\omega_2\right)+a'b\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\Phi\left(\omega_3\right)\right]}$$

and finally,

$$\boxed{[4.26]:\quad E\left(u|\varepsilon\right)=\frac{1}{a+b}-\frac{\sigma_v ab'\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\left[\omega_2\Phi\left(-\omega_2\right)-\phi\left(\omega_2\right)\right]}{ab'\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\Phi\left(-\omega_2\right)+a'b\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\Phi\left(\omega_3\right)}}$$

E.6.The conditional expected value $E\left(\exp\{\pm u\}|\varepsilon\right)$.

$$f_{\varepsilon,u}\left(\varepsilon,u\right)=\exp\left\{-\left(a+b\right)u\right\}\left(\Omega_4\left[\Phi\left(\omega_4+u/\sigma_v\right)-\Phi\left(\omega_4\right)\right]+\Omega_3\cdot\Phi\left(\omega_3\right)\right)$$

$$\Omega_3\equiv a'b\exp\left\{\frac{1}{2}\omega_3^2-\frac{\varepsilon^2}{2\sigma_v^2}\right\},\quad \Omega_4\equiv ab'\exp\left\{\frac{1}{2}\omega_4^2-\frac{\varepsilon^2}{2\sigma_v^2}\right\},\omega_4=\frac{\varepsilon}{\sigma_v}-\left(a+b-b'\right)\sigma_v$$

We have

$$E\left(\exp\{\pm u\}|\varepsilon\right)=\frac{1}{f_\varepsilon\left(\varepsilon\right)}\int_0^\infty\exp\{\pm u\}f_{\varepsilon,u}\left(\varepsilon,u\right)du$$

$$=\frac{1}{f_\varepsilon\left(\varepsilon\right)}\int_0^\infty\exp\{\pm u\}\exp\left\{-\left(a+b\right)u\right\}\cdot\left[\Omega_3\cdot\Phi\left(\omega_3\right)-\Omega_4\Phi\left(\omega_4\right)\right]du$$

$$+\frac{1}{f_\varepsilon\left(\varepsilon\right)}\int_0^\infty\exp\{\pm u\}\cdot\Omega_4\cdot\exp\left\{-\left(a+b\right)u\right\}\Phi\left(\omega_4+u/\sigma_v\right)du$$

$$= \frac{1}{f_\varepsilon(\varepsilon)} \int_0^\infty \exp\left\{\left[\pm 1 - (a+b)\right]u\right\} \cdot \left[\Omega_3 \cdot \Phi(\omega_3) - \Omega_4 \Phi(\omega_4)\right] du$$

$$+ \frac{1}{f_\varepsilon(\varepsilon)} \int_0^\infty \exp\left\{\left[\pm 1 - (a+b)\right]u\right\} \cdot \Omega_4 \cdot \Phi\left(\omega_4 + u/\sigma_v\right) du$$

$$E\left(\exp\{\pm u\} | \varepsilon\right) = \frac{\left[\Omega_3 \cdot \Phi(\omega_3) - \Omega_4 \Phi(\omega_4)\right]}{f_\varepsilon(\varepsilon)} \int_0^\infty \exp\left\{-(a+b\mp 1)u\right\} du$$

$$+ \frac{\Omega_4}{f_\varepsilon(\varepsilon)} \int_0^\infty \exp\left\{-(a+b\mp 1)u\right\} \cdot \Phi\left(\omega_4 + u/\sigma_v\right) du$$

For $E\left(\exp\{+u\} | \varepsilon\right)$ to converge we require $a+b>1$. Given this,

$$E\left(\exp\{\pm u\} | \varepsilon\right) = \frac{\left[\Omega_3 \cdot \Phi(\omega_3) - \Omega_4 \Phi(\omega_4)\right]}{f_\varepsilon(\varepsilon)} \frac{1}{(a+b)\mp 1}$$

$$+ \frac{\Omega_4}{f_\varepsilon(\varepsilon)} \int_0^\infty \exp\left\{-(a+b\mp 1)u\right\} \cdot \Phi\left(\omega_4 + u/\sigma_v\right) du$$

For the remaining integral, denote it I_1 we apply again the transformation

$$y = \omega_4 + \frac{u}{\sigma_v} \Rightarrow \begin{cases} u = \sigma_v y - \sigma_v \omega_4 \\ du = \sigma_v dy \\ u = 0 \Rightarrow y = \omega_4 \\ u = \infty \Rightarrow y = \infty \end{cases} \qquad \text{So,}$$

$$I_1 = \int_0^\infty \exp\left\{-(a+b\mp 1)u\right\} \cdot \Phi\left(\omega_4 + u/\sigma_v\right) du = \int_{\omega_4}^\infty \exp\left\{-(a+b\mp 1)\left(\sigma_v y - \sigma_v \omega_4\right)\right\} \cdot \Phi(y) \sigma_v dy$$

$$= \sigma_v \exp\left\{(a+b\mp 1)\sigma_v \omega_4\right\} \int_{\omega_4}^\infty \exp\left\{-(a+b\mp 1)\sigma_v y\right\} \cdot \Phi(y) dy$$

Using the same formula 101,000 from Owen as before, we have

$$\int_{\omega_4}^{\infty} \exp\left\{-(a+b\mp 1)\sigma_v y\right\}\cdot\Phi(y)\,dy = \frac{\exp\left\{\frac{1}{2}(a+b\mp 1)^2\sigma_v^2\right\}}{(a+b\mp 1)\sigma_v} + \frac{\exp\left\{-(a+b\mp 1)\sigma_v\omega_4\right\}}{(a+b\mp 1)\sigma_v}\Phi(\omega_4)$$

$$-\frac{\exp\left\{\frac{1}{2}(a+b\mp 1)^2\sigma_v^2\right\}\Phi\left(\omega_4+(a+b\mp 1)\sigma_v\right)}{(a+b\mp 1)\sigma_v}$$

while

$$\omega_4 = \frac{\varepsilon}{\sigma_v} - (a+b-b')\sigma_v = \omega_2 - (a+b)\sigma_v \;\Rightarrow \omega_4 + (a+b\mp 1)\sigma_v = \omega_2 \mp \sigma_v$$

So

$$I_1 = \quad \sigma_v \exp\left\{(a+b\mp 1)\sigma_v\omega_4\right\}\frac{\exp\left\{\frac{1}{2}(a+b\mp 1)^2\sigma_v^2\right\}}{(a+b\mp 1)\sigma_v}$$

$$+\,\sigma_v \exp\left\{(a+b\mp 1)\sigma_v\omega_4\right\}\frac{\exp\left\{-(a+b\mp 1)\sigma_v\omega_4\right\}}{(a+b\mp 1)\sigma_v}\Phi(\omega_4)$$

$$-\sigma_v \exp\left\{(a+b\mp 1)\sigma_v\omega_4\right\}\frac{\exp\left\{\frac{1}{2}(a+b\mp 1)^2\sigma_v^2\right\}\Phi(\omega_2\mp\sigma_v)}{(a+b\mp 1)\sigma_v}$$

$$=\sigma_v \exp\left\{(a+b\mp 1)\sigma_v\omega_4\right\}\frac{\exp\left\{\frac{1}{2}(a+b\mp 1)^2\sigma_v^2\right\}}{(a+b\mp 1)\sigma_v}\left[1-\Phi(\omega_2\mp\sigma_v)\right] + \frac{\sigma_v\Phi(\omega_4)}{(a+b\mp 1)\sigma_v}$$

Also note that we can write

$$\exp\left\{(a+b\mp 1)\sigma_v\omega_4\right\}\exp\left\{\frac{1}{2}(a+b\mp 1)^2\sigma_v^2\right\} =$$

$$=\exp\left\{\frac{1}{2}\left(\omega_4+(a+b\mp 1)\sigma_v\right)^2\right\}\exp\left\{-\frac{1}{2}\omega_4^2\right\} = \exp\left\{\frac{1}{2}(\omega_2\mp\sigma_v)^2-\frac{1}{2}\omega_4^2\right\}$$

So

$$I_1 = \frac{\exp\left\{\tfrac{1}{2}\left(\omega_2 \mp \sigma_v\right)^2 - \tfrac{1}{2}\omega_4^2\right\}\Phi\left(\pm\sigma_v - \omega_2\right) + \Phi\left(\omega_4\right)}{\left(a+b\mp 1\right)}$$

Going back to the full expression

$$E\left(\exp\{\pm u\}|\varepsilon\right) = \frac{\left[\Omega_3 \cdot \Phi\left(\omega_3\right) - \Omega_4 \Phi\left(\omega_4\right)\right]}{f_\varepsilon\left(\varepsilon\right)} \frac{1}{\left(a+b\right)\mp 1}$$

$$+\frac{\Omega_4}{f_\varepsilon\left(\varepsilon\right)} \frac{\exp\left\{\tfrac{1}{2}\left(\omega_2 \mp \sigma_v\right)^2 - \tfrac{1}{2}\omega_4^2\right\}\Phi\left(\pm\sigma_v - \omega_2\right) + \Phi\left(\omega_4\right)}{\left(a+b\mp 1\right)}$$

$$E\left(\exp\{\pm u\}|\varepsilon\right) = \frac{\Omega_3 \cdot \Phi\left(\omega_3\right)}{f_\varepsilon\left(\varepsilon\right)\left[\left(a+b\right)\mp 1\right]} + \frac{\Omega_4}{f_\varepsilon\left(\varepsilon\right)} \frac{\exp\left\{\tfrac{1}{2}\left(\omega_2 \mp \sigma_v\right)^2 - \tfrac{1}{2}\omega_4^2\right\}\Phi\left(\pm\sigma_v - \omega_2\right)}{\left(a+b\mp 1\right)}$$

We have $$\frac{\Omega_3}{f_\varepsilon\left(\varepsilon\right)} = \frac{\left(a+b\right)a'b\exp\left\{\tfrac{1}{2}\omega_3^2\right\}}{\left[ab'\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\Phi\left(-\omega_2\right) + a'b\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\Phi\left(\omega_3\right)\right]}$$

while

$$\frac{\Omega_4 \exp\left\{\tfrac{1}{2}\left(\omega_2 \mp \sigma_v\right)^2 - \tfrac{1}{2}\omega_4^2\right\}}{f_\varepsilon\left(\varepsilon\right)} = \frac{ab'\exp\left\{\dfrac{1}{2}\omega_4^2 - \dfrac{\varepsilon^2}{2\sigma_v^2}\right\}\exp\left\{\tfrac{1}{2}\left(\omega_2 \mp \sigma_v\right)^2 - \tfrac{1}{2}\omega_4^2\right\}}{\dfrac{\exp\left\{-\tfrac{1}{2}\left(\varepsilon/\sigma_v\right)^2\right\}}{a+b}\left[ab'\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\Phi\left(-\omega_2\right) + a'b\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\Phi\left(\omega_3\right)\right]}$$

$$= \frac{\left(a+b\right)ab'\exp\left\{\tfrac{1}{2}\left(\omega_2 \mp \sigma_v\right)^2\right\}}{\left[ab'\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\Phi\left(-\omega_2\right) + a'b\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\Phi\left(\omega_3\right)\right]}$$

So

$$[4.29]:\ E\left(\exp\{\pm u\}|\varepsilon\right) =$$

$$= \frac{\left(a+b\right)}{\left(a+b\mp 1\right)} \frac{ab'\exp\left\{\tfrac{1}{2}\left(\omega_2 \mp \sigma_v\right)^2\right\}\Phi\left(\pm\sigma_v - \omega_2\right) + a'b\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\cdot\Phi\left(\omega_3\right)}{ab'\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\Phi\left(-\omega_2\right) + a'b\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\Phi\left(\omega_3\right)}$$

E.7.The conditional expected value $E\left(\exp\{w-u\}|\varepsilon\right)$.

We want to calculate

$$E\left(\exp\{w-u\}|\varepsilon\right)=E\left(\exp\{z\}|\varepsilon\right)=\int_{-\infty}^{\infty}\exp\{z\}\,f_{z|\varepsilon}\left(z|\varepsilon\right)dz=\frac{1}{f_{\varepsilon}\left(\varepsilon\right)}\int_{-\infty}^{\infty}\exp\{z\}\,f_{\varepsilon,z}\left(\varepsilon,z\right)dz$$

But since $\varepsilon=v+z$ it follows that $f_{\varepsilon,z}\left(\varepsilon,z\right)=f_{v,z}\left(\varepsilon-z,z\right)$. Due to independence between v and z and the branching density of z we have

$$\int_{-\infty}^{\infty}\exp\{z\}\,f_{\varepsilon,z}\left(\varepsilon,z\right)dz=\int_{-\infty}^{0}\frac{1}{\sigma_v\sqrt{2\pi}}\frac{ab'}{a+b}\exp\left\{\left(b'+1\right)z\right\}\exp\left\{-\tfrac{1}{2\sigma_v^2}\left(\varepsilon-z\right)^2\right\}dz$$

$$+\int_{0}^{\infty}\frac{1}{\sigma_v\sqrt{2\pi}}\frac{ba'}{a+b}\exp\left\{\left(1-a'\right)z\right\}\exp\left\{-\tfrac{1}{2\sigma_v^2}\left(\varepsilon-z\right)^2\right\}dz$$

$$=\frac{1}{\sigma_v\sqrt{2\pi}}\frac{ab'}{a+b}\exp\left\{-\tfrac{1}{2\sigma_v^2}\varepsilon^2\right\}\int_{-\infty}^{0}\exp\left\{-\tfrac{1}{2\sigma_v^2}z^2\right\}\exp\left\{\left(b'+1+\frac{\varepsilon}{\sigma_v^2}\right)z\right\}dz$$

$$+\frac{1}{\sigma_v\sqrt{2\pi}}\frac{ba'}{a+b}\exp\left\{-\tfrac{1}{2\sigma_v^2}\varepsilon^2\right\}\int_{0}^{\infty}\exp\left\{-\tfrac{1}{2\sigma_v^2}z^2\right\}\exp\left\{\left(\frac{\varepsilon}{\sigma_v^2}+1-a'\right)z\right\}dz$$

Swap the limits of integration on the first Integral

$$=\frac{1}{\sigma_v\sqrt{2\pi}}\frac{ab'}{a+b}\exp\left\{-\tfrac{1}{2\sigma_v^2}\varepsilon^2\right\}\int_{0}^{\infty}\exp\left\{-\tfrac{1}{2\sigma_v^2}z^2\right\}\exp\left\{-\left(b'+\frac{\varepsilon}{\sigma_v^2}+1\right)z\right\}dz$$

$$+\frac{1}{\sigma_v\sqrt{2\pi}}\frac{ba'}{a+b}\exp\left\{-\tfrac{1}{2\sigma_v^2}\varepsilon^2\right\}\int_{0}^{\infty}\exp\left\{-\tfrac{1}{2\sigma_v^2}z^2\right\}\exp\left\{\left(\frac{\varepsilon}{\sigma_v^2}-a'+1\right)z\right\}dz$$

The integrals evaluate to

$$= \frac{1}{\sigma_v\sqrt{2\pi}}\frac{ab'}{a+b}\exp\left\{-\tfrac{1}{2\sigma_v^2}\varepsilon^2\right\}\frac{\sigma_v\sqrt{\pi}}{\sqrt{2}}\exp\left\{\frac{\sigma_v^2}{2}\left(b'+\frac{\varepsilon}{\sigma_v^2}+1\right)^2\right\}\left[2-2\Phi\left(\frac{\varepsilon}{\sigma_v}+\sigma_v b'+\sigma_v\right)\right]$$

$$+\frac{1}{\sigma_v\sqrt{2\pi}}\frac{ba'}{a+b}\exp\left\{-\tfrac{1}{2\sigma_v^2}\varepsilon^2\right\}\frac{\sigma_v\sqrt{\pi}}{\sqrt{2}}\exp\left\{\frac{\sigma_v^2}{2}\left(\frac{\varepsilon}{\sigma_v^2}-a'+1\right)^2\right\}\left[2-2\Phi\left(-\frac{\varepsilon}{\sigma_v}+\sigma_v a'-\sigma_v\right)\right]$$

Simplifying and using the shorthands $\omega_2 \equiv \frac{\varepsilon}{\sigma_v}+b'\sigma_v, \quad \omega_3 \equiv \frac{\varepsilon}{\sigma_v}-a'\sigma_v$

$$\int_{-\infty}^{\infty}\exp\{z\}\, f_{\varepsilon,z}(\varepsilon,z)\,dz = \frac{ab'}{a+b}\exp\left\{-\tfrac{1}{2\sigma_v^2}\varepsilon^2\right\}\exp\left\{\frac{1}{2}(\omega_2+\sigma_v)^2\right\}\left[1-\Phi(\omega_2+\sigma_v)\right]$$

$$+\frac{ba'}{a+b}\exp\left\{-\tfrac{1}{2\sigma_v^2}\varepsilon^2\right\}\exp\left\{\frac{1}{2}(\omega_3+\sigma_v)^2\right\}\left[1-\Phi(-\omega_3-\sigma_v)\right]$$

$$\int_{-\infty}^{\infty}\exp\{z\}\, f_{\varepsilon,z}(\varepsilon,z)\,dz = \frac{ab'}{a+b}\exp\left\{-\tfrac{1}{2\sigma_v^2}\varepsilon^2\right\}\exp\left\{\frac{1}{2}(\omega_2+\sigma_v)^2\right\}\Phi(-\omega_2-\sigma_v)$$

$$+\frac{ba'}{a+b}\exp\left\{-\tfrac{1}{2\sigma_v^2}\varepsilon^2\right\}\exp\left\{\frac{1}{2}(\omega_3+\sigma_v)^2\right\}\Phi(\omega_3+\sigma_v)$$

Then, dividing by the composite error density we get

$$[4.30]:\ E\left(\exp\{w-u\}\,|\,\varepsilon\right)=$$

$$=\frac{ab'\exp\left\{\tfrac{1}{2}(\omega_2+\sigma_v)^2\right\}\Phi(-\omega_2-\sigma_v)\ +\ a'b\exp\left\{\tfrac{1}{2}(\omega_3+\sigma_v)^2\right\}\Phi(\omega_3+\sigma_v)}{ab'\exp\left\{\tfrac{1}{2}\omega_2^2\right\}\Phi(-\omega_2)+a'b\exp\left\{\tfrac{1}{2}\omega_3^2\right\}\Phi(\omega_3)}$$

F. Bounds for the non-identifiable parameters and marginal moments.

We examine the signs of $\operatorname{Cov}(w,u)$ and the statistic $T=(1-m)a'-mb'$.

Case A. $\operatorname{Cov}(w,u)>0,\ T>0$

We have

$$\left\{\begin{array}{l} \operatorname{Cov}(w,u)>0 \Rightarrow a'b'>ab \Rightarrow \dfrac{a'}{a}>\dfrac{b}{b'} \\ \\ T>0 \Rightarrow (1-m)a'>mb' \Rightarrow ba'>ab' \Rightarrow \dfrac{a'}{a}>\dfrac{b'}{b} \end{array}\right. \Rightarrow (a'/a)^2>1 \Rightarrow a'>a$$

Since moreover $a=\dfrac{m}{1-m}b$ we also obtain $\dfrac{1-m}{m}a'>b$.

Summing the two we obtain

$$a+b \ < \ a' + \frac{1-m}{m}a' \Rightarrow a+b<\frac{a'}{m}.$$

At the same time we also have

$$\operatorname{Cov}(w,u)>0 \ \Rightarrow \frac{1}{(a+b)^2}>\frac{m(1-m)}{a'b'} \Rightarrow \ a+b<\left(\frac{a'b'}{m(1-m)}\right)^{1/2}$$

Combining we get

$$\operatorname{Cov}(w,u)>0,\ T>0 \ \Rightarrow \ a+b \ <\min\left\{\frac{a'}{m},\left(\frac{a'b'}{m(1-m)}\right)^{1/2}\right\}$$

Case B. $\operatorname{Cov}(w,u)>0,\ T<0$

We have

$$\left\{\begin{array}{l} \mathrm{Cov}(w,u)>0 \Rightarrow a'b'>ab \Rightarrow \dfrac{a'}{a}>\dfrac{b}{b'} \\ \\ T<0 \Rightarrow (1-m)a'<mb' \Rightarrow ba'<ab' \Rightarrow \dfrac{a'}{a}<\dfrac{b'}{b} \end{array}\right. \Rightarrow \frac{b}{b'}<\frac{b'}{b} \Rightarrow b<b'$$

Since moreover $b=\dfrac{1-m}{m}a$ we also obtain $b=\dfrac{1-m}{m}a<b' \Rightarrow a<\dfrac{m}{1-m}b'$.

Summing the two we obtain

$$a+b < b'+\frac{m}{1-m}b' \Rightarrow a+b<\frac{b'}{1-m}.$$

In this subcase too it holds that

$$\mathrm{Cov}(w,u)>0 \Rightarrow \frac{1}{(a+b)^2}>\frac{m(1-m)}{a'b'} \Rightarrow a+b<\left(\frac{a'b'}{m(1-m)}\right)^{1/2}$$

Combining we get

$$\mathrm{Cov}(w,u)>0,\ T<0 \Rightarrow a+b<\min\left\{\frac{b'}{1-m},\left(\frac{a'b'}{m(1-m)}\right)^{1/2}\right\}$$

Case C. $\mathrm{Cov}(w,u)<0,\ T>0$

We have

$$\left\{\begin{array}{l} \mathrm{Cov}(w,u)<0 \Rightarrow a'b'<ab \Rightarrow \dfrac{a'}{a}<\dfrac{b}{b'} \\ \\ T>0 \Rightarrow (1-m)a'>mb' \Rightarrow ba'>ab' \Rightarrow \dfrac{a'}{a}>\dfrac{b'}{b} \end{array}\right. \Rightarrow \frac{b'}{b}<\frac{b}{b'} \Rightarrow b>b'$$

Since moreover $b=\dfrac{1-m}{m}a$ we also obtain $b=\dfrac{1-m}{m}a>b' \Rightarrow a>\dfrac{m}{1-m}b'$.

Summing the two we obtain $\quad a+b > b' + \dfrac{m}{1-m}b' \Rightarrow a+b > \dfrac{b'}{1-m}$.

In this subcase it holds that

$$\operatorname{Cov}(w,u)<0 \Rightarrow \frac{1}{(a+b)^2} < \frac{m(1-m)}{a'b'} \Rightarrow a+b > \left(\frac{a'b'}{m(1-m)}\right)^{1/2}$$

Combining we get $\quad \operatorname{Cov}(w,u)<0,\ T>0 \Rightarrow a+b > \max\left\{\dfrac{b'}{1-m}, \left(\dfrac{a'b'}{m(1-m)}\right)^{1/2}\right\}$

Case D. $\operatorname{Cov}(w,u)<0,\ T<0$

We have

$$\begin{cases} \operatorname{Cov}(w,u)<0 \Rightarrow a'b' < ab \Rightarrow \dfrac{a'}{a} < \dfrac{b}{b'} \\ \qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow (a'/a)^2 < 1 \Rightarrow a' < a \\ T<0 \Rightarrow (1-m)a' < mb' \Rightarrow ba' < ab' \Rightarrow \dfrac{a'}{a} < \dfrac{b'}{b} \end{cases}$$

Since moreover $a = \dfrac{m}{1-m}b$ we also obtain $\dfrac{1-m}{m}a' < b$.

Summing the two we obtain $\quad a+b > a' + \dfrac{1-m}{m}a' \Rightarrow a+b > \dfrac{a'}{m}$.

At the same time we also have

$$\operatorname{Cov}(w,u)<0 \Rightarrow \frac{1}{(a+b)^2} < \frac{m(1-m)}{a'b'} \Rightarrow a+b > \left(\frac{a'b'}{m(1-m)}\right)^{1/2} .$$

Combining we get $\quad \operatorname{Cov}(w,u)<0,\ T<0 \Rightarrow a+b > \max\left\{\dfrac{a'}{m}, \left(\dfrac{a'b'}{m(1-m)}\right)^{1/2}\right\}$.

Section II.

A. Proof of Lemma 4.1.

Lemma 4.1: ***Let*** *$X_i, i=1,\dots,m$ be continuous random variables with marginal distribution functions $F_i(x_i)$, $i=1,\dots,m$, and Copula $C_{X_1 \cdots X_m}\left(F_1(X_1),\dots,F_m(X_m)\right)$. Let Φ be the standard normal distribution function, and Φ^{-1} its inverse. Let Φ_m be the multivariate standard normal distribution function of dimension m.*

Then: if the random variables $\Phi^{-1}\left(F_i(X_i)\right)$, $i=1,\dots,m$ follow jointly a Multivariate Normal (MVN) distribution, it holds that

$$C_{X_1 \cdots X_m}\left(F_1(X_1),\dots,F_m(X_m)\right)=\Phi_m\left(\Phi^{-1}\left(F_1(X_1)\right),\dots,\Phi^{-1}\left(F_m(X_m)\right)\right) \qquad [1]$$

Proof. Consider the uniform $U(0,1)$ random variables $U_i=F_i(X_i)$, $i=1,\dots,m$ with distribution functions $G_i(u_i)$, $i=1,\dots,m$. By Theorem 2.4.3 in Nelsen (2006) p. 25, strictly increasing transformations of random variables result in the transformed variables having the same Copula. The transformation $U_i=F_i(X_i)$, $i=1,\dots,m$ is strictly increasing. So for the Copula of the U_i's we have

$$C_{U_1 \cdots U_m}\left(G_1(U_1),\dots,G_m(U_m)\right)=C_{X_1 \cdots X_m}\left(G_1(U_1),\dots,G_m(U_m)\right) \qquad [2]$$

Moreover, it holds that $G_i(u_i)=u_i$, $i=1,\dots,m$. Then

$$C_{U_1 \cdots U_m}\left(G_1(U_1),\dots,G_m(U_m)\right)=C_{X_1 \cdots X_m}\left(U_1,\dots,U_m\right)=C_{X_1 \cdots X_m}\left(F_1(X_1),\dots,F_m(X_m)\right) \qquad [3]$$

Namely, the Copula of the distribution functions of the random variables (viewed as random variables themselves) is identical to the Copula of the random variables.

Consider now a strictly increasing transformation of the U_i variables, $Z_i = \Phi^{-1}\left(U_i\right)$, with distribution functions $\Phi\left(z_i\right)$, $i = 1,...,m$. These are standard Normals of course. By the same theorem of Nelsen's, we have for *their* Copula that (analogously to [2])

$$C_{Z_1 \cdots Z_m}\left(\Phi\left(Z_1\right),...,\Phi\left(Z_m\right)\right) = C_{U_1 \cdots U_m}\left(\Phi\left(Z_1\right),...,\Phi\left(Z_m\right)\right) \qquad [4]$$

From eq. [2] we have

$$C_{U_1 \cdots U_m}\left(\Phi\left(Z_1\right),...,\Phi\left(Z_m\right)\right) = C_{X_1 \cdots X_m}\left(\Phi\left(Z_1\right),...,\Phi\left(Z_m\right)\right)$$

Equating with the previous expression we get

$$C_{Z_1 \cdots Z_m}\left(\Phi\left(Z_1\right),...,\Phi\left(Z_m\right)\right) = C_{X_1 \cdots X_m}\left(\Phi\left(Z_1\right),...,\Phi\left(Z_m\right)\right)$$

But

$$C_{X_1 \cdots X_m}\left(\Phi\left(Z_1\right),...,\Phi\left(Z_m\right)\right) = C_{X_1 \cdots X_m}\left(\Phi\left(\Phi^{-1}\left(F_1\left(X_1\right)\right)\right),...,\Phi\left(\Phi^{-1}\left(F_m\left(X_m\right)\right)\right)\right)$$

$$= C_{X_1 \cdots X_m}\left(F_1\left(X_1\right),...,F_m\left(X_m\right)\right)$$

So $C_{Z_1 \cdots Z_m}\left(\Phi\left(Z_1\right),...,\Phi\left(Z_m\right)\right) = C_{X_1 \cdots X_m}\left(F_1\left(X_1\right),...,F_m\left(X_m\right)\right)$ [5]

This tells us that the Copula of the $Z_i = \Phi^{-1}\left(F_1\left(X_1\right)\right)$ variables is *identical* to the Copula of the X_i variables, in the sense of having the exact same functional form, and so reflecting the exact same dependence structure (*and* parameters).

Now, impose the sufficient condition of the Lemma, and *assume* that the Z_i variables follow a Multivariate Normal Distribution, $\Phi_m\left(Z_1,...,Z_m\right)$. By Sklar's theorem this will be equal to the associated unique Copula,

$$\Phi_m\left(Z_1,...,Z_m\right)=C_{Z_1\cdots Z_m}\left(\Phi\left(Z_1\right),...,\Phi\left(Z_m\right)\right)$$

Equating this with [5] we obtain

$$\Phi_m\left(Z_1,...,Z_m\right)=C_{X_1\cdots X_m}\left(F_1\left(X_1\right),...,F_m\left(X_m\right)\right)$$

and since

$$\Phi_m\left(Z_1,...,Z_m\right)=\Phi_m\left(\Phi^{-1}\left(U_1\right),...,\Phi^{-1}\left(U_m\right)\right)=\Phi_m\left(\Phi^{-1}\left(F_1\left(X_1\right)\right),...,\Phi^{-1}\left(F_m\left(X_m\right)\right)\right)$$

we arrive at

$$C_{X_1\cdots X_m}\left(F_1\left(X_1\right),...,F_m\left(X_m\right)\right)=\Phi_m\left(\Phi^{-1}\left(F_1\left(X_1\right)\right),...,\Phi^{-1}\left(F_m\left(X_m\right)\right)\right)\quad \text{QED.}$$

B. Proof of eq. [4.54].

Let two random variables X,Z with distribution functions F_X,F_Z respectively, and consider the transformations $\Phi^{-1}\left(F_X\left(X\right)\right)$, $\Phi^{-1}\left(F_Z\left(Z\right)\right)$. These are two standard Normal random variables. The absolute correlation coefficient of the transformed variables is

$$\left|\rho\left(\Phi^{-1}\left(F_X\left(X\right)\right),\ \Phi^{-1}\left(F_Z\left(Z\right)\right)\right)\right| \ \le\ \rho_M\left(X,Z\right)\equiv\sup_{h,g}\left\{\rho\left[h\left(X\right),g\left(Z\right)\right]\right\}$$

where h,g are any transformations of the random variables (including the identity transformation). Using the symbol $\circ$ to denote function composition, we have

$$\rho_M\left(X,Z\right)=\rho_M\left(F_X^{-1}\circ\Phi\circ\Phi^{-1}\circ F_X\left(X\right),F_Z^{-1}\circ\Phi\circ\Phi^{-1}\circ F_Z\left(Z\right)\right)$$

$$=\rho_M\left(\left(F_X^{-1}\circ\Phi\right)\circ\Phi^{-1}\left(F_X\left(X\right)\right),\left(F_Z^{-1}\circ\Phi\right)\circ\Phi^{-1}\left(F_Z\left(Z\right)\right)\right)$$

Let h^*, g^* be the argsup of this ρ_M. Then we can write

$$\rho_M\left(\left(F_X^{-1}\circ\Phi\right)\circ\Phi^{-1}\left(F_X\left(X\right)\right),\left(F_Z^{-1}\circ\Phi\right)\circ\Phi^{-1}\left(F_Z\left(Z\right)\right)\right)$$

$$=\rho\left\{h^*\left[\left(F_X^{-1}\circ\Phi\right)\circ\Phi^{-1}\left(F_X\left(X\right)\right)\right],g^*\left[\left(F_Z^{-1}\circ\Phi\right)\circ\Phi^{-1}\left(F_Z\left(Z\right)\right)\right]\right\}$$

$$=\rho\left\{\left(h^*\circ F_X^{-1}\circ\Phi\right)\circ\Phi^{-1}\left(F_X\left(X\right)\right),\left(g^*\circ F_Z^{-1}\circ\Phi\right)\circ\Phi^{-1}\left(F_Z\left(Z\right)\right)\right\}$$

We argue that

$$\rho\left\{\left(h^*\circ F_X^{-1}\circ\Phi\right)\circ\Phi^{-1}\left(F_X\left(X\right)\right),\left(g^*\circ F_Z^{-1}\circ\Phi\right)\circ\Phi^{-1}\left(F_Z\left(Z\right)\right)\right\}$$

$$\leq\rho_M\left[\Phi^{-1}\left(F_X\left(X\right)\right),\Phi^{-1}\left(F_Z\left(Z\right)\right)\right]\equiv\sup_{k,l}\left\{\rho\left[k\left(\Phi^{-1}\left(F_X\left(X\right)\right)\right),l\left(\Phi^{-1}\left(F_Z\left(Z\right)\right)\right)\right]\right\}$$

$$=\sup_{k,l}\left\{\rho\left[k\circ\Phi^{-1}\left(F_X\left(X\right)\right),l\circ\Phi^{-1}\left(F_Z\left(Z\right)\right)\right]\right\}$$

This inequality holds because we can choose $k=\left(h^*\circ F_X^{-1}\circ\Phi\right)$, $l=\left(g^*\circ F_Z^{-1}\circ\Phi\right)$ and obtain the equality, without necessarily obtain the supremum. But if $\Phi^{-1}\left(F_X\left(X\right)\right),\Phi^{-1}\left(F_Z\left(Z\right)\right)$ are jointly Normal, then it can be proven (see Klaassen and Wellner 1997 theorem 6.1) that

$$\rho_M\left[\Phi^{-1}\left(F_X\left(X\right)\right),\ \Phi^{-1}\left(F_Z\left(Z\right)\right)\right]=\left|\rho\left(\Phi^{-1}\left(F_X\left(X\right)\right),\ \Phi^{-1}\left(F_Z\left(Z\right)\right)\right)\right|$$

Linking relations, we have arrived at the sandwich inequality

$$\left|\rho\left(\Phi^{-1}\left(F_X\left(X\right)\right),\ \Phi^{-1}\left(F_Z\left(Z\right)\right)\right)\right|\leq\rho_M\left(X,Z\right)\leq\left|\rho\left(\Phi^{-1}\left(F_X\left(X\right)\right),\ \Phi^{-1}\left(F_Z\left(Z\right)\right)\right)\right|$$

which proves what we wanted to prove.--

Chapter 6

Re-visiting the production frontier: the contribution of management to production, the "wrong skewness" problem, and a two tier-stochastic frontier model to measure them.

Technical Appendix

A. The Cost-minimization problem of a price-taking firm with management.

We examine the problem

$$\min_{m,x_1,x_2} C = r_m m + r_1 x_1 + r_2 x_2$$

$$\text{s.t.} \quad g\left(m, x_1, x_2\right) = e^{h(m)} F\left(x_1, x_2\right) = \bar{Q}$$

and given prices. We will use subscripts to denote partial derivatives. We make the standard assumptions of positive marginal products, $F_1, F_2 > 0$, diminishing returns $F_{11}, F_{22} < 0$, and complementarity of conventional inputs, $F_{12} \geq 0$.

The Lagrangean is

$$\Lambda = r_m m + r_1 x_1 + r_2 x_2 \; + \; \lambda\left(\bar{Q} - e^{h(m)} F\left(x_1, x_2\right)\right)$$

The first-order conditions are

$$\frac{\partial \Lambda}{\partial m} = r_m - \lambda h'\left(m\right) e^{h(m)} F\left(x_1, x_2\right) = 0 \;, \quad \frac{\partial \Lambda}{\partial x_1} = r_1 - \lambda e^{h(m)} F_1\left(x_1, x_2\right) = 0$$

$$\frac{\partial \Lambda}{\partial x_2} = r_2 - \lambda e^{h(m)} F_2\left(x_1, x_2\right) = 0$$

From the first condition, it is evident that at the optimum, $h'(m) > 0$.

To examine the second-order conditions we form the bordered Hessian matrix

$$\bar{H} = \begin{bmatrix} 0 & g_m & g_1 & g_2 \\ g_m & \Lambda_{mm} & \Lambda_{m1} & \Lambda_{m2} \\ g_1 & \Lambda_{m1} & \Lambda_{11} & \Lambda_{12} \\ g_2 & \Lambda_{m2} & \Lambda_{12} & \Lambda_{22} \end{bmatrix}$$

Omitting the arguments of the functions for compactness, the elements of this matrix are

$$g_m = h'e^h F, \quad g_1 = e^h F_1, \quad g_2 = e^h F_2$$

$$\Lambda_{mm} = -\lambda e^h F\left(h'' + (h')^2\right), \quad \Lambda_{m1} = -\lambda h' e^h F_1, \quad \Lambda_{m2} = -\lambda h' e^h F_2$$

$$\Lambda_{11} = -\lambda e^h F_{11}, \quad \Lambda_{12} = -\lambda e^h F_{12}, \quad \Lambda_{22} = -\lambda e^h F_{22}$$

For sufficiency we require the principal minors to be strictly negative (see Chiang 1984, p. 385) , of $\bar{H}$ but also of its permutations since the order we positioned the three decision variables is arbitrary and could be different:

$$\left|\bar{H}_2\right| = \begin{vmatrix} 0 & g_m & g_1 \\ g_m & \Lambda_{mm} & \Lambda_{m1} \\ g_1 & \Lambda_{m1} & \Lambda_{11} \end{vmatrix} < 0, \qquad \left|\bar{H}_3\right| = \begin{vmatrix} 0 & g_m & g_1 & g_2 \\ g_m & \Lambda_{mm} & \Lambda_{m1} & \Lambda_{m2} \\ g_1 & \Lambda_{m1} & \Lambda_{11} & \Lambda_{12} \\ g_2 & \Lambda_{m2} & \Lambda_{12} & \Lambda_{22} \end{vmatrix} < 0$$

Permutations will be examined for the $\left|\bar{H}_2\right|$ principal minor. For the $\left|\bar{H}_3\right|$ any re-arrangement that brings the rows and columns in border-hessian form is completed by moving rows and columns an even number of times, so we would get the same value for the determinant.

We have,

$$\left|\bar{H}_2\right| = \begin{vmatrix} 0 & g_m & g_1 \\ g_m & \Lambda_{mm} & \Lambda_{m1} \\ g_1 & \Lambda_{m1} & \Lambda_{11} \end{vmatrix} = \begin{vmatrix} 0 & h'e^hF & e^hF_1 \\ h'e^hF & -\lambda e^hF\left(h''+(h')^2\right) & -\lambda h'e^hF_1 \\ e^hF_1 & -\lambda h'e^hF_1 & -\lambda e^hF_{11} \end{vmatrix}$$

Applying standard properties of determinants we have

$$\left|\bar{H}_2\right| = e^{3h}\begin{vmatrix} 0 & h'F & F_1 \\ h'F & -\lambda F\left(h''+(h')^2\right) & -\lambda h'F_1 \\ F_1 & -\lambda h'F_1 & -\lambda F_{11} \end{vmatrix}$$

$$= e^{3h}\left[-h'F\left(-h'F\lambda F_{11}+\lambda h'F_1^2\right)+F_1\left(-\lambda(h')^2FF_1+\lambda FF_1\left(h''+(h')^2\right)\right)\right]$$

$$= e^{3h}\left[\lambda(h')^2F\left(FF_{11}-F_1^2\right)+F_1\left(-\lambda(h')^2FF_1+\lambda FF_1h''+\lambda(h')^2FF_1\right)\right]$$

$$= \lambda e^{3h}F\left[(h')^2\left(FF_{11}-F_1^2\right)+h''F_1^2\right]$$

Given our assumptions, $FF_{11}-F_1^2<0$ so a sufficient condition is $h''\leq 0$.

For the first permutation, we need to examine

$$\left|\bar{H}_2\right|_{m,2} = \begin{vmatrix} 0 & g_m & g_2 \\ g_m & \Lambda_{mm} & \Lambda_{m2} \\ g_2 & \Lambda_{m2} & \Lambda_{22} \end{vmatrix}$$ which immediately leads to the conditions

$$\left\{FF_{22}-F_2^2<0, \quad h''\leq 0\right\}$$

which are the same as before.

For the final permutation we examine

$$\left|\bar{H}_2\right|_{1,2} = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & \Lambda_{11} & \Lambda_{12} \\ g_2 & \Lambda_{12} & \Lambda_{22} \end{vmatrix} = \begin{vmatrix} 0 & e^h F_1 & e^h F_2 \\ e^h F_1 & -\lambda e^h F_{11} & -\lambda e^h F_{12} \\ e^h F_2 & -\lambda e^h F_{12} & -\lambda e^h F_{22} \end{vmatrix}$$

$$= e^{3h} \begin{vmatrix} 0 & F_1 & F_2 \\ F_1 & -\lambda F_{11} & -\lambda F_{12} \\ F_2 & -\lambda F_{12} & -\lambda F_{22} \end{vmatrix} = e^{3h}\left[-F_1\left(-\lambda F_{22} F_1 + \lambda F_{12} F_2\right) + F_2\left(-\lambda F_{12} F_1 + \lambda F_{11} F_2\right)\right]$$

$$= e^{3h}\left[\lambda F_{22} F_1^2 - \lambda F_{12} F_1 F_2 - \lambda F_{12} F_1 F_2 + \lambda F_{11} F_2^2\right] < 0$$

The expression is negative given the assumptions on the derivatives of the production function with respect to the traditional inputs. So no additional condition arises here.

Turning to $\left|\bar{H}_3\right|$ we have

$$\left|\bar{H}_3\right| = \begin{vmatrix} 0 & g_m & g_1 & g_2 \\ g_m & \Lambda_{mm} & \Lambda_{m1} & \Lambda_{m2} \\ g_1 & \Lambda_{m1} & \Lambda_{11} & \Lambda_{12} \\ g_2 & \Lambda_{m2} & \Lambda_{12} & \Lambda_{22} \end{vmatrix} = -g_m \begin{vmatrix} g_m & g_1 & g_2 \\ \Lambda_{m1} & \Lambda_{11} & \Lambda_{12} \\ \Lambda_{m2} & \Lambda_{12} & \Lambda_{22} \end{vmatrix} + g_1 \begin{vmatrix} g_m & g_1 & g_2 \\ \Lambda_{mm} & \Lambda_{m1} & \Lambda_{m2} \\ \Lambda_{m2} & \Lambda_{12} & \Lambda_{22} \end{vmatrix} - g_2 \begin{vmatrix} g_m & g_1 & g_2 \\ \Lambda_{mm} & \Lambda_{m1} & \Lambda_{m2} \\ \Lambda_{m1} & \Lambda_{11} & \Lambda_{12} \end{vmatrix}$$

Calculating one by one we have

$$-g_m \begin{vmatrix} g_m & g_1 & g_2 \\ \Lambda_{m1} & \Lambda_{11} & \Lambda_{12} \\ \Lambda_{m2} & \Lambda_{12} & \Lambda_{22} \end{vmatrix} = -h' e^h F \begin{vmatrix} h' e^h F & e^h F_1 & e^h F_2 \\ -\lambda h' e^h F_1 & -\lambda e^h F_{11} & -\lambda e^h F_{12} \\ -\lambda h' e^h F_2 & -\lambda e^h F_{12} & -\lambda e^h F_{22} \end{vmatrix} = -(h')^2 \lambda^2 e^{4h} F \begin{vmatrix} F & F_1 & F_2 \\ F_1 & F_{11} & F_{12} \\ F_2 & F_{12} & F_{22} \end{vmatrix}$$

Turning to the second,

$$g_1\begin{vmatrix} g_m & g_1 & g_2 \\ \Lambda_{mm} & \Lambda_{m1} & \Lambda_{m2} \\ \Lambda_{m2} & \Lambda_{12} & \Lambda_{22} \end{vmatrix} = e^h F_1 \begin{vmatrix} h'e^h F & e^h F_1 & e^h F_2 \\ -\lambda e^h F\left(h''+(h')^2\right) & -\lambda h' e^h F_1 & -\lambda h' e^h F_2 \\ -\lambda h' e^h F_2 & -\lambda e^h F_{12} & -\lambda e^h F_{22} \end{vmatrix}$$

$$= \lambda^2 e^{4h} F_1 \begin{vmatrix} h'F & F_1 & F_2 \\ \left(h''+(h')^2\right)F & h'F_1 & h'F_2 \\ h'F_2 & F_{12} & F_{22} \end{vmatrix}$$

$$= \lambda^2 e^{4h} F_1 \begin{vmatrix} h'F & F_1 & F_2 \\ (h')^2 F & h'F_1 & h'F_2 \\ h'F_2 & F_{12} & F_{22} \end{vmatrix} - \lambda^2 e^{4h} F_1 h'' F \begin{vmatrix} F_1 & F_2 \\ F_{12} & F_{22} \end{vmatrix}$$

$$= (h')^2 \lambda^2 e^{4h} F_1 \begin{vmatrix} F & F_1 & F_2 \\ F & F_1 & F_2 \\ F_2 & F_{12} & F_{22} \end{vmatrix} - \lambda^2 e^{4h} F_1 h'' F \begin{vmatrix} F_1 & F_2 \\ F_{12} & F_{22} \end{vmatrix}$$

The first determinant has two identical rows so it equals zero. Therefore

$$g_1\begin{vmatrix} g_m & g_1 & g_2 \\ \Lambda_{mm} & \Lambda_{m1} & \Lambda_{m2} \\ \Lambda_{m2} & \Lambda_{12} & \Lambda_{22} \end{vmatrix} = -\lambda^2 e^{4h} F_1 h'' F \begin{vmatrix} F_1 & F_2 \\ F_{12} & F_{22} \end{vmatrix}$$

Finally

$$-g_2\begin{vmatrix} g_m & g_1 & g_2 \\ \Lambda_{mm} & \Lambda_{m1} & \Lambda_{m2} \\ \Lambda_{m1} & \Lambda_{11} & \Lambda_{12} \end{vmatrix} = -e^h F_2 \begin{vmatrix} h'e^h F & e^h F_1 & e^h F_2 \\ -\lambda e^h F\left(h''+(h')^2\right) & -\lambda h' e^h F_1 & -\lambda h' e^h F_2 \\ -\lambda h' e^h F_1 & -\lambda e^h F_{11} & -\lambda e^h F_{12} \end{vmatrix}$$

$$= -\lambda^2 e^{4h} F_2 \begin{vmatrix} h'F & F_1 & F_2 \\ F\left(h''+(h')^2\right) & h'F_1 & h'F_2 \\ h'F_1 & F_{11} & F_{12} \end{vmatrix}$$

$$= -\lambda^2 e^{4h} F_2 \begin{vmatrix} h'F & F_1 & F_2 \\ (h')^2 F & h'F_1 & h'F_2 \\ h'F_1 & F_{11} & F_{12} \end{vmatrix} + \lambda^2 e^{4h} F_2 F h'' \begin{vmatrix} F_1 & F_2 \\ F_{11} & F_{12} \end{vmatrix}$$

$$= -(h')^2 \lambda^2 e^{4h} F_2 \begin{vmatrix} F & F_1 & F_2 \\ F & F_1 & F_2 \\ F_1 & F_{11} & F_{12} \end{vmatrix} + \lambda^2 e^{4h} F_2 F h'' \begin{vmatrix} F_1 & F_2 \\ F_{11} & F_{12} \end{vmatrix}$$

$$= \ 0 \ + \ \lambda^2 e^{4h} F_2 F h'' \begin{vmatrix} F_1 & F_2 \\ F_{11} & F_{12} \end{vmatrix}$$

Collecting results we have

$$\left|\bar{H}_3\right| = -(h')^2 \lambda^2 e^{4h} F \begin{vmatrix} F & F_1 & F_2 \\ F_1 & F_{11} & F_{12} \\ F_2 & F_{12} & F_{22} \end{vmatrix} - \lambda^2 e^{4h} F_1 h'' F \begin{vmatrix} F_1 & F_2 \\ F_{12} & F_{22} \end{vmatrix} + \lambda^2 e^{4h} F_2 F h'' \begin{vmatrix} F_1 & F_2 \\ F_{11} & F_{12} \end{vmatrix}$$

Ignoring the common strictly positive terms, $\text{sign}\left\{\left|\bar{H}_3\right|\right\}$ has the sign of the expression

$$-(h')^2 \begin{vmatrix} F & F_1 & F_2 \\ F_1 & F_{11} & F_{12} \\ F_2 & F_{12} & F_{22} \end{vmatrix} - F_1 h'' \begin{vmatrix} F_1 & F_2 \\ F_{12} & F_{22} \end{vmatrix} + F_2 h'' \begin{vmatrix} F_1 & F_2 \\ F_{11} & F_{12} \end{vmatrix}$$

For the first component that does not involve h'' we have

$$-(h')^2 \begin{vmatrix} F & F_1 & F_2 \\ F_1 & F_{11} & F_{12} \\ F_2 & F_{12} & F_{22} \end{vmatrix} = -(h')^2 F \left(F_{11}F_{22} - F_{12}^2\right) + (h')^2 \begin{vmatrix} 0 & F_1 & F_2 \\ F_1 & F_{11} & F_{12} \\ F_2 & F_{12} & F_{22} \end{vmatrix}$$

Note that the second term has the bordered Hessian determinant of the conventional production function. So under the assumption that $F\left(x_1, x_2\right)$ is strictly quasi-concave, this term will be negative. Strict quasi concavity incorporates the standard assumptions of positive marginal products, $F_1, F_2 > 0$, diminishing returns $F_{11}, F_{22} < 0$ and $F_{12} \geq 0$

Under strict quasi-concavity of $F\left(x_1, x_2\right)$, a sufficient condition for the whole expression to be negative is that the first term is non-positive. Since at the optimum $h' > 0$ this requires $F_{11}F_{22} - F_{12}^2 \ \geq 0$. This is also the condition that the conventional production function $F(x_1, x_2)$ is concave, which accommodates constant and decreasing returns to scale in (x_1, x_2), when $F(x_1, x_2)$ is a homogeneous function.

Turning to the terms of $|\bar{H}_3|$ that involve h'' we have

$$-F_1 h'' \begin{vmatrix} F_1 & F_2 \\ F_{12} & F_{22} \end{vmatrix} + F_2 h'' \begin{vmatrix} F_1 & F_2 \\ F_{11} & F_{12} \end{vmatrix} = h''\left(-F_1\left(F_1F_{22} - F_2F_{12}\right) + F_2\left(F_1F_{12} - F_2F_{11}\right)\right)$$

$$= h''\left(-F_1^2 F_{22} + F_1F_2F_{12} + F_2F_1F_{12} - F_2^2 F_{11}\right)$$

All the terms in the parenthesis are positive, so the sign here depends solely on the sign of h''. It can also be zero since the previous component of $|\bar{H}_3|$ was found to be negative.

Combining, we see that under the usual assumptions on the conventional production function, the sufficient condition related to management is

$$h''(m) \leq 0.$$

B. The profit-maximization problem of a price-taking firm with management.

We examine the problem

$$\max_{m, x_1, x_2} \pi = e^{h(m)} F\left(x_1, x_2\right) - r_m m - r_1 x_1 - r_2 x_2$$

and given prices. We will use subscripts to denote partial derivatives. We make the standard assumptions of positive marginal products, $F_1, F_2 > 0$, diminishing returns $F_{11}, F_{22} < 0$, and $F_{12} \geq 0$.

The first-order conditions are

$$\frac{\partial \pi}{\partial m} = h'(m)e^{h(m)}F(x_1, x_2) - r_m = 0, \qquad \frac{\partial \pi}{\partial x_1} = e^{h(m)}F_1(x_1, x_2) - r_1 = 0,$$

$$\frac{\partial \pi}{\partial x_2} = e^{h(m)}F_2(x_1, x_2) - r_2 = 0$$

Here too at the optimum we will have $h'(m) > 0$.

To examine the second-order conditions we form the Hessian matrix

$$H = \begin{bmatrix} \pi_{mm} & \pi_{m1} & \pi_{m2} \\ \pi_{m1} & \pi_{11} & \pi_{12} \\ \pi_{m2} & \pi_{12} & \pi_{22} \end{bmatrix}$$. The elements of this matrix are

$$\pi_{mm} = e^h F\left(h'' + (h')^2\right), \quad \pi_{m1} = h'e^h F_1, \quad \pi_{m2} = h'e^h F_2$$

$$\pi_{11} = e^h F_{11}, \qquad \pi_{12} = e^h F_{12}, \qquad \pi_{22} = e^h F_{22}$$

We need all second partial derivative to be negative, so also $h'' + (h')^2 < 0$.

We need all symmetric permutations of the second principal minor to be positive,

$$\begin{vmatrix} \pi_{mm} & \pi_{m1} \\ \pi_{m1} & \pi_{11} \end{vmatrix} > 0, \begin{vmatrix} \pi_{mm} & \pi_{m2} \\ \pi_{m2} & \pi_{22} \end{vmatrix} > 0, \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{12} & \pi_{22} \end{vmatrix} > 0$$

$$\pi_{mm}\pi_{11} > (\pi_{m1})^2, \; \pi_{mm}\pi_{22} > (\pi_{m2})^2, \; \pi_{11}\pi_{22} > (\pi_{12})^2$$

$$\begin{cases} e^h F\left(h'' + (h')^2\right)e^h F_{11} > \left(h'e^h F_1\right)^2 \Rightarrow \; h''FF_{11} > (h')^2\left(F_1^2 - FF_{11}\right) \\ e^h F\left(h'' + (h')^2\right)e^h F_{22} > \left(h'e^h F_2\right)^2 \Rightarrow h''FF_{22} > (h')^2\left(F_2^2 - FF_{22}\right) \\ e^h F_{11} e^h F_{22} > \left(e^h F_{12}\right)^2 \Rightarrow F_{11}F_{22} - F_{12}^2 > 0 \end{cases}$$

The last condition requires that $F\left(x_1, x_2\right)$ is jointly strictly concave in $\left(x_1, x_2\right)$

$$\begin{vmatrix} e^h F\left(h'' + (h')^2\right) & h'e^h F_1 & h'e^h F_2 \\ h'e^h F_1 & e^h F_{11} & e^h F_{12} \\ h'e^h F_2 & e^h F_{12} & e^h F_{22} \end{vmatrix} = e^{3h} \begin{vmatrix} F\left(h'' + (h')^2\right) & h'F_1 & h'F_2 \\ h'F_1 & F_{11} & F_{12} \\ h'F_2 & F_{12} & F_{22} \end{vmatrix}$$

$$= e^{3h} \begin{vmatrix} F\left(h'' + (h')^2\right) & h'F_1 & h'F_2 \\ h'F_1 & F_{11} & F_{12} \\ h'F_2 & F_{12} & F_{22} \end{vmatrix} = (h')^2 e^{3h} \begin{vmatrix} 0 & F_1 & F_2 \\ F_1 & F_{11} & F_{12} \\ F_2 & F_{12} & F_{22} \end{vmatrix} + e^{3h} F\left(h'' + (h')^2\right) \begin{vmatrix} F_{11} & F_{12} \\ F_{12} & F_{22} \end{vmatrix}$$

The 3 X 3 determinant is the determinant of the bordered Hessian of $F\left(x_1, x_2\right)$, and it will be negative if $F\left(x_1, x_2\right)$ is strictly quasi-concave, which it is since we already require that $F\left(x_1, x_2\right)$ is strictly concave. The 2 X 2 determinant is positive because of strict concavity and $\left(h'' + (h')^2\right)$ is negative.

So in all, we require that $F\left(x_1, x_2\right)$ is strictly concave and that $h'' + (h')^2 < 0$, which certainly excludes $h'' = 0$.

C. Interactions between management and production inputs in a Cobb-Douglas production function - proof of equation [8] of the main text.

We want to prove $\quad Q = e^{h(m)} AK^a L^b, \quad a,b < 1 \Rightarrow \dfrac{dK/dm}{dL/dm} = \dfrac{K}{L}$

in a cost-minimization context. From the first order conditions derived earlier

$$\frac{\partial \Lambda}{\partial m} = r_m - \lambda h'(m) e^{h(m)} F(x_1, x_2) = 0 \ , \quad \frac{\partial \Lambda}{\partial x_1} = r_1 - \lambda e^{h(m)} F_1(x_1, x_2) = 0$$

$$\frac{\partial \Lambda}{\partial x_2} = r_2 - \lambda e^{h(m)} F_2(x_1, x_2) = 0$$

We have, say for the first input

$$\left\{ \begin{array}{l} r_m = \lambda h'(m) e^{h(m)} F(x_1, x_2) \Rightarrow \dfrac{r_m}{h'(m) F(x_1, x_2)} = \lambda e^{h(m)} \\ \\ \qquad r_1 = \lambda e^{h(m)} F_1(x_1, x_2) \Rightarrow \dfrac{r_m}{F_1(x_1, x_2)} = \lambda e^{h(m)} \end{array} \right. \Rightarrow \frac{r_m}{h'(m) F(x_1, x_2)} = \frac{r_1}{F_1(x_1, x_2)}$$

and so for both inputs $\left\{ \begin{array}{l} G_1 = r_1 h'(m) F(x_1, x_2) - r_m F_1(x_1, x_2) = 0 \\ \\ G_2 = r_2 h'(m) F(x_1, x_2) - r_m F_2(x_1, x_2) = 0 \end{array} \right.$

We map $x_1 = K, \ x_2 = L$. We have, omitting the function's arguments

$$\frac{dK/dm}{dL/dm} = \frac{-\dfrac{\partial G_1/\partial m}{\partial G_1/\partial K}}{-\dfrac{\partial G_2/\partial m}{\partial G_2/\partial L}} = \frac{\dfrac{-h'' r_K F}{r_K h' F_K - r_m F_{KK}}}{\dfrac{-h'' r_L F}{r_L h' F_L - r_m F_{LL}}} = \frac{r_K (r_L h' F_L - r_m F_{LL})}{r_L (r_K h' F_K - r_m F_{KK})}$$

This is meaningful only if $h'' < 0$. From the optimal relations we have

$\dfrac{h'F}{r_m} = \dfrac{F_K}{r_K} = \dfrac{F_L}{r_L}$ and we obtain, solving for r_m , $\quad r_m = r_K \dfrac{h'F}{F_K}, \quad r_m = r_L \dfrac{h'F}{F_L}.$

Inserting these into $\dfrac{dK/dm}{dL/dm}$ we have

$$\frac{dK/dm}{dL/dm}=\frac{r_K\left(r_L h'F_L-r_L\dfrac{h'F}{F_L}F_{LL}\right)}{r_L\left(r_K h'F_K-r_K\dfrac{h'F}{F_K}F_{KK}\right)}=\frac{F_L-\dfrac{F}{F_L}F_{LL}}{F_K-\dfrac{F}{F_K}F_{KK}}$$

For the first and second partial derivatives we have,

$$\frac{\partial F}{\partial K}=F_K=a\frac{F}{K},\quad F_{KK}=a\frac{F_K K-F}{K^2}=a\frac{aF-F}{K^2}=a(a-1)\frac{F}{K^2}=-(1-a)\frac{F_K}{K}$$

and $\quad F_L=b\dfrac{F}{L},\quad F_{LL}=-(1-b)\dfrac{F_L}{L}$

Using these we get

$$\frac{dK/dm}{dL/dm}=\frac{b\dfrac{F}{L}+\dfrac{F}{F_L}(1-b)\dfrac{F_L}{L}}{a\dfrac{F}{K}+\dfrac{F}{F_K}(1-a)\dfrac{F_K}{K}}=\frac{b\dfrac{F}{L}+\dfrac{F}{L}(1-b)}{a\dfrac{F}{K}+\dfrac{F}{K}(1-a)}=\frac{F/L}{F/K}=\frac{K}{L}$$

which is what we wanted to prove.

D. Conditional expected values for the 2TSF Generalized Exponential specification when $\varepsilon = w - u$.

We are initially considering the composite error term $\varepsilon = v + w - u$ with

$$v \sim \mathrm{N}\left(0, \sigma_v^2\right),\ w \sim GE\left(2, \theta_w, 0\right),\ \ u \sim GE\left(2, \theta_u, 0\right)$$

Writing $f_E\left(w; \theta_w\right)$ for the density of an Exponential random variable with scale parameter θ_w, we have

$$w \sim GE\left(2, \theta_w, 0\right) \Rightarrow f_w\left(w\right) = 2 f_E\left(w; \theta_w\right) - f_E\left(w; \theta_w / 2\right)$$

$$u \sim GE\left(2, \theta_u, 0\right) \Rightarrow f_u\left(u\right) = 2 f_E\left(u; \theta_u\right) - f_E\left(u; \theta_u / 2\right)$$

Suppose now that we obtain $\sigma_v^2 = 0$. Then $\varepsilon = v + w - u = w - u = z$

D.1. Conditional densities.

We are interested in the conditional densities $f_{w|\varepsilon} = \dfrac{f_{\varepsilon,w}\left(\varepsilon, w\right)}{f_\varepsilon\left(\varepsilon\right)}$ $f_{u|\varepsilon} = \dfrac{f_{\varepsilon,u}\left(\varepsilon, u\right)}{f_\varepsilon\left(\varepsilon\right)}$

a. $f_{w|\varepsilon} = \dfrac{f_{\varepsilon,w}\left(\varepsilon, w\right)}{f_\varepsilon\left(\varepsilon\right)}$

Note that $\varepsilon = w - u \Rightarrow u = w - \varepsilon$. We have

$$f_{w,u}\left(w,u\right) = f_w\left(w\right) f_u\left(u\right) = f_w\left(w\right) f_u\left(w - \varepsilon\right) = f_{\varepsilon,w}\left(\varepsilon, w\right)$$

So $f_{w|\varepsilon} = \dfrac{f_w(w)\, f_u(w-\varepsilon)}{f_\varepsilon(\varepsilon)}$

Note that from $u = w - \varepsilon \geq 0 \Rightarrow w \geq \varepsilon$. If $\varepsilon < 0$ this does not constrain the support of the random variable "w conditional on ε". But if $\varepsilon \geq 0$, it follows, and intuitively so, that we must have $w \geq \varepsilon$ because u is subtracted from w to obtain ε. This restricts the conditional support of w and so also the interval of integration over the conditional density. Since we want to obtain a single expression for all ε we can express the limit of integration as $\max\{\varepsilon, 0\}$.

b. Analogously, we have

$$f_{u|\varepsilon} = \frac{f_{\varepsilon,w}(\varepsilon,u)}{f_\varepsilon(\varepsilon)},\quad f_{w,u}(w,u) = f_w(w)\, f_u(u) = f_w(\varepsilon+u)\, f_u(u) = f_{\varepsilon,u}(\varepsilon,u)$$

So $f_{w|\varepsilon} = \dfrac{f_w(\varepsilon+u)\, f_u(u)}{f_\varepsilon(\varepsilon)}$

Here we have that $\varepsilon + u = w \geq 0 \Rightarrow u \geq -\varepsilon$. When $\varepsilon > 0$ this holds always, but when $\varepsilon \leq 0$ it will restrict the support of u and so also the interval of integration over the conditional density. Since we want to obtain a single expression for all ε we can express the limit of integration as $-\min\{\varepsilon, 0\}$

D.2. Conditional expected values.

D.2.1. $E\left(\exp\{\pm w\}|\varepsilon\right)$.

$$E\left(\exp\{\pm w\}|\varepsilon\right) = \int_{\max\{\varepsilon,0\}}^{\infty} \exp\{\pm w\}\, f_{w|\varepsilon}(w|\varepsilon)\,dw = \int_{\max\{\varepsilon,0\}}^{\infty} \exp\{\pm w\}\, \frac{f_w(w)\, f_u(w-\varepsilon)}{f_\varepsilon(\varepsilon)}\,dw$$

$$= \frac{1}{f_\varepsilon(\varepsilon)} \int_{\max\{\varepsilon,0\}}^{\infty} \exp\{\pm w\} \left[2 f_E(w;\theta_w) - f_E(w;\theta_w/2)\right]\left[2 f_E(w-\varepsilon;\theta_u) - f_E(w-\varepsilon;\theta_u/2)\right] dw$$

$$= \frac{1}{f_\varepsilon(\varepsilon)} \int_{\max\{\varepsilon,0\}}^{\infty} \exp\{\pm w\} \left[4 f_E(w;\theta_w) f_E(w-\varepsilon;\theta_u) - 2 f_E(w;\theta_w/2) f_E(w-\varepsilon;\theta_u)\right] dw$$

$$+ \frac{1}{f_\varepsilon(\varepsilon)} \int_{\max\{\varepsilon,0\}}^{\infty} \exp\{\pm w\} \left[-2 f_E(w;\theta_w) f_E(w-\varepsilon;\theta_u/2) + f_E(w;\theta_w/2) f_E(w-\varepsilon;\theta_u/2)\right] dw$$

$$= \frac{\exp\{\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)} \int_{\max\{\varepsilon,0\}}^{\infty} \exp\{\pm w\} \left[4 f_E(w;\theta_w) f_E(w;\theta_u) - 2 f_E(w;\theta_w/2) f_E(w;\theta_u)\right] dw$$

$$+ \frac{\exp\{2\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)} \int_{\max\{\varepsilon,0\}}^{\infty} \exp\{\pm w\} \left[-2 f_E(w;\theta_w) f_E(w;\theta_u/2) + f_E(w;\theta_w/2) f_E(w;\theta_u/2)\right] dw$$

$$= \frac{\exp\{\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)} \int_{\max\{\varepsilon,0\}}^{\infty} \exp\{\pm w\} \left[\frac{4}{\theta_w\theta_u} \exp\left\{-\frac{\theta_w+\theta_u}{\theta_w\theta_u} w\right\} - \frac{4}{\theta_w\theta_u} \exp\left\{-\frac{\theta_w+2\theta_u}{\theta_w\theta_u} w\right\}\right] dw$$

$$+ \frac{\exp\{2\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)} \int_{\max\{\varepsilon,0\}}^{\infty} \exp\{\pm w\} \left[-\frac{4}{\theta_w\theta_u} \exp\left\{-\frac{2\theta_w+\theta_u}{\theta_w\theta_u} w\right\} + \frac{4}{\theta_w\theta_u} \exp\left\{-\frac{2\theta_w+2\theta_u}{\theta_w\theta_u} w\right\}\right] dw$$

$$= \frac{\exp\{\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)} \frac{4}{\theta_w\theta_u} \int_{\max\{\varepsilon,0\}}^{\infty} \left[\exp\left\{-\frac{\theta_w+\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u} w\right\} - \exp\left\{-\frac{\theta_w+2\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u} w\right\}\right] dw$$

$$+ \frac{\exp\{2\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)} \frac{4}{\theta_w\theta_u} \int_{\max\{\varepsilon,0\}}^{\infty} \left[-\exp\left\{-\frac{2\theta_w+\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u} w\right\} + \exp\left\{-\frac{2\theta_w+2\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u} w\right\}\right] dw$$

$$
= \frac{\exp\{\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)} \frac{4}{\theta_w\theta_u} \frac{\theta_w\theta_u}{\theta_w+\theta_u-(\pm\theta_w\theta_u)} \int_{\max\{\varepsilon,0\}}^{\infty} \frac{\theta_w+\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u} \exp\left\{-\frac{\theta_w+\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u} w\right\} dw
$$

$$
- \frac{\exp\{\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)} \frac{4}{\theta_w\theta_u} \frac{\theta_w\theta_u}{\theta_w+2\theta_u-(\pm\theta_w\theta_u)} \int_{\max\{\varepsilon,0\}}^{\infty} \frac{\theta_w+2\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u} \exp\left\{-\frac{\theta_w+2\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u} w\right\} dw
$$

$$
- \frac{\exp\{2\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)} \frac{4}{\theta_w\theta_u} \frac{\theta_w\theta_u}{2\theta_w+\theta_u-(\pm\theta_w\theta_u)} \int_{\max\{\varepsilon,0\}}^{\infty} \frac{2\theta_w+\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u} \exp\left\{-\frac{2\theta_w+\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u} w\right\} dw
$$

$$
+ \frac{\exp\{2\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)} \frac{4}{\theta_w\theta_u} \frac{\theta_w\theta_u}{2\theta_w+2\theta_u-(\pm\theta_w\theta_u)} \int_{\max\{\varepsilon,0\}}^{\infty} \frac{2\theta_w+2\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u} \exp\left\{-\frac{2\theta_w+2\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u} w\right\} dw
$$

$$
= \frac{4}{f_\varepsilon(\varepsilon)} \frac{\exp\{\varepsilon/\theta_u\}}{\theta_w+\theta_u-(\pm\theta_w\theta_u)} \exp\left\{-\frac{\theta_w+\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u} \max\{\varepsilon,0\}\right\}
$$

$$
- \frac{4}{f_\varepsilon(\varepsilon)} \frac{\exp\{\varepsilon/\theta_u\}}{\theta_w+2\theta_u-(\pm\theta_w\theta_u)} \exp\left\{-\frac{\theta_w+2\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u} \max\{\varepsilon,0\}\right\}
$$

$$
- \frac{4}{f_\varepsilon(\varepsilon)} \frac{\exp\{2\varepsilon/\theta_u\}}{2\theta_w+\theta_u-(\pm\theta_w\theta_u)} \exp\left\{-\frac{2\theta_w+\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u} \max\{\varepsilon,0\}\right\}
$$

$$
+ \frac{4}{f_\varepsilon(\varepsilon)} \frac{\exp\{2\varepsilon/\theta_u\}}{2\theta_w+2\theta_u-(\pm\theta_w\theta_u)} \exp\left\{-\frac{2\theta_w+2\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u} \max\{\varepsilon,0\}\right\}
$$

So,

$$E\left(\exp\{\pm w\}|\varepsilon\right)=$$

$$=\frac{4\exp\{\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)}\left[\frac{\exp\left\{-\frac{\theta_w+\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u}\max\{\varepsilon,0\}\right\}}{\theta_w+\theta_u-(\pm\theta_w\theta_u)}-\frac{\exp\left\{-\frac{\theta_w+2\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u}\max\{\varepsilon,0\}\right\}}{\theta_w+2\theta_u-(\pm\theta_w\theta_u)}\right]$$

$$-\frac{4\exp\{2\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)}\left[\frac{\exp\left\{-\frac{2\theta_w+\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u}\max\{\varepsilon,0\}\right\}}{2\theta_w+\theta_u-(\pm\theta_w\theta_u)}-\frac{\exp\left\{-\frac{2\theta_w+2\theta_u-(\pm\theta_w\theta_u)}{\theta_w\theta_u}\max\{\varepsilon,0\}\right\}}{2\theta_w+2\theta_u-(\pm\theta_w\theta_u)}\right]$$

For $E\left(\exp\{\pm w\}|\varepsilon\right)$ to exist, we require that $\theta_w+\theta_u-\theta_w\theta_u\geq 0\Rightarrow \theta_w+\theta_u\geq\theta_w\theta_u$, otherwise at least one of the above integrals does not converge. Given that usually, θ_w,θ_u are estimated as being smaller than unity the condition will be satisfied.

D.2.2. $E\left(\exp\{-u\}|\varepsilon\right)$.

$$E\left(\exp\{-u\}|\varepsilon\right)=\int_{-\min\{\varepsilon,0\}}^{\infty}\exp\{-u\}\,f_{u|\varepsilon}\left(u|\varepsilon\right)du=\int_{-\min\{\varepsilon,0\}}^{\infty}\exp\{-u\}\frac{f_w(\varepsilon+u)\,f_u(u)}{f_\varepsilon(\varepsilon)}du$$

$$=\frac{1}{f_\varepsilon(\varepsilon)}\int_{-\min\{\varepsilon,0\}}^{\infty}\exp\{-u\}\left[2f_E(\varepsilon+u;\theta_w)-f_E(\varepsilon+u;\theta_w/2)\right]\left[2f_E(u;\theta_u)-f_E(u;\theta_u/2)\right]du$$

$$=\frac{1}{f_\varepsilon(\varepsilon)}\int_{-\min\{\varepsilon,0\}}^{\infty}\exp\{-u\}\left[4f_E(\varepsilon+u;\theta_w)\,f_E(u;\theta_u)-2f_E(\varepsilon+u;\theta_w)\,f_E(u;\theta_u/2)\right]du$$

$$+\frac{1}{f_\varepsilon(\varepsilon)}\int_{-\min\{\varepsilon,0\}}^{\infty}\exp\{-u\}\left[-2f_E(\varepsilon+u;\theta_w/2)\,f_E(u;\theta_u)+f_E(\varepsilon+u;\theta_w/2)\,f_E(u;\theta_u/2)\right]du$$

$$=\frac{\exp\{-\varepsilon/\theta_w\}}{f_\varepsilon(\varepsilon)}\int_{-\min\{\varepsilon,0\}}^{\infty}\exp\{-u\}\left[4f_E(u;\theta_w)f_E(u;\theta_u)-2f_E(u;\theta_w)f_E(u;\theta_u/2)\right]du$$

$$+\frac{\exp\{-2\varepsilon/\theta_w\}}{f_\varepsilon(\varepsilon)}\int_{-\min\{\varepsilon,0\}}^{\infty}\exp\{-u\}\left[-2f_E(u;\theta_w/2)f_E(u;\theta_u)+f_E(u;\theta_w/2)f_E(u;\theta_u/2)\right]du$$

$$=\frac{\exp\{-\varepsilon/\theta_w\}}{f_\varepsilon(\varepsilon)}\int_{-\min\{\varepsilon,0\}}^{\infty}\exp\{-u\}\left[\frac{4}{\theta_w\theta_u}\exp\left\{-\frac{\theta_w+\theta_u}{\theta_w\theta_u}u\right\}-\frac{4}{\theta_w\theta_u}\exp\left\{-\frac{2\theta_w+\theta_u}{\theta_w\theta_u}u\right\}\right]du$$

$$+\frac{\exp\{-2\varepsilon/\theta_w\}}{f_\varepsilon(\varepsilon)}\int_{-\min\{\varepsilon,0\}}^{\infty}\exp\{-u\}\left[-\frac{4}{\theta_w\theta_u}\exp\left\{-\frac{\theta_w+2\theta_u}{\theta_w\theta_u}u\right\}+\frac{4}{\theta_w\theta_u}\exp\left\{-\frac{2\theta_w+2\theta_u}{\theta_w\theta_u}u\right\}\right]du$$

$$=\frac{\exp\{-\varepsilon/\theta_w\}}{f_\varepsilon(\varepsilon)}\frac{4}{\theta_w\theta_u}\int_{-\min\{\varepsilon,0\}}^{\infty}\left[\exp\left\{-\frac{\theta_w+\theta_u+\theta_w\theta_u}{\theta_w\theta_u}u\right\}-\exp\left\{-\frac{2\theta_w+\theta_u+\theta_w\theta_u}{\theta_w\theta_u}u\right\}\right]du$$

$$+\frac{\exp\{-2\varepsilon/\theta_w\}}{f_\varepsilon(\varepsilon)}\frac{4}{\theta_w\theta_u}\int_{-\min\{\varepsilon,0\}}^{\infty}\left[-\exp\left\{-\frac{\theta_w+2\theta_u+\theta_w\theta_u}{\theta_w\theta_u}u\right\}+\exp\left\{-\frac{2\theta_w+2\theta_u+\theta_w\theta_u}{\theta_w\theta_u}u\right\}\right]du$$

$$=\frac{\exp\{-\varepsilon/\theta_w\}}{f_\varepsilon(\varepsilon)}\frac{4}{\theta_w\theta_u}\frac{\theta_w\theta_u}{\theta_w+\theta_u+\theta_w\theta_u}\int_{-\min\{\varepsilon,0\}}^{\infty}\frac{\theta_w+\theta_u+\theta_w\theta_u}{\theta_w\theta_u}\exp\left\{-\frac{\theta_w+\theta_u+\theta_w\theta_u}{\theta_w\theta_u}u\right\}du$$

$$-\frac{\exp\{-\varepsilon/\theta_w\}}{f_\varepsilon(\varepsilon)}\frac{4}{\theta_w\theta_u}\frac{\theta_w\theta_u}{2\theta_w+\theta_u+\theta_w\theta_u}\int_{-\min\{\varepsilon,0\}}^{\infty}\frac{2\theta_w+\theta_u+\theta_w\theta_u}{\theta_w\theta_u}\exp\left\{-\frac{2\theta_w+\theta_u+\theta_w\theta_u}{\theta_w\theta_u}u\right\}du$$

$$-\frac{\exp\{-2\varepsilon/\theta_w\}}{f_\varepsilon(\varepsilon)}\frac{4}{\theta_w\theta_u}\frac{\theta_w\theta_u}{\theta_w+2\theta_u+\theta_w\theta_u}\int_{-\min\{\varepsilon,0\}}^{\infty}\frac{\theta_w+2\theta_u+\theta_w\theta_u}{\theta_w\theta_u}\exp\left\{-\frac{\theta_w+2\theta_u+\theta_w\theta_u}{\theta_w\theta_u}u\right\}du$$

$$+\frac{\exp\{-2\varepsilon/\theta_w\}}{f_\varepsilon(\varepsilon)}\frac{4}{\theta_w\theta_u}\frac{\theta_w\theta_u}{2\theta_w+2\theta_u+\theta_w\theta_u}\int_{-\min\{\varepsilon,0\}}^{\infty}\frac{\theta_w\theta_u}{2\theta_w+2\theta_u+\theta_w\theta_u}\exp\left\{-\frac{2\theta_w+2\theta_u+\theta_w\theta_u}{\theta_w\theta_u}u\right\}du$$

$$= \frac{4\exp\{-\varepsilon/\theta_w\}}{f_\varepsilon(\varepsilon)} \frac{\exp\left\{\frac{\theta_w+\theta_u+\theta_w\theta_u}{\theta_w\theta_u}\min\{\varepsilon,0\}\right\}}{\theta_w+\theta_u+\theta_w\theta_u}$$

$$-\frac{4\exp\{-\varepsilon/\theta_w\}}{f_\varepsilon(\varepsilon)} \frac{\exp\left\{\frac{2\theta_w+\theta_u+\theta_w\theta_u}{\theta_w\theta_u}\min\{\varepsilon,0\}\right\}}{2\theta_w+\theta_u+\theta_w\theta_u}$$

$$-\frac{4\exp\{-2\varepsilon/\theta_w\}}{f_\varepsilon(\varepsilon)} \frac{\exp\left\{\frac{\theta_w+2\theta_u+\theta_w\theta_u}{\theta_w\theta_u}\min\{\varepsilon,0\}\right\}}{\theta_w+2\theta_u+\theta_w\theta_u}$$

$$+\frac{4\exp\{-2\varepsilon/\theta_w\}}{f_\varepsilon(\varepsilon)} \frac{\exp\left\{\frac{2\theta_w+2\theta_u+\theta_w\theta_u}{\theta_w\theta_u}\min\{\varepsilon,0\}\right\}}{2\theta_w+2\theta_u+\theta_w\theta_u}$$

So,

$$E\left(\exp\{-u\}|\varepsilon\right)=$$

$$=\frac{4\exp\{-\varepsilon/\theta_w\}}{f_\varepsilon(\varepsilon)}\left[\frac{\exp\left\{\frac{\theta_w+\theta_u+\theta_w\theta_u}{\theta_w\theta_u}\min\{\varepsilon,0\}\right\}}{\theta_w+\theta_u+\theta_w\theta_u}-\frac{\exp\left\{\frac{2\theta_w+\theta_u+\theta_w\theta_u}{\theta_w\theta_u}\min\{\varepsilon,0\}\right\}}{2\theta_w+\theta_u+\theta_w\theta_u}\right]$$

$$-\frac{4\exp\{-2\varepsilon/\theta_w\}}{f_\varepsilon(\varepsilon)}\left[\frac{\exp\left\{\frac{\theta_w+2\theta_u+\theta_w\theta_u}{\theta_w\theta_u}\min\{\varepsilon,0\}\right\}}{\theta_w+2\theta_u+\theta_w\theta_u}-\frac{\exp\left\{\frac{2\theta_w+2\theta_u+\theta_w\theta_u}{\theta_w\theta_u}\min\{\varepsilon,0\}\right\}}{2\theta_w+2\theta_u+\theta_w\theta_u}\right]$$

D.3.3. $E\left(h(m)\middle|\varepsilon\right)$

In order to examine the relation between management and the conventional inputs, we need to obtain $E\left(h(m)\middle|\varepsilon\right)=E\left(w\middle|z\right)$ in our case. The mathematical manipulations are the same as for the case $E\left(\exp\{\pm w\}\middle|\varepsilon\right)$ up to a point. Specifically we arrive at

$$E\left(w\middle|z\right)=\frac{\exp\{\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)}\int\limits_{\max\{\varepsilon,0\}}^{\infty} w\left[\frac{4}{\theta_w\theta_u}\exp\left\{-\frac{\theta_w+\theta_u}{\theta_w\theta_u}w\right\}-\frac{4}{\theta_w\theta_u}\exp\left\{-\frac{\theta_w+2\theta_u}{\theta_w\theta_u}w\right\}\right]dw$$

$$+\frac{\exp\{2\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)}\int\limits_{\max\{\varepsilon,0\}}^{\infty} w\left[-\frac{4}{\theta_w\theta_u}\exp\left\{-\frac{2\theta_w+\theta_u}{\theta_w\theta_u}w\right\}+\frac{4}{\theta_w\theta_u}\exp\left\{-\frac{2\theta_w+2\theta_u}{\theta_w\theta_u}w\right\}\right]dw$$

$$\Rightarrow E\left(w\middle|z\right)=\frac{\exp\{\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)}\frac{4}{\theta_w\theta_u}\int\limits_{\max\{\varepsilon,0\}}^{\infty} w\left[\exp\left\{-\frac{\theta_w+\theta_u}{\theta_w\theta_u}w\right\}-\exp\left\{-\frac{\theta_w+2\theta_u}{\theta_w\theta_u}w\right\}\right]dw$$

$$+\frac{\exp\{2\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)}\frac{4}{\theta_w\theta_u}\int\limits_{\max\{\varepsilon,0\}}^{\infty} w\left[-\exp\left\{-\frac{2\theta_w+\theta_u}{\theta_w\theta_u}w\right\}+\exp\left\{-\frac{2\theta_w+2\theta_u}{\theta_w\theta_u}w\right\}\right]dw$$

$$=\frac{\exp\{\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)}\frac{4}{\theta_w\theta_u}\frac{\theta_w\theta_u}{\theta_w+\theta_u}\int\limits_{\max\{\varepsilon,0\}}^{\infty} w\frac{\theta_w+\theta_u}{\theta_w\theta_u}\exp\left\{-\frac{\theta_w+\theta_u}{\theta_w\theta_u}w\right\}dw$$

$$-\frac{\exp\{\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)}\frac{4}{\theta_w\theta_u}\frac{\theta_w\theta_u}{\theta_w+2\theta_u}\int\limits_{\max\{\varepsilon,0\}}^{\infty} w\frac{\theta_w+2\theta_u}{\theta_w\theta_u}\exp\left\{-\frac{\theta_w+2\theta_u}{\theta_w\theta_u}w\right\}dw$$

$$-\frac{\exp\{2\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)}\frac{4}{\theta_w\theta_u}\frac{\theta_w\theta_u}{2\theta_w+\theta_u}\int\limits_{\max\{\varepsilon,0\}}^{\infty} w\frac{2\theta_w+\theta_u}{\theta_w\theta_u}\exp\left\{-\frac{2\theta_w+\theta_u}{\theta_w\theta_u}w\right\}dw$$

$$+\frac{\exp\{2\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)}\frac{4}{\theta_w\theta_u}\frac{\theta_w\theta_u}{2\theta_w+2\theta_u}\int\limits_{\max\{\varepsilon,0\}}^{\infty} w\frac{2\theta_w+2\theta_u}{\theta_w\theta_u}\exp\left\{-\frac{2\theta_w+2\theta_u}{\theta_w\theta_u}w\right\}dw$$

The four integrals are of the form

$$\int_a^\infty x\lambda \exp\{-\lambda x\}\,dx = \int_0^\infty x\lambda \exp\{-\lambda x\}\,dx - \int_0^a x\lambda \exp\{-\lambda x\}\,dx$$

$$= \frac{1}{\lambda} - \left[\int_0^a x \frac{d\left(1-\exp\{-\lambda x\}\right)}{dx} dx\right] = \frac{1}{\lambda} - \left[x\left(1-\exp\{-\lambda x\}\right)\Big|_0^a - \int_0^a \left(1-\exp\{-\lambda x\}\right) dx\right]$$

$$= \frac{1}{\lambda} - \left[a - a\exp\{-\lambda a\} - a + \int_0^a \exp\{-\lambda x\}\,dx\right]$$

$$= \frac{1}{\lambda} + a\exp\{-\lambda a\} - \int_0^a \exp\{-\lambda x\}\,dx = \frac{1}{\lambda} + a\exp\{-\lambda a\} + \frac{1}{\lambda}\exp\{-\lambda x\}\Big|_0^a$$

$$= \frac{1}{\lambda} + a\exp\{-\lambda a\} + \frac{1}{\lambda}\exp\{-\lambda a\} - \frac{1}{\lambda} = \left(a + \frac{1}{\lambda}\right)\exp\{-\lambda a\}$$

So

$$\begin{aligned}
E\left(w|\varepsilon\right) = {} & \frac{\exp\{\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)} \frac{4}{\theta_w+\theta_u}\left(\max\{\varepsilon,0\} + \frac{\theta_w\theta_u}{\theta_w+\theta_u}\right)\exp\left\{-\frac{\theta_w+\theta_u}{\theta_w\theta_u}\max\{\varepsilon,0\}\right\} \\
& - \frac{\exp\{\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)} \frac{4}{\theta_w+2\theta_u}\left(\max\{\varepsilon,0\} + \frac{\theta_w\theta_u}{\theta_w+2\theta_u}\right)\exp\left\{-\frac{\theta_w+2\theta_u}{\theta_w\theta_u}\max\{\varepsilon,0\}\right\} \\
& - \frac{\exp\{2\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)} \frac{4}{2\theta_w+\theta_u}\left(\max\{\varepsilon,0\} + \frac{\theta_w\theta_u}{2\theta_w+\theta_u}\right)\exp\left\{-\frac{2\theta_w+\theta_u}{\theta_w\theta_u}\max\{\varepsilon,0\}\right\} \\
& + \frac{\exp\{2\varepsilon/\theta_u\}}{f_\varepsilon(\varepsilon)} \frac{4}{2\theta_w+2\theta_u}\left(\max\{\varepsilon,0\} + \frac{\theta_w\theta_u}{2\theta_w+2\theta_u}\right)\exp\left\{-\frac{2\theta_w+2\theta_u}{\theta_w\theta_u}\max\{\varepsilon,0\}\right\}
\end{aligned}$$

--

Addendum to

TECHNICAL APPENDIX 3.III.

[Continuing section *C. The limiting distribution of the COLS/MM estimator*]

We first show that $n^{1/2}\hat{\mathbf{h}}_N = n^{-1/2}\sum_{i=1}^{n}\hat{\mathbf{h}}_i$ does *not* converge to the vector holding the central sample moments or cumulants of the error term centered on the true value, only some of its elements do. We manipulate the expression for the OLS residual, to obtain zero-mean quantities,

$$\hat{\varepsilon}_{i,OLS} = \varepsilon_i - \mathbf{x}_i'\left(\hat{\beta}_{OLS}-\beta\right) = \varepsilon_i - E\left(\varepsilon_i\right) - \left[\mathbf{x}_i'\left(\hat{\beta}_{OLS}-\beta\right) - E\left(\varepsilon_i\right)\right] = \tilde{\varepsilon}_i - \left[\mathbf{x}_i'\left(\hat{\beta}_{OLS}-\beta\right) - E\left(\varepsilon_i\right)\right]$$

We have $\mathbf{x}_i'\left(\hat{\beta}_{OLS}-\beta\right) - E\left(\varepsilon_i\right) = \mathbf{x}_i'\left(\mathrm{X}'\mathrm{X}\right)^{-1}\mathrm{X}'\boldsymbol{\varepsilon} - E\left(\varepsilon_i\right)$. This equals

$$\mathbf{x}_i'\left(\mathrm{X}'\mathrm{X}\right)^{-1}\mathrm{X}'\left(\boldsymbol{\varepsilon} - \mathbf{1}\cdot E\left(\varepsilon_i\right)\right) = \mathbf{x}_i'\left(\mathrm{X}'\mathrm{X}\right)^{-1}\mathrm{X}'\tilde{\boldsymbol{\varepsilon}}$$

because $\left(\mathrm{X}'\mathrm{X}\right)^{-1}\mathrm{X}'\cdot\mathbf{1}\cdot E\left(\varepsilon_i\right) = \left(1,\mathbf{0}\right)' E\left(\varepsilon_i\right) = E\left(\varepsilon_i\right)$ (see pp 338-339). We can also write this as

$$\mathbf{x}_i'\left(\hat{\beta}_{OLS}-\beta\right) - E\left(\varepsilon_i\right) = \mathbf{x}_i'\left(\hat{\beta}^*_{OLS}-\beta\right),\quad \operatorname{plim}\left(\hat{\beta}^*_{OLS}-\beta\right) = \mathbf{0}$$. So finally,

$$\hat{\varepsilon}_{i,OLS} = \tilde{\varepsilon}_i - \mathbf{x}_i'\left(\hat{\beta}^*_{OLS}-\beta\right) = \tilde{\varepsilon}_i - \mathbf{x}_i'\left(\mathrm{X}'\mathrm{X}\right)^{-1}\mathrm{X}'\tilde{\boldsymbol{\varepsilon}}.$$

Consider first the equation **related to the 2nd cumulant/central moment.**

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\left(c_2\right)^{-1}\hat{\varepsilon}^2_{i,OLS} - \mu_2\right) =$$

$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(c_2\right)^{-1}\left(\tilde{\varepsilon}_i^2 - 2\tilde{\varepsilon}_i\,\mathbf{x}_i'\left(\hat{\beta}^*_{OLS}-\beta\right) + \left(\hat{\beta}^*_{OLS}-\beta\right)'\mathbf{x}_i\mathbf{x}_i'\left(\hat{\beta}^*_{OLS}-\beta\right)\right) - \mu_2\right]$$

$$=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\left(c_2\right)^{-1}\tilde{\varepsilon}_i^2-\mu_2\right)\ -\ \frac{2}{c_2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\tilde{\varepsilon}_i\,\mathbf{x}_i'\left(\hat{\beta}_{OLS}^{*}-\beta\right)\ +\ \frac{1}{c_2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\hat{\beta}_{OLS}^{*}-\beta\right)'\mathbf{x}_i\mathbf{x}_i'\left(\hat{\beta}_{OLS}^{*}-\beta\right)$$

We have that $c_2 \to 1$ and it can be ignored. So

$$\ldots=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\tilde{\varepsilon}_i^2-\mu_2\right)\ -\ 2\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{\varepsilon}_i\,\mathbf{x}_i'\right)\left[\sqrt{n}\left(\hat{\beta}_{OLS}^{*}-\beta\right)\right]+\ \left(\hat{\beta}_{OLS}^{*}-\beta\right)'\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i'\right)\left[\sqrt{n}\left(\hat{\beta}_{OLS}^{*}-\beta\right)\right]$$

$$\to\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\tilde{\varepsilon}_i^2-\mu_2\right)\ -\ 2E\left(\tilde{\varepsilon}_i\right)E\left(\mathbf{x}_i'\right)\cdot O_p\left(1\right)+\ o_p\left(1\right)\cdot O_p\left(1\right)\cdot O_p\left(1\right)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\tilde{\varepsilon}_i^2-\mu_2\right)+0+0$$

where we have used the non-correlation between the error term and the regressors. So here

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(c_2\right)^{-1}\hat{\varepsilon}_{i,OLS}^2-\mu_2\right]\ -\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\tilde{\varepsilon}_i^2-\mu_2\right)\xrightarrow{p}0$$

We turn now to the equation **related to the 3d cumulant/central moment.**

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{c_3}\hat{\varepsilon}_{i,OLS}^3-\kappa_3\left(\varepsilon\right)\right)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{c_3}\left(\tilde{\varepsilon}_i-\mathbf{x}_i'\left(\hat{\beta}_{OLS}^{*}-\beta\right)\right)^3-\kappa_3\left(\varepsilon\right)\right)$$

$$\begin{aligned}=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\tilde{\varepsilon}_i^3-\kappa_3\left(\varepsilon\right)\right)-3\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{c_3}\tilde{\varepsilon}_i^2\mathbf{x}_i'\left(\hat{\beta}_{OLS}^{*}-\beta\right)\right)\\ +\frac{3}{c_3}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\tilde{\varepsilon}_i\left(\hat{\beta}_{OLS}^{*}-\beta\right)'\mathbf{x}_i\mathbf{x}_i'\left(\hat{\beta}_{OLS}^{*}-\beta\right)\\ +\ \frac{1}{c_3}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\hat{\beta}_{OLS}^{*}-\beta\right)'\mathbf{x}_i\left(\hat{\beta}_{OLS}^{*}-\beta\right)'\mathbf{x}_i\mathbf{x}_i'\left(\hat{\beta}_{OLS}^{*}-\beta\right)\end{aligned}$$

Here too, $c_3 \to 1$ and it can be ignored. So

$$..=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\tilde{\varepsilon}_i^3-\kappa_3\left(\varepsilon\right)\right)-3\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{\varepsilon}_i^2\mathbf{x}_i'\right)\left[\sqrt{n}\left(\hat{\beta}^*_{OLS}-\beta\right)\right]$$

$$+\ 3\left(\hat{\beta}^*_{OLS}-\beta\right)'\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{\varepsilon}_i\,\mathbf{x}_i\mathbf{x}_i'\right)\left[\sqrt{n}\left(\hat{\beta}^*_{OLS}-\beta\right)\right]$$

$$+\ \left(\hat{\beta}^*_{OLS}-\beta\right)'\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\left(\hat{\beta}^*_{OLS}-\beta\right)'\mathbf{x}_i\mathbf{x}_i'\right)\left[\sqrt{n}\left(\hat{\beta}^*_{OLS}-\beta\right)\right]$$

For the same reasons as before, the 3d and 4th term become negligible asymptotically. But not the 2nd term. So we obtain

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{c_3}\hat{\varepsilon}^3_{i,OLS}-\kappa_3\left(\varepsilon\right)\right]-\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\tilde{\varepsilon}_i^2-\kappa_3\left(\varepsilon\right)\right)-3\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{\varepsilon}_i^2\mathbf{x}_i'\right)\left[\sqrt{n}\left(\hat{\beta}^*_{OLS}-\beta\right)\right]\right\}\xrightarrow{p}0$$

So for this element of the $n^{1/2}\hat{\mathbf{h}}_N=n^{-1/2}\sum_{i=1}^{n}\hat{\mathbf{h}}_i$ vector, the limiting random variable is *not* the central moment centered on the true value.

Limiting Variance of the COLS/MM estimator.

Given the previous result, we will not obtain an explicit expression for the limiting variance, but we will show that it can be consistently estimated.

The variance of $n^{1/2}\hat{\mathbf{h}}_N=n^{-1/2}\sum_{i=1}^{n}\hat{\mathbf{h}}_i$ is

$$\mathrm{Var}\left(n^{-1/2}\sum_{i=1}^{n}\hat{\mathbf{h}}_i\right)\ =\ E\left(\hat{\mathbf{h}}_i\hat{\mathbf{h}}_i{}'\right)\ +\ \left(n-1\right)E\left(\hat{\mathbf{h}}_i\hat{\mathbf{h}}_k{}'\right)$$

(since the $\hat{\mathbf{h}}_i$ vectors are zero-mean). Due to uniform integrability, the variance of the limiting distribution will be the limit of the above expression.

We show that $\lim\limits_{n\to\infty}\left[\left(n-1\right)E\left(\hat{\mathbf{h}}_i\hat{\mathbf{h}}_k{}'\right)\right]=\mathbf{0}$.

Ignoring the bias-correction terms that will become unity at the limit, consider indicatively the upper left diagonal term of $(n-1)E\left(\hat{\mathbf{h}}_i\hat{\mathbf{h}}_k{}'\right)$, which is

$(n-1)E\left[\left(\hat{\varepsilon}_{i,OLS}^2-\mu_2\right)\left(\hat{\varepsilon}_{k,OLS}^2-\mu_2\right)\right]$. We have

$$
\begin{aligned}
&\left(\hat{\varepsilon}_{i,OLS}^2-\mu_2\right)\left(\hat{\varepsilon}_{k,OLS}^2-\mu_2\right)\\
&\quad=\left(\left(\tilde{\varepsilon}_i^2-\mu_2\right)-2\tilde{\varepsilon}_i\,\mathbf{x}_i'\left(\hat{\beta}_{OLS}^*-\beta\right)+\left(\hat{\beta}_{OLS}^*-\beta\right)'\mathbf{x}_i\mathbf{x}_i'\left(\hat{\beta}_{OLS}^*-\beta\right)\right)\\
&\qquad\qquad\times\left(\left(\tilde{\varepsilon}_k^2-\mu_2\right)-2\tilde{\varepsilon}_k\mathbf{x}_k'\left(\hat{\beta}_{OLS}^*-\beta\right)+\left(\hat{\beta}_{OLS}^*-\beta\right)'\mathbf{x}_k\mathbf{x}_k'\left(\hat{\beta}_{OLS}^*-\beta\right)\right)
\end{aligned}
$$

So the product is decomposed into the following nine terms (we use n instead of $n-1$ that makes no difference asymptotically):

$$
\begin{aligned}
\ldots=\;& nE\left[\left(\tilde{\varepsilon}_i^2-\mu_2\right)\left(\tilde{\varepsilon}_k^2-\mu_2\right)\right]\;-nE\left[\left(\tilde{\varepsilon}_i^2-\mu_2\right)2\tilde{\varepsilon}_k\mathbf{x}_k'\left(\hat{\beta}_{OLS}^*-\beta\right)\right]\\
&+\;nE\left[\left(\tilde{\varepsilon}_i^2-\mu_2\right)\left(\hat{\beta}_{OLS}^*-\beta\right)'\mathbf{x}_k\mathbf{x}_k'\left(\hat{\beta}_{OLS}^*-\beta\right)\right]\\
&-nE\left[2\tilde{\varepsilon}_i\,\mathbf{x}_i'\left(\hat{\beta}_{OLS}^*-\beta\right)\left(\tilde{\varepsilon}_k^2-\mu_2\right)\right]\;+\;nE\left[2\tilde{\varepsilon}_i\,\mathbf{x}_i'\left(\hat{\beta}_{OLS}^*-\beta\right)2\tilde{\varepsilon}_k\mathbf{x}_k'\left(\hat{\beta}_{OLS}^*-\beta\right)\right]\\
&-\;nE\left[2\tilde{\varepsilon}_i\,\mathbf{x}_i'\left(\hat{\beta}_{OLS}^*-\beta\right)\left(\hat{\beta}_{OLS}^*-\beta\right)'\mathbf{x}_k\mathbf{x}_k'\left(\hat{\beta}_{OLS}^*-\beta\right)\right]\\
&+\;nE\left[\left(\hat{\beta}_{OLS}^*-\beta\right)'\mathbf{x}_i\mathbf{x}_i'\left(\hat{\beta}_{OLS}^*-\beta\right)\left(\tilde{\varepsilon}_k^2-\mu_2\right)\right]\\
&-\;nE\left[\left(\hat{\beta}_{OLS}^*-\beta\right)'\mathbf{x}_i\mathbf{x}_i'\left(\hat{\beta}_{OLS}^*-\beta\right)2\tilde{\varepsilon}_k\mathbf{x}_k'\left(\hat{\beta}_{OLS}^*-\beta\right)\right]\\
&+\;nE\left[\left(\hat{\beta}_{OLS}^*-\beta\right)'\mathbf{x}_i\mathbf{x}_i'\left(\hat{\beta}_{OLS}^*-\beta\right)\left(\hat{\beta}_{OLS}^*-\beta\right)'\mathbf{x}_k\mathbf{x}_k'\left(\hat{\beta}_{OLS}^*-\beta\right)\right]
\end{aligned}
$$

We examine terms separately.

<u>Term 1:</u> $nE\left[\left(\tilde{\varepsilon}_i^2-\mu_2\right)\left(\tilde{\varepsilon}_k^2-\mu_2\right)\right]=nE\left(\tilde{\varepsilon}_i^2-\mu_2\right)E\left(\tilde{\varepsilon}_k^2-\mu_2\right)=n\cdot 0\cdot 0=0$

<u>Terms 2,4</u>:

$$nE\left[\left(\tilde{\varepsilon}_i^2-\mu_2\right)2\tilde{\varepsilon}_k\mathbf{x}_k'\left(\hat{\beta}_{OLS}^*-\beta\right)\right]=nE\left[\left(\tilde{\varepsilon}_i^2-\mu_2\right)2\tilde{\varepsilon}_k\mathbf{x}_k'\left(\mathrm{X}'\mathrm{X}\right)^{-1}\mathrm{X}'\tilde{\boldsymbol{\varepsilon}}\right]$$

$$=2nE\left[\mathbf{x}_k'\left(\mathrm{X}'\mathrm{X}\right)^{-1}\mathrm{X}'\tilde{\boldsymbol{\varepsilon}}\tilde{\varepsilon}_k\left(\tilde{\varepsilon}_i^2-\mu_2\right)\right]=2nE\left[\mathbf{x}_k'\left(\mathrm{X}'\mathrm{X}\right)^{-1}\mathrm{X}'E\left(\tilde{\boldsymbol{\varepsilon}}\tilde{\varepsilon}_k\left(\tilde{\varepsilon}_i^2-\mu_2\right)\middle|\mathrm{X}\right)\right]$$

$$=2nE\left[\mathbf{x}_k'\left(\mathrm{X}'\mathrm{X}\right)^{-1}\mathrm{X}'E\left(\tilde{\boldsymbol{\varepsilon}}\tilde{\varepsilon}_k\left(\tilde{\varepsilon}_i^2-\mu_2\right)\right)\right]$$

The vector $\tilde{\boldsymbol{\varepsilon}}\tilde{\varepsilon}_k\left(\tilde{\varepsilon}_i^2-\mu_2\right)$ contains products of the form

$\tilde{\varepsilon}_\ell\tilde{\varepsilon}_k\left(\tilde{\varepsilon}_i^2-\mu_2\right),\ \tilde{\varepsilon}_k^2\left(\tilde{\varepsilon}_i^2-\mu_2\right),\ \tilde{\varepsilon}_k\left(\tilde{\varepsilon}_i^3-\tilde{\varepsilon}_i\,\mu_2\right)$

and we have

$$E\left[\tilde{\varepsilon}_\ell\tilde{\varepsilon}_k\left(\tilde{\varepsilon}_i^2-\mu_2\right)\right]=\quad E\left(\tilde{\varepsilon}_\ell\right)E\left(\tilde{\varepsilon}_k\right)E\left(\tilde{\varepsilon}_i^2-\mu_2\right)=0\cdot 0\cdot 0=0$$

$$E\left[\tilde{\varepsilon}_k^2\left(\tilde{\varepsilon}_i^2-\mu_2\right)\right]=E\left(\tilde{\varepsilon}_k^2\right)E\left(\tilde{\varepsilon}_i^2-\mu_2\right)=E\left(\tilde{\varepsilon}_k^2\right)\cdot 0=0$$

$$E\left[\tilde{\varepsilon}_k\left(\tilde{\varepsilon}_i^3-\tilde{\varepsilon}_i\,\mu_2\right)\right]=E\left(\tilde{\varepsilon}_k\right)E\left(\tilde{\varepsilon}_i^3-\tilde{\varepsilon}_i\,\mu_2\right)=0\cdot E\left(\tilde{\varepsilon}_i^3-\tilde{\varepsilon}_i\,\mu_2\right)=0$$

For term 4, just switch the observation indices. So the **2nd and 4th terms are zero**.

<u>Terms 3,7:</u>

$$nE\left[\left(\tilde{\varepsilon}_i^2-\mu_2\right)\left(\hat{\beta}_{OLS}^*-\beta\right)'\mathbf{x}_k\mathbf{x}_k'\left(\hat{\beta}_{OLS}^*-\beta\right)\right]=E\left[\left(\tilde{\varepsilon}_i^2-\mu_2\right)\left[\sqrt{n}\left(\hat{\beta}_{OLS}^*-\beta\right)'\right]\mathbf{x}_k\mathbf{x}_k\left[\sqrt{n}\left(\hat{\beta}_{OLS}^*-\beta\right)\right]\right]$$

At the limit, $\sqrt{n}\left(\hat{\beta}_{OLS}^*-\beta\right)=\left(\tfrac{1}{n}\mathrm{X}'\mathrm{X}\right)^{-1}\left(\dfrac{1}{\sqrt{n}}\mathrm{X}'\tilde{\boldsymbol{\varepsilon}}\right)$ is independent of any single element

$\left(\tilde{\varepsilon}_i^2-\mu_2\right)$ so

$$E\left[\left(\tilde{\varepsilon}_i^2-\mu_2\right)\left[\sqrt{n}\left(\hat{\beta}_{OLS}^*-\beta\right)'\right]\mathbf{x}_k\mathbf{x}_k'\left[\sqrt{n}\left(\hat{\beta}_{OLS}^*-\beta\right)\right]\right]=$$

$$=E\left(\tilde{\varepsilon}_i^2-\mu_2\right)\cdot E\left\{\left[\sqrt{n}\left(\hat{\beta}_{OLS}^*-\beta\right)'\right]\mathbf{x}_k\mathbf{x}_k'\left[\sqrt{n}\left(\hat{\beta}_{OLS}^*-\beta\right)\right]\right\}=0$$

Switching indices, the same result holds for the **7th term**.

<u>**Terms 6,8:**</u> $nE\left[2\tilde{\varepsilon}_i\,\mathbf{x}_i'\left(\hat{\beta}_{OLS}^*-\beta\right)\left(\hat{\beta}_{OLS}^*-\beta\right)'\mathbf{x}_k\mathbf{x}_k'\left(\hat{\beta}_{OLS}^*-\beta\right)\right]$

$$=E\left[2\tilde{\varepsilon}_i\,\mathbf{x}_i'\left(\hat{\beta}_{OLS}^*-\beta\right)\left[\sqrt{n}\left(\hat{\beta}_{OLS}^*-\beta\right)'\right]\mathbf{x}_k\mathbf{x}_k'\left[\sqrt{n}\left(\hat{\beta}_{OLS}^*-\beta\right)\right]\right]\to 0$$

because $\left(\hat{\beta}_{OLS}^*-\beta\right)=o_p(1)$. Switching indices, we obtain the same result for the **8th**.

<u>**Term 5:**</u>

$$nE\left[2\tilde{\varepsilon}_i\,\mathbf{x}_i'\left(\hat{\beta}_{OLS}^*-\beta\right)2\tilde{\varepsilon}_k\mathbf{x}_k'\left(\hat{\beta}_{OLS}^*-\beta\right)\right]=4E\left[\tilde{\varepsilon}_i\,\tilde{\varepsilon}_k\mathbf{x}_i'\left[\sqrt{n}\left(\hat{\beta}_{OLS}^*-\beta\right)\right]\mathbf{x}_k'\left[\sqrt{n}\left(\hat{\beta}_{OLS}^*-\beta\right)\right]\right]$$

The term inside the expected value is bounded. Due to regressors exogeneity, the fact that $\sqrt{n}\left(\hat{\beta}_{OLS}^*-\beta\right)$ is independent of any single element $\tilde{\varepsilon}_i,\tilde{\varepsilon}_k$, and the i.i.d. sample assumption, we have $\ldots=4E\left[\mathbf{x}_i'\left[\sqrt{n}\left(\hat{\beta}_{OLS}^*-\beta\right)\right]\mathbf{x}_k'\left[\sqrt{n}\left(\hat{\beta}_{OLS}^*-\beta\right)\right]\right]E\left(\tilde{\varepsilon}_i\right)E\left(\tilde{\varepsilon}_k\right)=0$

<u>**Term 9:**</u> $nE\left[\left(\hat{\beta}_{OLS}^*-\beta\right)'\mathbf{x}_i\mathbf{x}_i'\left(\hat{\beta}_{OLS}^*-\beta\right)\left(\hat{\beta}_{OLS}^*-\beta\right)'\mathbf{x}_k\mathbf{x}_k'\left(\hat{\beta}_{OLS}^*-\beta\right)\right]$

$$=E\left[\left(\hat{\beta}_{OLS}^*-\beta\right)'\mathbf{x}_i\mathbf{x}_i'\left[\sqrt{n}\left(\hat{\beta}_{OLS}^*-\beta\right)\right]\left(\hat{\beta}_{OLS}^*-\beta\right)'\mathbf{x}_k\mathbf{x}_k'\left[\sqrt{n}\left(\hat{\beta}_{OLS}^*-\beta\right)\right]\right]\to 0$$

... because it is a product of $O_p(1)$ and $o_p(1)$ terms.

Analogous relations and results hold for the other elements of the matrix $(n-1)E\left(\hat{\mathbf{h}}_i\hat{\mathbf{h}}_k{}'\right)$ so we obtain $\lim_{n\to\infty}\left[(n-1)E\left(\hat{\mathbf{h}}_i\hat{\mathbf{h}}_k{}'\right)\right]=\mathbf{0}$. Therefore the variance of the limiting distribution is $\lim_{n\to\infty}E\left(\hat{\mathbf{h}}_i\hat{\mathbf{h}}_i{}'\right)=E\left(\mathbf{h}_i\mathbf{h}_i{}'\right)$, which can be consistently estimated by the corresponding sample moments.--

ATHENS UNIVERSITY OF ECONOMICS & BUSINESS
ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ