Department of International and European Economic Studies

MSc in International Economics and Finance

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TITLE

International Portfolio Management with Options Using CVaR Model

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ABSTRACT

This thesis develops an optimization framework for multicurrency asset allocation problems using as risk measure the Conditional Value-at-Risk (CVaR). The aim is to create an optimal internationally diversified portfolio, as well as to mitigate the exposure to risks. More precisely, we focus only on the control of the market risk by incorporating European put options in the portfolio optimization model. In this survey two different models are presented. The first one is a simple model of international portfolio management. The second one is the first model augmented with options in order to hedge the market risk. By including put options in the portfolio optimization model we observe, after extensive computational experiments, that there is better performance than in the simple model.

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1. *Introduction*

Active management of international portfolio is of interest to multinational firms, financial intermediaries, institutions such as banks, insurance firms, etc. and individual investors. Investments in foreign securities are becoming available to a wide range of investors. This happens because of liberalization of markets.

International investment provides some benefits. One benefit is the higher profits that the investors may have in the event of favorable performance of foreign markets. One more benefit is that there is a wider scope for diversification and there are also potential benefits of international diversification which were recognized earlier by Grubel (1968), Levy and Sarnat (1970), Solnik (1974) and others.

The aim of asset managers is to select investment portfolios that yield the maximum possible return and at the same time ensuring a preferable level of risk exposure. Investors want to protect their portfolios from risk. The risk provokes potential losses in portfolio value due to possible reductions in the market value of financial assets. This in its turn comes from changes in asset prices, interest rates, foreign exchange rates and many others. By doing diversification into multiple assets, investors manage to eliminate unsystematic risk (or else called diversifiable risk or idiosyncratic risk), that is, potential severe losses from any individual asset.

It is important to point out that improvements in risk-return profile of investors are achieved by holding internationally diversified portfolio. The reason is that there are low correlations between international assets. Moreover, international investments provide continuously a wider scope for portfolio diversification than is available in a domestic market. However, neither domestic nor international diversification can mitigate systematic market risk (alternatively undiversifiable risk). Eun and Resnick (1988, 1994) point out that international diversification decisions should take into account the exposure to currency risk, see also Driessen and Leven (2007) and Bai and Green (2010).

More precisely, international diversification entails exposure to currency risk and to market risks, associated with the securities which are included in the portfolio that should be managed. The volatility of return from a foreign security not only depends on the differential change of its price within any given period, which is the local return, but also it depends on the variation of the foreign exchange rate to the reference currency, as well as on the correlation between these two.

Therefore, investors should do hedging in order to reduce the overall risk exposure, especially the currency risk of their international portfolios. A great part of the literature studies the advantages of hedging exchange rate risk, see for example Perold and Schulman (1988), Jorion (1989), Black (1990), Glen and Jorion (1993), Abken and Shrikhande (1997) and others. This literature presents different views of the hedging of currency risk in international portfolios. Each study of the above, takes into consideration various factors, as for example, if the investor is risk averse, the time horizon of the investor's decision, if the investment strategy is passive or active, the investment opportunity set, the distribution of financial asset returns and exchange rates, the base currency of the investor and the market conditions. This means that there are many alternative hedging strategies that depend on the above factors. However, despite the general agreement that currency hedging can reduce the portfolio risk, there is no consensus on a universally optimal currency hedging rule.

Currency risk is typically hedged via forward contracts. That is, the most popular strategy is to incorporate in the portfolio management models decisions for optimal selection of currency forwards in order to mitigate the currency risk. Nevertheless, there are also quantos or currency options as alternatives to control the currency risk.

On the other hand, options are used in order to protect the international portfolio and also the domestic portfolio, from the market risk. In other words, options are convenient to protect the value of positions in risky assets in the event of specific variations of market prices, see for example Ahn et al. (1999) who investigate the problem of covering the market risk of a given exposure in a risky asset by minimizing its Value-at-Risk (VaR) using options, Liu and Pan (2003) who investigate optimal investment strategies given investor access not only to bonds and stocks but also to derivatives, Blomvall and Lindberg (2003) who examine portfolio which are composed of a stock index, call options on this index and a risk-free asset and they show that for a given level of risk they achieve a higher return using options. And many others who have dealt with application of options in order to hedge risks.

In this work, we consider international portfolio of stock and bond indices from various countries and as it was mentioned before, these international portfolio is exposed to market risks of positions in stock indices in each country as well as to currency risk of foreign investments. The goal is to do the optimal diversification of assets and the correct risk control decisions. For this reason, we use stochastic programs in order to minimize the losses tail over the holding period while, simultaneously, we set a target level for expected return. In this way we achieve a tradeoff between total portfolio risk and reward.

Hence, we focus on the development and implementation of a model that employs the Conditional Value-at-Risk (CVaR) risk measure, as many others have already presented on their studies, see for example Topaloglou et al. (2002). CVaR risk measure is developed by Rockafeller and Uryasev (2000 and 2002). Our motivation for applying a CVaR model is coming from the event that the returns of international assets and proportional changes of exchange rates exhibit asymmetric distributions. Empirical evidence that supports this assertion is given in sub-section 3.1.

Therefore, CVaR is a more appropriate metric than alternative risk measures, such as variance or Value-at-Risk (VaR). It is suitable for skewed distributions and its aim is to control the tail of portfolio's loss distribution. Moreover, the most important is that CVaR satisfies the axioms of the coherent risk measures as we can see from Artzner et al. (1997 and 1999).

This study is based on the model that is presented in survey of Topaloglou et al. (2011). More specifically, we incorporate in the international portfolio management model only European put options in order to control the market risk exposure of holdings in domestic and foreign stock indices. Black-Scholes model is used for pricing of European put options.

We generate scenarios of asset returns using the historical data, and with their associated probabilities we establish the necessary inputs to the optimization model that determines portfolio composition. This parametric optimization model, which employs the CVaR as risk measure, trades off expected portfolio return against the relevant risk measure.

In this way, we create two efficient risk-return frontiers which are referred in different periods for the first model that does not include options. Subsequently, we carry out backtesting experiments for the model that does not contain options and we compare three types of investors. We also carry out backtesting experiments for the model that includes options. Finally we compare the two backtestings and we observe that the returns of portfolio that includes put options are greater that these of portfolio without option. This means that the use of options in the portfolio can significantly reduce the downside risk and improve the reward.

The rest of the paper is organized as follows. In Section 2 we discuss the different types of risk. In Section 3 we present the risk measures and their models. In Section4 we explain the derivatives and the differences between the most common derivatives. In Section 5 we examine the statistical characteristics of the data used in the models, we present the two models (model with options and model without options), we describe the numerical experiments and analyze the results. Finally, in Section 6 we discuss the conclusions and the directions for further research.

The contributions of our work are the development of a CVaR model for optimal selection of international portfolios with and without option hedging decisions within the portfolio selection model and the empirical comparison of these models.

2. *Different Types of Risk*

Risk is defined as the chance that an investment's actual return will be different than expected. This includes the possibility of losing some or all of the original investment.

A fundamental idea in finance is the relationship between risk and return. The greater the amount of risk that an investor is willing to take on, the greater the potential return. The reason for this is that investors need to be compensated for taking on additional risk.

This means that there is always a risk/return tradeoff in investing. Lower returns are usually associated with lower risk investments. Higher potential returns are associated with investments of higher risk as most investors expected to be compensated for taking on additional risk. Risk lovers, however, go against this principle: they acquire investments of higher risk with a lower expected return.

More specifically, risk lover is an investor who is willing to take on additional risk for an investment that has a relatively low expected return. This contrast with the typical investor mentality-risk aversion. Risk averse investors tend to take on increased risk only if they are warranted by the potential for higher returns.

Namely, a risk averse investor dislikes risk, and therefore will stay away from adding high-risk stocks or investments to their portfolio and in turn will often lose out on higher rates of return. Investors looking for "safer" investment will generally stick to index funds and governments bonds, which generally have lower returns.

Moreover, there is another type of investor which is the risk neutral who is indifferent to risk. More precisely, the risk neutral investor would be in the middle of the continuum represented by risk lovers investors at one end, and risk averse investors at the other extreme. This type of investor is more concerned about the expected return on his or her investment, not on the risk he or she may be taking on.

As mentioned at the beginning, risk can be referred as the chances of having an unexpected or negative outcome. Any action or activity that leads to loss of any type can be termed as risk.

There are many different types of risk that an investor might face and needs to overcome. The two basic types of risk are the following:

- Systematic Risk
- Unsystematic Risk

<u>Systematic Risk</u>

This risk inherent to the entire market or an entire market segment. Systematic risk, also known as "undiversifiable risk", "volatility" or "market risk", affects the overall market, not just a particular stock or industry. This type of risk is both unpredictable and impossible to completely avoid. It cannot be mitigated through diversification, only through hedging or by using the right asset allocate

Unsystematic Risk

Company- or industry- specific hazard that is inherent in each investment. Unsystematic risk also known as "non-systematic risk", "specific risk", "diversifiable risk" or "residual risk" and can be reduced through diversification.

Diversification is a risk management technique that mixes a wide variety of investments within a portfolio. The rationale behind this technique contends that a portfolio of different kinds of investments will, on average, yield higher returns and pose a lower risk than any individual investment found within the portfolio. Diversification strives to smooth out unsystematic risk events in a portfolio so that the positive performance of some investments will neutralize the negative performance of others. Therefore, the benefits of diversification will hold only if the securities in the portfolio are not perfectly correlated.

Now, that we have determined the fundamental types of risk, let's look at more specific types of risk, particularly when we talk about stocks and bonds.

The types of systematic risk are listed below:

- Interest Rate Risk
- Market Risk
- Purchasing Power or Inflationary Risk

(In context of types of risk in finance, purchasing power risk and inflationary risk are same)

And the types of unsystematic risk are listed below:

- Business or Liquidity Risk

(In the context of types of risk in finance, business and liquidity risk are same)

- Financial or Credit Risk

(In the context of types of risk in finance, financial risk and credit risk are same)

- Operational Risk

More specifically, the meaning of every specific type of risk is described as follow:

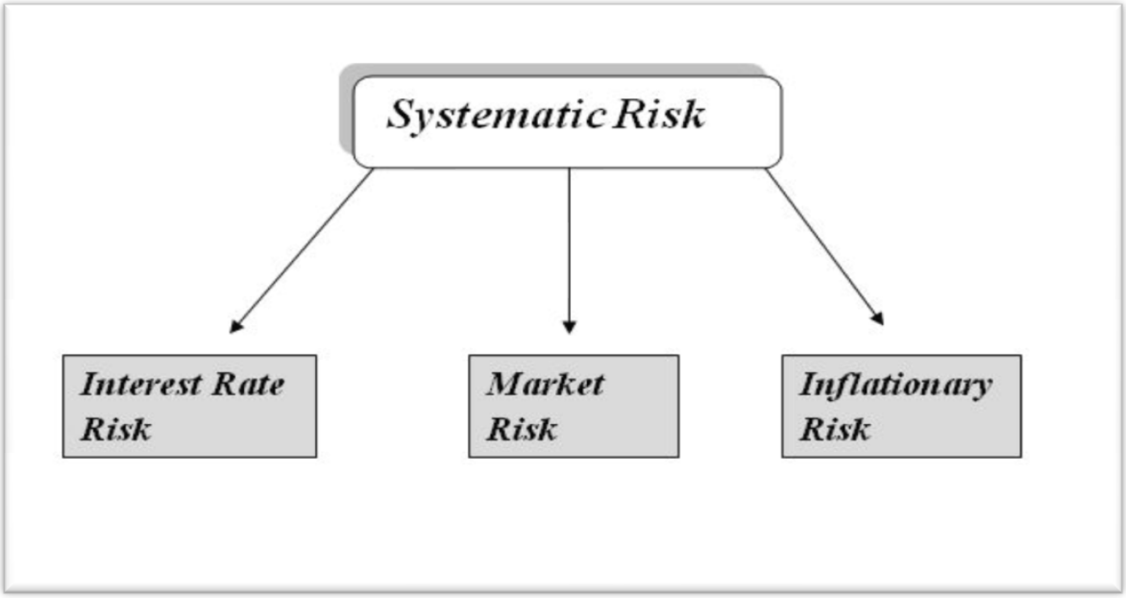

Figure 1: The types of systematic risk

Interest Rate Risk

Interest rate risk is the risk that an investment's value will change as a result of a change in interest rates from time to time. When interest rates rise after you lock in your money, meaning you do not earn as much on your money as you would have if you had invested at the higher rate.

This type of risk is most apparent in the bond market because bonds are are issued at specific interest rates. Generally, a rise in interest rates will cause a decline in market prices of existing bonds, while a decline in interest rates tends to cause bond prices to rise.

The types of interest rate risk are listed below:

- Price Risk: It arises due to the possibility that the price of the shares, commodity, investment, etc. may decline or increase in the future.
- Reinvestment Rate Risk: It results from the fact that the interest or dividend earned from an investment cannot be reinvested with the same rate of return as it was acquiring earlier.

Market Risk

This is the most familiar of all risks. Also referred to as volatility, which is a statistical measure of the dispersion of returns for a given security or market index. Market risk is the day-to-day fluctuations in a stock's price. It applies mainly to stocks and options. Because market movement is the reason why people can make money from stocks, volatility is essential for returns, and the more unstable the investment the more chance there is that it will experience a dramatic change in either direction.

The meaning of different types of market risk is as follow:

- Absolute Risk: It is without any content. For example, if a coin is tossed, there is fifty percentage chance of getting a head and vice-versa.
- Relative Risk: It is the assessment or evaluation of risk at different levels of business functions. For example, a relative risk from a foreign exchange fluctuation may be higher if the maximum sales accounted by an organization are of export sales.
- Directional Risks: They are those risks where the loss arises from an exposure to the particular assets of a market. This means that the directional risk is caused due to movement in stock price, interest rates and more. For example, an investor holding some shares experience a loss when the market price of those shares falls down.
- Non-Directional Risk: It arises where the method of trading is not consistently followed by the trader. For example, the dealer will buy and sell the share simultaneously to mitigate the risk.
- Basis Risk: It is due to the possibility of loss arising from imperfectly matched risks. For example, the risks which are in offsetting positions in two related but non-identical markets.
- Volatility Risk: It is of a change in the price of securities as a result of changes in the volatility of a risk-factor. For example, it applies to the portfolios of derivative instruments, where the volatility of its underlying is a major influence of prices.

Purchasing power or inflationary risk

Purchasing power risk is also known as inflation risk. It is so, since it emanates from the fact that it affects a purchasing power adversely. It is not desirable to invest in securities during an inflationary period.

The meaning of the two different types of purchasing power/ inflationary risk is as follow:

- Demand Inflation Risk: It is arises due to increase in price, which results from an excess of demand over supply. It occurs when supply falls to cope with the demand and hence cannot expand anymore. In other words, demand inflation occurs when production factors are under maximum utilization.

- Cost Inflation Risk: It arises due to sustained increase in the prices of goods and services. It is actually caused by higher production cost. A high cost of production inflates the final price of finished goods consumed by people.

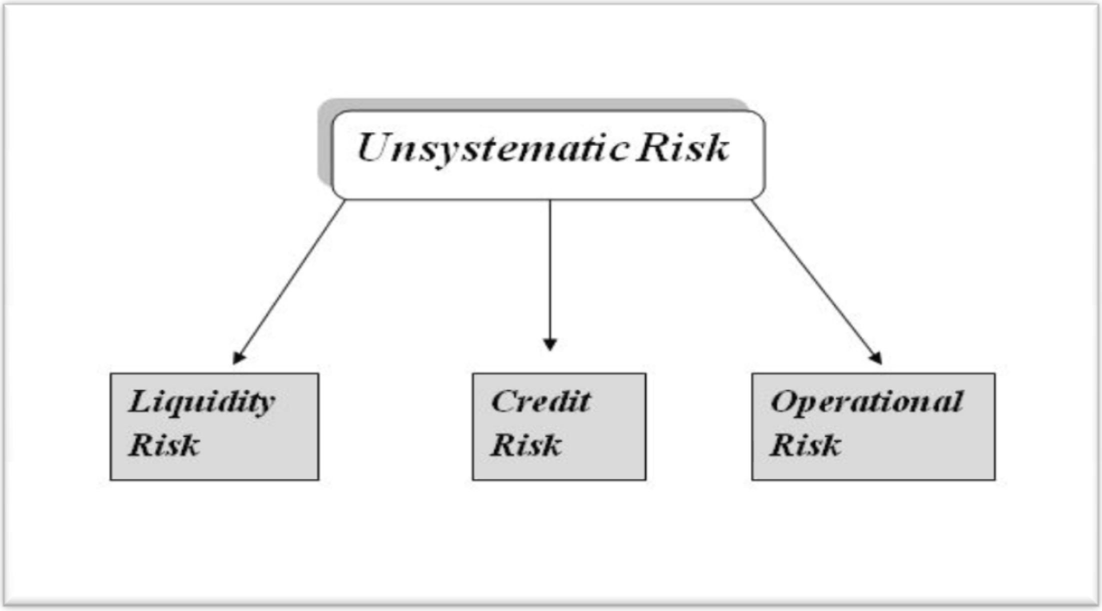

Figure 2: The types of unsystematic risk

Business or Liquidity Risk

Business risk is also known as liquidity risk. It is so, since it originates from the sale and purchase of securities affected by business cycles, technological changes, etc.

The types of business or liquidity risk are listed below:

- Asset Liquidity Risk: It arises either due to insufficient buyers or insufficient sellers against sell orders and buy orders respectively. For example, assets sold at a lesser value than their book value.
- Funding Liquidity Risk: It exists for not having an access to the sufficient funds to make a payment on time. For example, when commitments made to customers are not fulfilled as discussed in the service level agreements.

Financial or Credit Risk

Financial risk is also known as credit risk. This type of risk arises due to change in the capital structure of the organization. The capital structure mainly comprises of three ways by which funds are sourced for the projects. These are as follows:

- Owned funds. For example, share capital
- Borrowed funds. For example, loan funds.
- Retained earnings. For example, reserve and surplus.

The types of financial or credit risk are listed below:

- Exchange Rate Risk: It is also called as exposure rate risk. It is a form of financial risk that arises from a potential change seen in the exchange rate of one country's currency in relation to another country's currency and vise-versa. This means that when investing in foreign countries you must consider the fact that currency exchange rates can change the price of the asset as well. As an example, if you are a resident of America and invest in some Canadian stock in Canadian dollars, even if the share value appreciates, you may lose money if the Canadian dollar depreciates in relation to the American dollar.
- Recovery Rate Risk: It is an often neglected aspect of a credit-risk analysis. The recovery rate is normally needed to be evaluated.
- Sovereign Risk: It is associated with the government. Here, a government is unable to meets its loan obligations and break a promise on loans it guarantees.
- Settlement Risk: On the other hand it is arises when one party makes the payment while the other party fails to fulfill the obligations.

Operational Risk

Operational risks are the business risks failing due to human errors. This risk will change from industry to industry. It occurs due to breakdowns in the internal procedures, people, policies and systems.

The types of operational risk are listed below:

- Model Risk: It is involved in using various models to value financial securities. It is due to probability of loss resulting from the weaknesses in the financial model used in assess and managing a risk.

- <u>People Risk:</u> It is arises when people do not follow the organization's procedures, practices and/or rules. That is, they deviate from their expected behavior.
- <u>Legal Risk:</u> This risk arises when parties are not lawfully competent to enter an agreement among themselves. Furthermore, this relates to the regulatory risk, where a transaction could conflict with a government policy or particular legislation might be emended in the future with retrospective effect.
- <u>Political Risk</u>: This is the risk that arises in connection with uncertainty about a country's political environment and the stability of its economy. This uncertainty may has an unfavorable impact on an investor. It is especially prevalent in the third-world countries.

So these are some basic types of risk seen in the domain of finance and as you can see, there are several types of risk that a smart investor should consider and pay careful attention to.

3. Risk Measures

In finance, we are often exposed to risk in capital, whether as investor, traders or corporations. It seems therefore useful to quantify the riskiness of our position and hence to decide if it is acceptable or not.

More specifically, asset managers aim to select investment portfolios that yield the maximum possible return, while at the same time ensuring an acceptable level or risk exposure. Risk derives from potential losses in portfolio value due to possible reductions in the market value of financial assets resulting from changes in equity prices, interest rates, foreign exchange rates, credit rating of security issuers, etc. As a consequence, measures of risk have a crucial role in optimization under uncertainty, especially in coping with the losses that might be incurred in finance or the insurance industry.

So, several classes of risk measure were proposed in literature and we will analyze some of them in the next sections.

3.1 Mean-Variance

In the pre-Markowitz era financial risk was considered as a correcting factor of expected return and risk-adjusted returns were defined on an ad-hoc basis. These primitive measures had the advantage of allowing an immediate preferential order of all investments.

In 1950's Markowitz formalizes the portfolio selection problem in terms of two criteria to be optimized: the mean which was the expected outcome to be maximized and the second criteria was the risk, a scalar measure of the volatility of outcomes to be minimized.

The original formulation of Markowitz adopts the variance as risk measure ($Var(x) = E\big(x - E(x)\big)^2 = E(x^2) - \big(E(x)\big)^2$). This risk measure associated with the return of each investment by means of the deviation from the mean of the return distribution. In the case of a combination of assets, that is to say in the case of portfolio, he gauges the risk level via the covariance between all pairs of investments ($Cov(x,y) = E\big[\big(x - E(x)\big)\big(y - E(y)\big)\big] = E(xy) - E(x)E(y)$).

It is useful to note that mean-variance analysis is appropriate when returns are normally distributed. However, in actual applications, returns are typically not normally distributed. Moreover, the mean-variance model ignores the risk which associated with extreme events that occur with a very low probability. This means that this model does not take into account the tail of distribution.

3.2 Beta

In the 1960's, after the mean-variance approach, the concept of beta (β) was introduced. This development was motivated by computational reasons. The measure of the linear dependence between the return of each security and that of the market, β, or else beta, led to the development of the main pricing models like CAPM and APT, where CAPM is the following equation:

$$E(R_i) = R_f + [E(R_m) - R_f]\beta_i$$

Where:
R_f is the risk-free asset
$E(R_i)$ is the expected return of asset i of the portfolio i
$E(R_m)$ is the expected return of market
β_i is the beta, i.e., is the probability factor and shows the sensitivity to market risk. Beta is defined as $\beta_\iota = \frac{\sigma_{im}}{\sigma_{m^2}}$.
We can see that there is a linear relation between return and beta.

However, the problem of these models is that they have been developed in a "normal world", while extendible to heavy-tailed distribution and so this leads to misleading results.

3.3 Mean Absolute Deviation (MAD)

Another risk measure is the Mean Absolute Deviation, or MAD for short, and it has the following function:

$$MAD(x) = \mathcal{E}[|R(x;\tilde{r}) - R(x;\bar{r})|]$$

This risk function assumes that investors consider equally undesirable downside risk and upside deviations of the portfolio return $R(x;\tilde{r})$, which is more precisely a linear function with coefficients the asset returns, i.e., $R(x;\tilde{r}) = \sum_{i=1}^{n} \tilde{r}_i \tilde{x}_i$, from its mean value $R(x;\bar{r})$.

In other words, MAD is defined as the mean absolute deviation of portfolio return from its expected value. Moreover, it is useful to note that the asset allocation x is in percentage of total wealth when calculating portfolio return and in nominal amounts when calculating total portfolio value.

When asset returns are given by discrete and finite scenario set the model is formulated as a linear program. Also, when returns are normally distributed, the Variance and MAD risk measures are equal, within a constant. In this case, the solution of a Mean Absolute Deviation (MAD) is equivalent to the solution of the Markowitz Mean-Variance model.

Before we present the model of MAD, it would proper to refer that a constraint which is always present in all of these models is the budget constraint, $V_0 = \sum_{i=1}^{n} P_{0i} x_i$, and the non-negativity of the variables x, so that short sales are not allowed.

The following model trades off the portfolio Mean Absolute Deviation against its expected value subject to the condition that the expected value of the portfolio exceeds μV_0 parametrized by the target return μ (Zenios, 2008).

$$Minimize \quad \mathcal{E}[\left|V\left(x;\tilde{P}\right) - V(x;\bar{P})\right|]$$

$$subject\ to: \quad V\left(x;\tilde{P}\right) \geq \mu V_0$$

$$\sum_{i=1}^{n} P_{0i}\, x_i = V_0$$

$$x \in X$$

And if we substitute the objective function which in the discrete scenario setting takes the form:

$$MAD(x) = \sum_{l \in \Omega} p^l \, |V(x;P^l) - V(x;\bar{P})|$$

Therefore, we can rewrite the above model as following:

$$Minimize \qquad \sum_{l \in \Omega} p^l \, |V(x;P^l) - V(x;\bar{P})|$$

$$subject\ to: \quad \sum_{i=1}^{n} \bar{P}_i x_i \geq \mu V_0$$

$$\sum_{i=1}^{n} P_{0i}\, x_i = V_0$$

$$x \in X$$

However, there is a problem in this model because the objective function is not linear and so the solution of the model is not possible with linear programming. This difficulty can be overcome with a reformulation of the model and for this reason we introduce two variables, $\tilde{y}_+$ and $\tilde{y}_-$, in order to measure, respectively, the positive and negative deviations of the portfolio value from its mean. We can write the deviation function as:

$$V\left(x;\tilde{P}\right) - V(x;\bar{P}) = \tilde{y}_+ - \tilde{y}_-$$

where:

$$\tilde{y}_+ = \max\left[0, V\left(x; \tilde{P}\right) - V(x; \bar{P})\right]$$
$$\tilde{y}_- = \max\,[0, V(x; \bar{P}) - V\left(x; \tilde{P}\right)]$$

In the discrete scenario setting, we can express the definitions above as a system of inequalities

$$y_+^l \geq V(x; P^l) - V(x; \bar{P})$$
$$y_+^l \geq 0$$
$$y_-^l \geq V(x; \bar{P}) - V(x; P^l)$$
$$y_-^l \geq 0$$

for all $l \in \Omega$.

Moreover the objective function of MAD model can be written as $\sum_{l \in \Omega} p^l\, y^l$ and also we have defined the auxiliary variable y^l such that $y^l \geq y_+^l$ and $y^l \geq y_-^l$, for all $l \in \Omega$.

Hence, in this case, i.e., using the auxiliary variable y^l, MAD can be optimized by the following linear program.

$$Minimize \qquad \sum_{l \in \Omega} p^l y^l$$

$$subject\ to: \quad \sum_{i=1}^{n} \bar{P}_i x_i \geq \mu V_0$$

$$y^l \geq V(x; P^l) - V(x; \bar{P})$$
$$y^l \geq V(x; \bar{P}) - V(x; P^l)$$

$$\sum_{i=1}^{n} P_{0i} x_i = V_0$$

$$x \in X$$

3.4 Value-at-Risk (VaR)

Historically, the most commonly used risk measure is the variance (standard deviation) of a portfolio's return. However, as mentioned above, variance is not a satisfactory measure.

In 1994, the concept of Value-at-Risk, or VaR for short, was introduced dynamically. This type of risk measure has received greater acceptance in practice because it answered to a very basic question: How much one can expect to lose in one day, week, month or year with a given probability? In other words, what is the percentage of the value of the investment that is at risk?

This means that Value-at-Risk is an estimate of the maximum potential loss in value of a portfolio of financial instruments with a certain confidence level, which a dealer would experience during a standardized period. In other words, with a certain probability, losses will not exceed VaR.

So, VaR answers the following question: What is the maximum loss with a given confidence level,$\alpha * 100\%$, over a given horizon? Its calculation also reveals that with probability $(1-\alpha) * 100\%$ the losses will exceed VaR. More precisely the Value-at-Risk of a portfolio at the α probability level is the level α-percentile of the losses of te portfolio, i.e., the lowest possible value ζ such that the probability of losses less or equal to VaR is greater or equal to $\alpha * 100\%$. It is given as:

$$VaR(x; a) = \min\{\zeta \epsilon \mathbb{R} | \psi(x, \zeta) \geq \alpha\}$$

Where:
x is the portfolio of assets,
$\psi(x, \zeta)$ is the probability function, i.e. , the probability that the loss function does not exceed some threshold value ζ is given, in the discrete scenario setting, by the probability function

$$\psi(x, \zeta) = \sum_{\{l \epsilon \Omega | L(x; P^l) \leq \zeta\}} p^l$$

Also, we have to note that $l \epsilon \Omega$ are indices of plausible scenarios with probability of realization p^l and P^l is the price of the security under scenario l.

And if the current value of portfolio is V_0 then the losses in portfolio value are given by the loss function: $L(x; P^l) = V_0 - V(x; \tilde{P})$. Positive values of the loss function correspond to downside risk, while negative values of the loss function correspond to gains. Moreover, the percentile ζ is the left endpoint of the non-empty interval consisting of the values ζ such that $\psi(x, \zeta) = \alpha$ (Zenios, 2008).

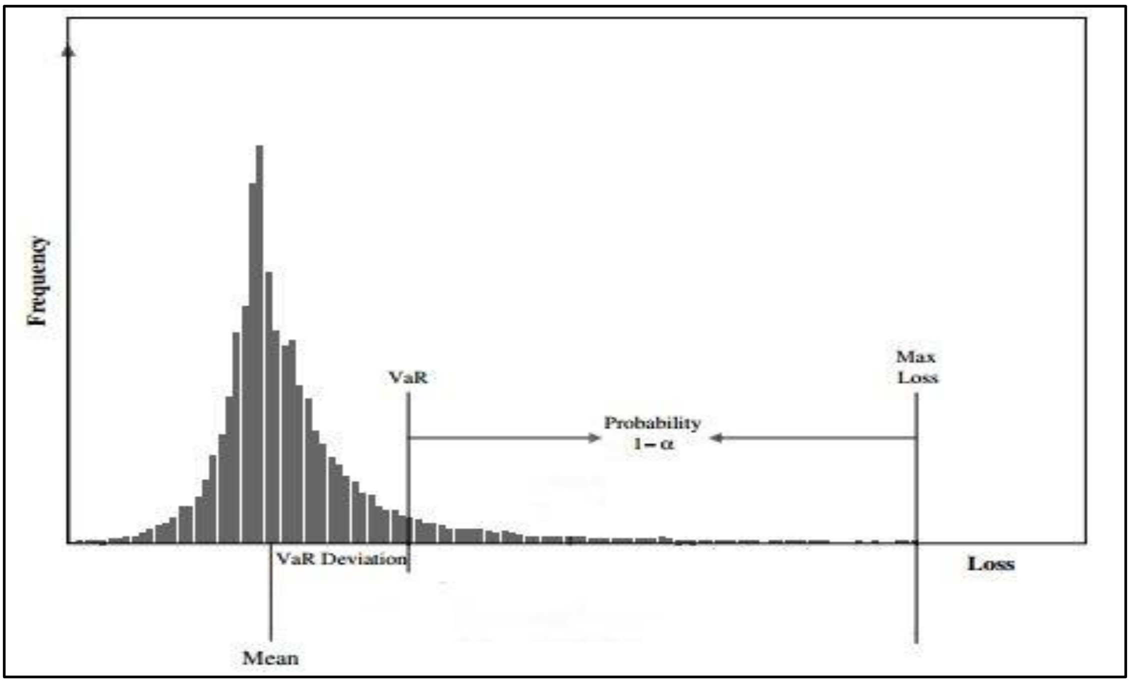

Illustration 1: Illustrates the concepts involved in the calculation of VaR.

The dependence of VaR on the confidence level α is sometimes made explicit by referring to α −VaR. For example, if α is 95%, we want to calculate the 0.95-VaR.

Despite of its popular use in risk measurement, VaR is not used in the mathematical models for optimal portfolio selection. The reason is that VaR ignores the potential loss beyond the given confidence level. Namely, VaR does not provide any information about the amount of loss exceeding VaR. Such losses can be catastrophic.

Furthermore, VaR has some serious limitations, such as lack of sub-additivity and convexity. More exactly, non-sub-additivity implies that portfolio diversification may lead to an increase risk instead of decreasing, and non-convexity makes it impossible to use VaR in optimization problems. VaR is coherent only when it is based on the standard deviation of normal distributions. Moreover, VaR is difficult to optimize when it is calculated from scenarios.

3.5 Conditional-Value-at-Risk (CVaR)

In contrast to the previous section, a commonly used risk measure in recent year, Conditional-Value-at-Risk, or CVaR for short and developed by Rockafellar and Uryasev (2002), is, in fact, a coherent risk measure.

Coherent risk measure $\rho: X \rightarrow R$, is a risk measure which satisfies the following conditions (Szegö, 2002):

- Positive homogeneity $\Longrightarrow \rho: (\lambda x) = \lambda\rho(x)$ for all random variables x and all positive real numbers λ.

- Sub-additivity $\Longrightarrow \rho(x + y) \leq \rho(x) + \rho(y)$ for all random variables x and y.
  This means that the risk of portfolio which is consisted of these two assets is less than the risk of individual asset.

- Monotonicity $\Longrightarrow x \leq y$ implies $\rho(x) \leq \rho(y)$ for all random variables x and y.

- Transitional invariance $\Longrightarrow \rho(x + ar_0) = \rho(x) - a$ for all random variables x and real numbers α, and all riskless rates r_0.

Some authors have replaced the first two conditions of coherence with the condition that the risk measure be convex which means that:

$$\rho(\lambda x + (1-\lambda)y) \leq \lambda\rho(x) + (1-\lambda)\rho(y), \quad 0 \leq \lambda \leq 1$$

Therefore, CVaR is a measure of risk that goes beyond the information revealed by VaR is the expected value of the losses that exceed VaR. This measure, for continuous distributions, is also known as Mean Excess Loss, Expected Shortfall or Tail VaR. However, for discrete distributions, CVaR may differ from Expected Shortfall.

For general distributions the CVaR is defined as a weighted average of VaR and the expected losses that are strictly greater than VaR. For discrete distributions CVaR of the losses of the portfolio is the expected value of the losses, conditioned on the losses being in excess of VaR.

If ζ is the VaR at the $a * 100\%$ level, then the CVaR is given by the expression:

$$CVaR(x; a) = \mathcal{E}\{L(x; P^l) | L(x; P^l) > \zeta\}$$

And in the discrete scenario setting we have

$$CVaR(x; a) = \frac{\sum_{\{l\epsilon\Omega | L(x;P^l) > \zeta\}} p^l L(x; P^l)}{\sum_{\{l\epsilon\Omega | L(x;P^l) > \zeta\}} p^l}$$

Here ζ is the a-VaR and its value depends on a. Under a technical condition that the probability of scenarios with losses strictly greater than ζ is exactly equal $1 - a$, i.e., $\psi(x, \zeta) = \alpha$, we have

$$CVaR(x;a) = \frac{\sum_{\{l\epsilon\Omega|L(x;P^l)>\zeta\}} p^l L(x;P^l)}{1-a}$$

All the above-mentioned are demonstrated at the Illustration below.

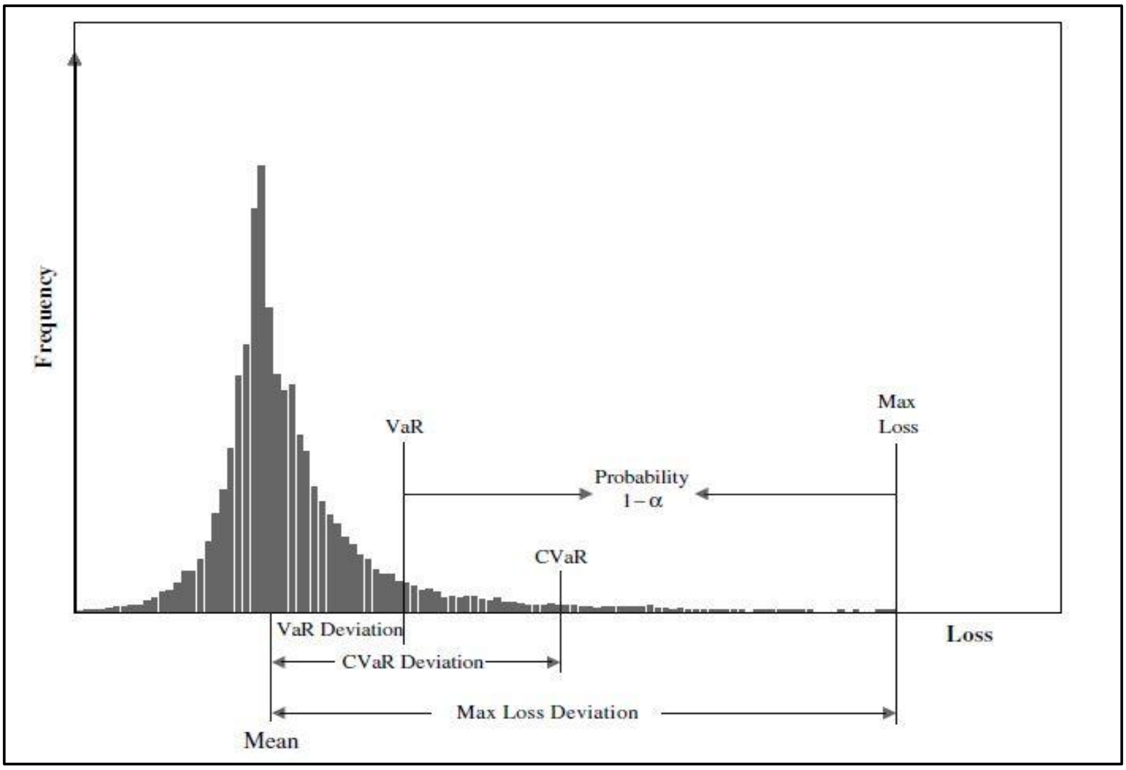

Illustration 2: General figure of VaR and CVaR.

Moreover, we will see that when we have discrete distribution CVaR can be optimized using a linear programming (LP) techniques, allowing handling of portfolios with a large number of instruments and scenarios.

The formulation of a linear model is facilitated with the use of auxiliary variables. Let define:

$$\tilde{y}_+ = \max\,[0, L(x;\tilde{P}) - \zeta]$$

where $\tilde{y}_+$ is equal to zero when the losses are less than the VaR (i.e. ζ) and it is equal to the excess loss when the losses exceed ζ.

In the discrete scenario setting we have

$$\tilde{y}_+^l = \max[0, L(x;\tilde{P}) - \zeta], \text{ for all } l\epsilon\Omega$$

Using the definition of $\tilde{y}_+^l$ we can write:

$$\begin{aligned}\sum_{l\in\Omega} p^l y_+^l &= \sum_{\{l\in\Omega | \mathrm{L}(\mathrm{x};P^l)<\zeta\}} p^l y_+^l + \sum_{\{l\in\Omega | \mathrm{L}(\mathrm{x};\mathrm{P}^l)\geq\zeta\}} p^l y_+^l \\ &= 0 + \sum_{\{l\in\Omega | \mathrm{L}(\mathrm{x};\mathrm{P}^l)\geq\zeta\}} p^l (L(x;P^l) - \zeta) \\ &= \sum_{\{l\in\Omega | \mathrm{L}(\mathrm{x};\mathrm{P}^l)\geq\zeta\}} p^l L(x;P^l) - \zeta \sum_{\{l\in\Omega | \mathrm{L}(\mathrm{x};\mathrm{P}^l)\geq\zeta\}} p^l \\ &= \sum_{\{l\in\Omega | \mathrm{L}(\mathrm{x};\mathrm{P}^l)\geq\zeta\}} p^l L(x;P^l) - \zeta\,(1-\alpha)\end{aligned}$$

Dividing both sides by (1-α) and rearranging terms we get

$$\zeta + \frac{\sum_{l\in\Omega} p^l y_+^l}{1-\alpha} = \frac{\sum_{\{l\in\Omega | \mathrm{L}(\mathrm{x};\mathrm{P}^l)\geq\zeta\}} p^l L(x;P^l)}{1-\alpha}$$

We minimize CVaR subject to the condition that the expected value of the portfolio exceeds some target μV_0, and the constraint which a portfolio consists a position x_i with the current prices P_{0i} and it has an initial value $V_0 = \sum_{i=1}^{n} P_{0i} x_i$. This is a budget constraint stipulating that the total holdings in the portfolio at current market prices are the initial endowment. The future value of the portfolio is given by:

$$V(x;\check{P}) = \sum_{i=1}^{n} \tilde{P}_i x_i$$

and its mean value is given by:

$$V(x;\bar{P}) = \sum_{i=1}^{n} \bar{P}_i x_i$$

And finally, subject to the constraint which may be imposed on the asset allocation of the form $x \epsilon X$, where X denotes the set of feasible solutions.

So the problem of the minimizing CVaR is written as:

$$Minimize \quad CVaR(x; a)$$

$$subhect\ to: \quad \mathcal{E}[V(x; \tilde{P})] \geq \mu V_0$$

$$\textstyle\sum_{i=1}^{n} P_{0i} x_i = V_0$$

$$x \epsilon X$$

And using the definition of CVaR, we can write the above problem as follows:

$$Minimize \qquad \zeta + \frac{1}{1-\alpha} \sum_{l \epsilon \Omega} p^l \, y_+^l$$

$$subject\ to: \qquad \sum_{i=1}^{n} \bar{P}_i \, x_i \geq \mu V_0$$

$$\begin{aligned} y_+^l &\geq L(x; P^l) - \zeta, && \text{for all } l \in \Omega \\ y_+^l &\geq 0\,, && \text{for all } l \in \Omega \end{aligned}$$

$$\sum_{i=1}^{n} P_{oi} \, x_i = V_0$$

$$x \in X$$

Solution of this model gives us the minimum CVaR, and the VaR value ζ^+ corresponding to the minimum CVaR portfolio. Moreover, there is an equivalent formulation of the above model in which we can maximize expected portfolio value subject to a constraint on CVaR (Zenios, 2008).

3.6 Expected Utility Maximization

One more risk measure is the expected utility. Investors are restricted in their choices to the selection of a model and to the setting of a target return. The expected utility maximization model allows users to optimize according to their own criteria when trading off risk and returns (rewards), which means that this model allows the users to select a unique portfolio that optimizes the users' utility function (Zenios, 2008).

We will define the expected utility maximization model using portfolio return $R(x;\tilde{r}) = \sum_{i=1}^{n} \tilde{r}_i x_i$, where x is, as mentioned earlier in MAD section, the asset allocations which are in percentage of total wealth when calculating portfolio return. In the scenario setting, portfolio return becomes $R(x;r^l) = \sum_{i=1}^{n} r_i^l x_i$, and the model can be written as follows:

$$\begin{aligned} &\mathit{Maximize} && \sum_{l\in\Omega} p^l \mathcal{U}\Big(R(x;r^l)\Big) \\ &\mathit{subject\ to}: && R(x;r^l) = \sum_{i=1}^{n} r_i^l x_i \\ & && \sum_{i=1}^{n} x_i = 1 \\ & && x \in X \end{aligned}$$

3.7 Regret Model

In the regret model we try to minimize the regret function:

$$G(x;\tilde{P},\tilde{g}) = V(x;\tilde{P}) - \tilde{g}$$

where $\tilde{g}$ is the random target value for the portfolio and $V(x;\tilde{P})$ is the portfolio values.
If $V(x;\tilde{P}) > \tilde{g}$, then we have gains and if $V(x;\tilde{P}) < \tilde{g}$, then we have losses. Both gains and losses are measured vis-$\dot{a}$-vis a target which serves as the benchmark.

Like the MAD model, we introduce again the same auxiliary variables,

$$\tilde{y}_+ = \max\,[0, V(x;\tilde{P}) - \tilde{g}]$$
$$\tilde{y}_- = \max\,[0, \tilde{g} - V(x;\tilde{P})]$$

in order to measure the positive and negative deviations of the portfolio value from the target, respectively. And so we can write the regret function as:

$$V(x;\tilde{P}) - \tilde{g} = \tilde{y}_+ - \tilde{y}_-$$

More specifically, the variable $\tilde{y}_+$ measures the upside regret when the portfolio outperforms the target and the variable $\tilde{y}_-$ measures the downside regret.

In the discrete scenario setting we can express the definitions above as systems of inequalities:

$$y_+^l \geq V(x; P^l) - g^l,$$
$$y_+^l \geq 0,$$

$$y_-^l \geq g^l - V(x; P^l),$$
$$y_-^l \geq 0,$$

$$for\ all\ l \in \Omega$$

It is customary to think of downside regret as the measure of risk and of upside regret as a measure of the portfolio reward.

Consider now the minimization of expected downside regret, with the requirement that the expected value of the portfolio exceeds μV_0, and constraints such as those imposed when minimizing the Mean Absolute Deviation risk function. We have the following model that trades off the risk measure of expected downside regret against expected value.

$$Minimize \quad \mathcal{E}[\tilde{y}_-]$$

$$subject\ to: \quad \mathcal{E}[V(x; \bar{P})] \geq \mu V_0$$

$$\sum_{i=1}^n P_{0i}\, x_i = V_0$$

$$x \in X$$

In the discrete scenarios setting, using the definition of y_-^l, we can formulate the regret minimization problem as the following linear program.

$$Minimize \quad \sum_{l \in e\Omega} p^l y_-^l$$

$$subject\ to: \quad \sum_{i=1}^n \bar{P}_i x_i \geq \mu V_0$$

$$y_-^l \geq g^l - V(x; P^l), \qquad for\ all\ l \in \Omega$$
$$y_-^l \geq 0, \qquad for\ all\ l \in \Omega$$

$$\sum_{i=1}^n P_{0i} x_i = V_0$$

$$x \in X$$

So this is the minimization of the expected downside regret.

Furthermore, it is important to add here that the Regret Mode is modeled with an exogenously given target whereas the Mean Absolute Deviation Model the target is endogenous, depending on the portfolio. This means that Regret Model measures risk as a piece-wise linear function of the portfolio deviations from a random target. The model trades off risk against the portfolio reward which is given by the expected value of the portfolio. The expected portfolio value is calculated independently of the random target.

This view is inconsistent with enterprise wide risk management, since the reward of the portfolio must be measured vis-$\grave{a}$-vis the random target. For example, if the portfolio value is always equal to a target liability then the portfolio has neither risk nor reward because risk is manifested when portfolio values are below the target and the opposite happens for the reward (Zenios, 2008).

3.8 Put/Call Efficient Frontier

For the above reason, now we will present a model that trades off the portfolio downside (risk) against the portfolio upside (reward) taking into account the random target.

The upside potential has identical payoffs to a call option on the future portfolio value relative to the target. More specifically, when the portfolio value is below the target there is zero upside potential, and the call option is out-of-the-money. Moreover, when the portfolio value exceeds the target the upside potential is precisely the payoff of a call that is in-the-money. Similarly, the downside payoffs are identical of those of a short position in a put option (i.e. short position has someone who sell or write the option) on the future portfolio value relative to the target.

So the portfolio call value is the expected upside and the put value is the expected downside. Portfolio that achieves the higher call value for a given put value are called Put/Call Efficient. Alternatively, we define a portfolio as Put/Call Efficient if it achieves the lowest put value for a given call value.

The deviations of the portfolio value from the random target $\tilde{g}$ are expressed again using variables $\tilde{y}_+$ and $\tilde{y}_-$ as $(x;\tilde{P}) - \tilde{g} = \tilde{y}_+ - \tilde{y}_-$, which is identical to the regret function and $\tilde{y}_+$, $\tilde{y}_-$ are defined with the same way, except how we make an interesting observation. The auxiliary variable $\tilde{y}_+$ measures the upside potential of the portfolio to outperform the target. Also, the auxiliary variable $\tilde{y}_-$ measures the downside risk and has the same payoff as a short position in an European put option (for the meaning of call and put option see the Section 4).

The following model traces the Put/Call Efficient Frontier for put values parametrized by ω.

$$Maximize \qquad \mathcal{E}[\tilde{y}_+]$$

$$subject\ to: \qquad \mathcal{E}[\tilde{y}_-] \leq \omega$$

$$\sum_{i=1}^{n} P_{0i} x_i = V_0$$

$$x \in X$$

As we can see, we start with a linear programming formulation for this model without any constraints of the form $x \in X$ (Zenios, 2008).

Now we formulate a linear program for tracing the efficient frontier in the above model. The linear expression for the portfolio value is:

$$V(x; P^l) = \sum_{i=1}^{n} P_i^l x_i$$

Let define I^l to be the total return of the benchmark portfolio. Then our random target is $g^l = V^0 I^l$ and substituting for V^0 from the budget constraint $V^0 = \sum_{i=1}^{n} P_{0i} x_i$, we will get:

$$g^l = \sum_{i=1}^{n} P_{0i} I^l x_i$$

Hence, the equation which we write in the beginning and it was identical with the regret function, now can be expressed in the following discrete form:

$$\sum_{i=1}^{n} P_i^l x_i - \sum_{i=1}^{n} P_{0i} I^l x_i = y_+^l - y_-^l \quad , for\ all\ l \in \Omega$$

Therefore, the model of Put/Call Efficient portfolio can be formulated as follows.

$$Maximize \quad \sum_{l \in \Omega} p^l y_+^l$$

$$subject\ to: \quad \sum_{l \in \Omega} p^l y_-^l \leq \omega$$

$$y_+^l - y_-^l - \sum_{i=1}^{n} (P_i^l - P_{0i} I^l) x_i = 0$$

$$y_+^l, y_-^l \geq 0, \ \ for\ all\ l \in \Omega$$

3.9 Some more correct measure of risk

Finally, it would be proper to point out that there are additional risk measures which satisfy the conditions which make VaR an inadequate risk measure. For example, it does not measure losses exceeding VaR, it is non-subadditive and non-convex. It may also provide conflicting results at different confidence levels and a reduction of VaR may lead to stretch the tail exceeding VaR. These are some of the important conditions that make Var an inappropriate risk measure.

3.9.1 Worst Conditional Expectation (WCE)

It is in fact a coherent measure, and more specifically WCE is sub-additive, but very useful only in a theoretical setting since it requires the knowledge of the whole underlying probability space.

3.9.2 Tail Conditional Expectation (TCE)

Tail Conditional Expectation (TCE), also known as Tail Value at Risk (TVaR) or Conditional Tail Expectation (CTE), is a risk measure associated the more general Value-at-Risk (VaR). It quantifies the expected value of the loss given that an event outside a given probability level has occurred.

Moreover, TCE lends itself naturally to practical applications, in contrast to WCE, but it is not coherent, because it is not sub-additive, as in the case of VaR.

For this reason there was a need to construct a risk measure which is both coherent and easy to compute and to estimate. This risk measure is the Expected Shortfall (ES) as presented below.

3.9.3 Expected Shortfall (ES)

Expected Shortfall (ES) is a risk measure, a concept used in finance and more specifically in the field of financial risk measure to evaluate the market risk or credit risk of a portfolio. It is an alternative to VaR that is more sensitive to shape of the loss distribution in the tail of the distribution.

The definition of ES at a specified level a is the literal mathematical transcription of the concept “average loss in the worst $a * 100\%$ cases”. Expected Shortfall is also called Conditional Value at Risk (CVaR), Average Value at Risk (AVaR) and Expected Tail Loss (ETL). Some authors consider that ES and CVaR are two different names for thr same object. However, this happens only in the case of continuous random variables, i.e., only in that case the definition of ES coincides with that of CVaR.

3.9.4 Spectral Risk Measure (SRMs)

Last but not least, we have the Spectral Risk Measures which are a generalization of the previous risk measures, in which the distribution function is pre-multiplied with an admissible risk aversion function. This risk aversion function allows to introduce a subjective risk weight.

In other words, one of the nice features of SRMs is that they relate the risk measure to the user's risk-aversion, in effect, the spectral risk measure is weighted average of the quantiles of a loss distribution, the weights of which depend on the user's risk aversion.

An advantage of spectral measure is the way in which they can be related to risk aversion and particularly to a utility function, through the risk weights. However, a great part of the literature gives very little guidance on the choice of risk aversion function or on the question of what suitable risk aversion function might entail. For this reason, there were some authors who wanted to investigate this issue further and so they examined alternative SRMs based on alternative underlying utility functions.
For example, the 'exponential SMRs' which based on an exponential utility function or the 'power SRMs' which based on a power utility function.

4. *Derivatives*

In finance, a derivative is a contract that derives its value from the performance of an underlying entity. This underlying entity can be an asset, index, or interest rate and is often called the "underlying".

There are many types of derivative but the most common are the futures contracts, the forward contracts and the options (put and call options).

Derivatives can be used for a number of purposes including insuring against price movements (hedging), increasing exposure to price movements for speculation or getting access to otherwise hard to trade assets or markets (arbitrage)

That is to say, the three more important reasons that investors use derivatives are more analytically the following:

I. Hedging

Portfolio managers, individual investors and corporations use hedging techniques to reduce their exposure to various risks. More specifically, hedging against investment risk means strategically using instruments in the market to offset the risk of any adverse price movements. In other words, investors hedge one investment by making another. Hedging techniques generally involve the use of complicated financial instruments of which the two most common are the options and the futures contracts

Moreover, the most common hedging strategy is to take short position in futures contracts or long position in put options, when we are unsure of what the market will do, i.e., when we want to hedge the market risk. The reason why we take futures or put options for hedging the market risk, it will be analyzed in the next sub-sections.

II. Speculation

Speculation involves trading a financial instrument involving high risk, in expectation of significant returns. The motive is to take maximum advantage from fluctuations in the market.

With speculation, the risk of loss most or all of the initial outlay is more than offset by the possibility of a huge gain, otherwise, there would be very little motivation to speculate.

Moreover, speculation is often confused with gambling but the key difference is that speculation is generally tantamount to taking a calculated risk and is not depend on pure chance, whereas gambling depends on totally random outcomes or chance.

III. Arbitrage

Arbitrage exists as a result of market inefficiencies. This means that an investor has the opportunity to buy in one market and simultaneously to sell in another, profiting from a temporary difference in the prices in these two different markets. This is considered riskless profit for the investor. For this reason, if all markets were perfectly efficient, there would never be any arbitrage opportunities. However, markets seldom remain perfect.

In this chapter we will analyze the most common derivatives, as we mentioned in the beginning, which are the futures and forward contracts, as well the call and put options.

4.1 Futures Contracts

A future contract is a particularly simple derivative. More precisely, it is a contract between two parties where both parties agree to buy and sell an asset and at a predetermined price, at a specified date in future. In other words, futures contract is an agreement between two parties. One part is the buyer in the futures contract and he/she is known as to hold a long position and the other part is the seller in the futures contract and he/she is
said to be having short position. As the two parties to the contract do not necessarily know each other, the exchange provides a mechanism that gives the two parties a guarantee that the contract will be honored.

The financial instrument, as for example, stock, commodity, index, currency and others, on which a derivative’s price is based, is said underlying asset. The underlying asset in a futures contract could be stocks, commodities, currencies, interest rates and bonds. When, for example, the asset is commodity, there may be quite a variation in the quality of what is available in the marketplace. For this reason, when the financial assets in futures are specified, it is important to be well defined.

Moreover, it is proper to point out that the futures contracts, are held at a recognized stock exchange and for this reason they are a standardized contract. In other words, a future contract takes place on an organized exchange where all of the contract’s terms and conditions are formalized. More specifically, the terms that are standardized include price, date, quantity, trading procedures and place of delivery.

The exchange acts as mediator and facilitator between the parties. This means that if two investors get in touch with each other directly and agree to trade an asset in the future for a certain price, there is obvious risk. The reason is that one of the investors may regret the deal and try to back out or alternatively the investor may simply not has the financial resources to honor the agreement. Therefore, it is very important the role of the stock exchange which organizes trading so that contract defaults are avoided.

Hence, in the beginning both parties are required by the exchange to put beforehand a nominal account as part of contract, known as the margin. Furthermore, since the futures prices are bound to change every day, the differences in prices are settled on daily basis from the margin. In other words, the broker will require from the investor to deposit funds in a margin account. The amount that must be deposited at the time the contract is entered into is known as the initial margin. At the end of each trading day, the margin account is adjusted to reflect the investor's gain or loss. This process is called marking to market. A trade is first marked to market at the close of the day on which it takes place. It is then marked to market at the close of trading on each subsequent day.

Also, the investor has the right to withdraw any balance in the margin account in excess of the initial margin. To ensure that the balance in the margin account never becomes negative a maintenance margin, which is somewhat lower that the initial margin, is set. If the balance in the margin account falls below the maintenance margin, the investor receives a margin call and is expected to top up the margin account to the initial marginal level the next day. The extra funds deposited are known as variation margin. If the investor does not provide the variation margin, the broker closes out the position.

However, the vast majority of futures contracts do not lead to delivery. The reason is that most traders choose to close out their positions before the delivery period which is specified in the contract. This means that the trader has to enter into the opposite type of trade from the original one in order to close out a position. In other words, the buyers of the futures contracts usually sell their contracts before the delivery date, thus offsetting their position.

As we mentioned before, a futures contract obliges its purchaser to buy a given amount of a specified asset at a particular time in the future, which is known as the delivery date, at a predetermined price. Similarly, the seller of the contract is obliged to deliver the asset at the predetermined price

While a futures contract may be used by a buyer or seller to hedge other position in the same asset, price changes in the asset after the futures contract agreement are made provide gains to one party at the expense of the other. If, for example, the price of the underlying asset increases after the agreement is made, the buyer gains at the expense of the seller. Similarly, if the price of the asset decreases, the seller gains at the expense of the buyer. And this is indicated by the following figure:

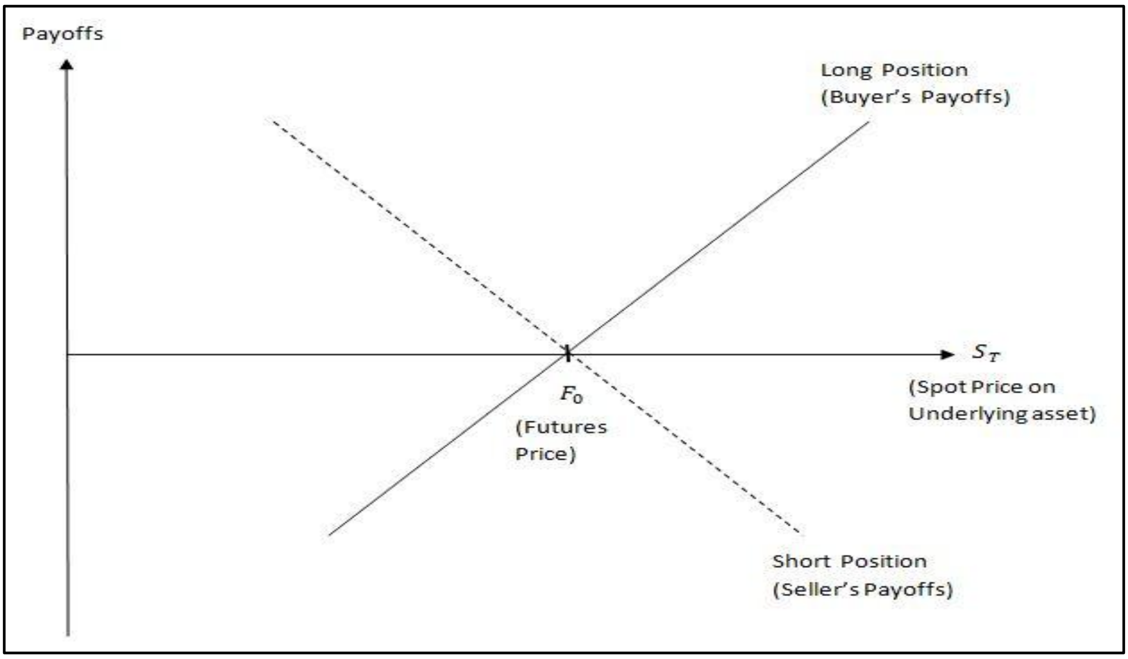

Figure 3: Cash flows on futures contracts

Finally, futures contracts can be used to hedge against risk or speculate the prices. Concerning hedging of an equity it would be proper to use futures contracts. More specifically, a hedge using index futures removes the risk arising from market moves, i.e., we desire to hedge the systematic-risk, and leave the hedger exposed only to the performance of the portfolio relative to the market. Another reason for hedging is that hedger is planning to hold a portfolio for a long period of time and requires short-term protection in an uncertainty market situation.

Therefore, futures contracts are one of the most common derivatives used to hedge risk. The main reason that investors use futures contracts is to offset their risk exposures and limit themselves from any fluctuations in price.

4.2 Forward Contracts

Fundamentally, forward and futures contracts have the same function. This means that both types of contracts allow people to buy or sell specific type of asset at a specific time at a given price. However, it is in the specific details that these contract differ.

Unlike standard futures contracts, a forward contract can be customized to any commodity, amount and delivery date, i.e., it is not a standardized contract like the futures. Forward contracts are private agreements between the two parties and are not so rigid in their stated terms and conditions.

Because forward contracts are private agreements, there is always a chance that a part may default on its side of the agreement. This means that the forward contracts do not trade on a centralized exchange and therefore regarded as over-the-counter (OTC) market, usually between two financial institutions or between a financial institution and one of its clients. And while their OTC nature makes it easier to customize terms, the lack of a centralized clearinghouse also gives rise to a higher degree of default risk. As a result, forward contracts are not as easily available to the retail investor as futures contracts.

Moreover, it is very important to mention that the price of a futures contract resets to zero at the end of every day because daily profits and losses, based on the prices of the underlying asset, are exchanged by traders via their margin accounts. This process is called marking to the market, as it was referred before, in the previous sub-section concerning the futures contracts. In contrast, a forward contract starts to become less or more valuable over time until the maturity date, which is the only time when either contracting party profits or loses. In other words, under the forward contract, the whole gain or loss is realized at the end of the life of the contract and under the futures contract, the gain or loss is realized day by day because of the daily settlement procedures.

More precisely, while the net settlement is the same under the two contracts, the timing of the settlements is different, i.e., on the forward contract, the settlement occurs at maturity and on the other hand, on the futures contract, the profits or losses are recommended each period. This is the reason that can lead to different prices for the two types of contracts.

Finally, it is useful to refer that at the time the forward contract is entered into, the delivery price, which is the price in a forward contract, is chosen so that the value of the forward contract on both sides is zero. This means that it costs nothing to take either a long or a short position. While, if an investor wishes to buy or sell in the futures market, it is required to post an initial margin in the form of cash, a portion of the full price.

4.3 Options

While the difference between a futures and a forward contract may be subtle, the difference between these contracts and options contracts is much greater.

Initially, an option is a financial instrument which value depends on the values of others, more basic underlying variables. There are two basic types of options. The first one is the call option which gives the holder the right to buy the underlying asset by a certain date for a certain price and the second one is the put option which gives the holder the right to sell the underlying asset by a certain date for a certain price. Also the former is similar to have a long position on a stock, i.e., to buy the stock and the latter is similar to have a short position on a stock, i.e., to sell the stock.

Hence the biggest difference between futures, forward contracts and options is that an option is a contract that gives the buyer the right, but not the obligation as happens to futures and forward contract, to buy or sell an underlying asset at a specific price on or before a certain date. In other words, in an option contract, the buyer is not obligated to fulfill his side of the bargain, which is to buy the asset at the agreed upon price in the case of a call option and sell the asset at the agreed upon price in the case of a put option.

The price at which an underlying asset can be purchased or sold is called exercise or strike price and the date that these contracts expire is known as the expiration date or maturity. So, all the options have a specific duration to their maturity. However, there are the American options that can be exercised at any time between the date of purchase and the expiration date, while the European options can be exercised only at the expiration date itself. Most of the options that are traded on exchanges are American, but European options are generally easier to analyze than American options. For this reason, everything that will be mentioned in this chapter, will concern only the European options only.

Furthermore, there are two sides to every option contract. On one side, is the investor/trader who has taken the long position (i.e., he bought the option) and on the other side, is the investor/trader who has taken a short position (i.e., he sold the option and also known as "written"). People who buy options are called holders and those who sell options are called issuers or writers. The writer of an option receives cash up front, because in order to buy an option the investor has to pay, but he or she has potential liabilities later. Also, the writer's profit or loss is the reverse of that of the purchaser of the option.

Hence, there are four types of option positions:

- A long position in a call option (long call)
- A long position in a put option (long put)
- A short position in a call option (short call)
- A short position in a put option (short put)

Moreover, there are options on stocks, currencies, stock indices and futures. In more details, as concerns the stock options, options trade on more than one thousand different stocks and one contract gives the holder the right to buy or sell 100 shares at the specified strike price. This contract size is convenient because the shares themselves are normally traded in lots of 100.

We have also the foreign currency options. Most currency options trading are now in the over-the-counter market, but there is some exchange trading. The major exchange for trading foreign currency options is in the United States which offers both European and American contracts on a variety of different currencies. The size of one contract depends on the currency.

Furthermore, many different index options currently trade throughout the world in both the over-the-counter market and the exchange-traded market. Most of the contracts are European. An exception is the OEX contract on the S&P100, which is American. One contract is usually to buy or sell 100 times the index at the specified strike price.

And finally, there are the futures options. When an exchange trades a particular futures contract it often also trades options on that contract. A futures option normally matures just before the delivery period in the futures contract. When a call option is exercised, the holder acquires from the writer to buy the underlying futures contract plus a cash amount equal to the excess of the futures price over the strike price, and via versa in a put option.

Now, let's analyze more precisely the two types of options:

In the exchange, for the rights conferred by the option, the option buyer has to pay the option seller a premium for carrying on the risk that comes with the obligation. The option premium depends on the strike price, volatility of the underlying, as well as on the remaining to expiration. The option premium is also known as option price and should not be confused with the strike price. For the premium of an option, the holder (buyer) locks in a specific strike price (K) in order to buy or sell the underlying asset.

4.3.1 Call Option

The call option gives the holder the right to buy the underlying asset. When the underlying security falls below the strike price ($S_T < K$) at the time of the option's expiration, the holder will clearly choose not to exercise. Because we analyze only the European options, the investor can exercise only on the expiration date. So, in this case, the option will simply be left to expire and the holder will lose the cost paid for the purchase of the option.

On the other hand, if the security's market price exceeds the strike price ($S_T > K$), the holder has the right to exercise the option and so he or she will realize a profit. The option will be exercised because the holder has interest to buy the underlying asset in the strike price which is smaller than the price it has in the market at the expiration date. Moreover, the gain grows linearly with the increase of the security's market price at the option's expiration time.

The payoff to the trader at maturity who has long position in a European call option is:

$$\max[S_T - K, 0]$$

and this reflects the fact that the option will be exercised if $S_T - K > 0 \Rightarrow S_T > K$ and will not be exercised if $S_T < K$.

The net profit is:

$$\max[S_T - K, 0] - c$$

where c is the price of the call option.

The payoff to the investor of a short position in a European call option is:

$$-\max\,[S_T - K, 0]$$

or

$$\min[K - S_T, 0]$$

Diagrammatically the net profits are shown below:

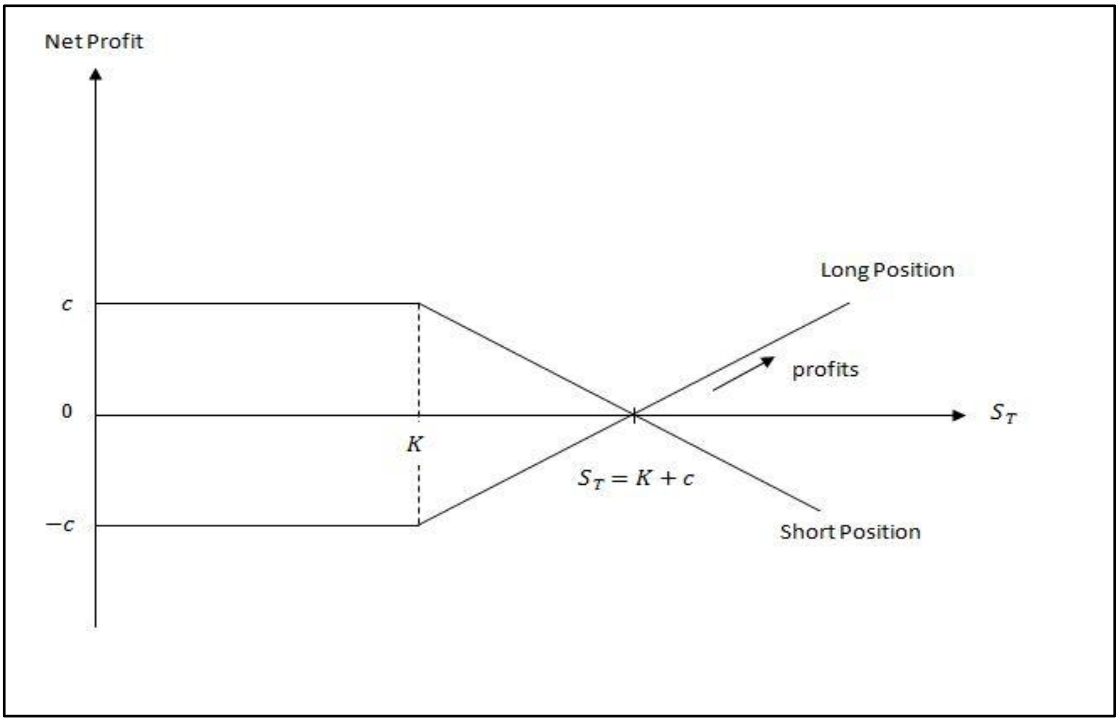

Figure 4: The net profits of a European call option

Hence, a long position in a call option accentuates the upside potential in the event of market upturns.

4.3.2 Put Option

Whereas the purchaser of a call option is hoping that the stock price will increase, the purchaser of a put option is hoping that it will decrease. This means that the option will not yield a payoff and will be left to expire, if the market price of the underlying asset exceeds the strike price at maturity (i.e., when $S_T > K$). However, in the event that the market price of the underlying asset falls below the strike price of the option at maturity, then the holder reaps an immediate profit (i.e., when $S_T < K$). In this case the option will be exercised and the holder reaps profits because he/she has the best interest to do this in order to sell the underlying asset more expensive than it is in the market.

Here the gain grows linearly with the decline of the security's market and so a put option can provide coverage against potential declines in the price of the underlying security.

The payoff to the investor at maturity who has a long position in a European put option is:

$$\max\,[K - S_T, 0]$$

and this reflects the fact that the put option will be exercised if $K - S_T > 0 \Rightarrow K > S_T$ and will not be exercised if $K < S_T$.

And the net profit, that the investor has, is:

$$\max[K - S_T, 0] - p$$

On the other hand, the payoff of someone who has a short position in a European put option is:

$$-\max\,[K - S_T, 0]$$

or

$$\min\,[S_T - K, 0]$$

Similarly, diagrammatically the net profits of two sides of put option shown below:

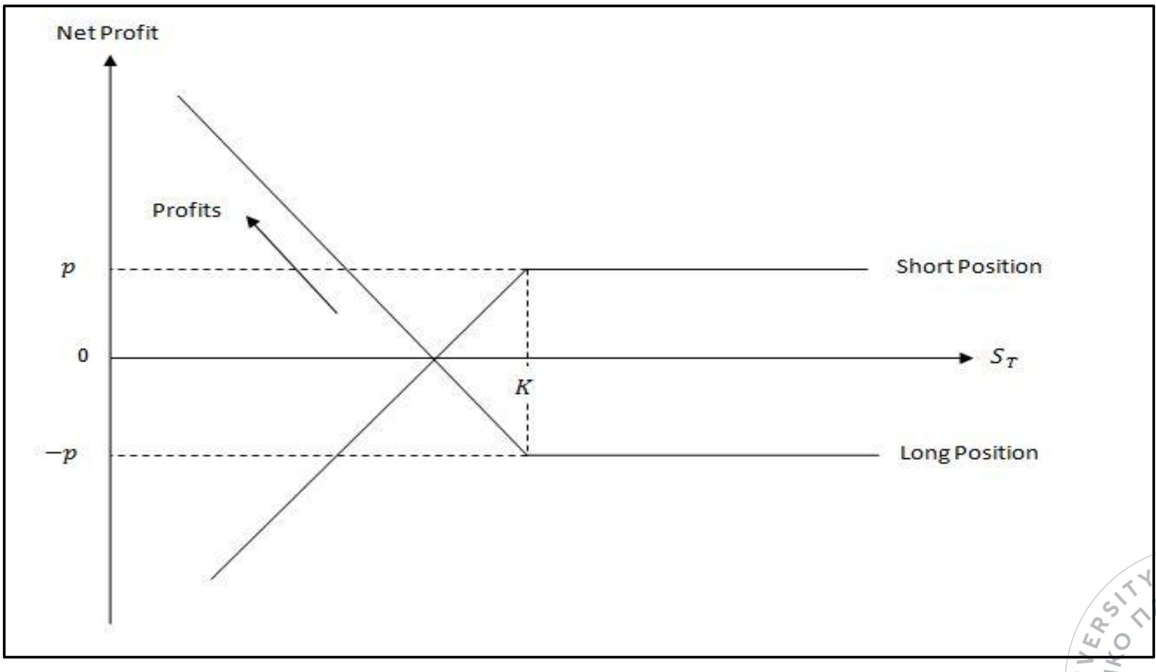

Figure 5: The net profits of a European put option

As we can observe from the two diagrams, the profit or loss of someone who has short position (writer) is the reverse of that for someone who has the long position in the option (holder).

4.3.3 Moneyness

There is different between the two basic types, as regards the existence of an option in-the-money (ITM), out-of-the-money (OTM) or at-the-money (ATM).

More analytically, the call option, is said to be in-the-money if the share price is above the strike price (i.e., it would be worth something to be exercised). Contrariwise, a put option is said to be in-the-money if the share price is less than strike price (i.e. it would be also worth something to be exercised). The amount by which an option is in-the-money is referred as intrinsic value.

Similarly, a call option is out-of-the-money if the asset price is less than the strike price and a put option is out-of-the-money if the asset price is greater than the strike price. In this case for both options it would be worth nothing to be exercised.

And finally, when a call option and a put option are at-the-money that means that the share price is equal to the strike price.

4.3.4 Trading Strategies involving Options

There are number of different trading strategies involving a single option on a stock and the stock itself. One of these strategies is the portfolio consists of a long position in a stock plus a short position in a call option. This is known as writing a covered call. The long position in the stock protects the investor from the payoff on the short call that becomes necessary if there is a sharp rise in the stock price. There is also the reverse of the above strategy which has a short position in a stock is combined with a long position in a call option. The other approach is the protective put which involves buying a put option on a stock and the stock itself and the inverse of this strategy.

Moreover, there are certain strategies that involve taking a position in two or more options. This means that these strategies involve combinations of positions in call and put options on the same underlying asset and each of them generates a different payoff profile. The choice among them is based on the investor's view regarding potential movements in the value of the underlying asset and his preferences for protection in the case of such movements.

One of the popular combinations is the straddle strategy which consists of long position in one call and one put option on the same underlying asset, with the same strike price and expiration date. A straddle is appropriate when an investor is expecting a large more in as stock price but does not know in which director the move will be. The bottom straddle or straddle purchase are the alternatives names of the straddle strategy and the top straddle or straddle write is the reverse position. To sell a call and a put option with the same strike price and expiration date is a highly risky strategy and the loss arising from a large move is unlimited.

The other popular combination is the strangle strategy which sometimes called the bottom vertical combination. This strategy consists of a long position in one call and a long position in one put option on the same underlying asset with the same expiration date but different strike price. This is the difference with the straddle strategy.

However, a strangle is a similar strategy to straddle because the investor again is betting that there will be a large price move, but is uncertain whether it will be an increase or decrease. Moreover, the stock price has to move farther in a strangle than in a straddle for the investor to make a profit. But the downside risk if there is only a small change in the value of the stock is less with the strangle than with the straddle, because the strangle is cheaper alternative than the straddle as the prices of its constituent options are lower than in the strangle. The reason is that the two options are out-of-the-money in the strangle strategy, while in the straddle strategy the two options are at-the-money, which means that they are more expensive.

This is the long strangle but the sale of a strangle is sometimes referred to as a top vertical combination and it can be appropriate for an investor who feels that large stock moves are unlikely. However, as with sale of a straddle, it is a risky strategy involving unlimited potential loss to the investor.

Furthermore, in the category of combinations of positions in options, we have also the strip and strap strategy. A strip strategy consists of a long position in one call and two puts with the same strike price and expiration date. An investor takes a long position in strip when he or she expects a large move in the stock price and move precisely when he or she considers a decrease in the index more likely than an increase and for this reason this strategy takes long position in two put options.

On the contrary, a strap strategy consists of a long position in two calls and one put. Here, an investor takes a long position in this strategy when he or she expects large moves in the stock price but considers an increase in the stock more likely than a decrease and for this reason this strategy takes a long position in two call options.

Finally, it is worth mentioning the spread trading strategies which involve taking a position in two or more options of the same type, i.e., taking a position in two or more call options or two or more put options, but with different strike price.

One of the most popular types of spreads is the bull spread. This strategy can be created by taking a long position in a call option on a stock with a certain strike price and taking a short position in a call option on the same stock with a higher strike price, however, both options have the same expiration date. A bull spread strategy limits the investor’s upside as well as downside risk. There are three types of bull spreads. The first one is to be both calls out-of-the-money, which is the most aggressive bull spread strategy. It costs very little and has a small probability of giving a relatively high payoff. The other type of strategy is to be the one call option in-the-money and the other call option out-of-the-money. And the last strategy is to be both calls in-the-money, which is more conservation in contrast to first type.

There is also the bear spread strategy in which an investor hopes that the stock price will decline in contrast to an investor who enters into a bull spread strategy and he/she hopes that the stock price will increase. Bear spread is a strategy composed of taking a long position in a put option with one strike price and taking a short position in a put option with another strike price which is less than the long put and this is in contrast to a bull spread strategy. Like bull spreads, bear spreads limit both the upside profit potential and the downside risk.

One more popular type of spreads is the butterfly spread strategy which involves positions in options with three different strike prices. It consists of a long position in a call option with a relatively low strike price, a long position in a call option with a relatively high strike price and a short position in two call options with a strike price halfway between the other two strike prices (i.e., between the low and the high strike price). So the butterfly spread leads to a profit if the stock price stays close to the strike prices, but gives rise to a small loss if there is a significant stock price in either direction. Therefore, it is an appropriate strategy for an investor who believes that large stock price moves are unlikely. Butterfly spreads can also be created using put options.

Finally, a butterfly spread can be shorted by following the reverse strategy. Taking a short position in the options with the high and the low strike price and taking a long position in the two options with the middle strike price. This strategy produces a modest profit if there is a significant movement in the stock price.

Generally, there are many more strategies but all the above are the most significant in trading strategies involving options. As we can understand, these are few of the ways in which options can be used to produce an interesting relationship between profit and stock price.

4.3.5 Why the investors use options

There are two main reasons why an investor would use options:

The first one reason is to speculate. In area of options we can think of speculation as betting on the movement of a security. The advantage of options is that you are not limited on making a profit only when the market goes up but you can make money when the market goes down or even sideways.

The other function of options is hedging. Options can be used to insure your investments against a downturn. Critics of options say that if you are so unsure of your stock choice that you need a hedge, you should not make the investment. On the other hand, there is no doubt that hedging can be useful for large institutions but also for individual investors.

4.3.6 *Hedging with options vs. futures*

Now we will see how we use futures and options in order to hedge the risk of our portfolio, which consists of assets and more precisely of shares on a stock index, and how they differ.

Let's assume that in the coming months, and while the portfolio is in our possession, the economy will not go up. Therefore the price of index will reduce and consequently the risk here lies in falling the value of our portfolio. In order to gain money we should take a short position in futures. So, by taking futures on the index, we manage to offset the downside risk.

On the other hand, if we want to take only options instead of futures contracts we should take a long position in put options. More specifically, we take put options because we want to make a profit when the index falls.

The most important different between futures and put options, in order to hedge the downside risk as shown in the following picture, is that by using futures we lock the value of the portfolio in V_T. While by using put options we have to make the question: "Up to which amount we are willing to lose for our portfolio?". The amount that we are willing to lose, denoted by L, has to do with how expensive are the put options which we want to buy.

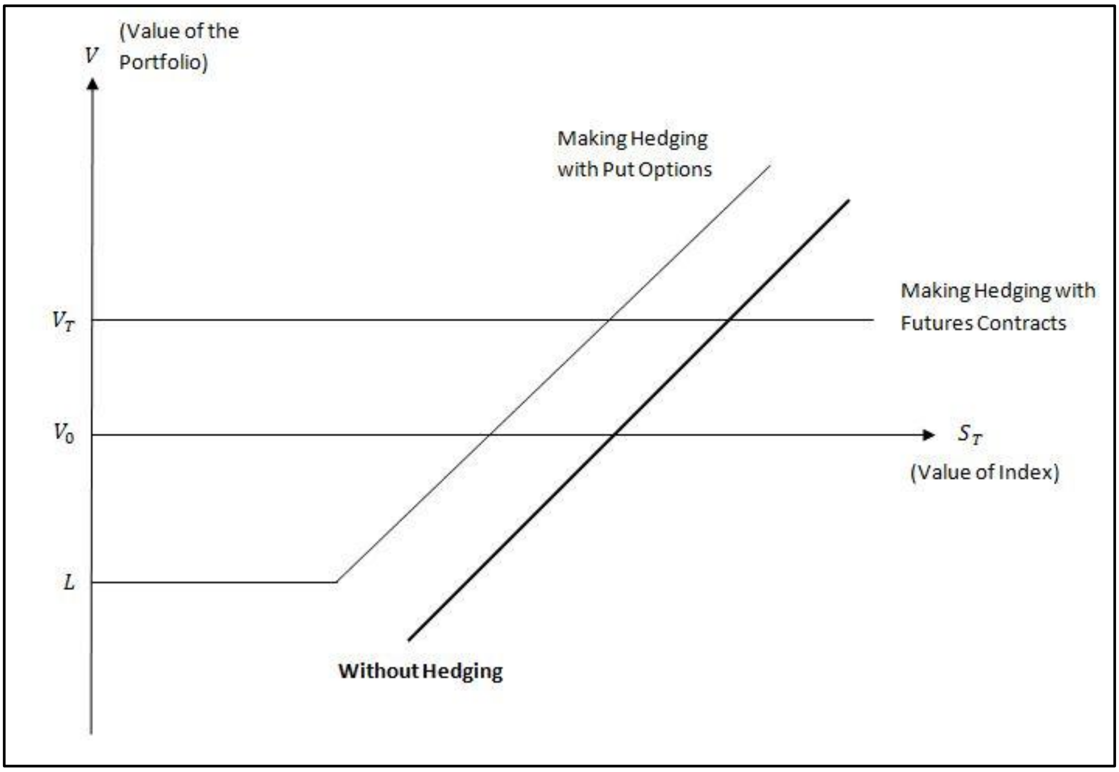

Figure 6: Differences between hedging with options, hedging with futures and without hedging

More precisely, if we do not do something in order to offset the risk, then the portfolio will move in the same way as the index. Contrary, if we take short position in futures, we manage to stabilize the value of our portfolio and this is illustrated in the above diagram by a horizontal line that begins from the point V_T of the vertical axis. And also, if we take long position in put options, then if the market does not go well, the portfolio value will drop as much as we have set. While, if the market goes well, then the portfolio value will rise and will move like the stock index.

4.3.7 Option Valuation

Although options valuation has been studied at least since the nineteenth century, the contemporary approach is based on the Black-Scholes model which was first published in 1973.

Investors should have a good understanding of the factors that determine the value of an option. These factors include the current stock price, the time to expiration, the volatility and the interest rates. If the share pays dividends, these factors will include also the cash dividends paid.

There are several options pricing models that use these parameters to determine the fair market value of the option. Binomial tree and Black-Scholes are the most popular approaches concerning the pricing of stock options. However, there are some important differences between these two approaches.

In the Binomial model, we create discrete distributions of asset price. In other words, this model breaks down the time to expiration into potentially a very large number of time intervals or steps. At each step it is assumed that the stock price will move up or down by an amount calculated using volatility and time to expiration. This produces a binomial distribution or binomial tree of underlying stock prices. The tree represents all the possible paths that the stock price could take during the life of the option. Moreover, the option prices at each step of the tree are calculated working back from the expiration to the present. The big advantage that the Binomial model has over the Black-Scholes model is that it can be used to accurately price of American options and this is because with the binomial tree it is possible to check at every point in an option's life for the possibility of early exercise.

While the Black-Scholes model is in continuous time, i.e., it shows movement in time and the main advantage of this model is speed. This means that it lets us to calculate a very large number of option prices in a very short time.

In this sub-section we will go deeper only in the Black-Scholes model for valuing European call and put options on a non-dividend-paying stock, because this exact model is used in our empirical application in the next section.

The Black-Scholes formulas for the prices at time zero of a European call option and put option on a non-dividend-paying stock are the following:

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

and

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

Where:

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \text{ , or } d_2 = d_1 - \sigma\sqrt{T}$$

The function $N(d)$ is the cumulative probability distribution function for a standardized normal distribution. More precisely, it is the probability that a variable with a standard normal distribution, $N(0,1)$, will be less than d. In other words $N(d_1) = Prob(z \leq d_1)$ and $N(-d_1) = 1 - N(d_1)$, where $z \sim N(0,1)$.

The variables are:

c = European call price
p = European put price
S_0 = stock price at time zero
K = strike price
r = continuously compounded risk-free interest rate
σ = stock price volatility
T = time to maturity of the option

Furthermore, it is very important to point out that the model is based on a normal distribution of underlying asset returns which is the same thing as saying that the underlying asset prices themselves are log normally distributed.

5. *Empirical application*

In this section, we deal with the international portfolio management which includes securities from different countries. Adding stock and bond indices from four different countries we manage to construct a well internationally diversified portfolio. Our goal is to rid from the unsystematic risk, which has to do with each individual asset, as it is mentioned in chapter 2.

However, an international portfolio is exposed and to other risks, apart from market risks, which should be taken into consideration. The primary risks are on the one hand, the market risks of positions in stock indices in the various countries and on the other hand, the currency risk of foreign investments. An appropriate way to control the total risk exposure is the hedging.

For this reason, in order to hedge the market risk of the underlying stock indices, options can be used. While currency risk can be mitigated by using currency forwards. There are also other derivatives that can be used in order to hedge the above risks. For example the quantos or the currency options for the currency risk and the futures for the market risk.

In our empirical application, it is used European put options to hedge only the market risk exposures of holdings in domestic and foreign stock indices.

5.1 Data Description

Our international portfolio is composed of stock and bond indices in four markets. So, we have collected market prices for two stock indices for each of the following four countries:S_1, S_2 for Greece, S_{usa1}, S_{usa2} for USA, S_{uk1}, S_{uk2} for United Kingdom and finally, S_{jp1}, S_{jp2} for Japan. Moreover, we have collected corresponding market prices for two government bond indices for each of the above countries, denoted as B_1, B_2 for Greece, B_{usa1}, B_{usa2} for USA, B_{uk1}, B_{uk2} for United Kingdom and B_{jp1}, B_{jp2} for Japan. The two bond indices have different maturity bands and more accurately the first one has 1-3 years (B_1, B_{usa1}, B_{uk1}, B_{jp1}) and the second one has 7-10 years (B_2, B_{usa2}, B_{uk2}, B_{jp2}) respectively for every country.

Furthermore, spot exchange rates have been collected for the corresponding currencies, which are the dollar for USA (US $), the pound for United Kingdom (UK £) and the yen for Japan (JAPANESEN ¥). These spot exchange rates are denoted as follows: $to€, £to€ and ¥to€. All the data has been collected by the Bloomberg and it is on a monthly time step, spanning the period from 31/12/1999 to 30/3/2012. This means that we have 148 observations for every index in each country.

From these market data, we compute the monthly returns for the stock and bond indices of each country (i.e. we compute the monthly local returns), as well as the monthly appreciation rate of the spot exchange rates. So we have 147 returns for every index and spot exchange rate in each of four countries.

Asset Class (Index)	Mean	Variance	Std. Dev.	Skewness	Kurtosis
S1	-0,0114	0,0084	0,0919	0,0517	0,8866
S2	-0,0136	0,0082	0,0905	-0,1275	0,4502
B1	-0,0093	0,0014	0,0379	-4,3607	25,2938
B2	-0,0089	0,0023	0,0483	-3,3508	17,9128
Susa1	0,0008	0,0022	0,0469	-0,4675	0,6766
Susa2	0,0076	0,0035	0,0590	-0,4011	0,9453
Busa1	0,0032	0,0000	0,0047	0,2646	0,8494
Busa2	0,0061	0,0004	0,0197	-0,0106	1,4809
Suk1	-0,0003	0,0018	0,0430	-0,5477	0,3634
Suk2	0,0018	0,0033	0,0575	-0,1280	3,6903
Buk1	0,0041	0,0000	0,0045	0,3782	1,2336
Buk2	0,0057	0,0002	0,0148	0,1182	0,9140
Sjp1	-0,0024	0,0029	0,0541	-0,4156	0,9514
Sjp2	0,0020	0,0027	0,0518	-0,2055	-0,3855
Bjp1	0,0005	0,0000	0,0013	0,3432	3,0635
Bjp2	0,0023	0,0001	0,0095	-0,9039	2,5994
Exchange Rate					
\$ to €	0,0024	0,0010	0,0314	-0,0838	0,8148
£ to €	0,0023	0,0007	0,0259	1,7929	11,2904
¥ to €	0,0011	0,0013	0,0362	-0,7902	3,0881

Table 1: Sample statistics of monthly local returns of stock and bond indices and monthly returns of spot exchange rates.

In the above table is presented the statistical characteristics of monthly local returns of all assets and exchange rates. As it can be observed the returns of all assets and exchange rates respectively do not follow normal distributions because kurtosis is different of 3 ($kurtosis \neq 3$) and skewness is different of 0 ($skewness \neq 0$).

More specifically, most of the domestic returns of the indices and the exchange rates exhibit considerable variance in comparison to their mean. Moreover, both the local returns of the indices and the returns of the spot exchange rates exhibit skewness. Skewness is a statistical measure which describes asymmetry from the normal distribution in a set of statistical data. It can come in the form of “negative skewness” or “positive skewness”, depending on whether data points are skewed to the left or to the right of the data average respectively. As illustrated in the above table, we have both negative skewness and positive skewness.

Furthermore, some of the random variables exhibit high kurtosis which means that the distribution has fatter tails than the normal distribution and such a distribution is called leptokurtic; it has a more acute peak around the mean. Some other random variables exhibit low kurtosis (less than 3) and therefore tend to have thinner tails. This distribution is called platykurtic; it has a lower, wider peak around the mean. Kurtosis is a statistical measure used to describe the distribution of observed data around the mean. Hence, none random variable does not have kurtosis exactly 3.

Consequently, all of these clearly have an influence on using the CVaR as risk measure in our empirical application. The reason is that the CVaR risk metric is suitable for skewed distribution as it is mentioned in chapter 3.

5.2 Portfolio optimization model without options

The problem of portfolio restructuring is viewed from the prospect of a European investor and more exactly from the prospect of an investor from Greece, who may hold assets denominated in multiple currencies. In other words, his or her portfolios are composed of two stock indices and two bond indices from various countries, and as it is referred in the subsection of data description, from the country of Greece, USA, UK and Japan. These portfolios are exposed to market and currency risk. However, in this sub-section it is presented the portfolio optimization model without using options in order to hedge the market risk.

All decisions are taken at the beginning ($t = 0$) of the planning period $[0, T]$, taking into account the scenarios that describe the reasonable changes in the values of the random variables during the holding period. In our model the length of the horizon is one month (i.e.$T = 1/12$).

The aim is to minimize the tail risk of the value of the revised portfolio at the end of the planning period, this means at the time T, for a covetable target expected return over the holding period.

At this point, it is proper to refer that the CVaR model, which is presented in the next sub-section, is solved and implemented in GAMS using OSL solver (Optimization Subroutine Library) for solving LP problems (Linear Programming). Therefore, by minimizing the CVaR risk measure, we manage to find the optimal international portfolio.

5.2.1 The Model without options

Now, we will present first the optimization model without including put options.

$$\min \quad z + \frac{1}{1-a} \sum_{n \epsilon N} Prob(n)\, y(n) \quad \equiv CVaR$$

Subject to:

$$h_l = \sum_{i \epsilon I_c} x_{ic}\, \pi_{ic} (1+\delta) + \sum_{c \epsilon C} x_c^e (1+d) \qquad \mathbf{(1)}$$

$$x_c^e e_c = \sum_{i \epsilon I_c} x_{ic} \pi_{ic} (1+\delta)\ , \quad \forall c \epsilon C \qquad \mathbf{(2)}$$

$$V_l^n = \sum_{i \epsilon I_l} w_{il} \pi_{il}^n + \sum_{c \epsilon C} \left\{ \left(\sum_{i \epsilon I_c} w_{ic} \, \pi_{ic}^n \right) \frac{1}{e_c^n} \right\}, \qquad \forall n \epsilon N \quad (\mathbf{3})$$

$$R_n = \frac{V_l^n - V_l^0}{V_l^0} \Rightarrow R_n = \frac{V_l^n}{V_l^0} - 1 \,, \quad \forall n \epsilon N \quad (\mathbf{4})$$

$$\bar{R} = \sum_{n \epsilon N} Prob(n) \, R_n \qquad (\mathbf{5})$$

$$\bar{R} \geq \mu \qquad (\mathbf{6})$$

$$y_n \geq L_n - z \;\; , \;\; \forall n \epsilon N \qquad (\mathbf{7})$$

$$y_n \geq 0 \,, \qquad \forall n \epsilon N \qquad (\mathbf{8})$$

$$L_n = -R_n \,, \qquad \forall n \epsilon N \qquad (\mathbf{9})$$

$$w_{ic} = x_{ic} \;\; , \qquad \forall i \epsilon I_c \,, \forall c \epsilon C_0 \qquad (\mathbf{10})$$

$$w_{ic} \geq 0, \;\; x_{ic} \geq 0 \,, \qquad \forall i \epsilon I_c \,, \forall c \epsilon C_0 \qquad (\mathbf{11})$$

Where:

z: is the VaR value of the portfolio losses during the holding period

$Prob(n)$: is the probability of scenario nϵN, where N is the set of scenarios

y_n: is the auxiliary variable which is used in order to define the CVaR and which specifically measures the excess loss beyond VaR in scenarios nϵN

a: is the prespecified confidence level for the CVaR measure

The objective function that we want to minimize is the CVaR risk measure.

The constraint ($\mathbf{1}$) is the cash balance condition at time $t = 0$ for the base currency l, which is the euro, because this model is viewed from the prospect of a European investor.

It is very useful to make the following clarification:

$l \epsilon C_0$: is the index of the base currency in the set of currencies

C: is the set of foreign currencies without the base currency

C_0: is the set of currencies including the base currency

I_l: is the set of financial assets denominated only in base currency (i.e. euro) and specifically in Greece these assets are four (two stock indices, one short-term bond index and one long-term bond index). This means that $I_l = 4$

I_c: is the set of financial assets denominated in currency $c \epsilon C_0$, i.e. , in each country, including the base country. Again the financial assets in each country are four as in Greece ($I_c = 4$)

Moreover, we have to identify the above variables as well as the parameters:

x_{il}: is the units of asset $i \epsilon I_l$ in the base currency purchased

π_{il}: is the current market price, in units of the base currency

So, $(x_{il}\pi_{il})$ shows us how much money (in euro) the investor gives in his/her market in order to purchase assets. This term is multiplied with (1+δ), where δ is the proportional transaction cost for purchases of assets.

x_c^e: is the amount of the base currency exchanged in the spot market for foreign currency $c \epsilon C$. This term also is multiplied with a transaction cost (d) which is for currency exchanges in the spot market.

The constraint (**2**) is the cash balance condition at time $t = 0$ for the foreign currencies $c \epsilon C$. The foreign currencies, as it was mentioned before, are dollar ($), pound (£) and yen (¥).

Where:

e_c: is the current spot exchange rate for currency $c \epsilon C$

x_{ic}: is the units of asset $i \epsilon I_c$ in currency $c \epsilon C$ purchased

π_{ic}: is the current market price, in units of the local currency

Furthermore, the constraint (**3**) shows the value of the portfolio at the end of the holding period under scenario $n \epsilon N$. The terminal value of the portfolio is in units of the base currency (i.e. euro). Hence, the assets that are contained in the portfolio and which are in foreign currencies are valued in terms of the base currency by applying the respective spot exchange rates at the end of the holding period.

In this constraint we have the following variables and parameters:

w_{ic}: is the units of asset $i \epsilon I_c$ in currency $c \epsilon C_0$ held in the revised portfolio

V_l^n: is the total value of the revised portfolio at the end of the holding period under scenario $n \epsilon N$. The total value is in units of the base currency

π_{ic}^n: is the market price of every asset i, in units of the local currency $c \epsilon C_0$ (i.e. including the base currency) at the end of the planning horizon under scenario $n \epsilon N$

e_c^n: is the spot exchange rate of $c \epsilon C$ at the end of the horizon under scenario $n \epsilon N$.In other words, it is the exchange rate for every foreign country at the end of the planning horizon under scenario n.

Moreover, the constraint (**4**) computes the portfolio return over the holding period under each scenario n.

Where:

$V_l^0 = h_l$: is the total value of the initial portfolio and this total value is also expressed in units of the base currency.
R_n: is the holding period return of the revised portfolio under scenario $n \epsilon N$

And the constraint (**5)** defines the expected return of the portfolio, which in constraint **(6)** is greater or equal than a minimum bound μ.

More strictly:

$\bar{R}$*:* is the expected holding period return of the revised international portfolio

μ: is the expected target return of the revised international portfolio, which is selected by the investor in accordance to his/her desire of risk

The constraint (**7**) determines the excess losses y_n, which is an auxiliary variable for the definition of CVaR and it measures the excess loss beyond the VaR in each scenario $n \epsilon N$. Also the constraint (**8**) shows that the auxiliary variable y_n is non-negative.

Additionally, the portfolio loss is computed by the constraint (**9**), over the helding period under each scenario and more specifically L_n is the portfolio loss during the planning horizon under scenario $n \epsilon N$.

Hence, all of these three equations (constraints (**7), (8)** and (**9**)) can be summarized in the following constraint:

$$y_n \geq -R_n - z$$

Finally, the equation (**10**) shows that the units of asset that we hold from each asset i of each country are equal to the units of asset from each asset i of each country that we purchase. And the constraint (**11),** which includes the two inequalities, ensures that the units of asset i that are purchased (x_{ic}), as well as the units of asset i that are held (w_{ic}) are non-negative.

Last but not least, as perceived by the above model, there are both deterministic and stochastic inputs. The deterministic inputs are the current prices of the securities and the current spot exchange rates. While, the stochastic inputs are the prices of the securities in each scenario $n\epsilon N$, (π_{ic}^n), and the spot exchange rates in each scenario $n\epsilon N$, (e_c^n). More precisely, the scenario dependent data together with the associated probabilities specify the joint distribution of the random variables at the end of the planning horizon.

By using the monthly returns of each asset $i\epsilon I_c$ in currency $c\epsilon C_0$ under scenario $n\epsilon N$, (r_{ic}^n), and the monthly returns of the spot exchange rate of currency $c\epsilon C$ under scenario $n\epsilon N$, (g_c^n), we can compute the corresponding outcomes of asset prices (π_{ic}^n) and spot exchange rate (e_c^n) at the end of the holding period for each scenario $n\epsilon N$.

This means that:

$$\pi_{ic}^n = (1 + r_{ic}^n)\pi_{ic}\,, \quad \forall c\epsilon C_0, \forall i\epsilon I_c\,, \forall n\epsilon N$$

and

$$e_c^n = (1 + g_c^n)e_c\,, \quad \forall c\epsilon C, \forall n\epsilon N$$

It is proper to refer that in our model the scenarios are coming from the historical data and we use equiprobable scenarios which means that $Prob(n) = 1/N$, $\forall n\epsilon N$ and more analytically in this model it is used $Prob(n) = 1/147$.

5.2.2 Empirical results

First, we construct two efficient frontiers of international stock and bond portfolios, by using the model that it was described before, with the CVaR risk measure. The first one is referred to period from 31/12/1999 to 30/3/2012, while the second one is referred to period from 31/12/1999 to 31/12/2009.

As it can be observed from the first efficient frontier (Fig.1), it has maximum expected return, 0,0025, which this corresponds to the top edge of the frontier. At this upper point there is portfolio with only one asset and this asset is from USA and more precisely, it is of the two stock indices (S_{usa2}).

Contrary, at the lower point there is a portfolio with the minimum risk,0,0351. This portfolio consists of many different assets from many countries (S_2 and B_1 from Greece, S_{usa2} and B_{usa2} from USA, B_{uk2} from United Kingdom and S_{jp2} from Japan).

Moreover, there are also intermediate points and each point of the efficient frontier presents a different combination return-risk. Hence, at these points there are efficient portfolios that consist of different proportions of assets in each point.

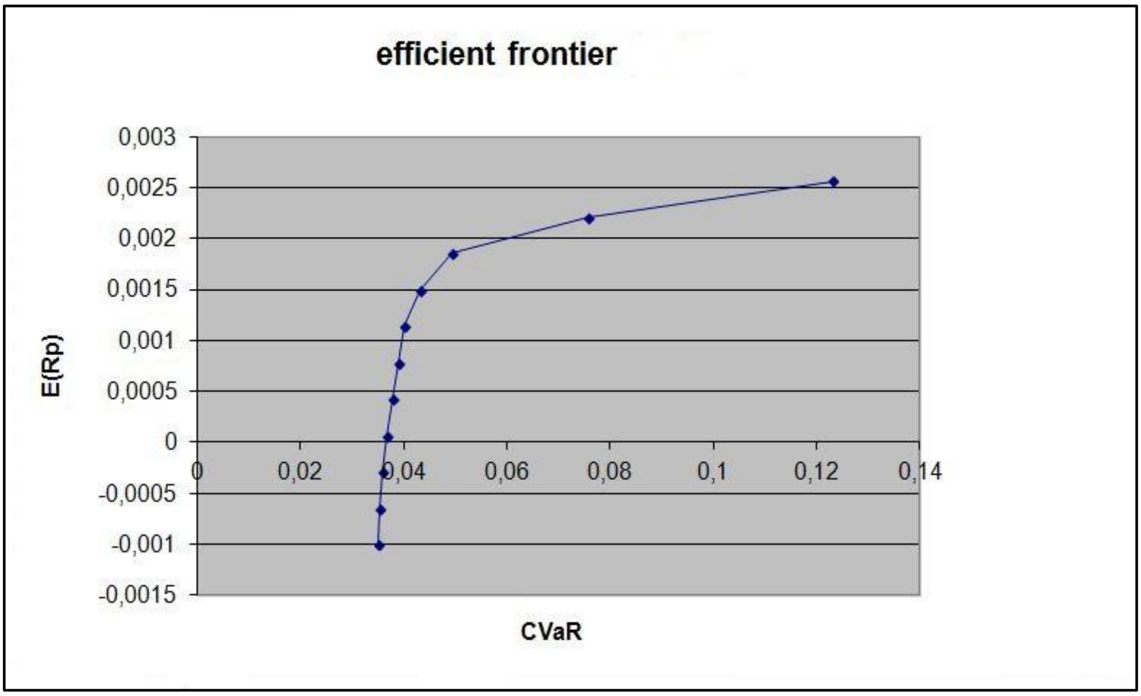

Figure 7: Risk-return efficient frontier of portfolios generated with the CVaR model. It is referred from 31/12/1999 to 30/3/2012.

Regarding the second efficient frontier (Fig.2) is concerned, it is referred to a shorter period which includes 121 observations (prices) from each asset in every country and so 120 returns. In this graph we can see that the upper point has maximum return, 0,00015, and the asset, which has this maximum expected return, is the same as in the first efficient frontier (S_{usa2}).

As far as the lower point has the minimum risk, which here is 0,013, and there is a portfolio which consists from four different assets (B_1 from Greece, S_{usa2} and B_{usa1} from USA and B_{jp2} from Japan). All the intermediate points represent portfolios with different proportions of assets and all these portfolios have different combinations return and risk, as it was mentioned before.

Therefore, the investor, who desires high return, and as a result he or she also has high risk (i.e. he/she is an aggressive investor), prefers the portfolios that are located in higher expected return of the efficient frontier. In other words, the investor chooses portfolios being at the highest points. On the contrary, the investor, who wants the minimum risk (i.e. he/she is a defensive investor), prefers the portfolios that are located in the lowest points of the efficient frontier. This means that this investor chooses portfolios that have low CVaR (risk).

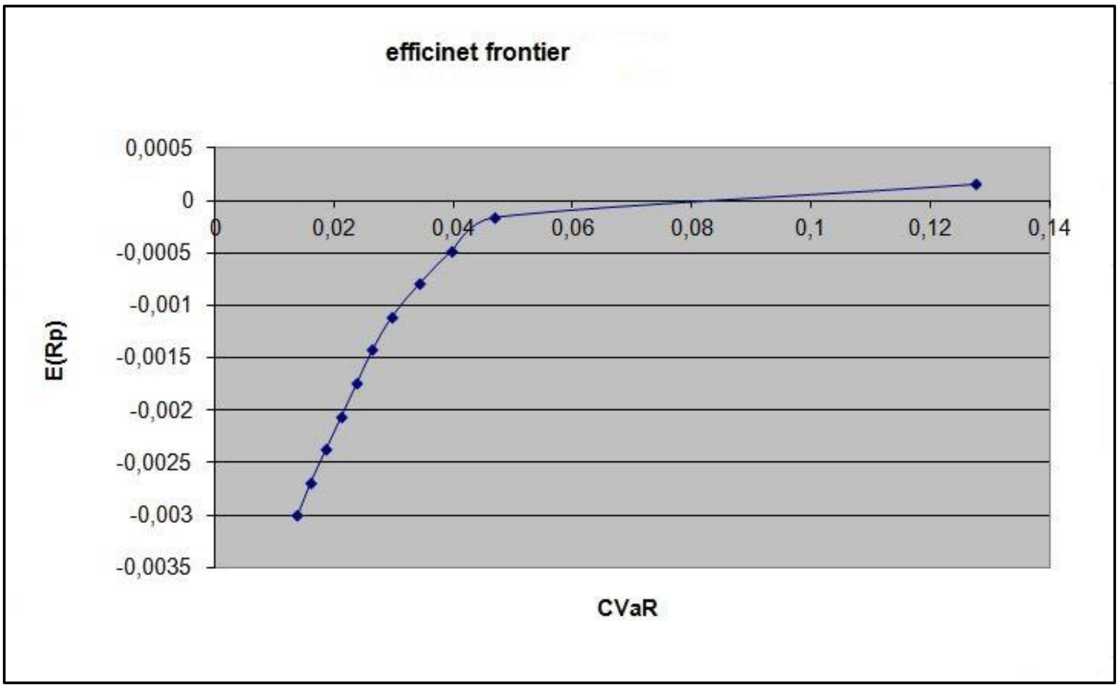

Figure 8: Risk-return efficient frontier of portfolios generated with the CVaR model. It is referred from 31/12/99 to 31/12/09.

Finally, we apply backtesting for the period from 31/12/2009 to 30/3/2012, i.e., for 27 months. Before we explain the process of backtesting and analyze the results, it would be correct to say what exactly the backtesting is.

Backtesting is the process of testing a trading strategy on prior time. Instead of applying a strategy for the time period forward, which could take years, a trader can do a simulation of his or her trading strategy on relevant past data in order to gauge its effectiveness. The backtesting is a jargon used in finance, and not only, to refer to testing a trading strategy or predictive model using existing historical data. It is a special type of cross-validation applied to time series data.

More precisely, backtesting is a key component of effective trading development. It is accomplished by reconstructing, with historical data, trades that would have occurred in the past using rules defined by a given strategy. The result offers statistics that can be used to gauge the effectiveness of the strategy. Using this data, traders can optimize and improve their strategies, find any technical or theoretical flaws and gain confidence in their strategy before applying it to the real markets. However, this procedure requires simulating past conditions with sufficient detail, making one limitation of backtesting the need for detailed historical data.

Hence, the backtesting assumes that what happened in the past, the same will happen in the future, and this assumption can cause potential risks for the strategy. Past performance does not necessarily guarantee futures returns. In other words, although the economic and financial situations change every day, if we consider that there is a world in which the past has a similarity with the present, the backtesting can be a useful tool analysis and forecast. Moreover, most technical-analysis strategies are tested with this approach.

By conducting back tests, the goal is to bring a more methodical, scientific method to stock market decisions and portfolio construction. Therefore, we resort to backtesting experiments on a rolling horizon basis for a more substantive comparison between different types of investors. We will deal with three types of investor. The first type is the aggressive investor who desires higher returns and so he/she will also have higher risk. The second type is the defensive investor who desires as low as possible risk and the third one is the investor who is somewhere between the first and the second type of investors. In other words the last investor wants neither the highest return nor the lowest risk. It is proper to note that all the investors are risk averse. This means that they do not desire the risk. Nevertheless, how much risk averse or not they are, depends on how aggressive or defensive investors they are.

The rolling horizon simulations cover the 27-month period as it was mentioned before. At each month, we use the historical data from the previous 121 observations (i.e. from 31/12/1999 to 31/12/2009, so 120 returns). More analytically, we run out CVaR model (in GAMS) at each month of the 27, using every time the previous 121 observations (asset prices). This means that we begin the first test in 31/12/2009 using the previous 120 historical returns. The optimization model is solved for every type of investors and we record the optimal portfolio. Then, using the next month's actual prices of the assets and the spot exchange rates, that compose the optimal portfolio, we calculate the real final value of this portfolio as well as the real final return.

In the sequel, we go one month front, i.e. 29/1/2010, using again the previous 121 observations (now from 31/1/2000 to 29/1/2010) as historical data. Now the real final value of the portfolio, which found from the previous test for each investor, will be used as initial value in the CVaR model for the new tests. In like manner the optimization model is solved for every investor's type and we compute the real final value of the optimal portfolio and the real final return. This process is continued for the remaining 27 months. The results of the backtesting for the CVaR model are depicted in Fig.3

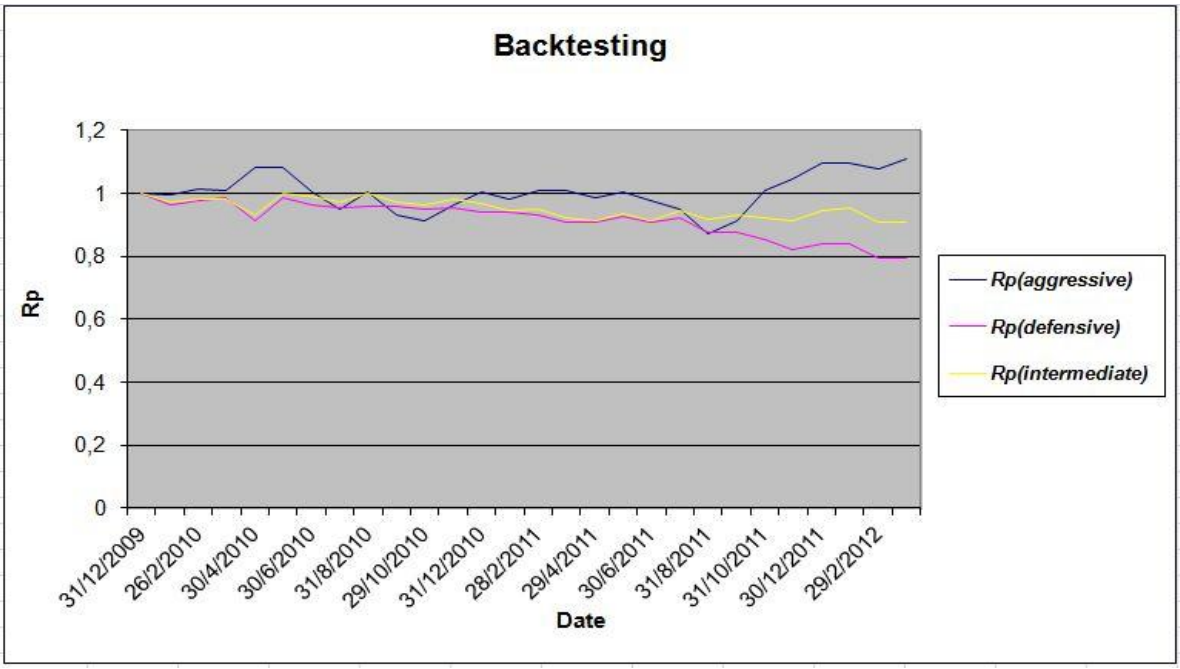

Figure 9: Backtesting experiments for comparison of three types of investor generated with the CVaR model without options.

The above graph presents the real final returns of the optimal international returns that are found from the back tests, for each of the three investor's types. As it can be observed and as it is expected, most of the time the aggressive investor has the highest real return compared to the other two investors. Moreover, this is the type of investor which has the largest falls

Generally, as we can see, he or she is the only investor who has so many fluctuations. The reason is that this type of investor has very high risk because he/she desires very high returns and thus he/she has sharp variations (blue line in Fig.3), which many times are even lower than the returns of defensive investor.

Now, regarding the defensive investor (red line in Fig.3), we can observe that he/she has the lowest real returns as it is expected because this type of investor wants to have the least possible risk and so he or she has lower returns than the other two investors.

And finally, the third type of investor (yellow line Fig.3) has real returns of his/her international portfolio that are mainly between the real returns of the first and second investor. This is also something expected to happen because as we mentioned before, the third investor is someone who desires to have return and risk somewhere between the other two extreme cases of investors.

5.3 Portfolio optimization model with options

In this subsection, we examine the previous model adding options and more precisely, using European put options in order to hedge the market risk.

The options are well-suited to protect the value of positions in risky assets in the event of specific variations of market prices. For this reason, we will use only put options because we want to protect from the downside risk, i.e. when we expect that the market will not go up and so the market values will fall.

Therefore, now we will present the complete model. We will optimize again the previous multi-asset international portfolio (of stock and bond indices from four countries) including also European put options on the riskier assets which are the stock indices.

Moreover, we have to mention that this complete CVaR model is solved with GAMS using OSL solver. Thus, minimizing the CVaR risk measure we manage to find the optimal international portfolio as we did with the previous model.

5.3.1 The Model with options

Now, it is presented the optimization model that incorporates simple options.

$$\min \quad z + \frac{1}{1-a}\sum_{n\epsilon N} Prob(n)\, y(n) \quad \equiv CVaR$$

Subject to:

$$h_l = \sum_{i\epsilon I_c} x_{ic}\, \pi_{ic}(1+\delta) + \sum_{c\epsilon C} x_c^e(1+d) + \sum_{i\epsilon IS_l} [nps(S_{il}, K_i) \cdot ps(S_{il}, K_i)] \qquad (\mathbf{1})$$

$$x_c^e e_c = \sum_{i\epsilon I_c} x_{ic}\pi_{ic}(1+\delta) + \sum_{i\epsilon IS_c} [nps(S_{ic}, K_i) \cdot ps(S_{ic}, K_i)]\,, \quad \forall c\epsilon C \qquad (\mathbf{2})$$

$$\begin{aligned} V_l^n = {} & \sum_{i\epsilon I_l} w_{il}\pi_{il}^n + \sum_{i\epsilon IS_l} \left[nps(S_{il}, K_i) \cdot max\,(K_i - S_{n,il}\,, 0\right] \\ & + \sum_{c\epsilon C} \left\{ \frac{1}{e_c^n} \left[\sum_{i\epsilon I_c} w_{ic}\, \pi_{ic}^n \right.\right. \\ & \left.\left. + \sum_{i\epsilon IS_c} \left[nps(S_{ic}, K_i) \cdot \max\,(K_i - S_{n,ic}\,, 0)\right] \right] \right\}, \qquad \forall n\epsilon N \qquad (\mathbf{3}) \end{aligned}$$

$$nps(S_{1c}, K_1) \cdot ps(S_{1c}, K_1) \le w_{1c}\pi_{1c}\,, \qquad \forall c\epsilon C_0 \qquad (\mathbf{4i})$$

$$nps(S_{2c}, K_2) \cdot ps(S_{2c}, K_2) \le w_{2c}\pi_{2c}\,, \qquad \forall c\epsilon C_0 \qquad (\mathbf{4ii})$$

$$R_n = \frac{V_l^n - V_l^0}{V_l^0} \Rightarrow R_n = \frac{V_l^n}{V_l^0} - 1\,, \quad \forall n\epsilon N \quad (\mathbf{5})$$

$$\bar{R} = \sum_{n \epsilon N} Prob(n)\, R_n \qquad (\mathbf{6})$$

$$\bar{R} \geq \mu \qquad (\mathbf{7})$$

$$y_n \geq L_n - z \ , \ \ \forall n \epsilon N \qquad (\mathbf{8})$$

$$y_n \geq 0\, , \qquad \forall n \epsilon N \qquad (\mathbf{9})$$

$$L_n = -R_n\, , \qquad \forall n \epsilon N \qquad (\mathbf{10})$$

$$w_{ic} = x_{ic} \ , \qquad \forall i \epsilon I_c\, , \forall c \epsilon C_0 \qquad (\mathbf{11})$$

$$w_{ic} \geq 0, \ \ x_{ic} \geq 0\, , \qquad \forall i \epsilon I_c\, , \forall c \epsilon C_0 \qquad (\mathbf{12})$$

As in the previous model the objective function (CVaR) minimizes the expected excess loss, beyond the $(1 - a)$ quantile, over the holding period $[0, T]$.

Constraints (**1**) and (**2**) are the cash balance conditions, at $t = 0$, for the base currency l, and the foreign currencies , $\forall c \epsilon C$, respectively.

Where:

$nps(S_{il}, K_i)$: is purchases of European put option $i \epsilon IS_l$ on stock index S_{il} in the base currency with exercise price K_i and maturity at time T. In other words, this variable is the number of European put options, which are at-the-money only, on the two stock indices of base country (Greece) with strike price K_i ($K_i = S_{il}$, at-the-money put options)

$ps(\, S_{il}, K_i)$: is the price of a European put option on stock index S_{il} of base country with exercise price K_i and maturity at time T

But more generally:

$nps(S_{ic}, K_i)$: is purchases of a European at-the-money put options $i \epsilon IS_c$ on stock index S_{ic} in currency $c \epsilon C_0$ with exercise price K_i and maturity at time T

$ps(\, S_{ic}, K_i)$: is the price of a European put option $i \epsilon IS_i$ on stock index S_{ic} of country $c \epsilon C_0$ with exercise price K_i and maturity at time T

Where T is the length of the planning horizon, S_{ic} is the price of each stock index in each currency $c\epsilon C_0$ and K_i is the strike price (or exercise price) of the put option in units of the respective currency and it is $K_i = S_{ic}$ because the put option that we choose to put in the model is only at-the-money.

Moreover, the constraint (**3**) is the final value of the international portfolio, in units of the base currency, and reflects the market value of all asset holdings at the end of the holding period and the option payoffs.

For the put options, the payoff is $\max(K - S_0, 0)$. Consequently, in our model the payoff is denoted as $\max(K_i - S_{n,ic}, 0)$ $\forall c\epsilon C_0$, where $S_{n,ic}$ is the price of the stock index S_{ic} in currency $c\epsilon C_0$ at the end of the planning period under scenario $n\epsilon N$.

In this complete model, we have a further two constraints, (**4i**) and (**4ii**), which depict that the expenditure for purchases of put options on each stock index in each market is less or equal to the expenditure of the corresponding stock index. All the remaining constraints are identical to the initial model (eq. (**5**), (**6**), (**7**), (**8**), (**9**), (**10**), (**11**) and (**12**)).

Here the deterministic inputs are the current prices of the assets, the current spot exchange rates and the prices of European put options on each stock index maturing at the time horizon. And the stochastic inputs are the prices of the assets and the spot exchange rates which determine the option payoffs at the horizon under each scenario.

This stochastic linear program determines the optimal asset mix across multiple markets and positions in put options on stock indices in each currency in order to mitigate exposure to the respective market risk.

Additionally, we should refer that the prices of the European put options, $ps(S_{ic}, K_i)$, on each stock index in each country, are calculated by the Black-Scholes formulas which are the following:

$$p = Ke^{-r_f T} N(-d_2) - S_{ic} N(-d_1), \ \forall i\epsilon I_c, \ \forall c\epsilon C_0$$

Where:

$$d_1 = \frac{\ln\left(\frac{S_{ic}}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}$$

As mentioned and analyzed in chapter 4.

5.3.2 Empirical results

We apply backtesting for the period from 31/12/2009 to 30/3/2012. Again we follow exactly the same procedure as we did in the previous model without options, i.e. we perform back test on a rolling horizon basis for each type of investor.

However, in this complete model, after backtesting experiments, it is observed that the results for the first investor (aggressive) and the third investor (who is somewhere between the first and the second investor's types) are not satisfactory and compatible with the real world. While for the second investor (defensive), the results are more satisfactory.

One reason for the above outcome is that we have put the additional constraints ((**4i**) and (**4ii)**) in the model with options, which are not so restricted and affect the results. More specifically, these constraints denote that expenditures for options are less or equal to expenditures for assets. This means that they allow model to choose a high number of assets ($w_{ic}\ \ \forall c \epsilon C_0$) and that is why they are taken too great values, that are not consistent with reality, when it is calculated the real final value of optimal international portfolios.

Nevertheless, we could correct this outcome adding other constraints instead of (**4i**) and (**4ii**). That is to change the above constraints putting only the following constraints:

$$nps(S_{1c}, K_1) \leq w_{1c}$$

and

$$nps(S_{2c}, K_2) \leq w_{2c}$$

These constraints mean that the number of put options that are purchased have to be less or equal to the number of assets. Then, we will see that the values will be limited greatly.

Another reason is that the pricing of put options with the Black-Scholes equation is not the best possible. This is happening because the Black-Scholes model does not take into consideration the distribution of underlying asset and it considers that underlying asset returns follow normal distribution. This implies that there is problem with bad option prices and this is because the returns of the data that are used in our models follow skewed distribution. Therefore, while the two models, which are used in the empirical application, take into account the skewed distribution of the data, it is placed in the second model prices of put options which are not correct, and this constitutes one more reason that we do not get such good results.

Due to the aforementioned two reasons the figure 4 depicts only the comparison of backtesting between the defensive investor without options (first model) and the defensive investor with options (second model).

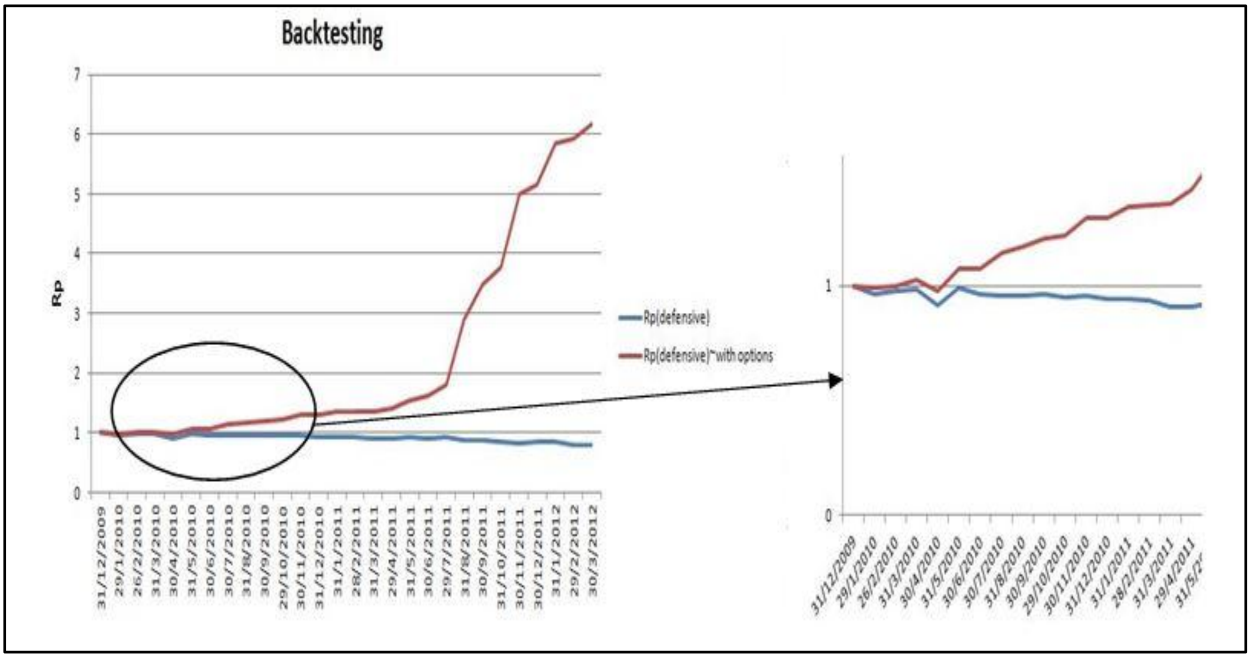

Figure 10: Backtesting experiments for comparison the same type of investor in two different models generated with the CVaR model with and without options respectively.

As we can observe from Fig.4, for the three first months approximately, both defensive investors have almost the same real returns of their international portfolios. After of these months, the two investors start to separate.

Let's remember that the 'blue line' from the backtesting of the second model with options and the 'red line' from the backtesting of the first model without options are exact the same lines since they describe the same defensive investor.

Moreover, as we can see from the above graph, the deviation of the two investors is very large and clear month by month. This is something that we expect, because adding put options in model improves the performance of the investment. In this way, we manage to protect the portfolio of the investor from market risk. By using put options on stock indices, the portfolio is hedged and gives higher expected and real portfolio's returns.

More precisely, put options are exercised when the price of the stock index falls. Thus the investor, who holds put options, gains the different between the strike price and the stock's price when the last declines and nothing when the price of stock index increases. This is the payoff of the put option which is also added in the equation of final value of the international portfolio and makes the difference between the investors of the two models.

6. *Conclusions*

We developed a stochastic optimization approach for managing international portfolios of financial assets, such as stock and bond indices in multiple currencies in a dynamic setting. This model determines optimal diversification to assets in multiple markets and it can select appropriate derivatives in order to mitigate exposures to market risk and currency risk. We focused on the market risk only and for this reason European put options on stock indices are used. This framework involves suitable portfolio optimization models for risk management.

Our implementation of the risk management optimization model uses the Conditional Value-at-Risk (CVaR) metric. We preferred the CVaR because it is a coherent risk measure and it is suitable for asymmetric and fat-tailed return distributions exhibited by market data of international stock and bond indices, as well as by exchange rates.

However, a different objective function could have been chosen for optimization. There are alternative objective functions to reflect the decision maker's risk preferences apart from CVaR risk measure (such as utility function or other suitable risk measures). Moreover, we could incorporate additional practical constraints such as managerial and regulatory requirements that can be expressed in terms of linear constraints on the decision variable, as for example limits on holdings on individual securities or group of assets.

In this survey two models were presented, the first one is without options and the other is with options. More specifically, in the second model we hedged the market risk adding in the initial model European put options on stock indices in each country. In these models the scenarios were coming from the historical data and equiprobable scenarios were used. Additionally, we calculated the prices of the European put options using the Black-Scholes formulas.

After extensive computational experiments using real market data for the stock and bond indices and for the spot exchange rates, it was shown that the unhedged portfolio exhibited worse performance than the portfolio in which it had occurred hedging. This means that the inclusion of explicit decisions for put options on the stock indices in the model enables the determination of optimal selective hedging decisions to control market risk exposures in each country and so it gave better performance.

Finally, this study could be extended to use a moment-matching method for scenario generation. This method effectively approximates the empirical distribution of the random variable or any alternative scenarios generation procedure which captures the views of an investor on the distribution of uncertain asset returns and exchange rates.

Furthermore, we could price the European options on the base of the postulated scenario sets of asset prices, as Topaloglou et all. (2008) did in their paper. They used two different methods. The first method determines an equivalent risk-neutral probability measure from the physical probability measure that is associated with the postulated price outcomes on the scenario tree. And the second method is based on series expansion of the distribution of asset log-returns. These proposed approaches can price European options at any node of the scenario tree, thus allowing transactions with options at any decision stage of a multi-period portfolio optimization model. Moreover, the above valuation procedures are notably more effective in those cases that the Black-Scholes formula is used since the last one is known for the systematically mispricing of the options.

Another extension could constitute the used of additional currency forwards or quantos in order to control the currency risk of foreign investments. We could also use European call options, apart from European put options, for speculative purposes to take bets on specific direction and magnitude of change in the price of the underlying asset.

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