https://doi.org/10.26219/heal.aueb.5591

ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ
ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

SCHOOL OF INFORMATION SCIENCES & TECHNOLOGY

DEPARTMENT OF STATISTICS

POSTGRADUATE PROGRAM

Control Charts for the Weibull Distribution

By

Pelagia S. Lagopati

A THESIS
Submitted to the Department of Statistics
of the Athens University of Economics and Business
in partial fulfilment of the requirements for
the degree of Master of Science in Statistics

Athens, Greece
June 2018

https://doi.org/10.26219/heal.aueb.5591

https://doi.org/10.26219/heal.aueb.5591

ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ
ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

ΣΧΟΛΗ ΕΠΙΣΤΗΜΩΝ & ΤΕΧΝΟΛΟΓΙΑΣ ΤΗΣ ΠΛΗΡΟΦΟΡΙΑΣ

ΤΜΗΜΑ ΣΤΑΤΙΣΤΙΚΗΣ

ΜΕΤΑΠΤΥΧΙΑΚΟ ΠΡΟΓΡΑΜΜΑ

ΔΙΑΓΡΑΜΜΑΤΑ ΕΛΕΓΧΟΥ ΓΙΑ ΤΗΝ ΚΑΤΑΝΟΜΗ WEIBULL

Πελαγία Σ. Λαγοπάτη

ΔΙΑΤΡΙΒΗ
Που υποβλήθηκε στο Τμήμα Στατιστικής
του Οικονομικού Πανεπιστημίου Αθηνών
ως μέρος των απαιτήσεων για την απόκτηση
Μεταπτυχιακού Διπλώματος Ειδίκευσης στη Στατιστική

Αθήνα
Ιούνιος 2018

https://doi.org/10.26219/heal.aueb.5591

https://doi.org/10.26219/heal.aueb.5591

DEDICATION

This thesis is dedicated to my parents and to my sister, who have always been a source of love, support and encouragement during the challenges of my university life.

https://doi.org/10.26219/heal.aueb.5591

https://doi.org/10.26219/heal.aueb.5591

ACKNOWLEDGEMENTS

I would first like to thank my thesis advisor professor Stelios Psarakis for his advice, guidance and understanding. His office was always open whenever I ran into a trouble spot or had a question about my research. I am also grateful to all the people who have contributed to this work in their own way.

Finally, I must express my profound gratitude to my family and to my boyfriend for providing me with unfailing support and continuous encouragement throughout my years of study and through the process of researching and writing this thesis. This accomplishment would not have been possible without them. Thank you.

https://doi.org/10.26219/heal.aueb.5591

 https://doi.org/10.26219/heal.aueb.5591

VITA

Pelagia S. Lagopati was born in Athens, Greece, on March 9, 1993. Between 2011 and 2016 she studied Mathematics at the National and Kapodistrian University of Athens and for one semester in 2015 she studied Mathematics as an exchange student at the Ludwig Maximilian University of Munich. In 2016-2017 she attended the postgraduate program in Statistics at Athens University of Economics and Business.

https://doi.org/10.26219/heal.aueb.5591

https://doi.org/10.26219/heal.aueb.5591

ABSTRACT

Pelagia Lagopati

CONTROL CHARTS FOR THE WEIBULL DISTRIBUTION

June 2018

The Weibull distribution is widely used because it is a positive stable distribution and because of its large variety of shapes, which makes it extremely flexible in fitting data. Control charts are a tool of Statistical Process Control, which are used to monitor quality variables from a process. When the process distribution is Weibull, the standard control charts can lead to misleading conclusions. In order to use the Shewhart X-bar, R and S control charts, alternative methods, which use asymmetric control limits, are proposed, such as the weighted variance and the skewness correction methods. Also control charts for monitoring the shape and scale parameter of the Weibull distribution are discussed. Moreover, EWMA control charts are proposed for detecting small process shifts and for monitoring the shape parameter, the lifetimes' mean or the quantiles of the Weibull distribution. CUSUM control charts are also used mostly for detecting small process shifts and in order to monitor Weibull distributed processes, various CUSUM control charts are presented, such as the Weibull CUSUM, the exponential CUSUM and the adjusted exponential CUSUM chart. For both EWMA and CUSUM charts, a transformation of exponential data to normal is suggested so that the standard control charts can be applied. Then, a mixed EWMA-CUSUM control chart is discussed, where the EWMA and the CUSUM statistics are combined for the Weibull distribution, as it can lead to better performance. Throughout this thesis, the performance of the control charts is evaluated using the Average Run Length (ARL) values. Finally, some illustrative examples verify some of the above conclusions and various control charts are presented using the R programming language.

https://doi.org/10.26219/heal.aueb.5591

https://doi.org/10.26219/heal.aueb.5591

ΠΕΡΙΛΗΨΗ

Πελαγία Λαγοπάτη

ΔΙΑΓΡΑΜΜΑΤΑ ΕΛΕΓΧΟΥ ΓΙΑ ΤΗΝ ΚΑΤΑΝΟΜΗ WEIBULL

Ιούνιος 2018

Η κατανομή Weibull χρησιμοποιείται ευρέως εξαιτίας της μεγάλης ποικιλίας σχημάτων της, που την κάνει πολύ ευέλικτη στην προσαρμογή διαφόρων δεδομένων. Τα διαγράμματα ελέγχου είναι ένα εργαλείο του Στατιστικού Ελέγχου Διεργασιών, που χρησιμοποιούνται για την παρακολούθηση των ποσοτικών μεταβλητών σε μία διαδικασία παραγωγής. Όταν η κατανομή που ακολουθούν τα δεδομένα είναι η Weibull, τότε η χρήση των διαγραμμάτων ελέγχου μπορεί να οδηγήσει σε λάθος συμπεράσματα. Για να μπορέσουν να χρησιμοποιηθούν σωστά τα διαγράμματα Shewhart X-bar, R και S, προτείνονται κάποιες εναλλακτικές μέθοδοι, οι οποίες χρησιμοποιούν ασύμμετρα όρια για τα διαγράμματα ελέγχου, καθώς και διαγράμματα για τον εντοπισμό αλλαγών στις παραμέτρους σχήματος και κλίμακας. Για τον εντοπισμό μικρών αλλαγών στη διαδικασία, χρησιμοποιούνται κυρίως τα διαγράμματα EWMA και CUSUM. Ορισμένα EWMA παρουσιάζονται για τον εντοπισμό αλλαγών στην παράμετρο σχήματος ή στη μέση τιμής της διαδικασίας. Για την περίπτωση δεδομένων που ακολουθούν τη Weibull, παρουσιάζονται επίσης διάφορα CUSUM διαγράμματα αλλά και ένας μετασχηματισμός τόσο για τα CUSUM όσο και για τα EWMA διαγράμματα, με βάση τον οποίο, τα εκθετικά δεδομένα προσεγγίζουν την κανονικότητα κι έτσι μπορούν να χρησιμοποιηθούν τα κλασικά διαγράμματα ελέγχου. Επιπλέον, προτείνεται ένας συνδυασμός των EWMA και CUSUM, ο οποίος μπορεί να οδηγήσει σε καλύτερη απόδοση στον έλεγχο της διαδικασίας. Η αξιολόγηση της απόδοσης των διαφόρων διαγραμμάτων γίνεται με τη χρήση του Average Run Length (ARL). Τέλος, μερικά παραδείγματα επιβεβαιώνουν τα παραπάνω συμπεράσματα και διάφορα διαγράμματα ελέγχου παρουσιάζονται χρησιμοποιώντας τη γλώσσα προγραμματισμού R.

https://doi.org/10.26219/heal.aueb.5591

https://doi.org/10.26219/heal.aueb.5591

TABLE OF CONTENTS

https://doi.org/10.26219/heal.aueb.5591

https://doi.org/10.26219/heal.aueb.5591

https://doi.org/10.26219/heal.aueb.5591

https://doi.org/10.26219/heal.aueb.5591

LIST OF TABLES

https://doi.org/10.26219/heal.aueb.5591

 https://doi.org/10.26219/heal.aueb.5591

LIST OF FIGURES

https://doi.org/10.26219/heal.aueb.5591

CHAPTER 1

INTRODUCTION

1.1 Statistical Quality Control: Brief History and Methods

Controlling and improving quality is of high importance in the business world for many organizations, companies and agencies. Nowadays competition between companies is very strong and the control, as well as the improvement of the products' or services' quality, can attract more customers and as a result the company will dominate its competitors. As quality improvement we refer to the reduction of variability in processes and products. Statistical Quality Control (SQC) refers to the use of statistical methods in monitoring and maintaining the quality of products or services. These methods provide the means of sampling, testing and evaluating a product and the control and improvement of the product is achieved through the information in those data.

With the development of production industries prior to 1900, Frederick W. Taylor had the idea of dividing the work into tasks, which led to a positive impact on the quality of the manufactured goods. A few years later, in 1924, Walter A. Shewhart of the Bell Telephone Laboratories was the one, who introduced the statistical control chart, which is considered the formal beginning of statistical quality control. By the middle of 1930s, statistical quality control methods were in use at Western Electric, which was an American electrical engineering and manufacturing company, but the value of statistical quality control was not widely recognized by industry.

World War II led to a wide use and acceptance of statistical quality control in manufacturing industries, as it made it clear, that statistical methods were of great importance for the control and improvement of the products' quality. In 1946 was founded the American Society of Quality

https://doi.org/10.26219/heal.aueb.5591

Control, which is an organization dedicated to promoting and advancing quality tools, principles and practices in workplaces and communities.

In the 1950s, designed experiments for product and process improvement were first introduced in the United States. The initial applications were in the chemical industry and in the late 1970s or early 1980s, these methods were spread outside this industry as well as in many Western companies. The methodology was also introduced in engineering, organizations in industry and academic engineering courses.

Three methods of SQC are the design of experiments, acceptance sampling and Statistical Process Control (SPC). The design of experiments is a systematic method to determine the relationship between multiple input factors, which affect a process, and the output product parameters (responses) of that process. In other words, it is a method used to find cause-and-effect relationships. The input factors can be controllable ($X_1, X_2, \ldots, X_p$) or uncontrollable ($Z_1, Z_2, \ldots, Z_p$). Whereas the latter cannot be changed, the first can be modified to improve product quality and optimize process performance. The production process transforms the input raw materials, into a finished product that has several quality characteristics. The output variable y is a quality characteristic, which is a measure of process and product quality. The relationship between the factors and the responses is shown in Figure 1 (Douglas C. Montgomery, Introduction to Statistical Quality Control, 6th Edition).

https://doi.org/10.26219/heal.aueb.5591

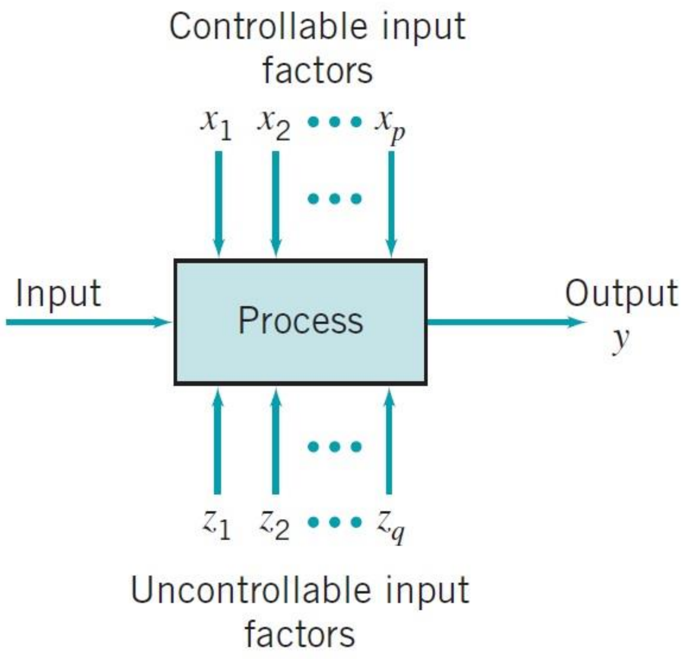


Figure 1. General Model of a process

The second method is acceptance sampling. According to this method, a lot or batch will be accepted or rejected based on the inspection of a sample from this batch. Acceptance sampling performed before the process usually involves sampling from the incoming raw materials, whereas after the process involves sampling from the final production.

The third method is Statistical Process Control (SPC). SPC is a valuable process, which employs statistical methods to monitor and control a process. What makes SPC different from acceptance sampling is that SPC is performed during the manufacturing process, rather than before or after the process. In other words, SPC uses tools to observe or monitor the performance of the production or service process in order to detect significant variations, that if not corrected, may later result in rejected product, before it happens. Thus, SPC is known as a prevention and prediction tool for quality. Variation in a process falls into one of the two following classes:

- Common causes or Non-assignable sources of variation, which refer to many sources of variation that consistently act on the process. They are natural in a system.
- Special causes or Assignable sources of variation, which refer to any factor causing variation that affects only some of the process output.

 https://doi.org/10.26219/heal.aueb.5591

They are a result of a specific change in the process and are related to some defect in the system.

The seven major tools of SPC, which are often called as “the magnificent seven”, are:

- Histogram or stem-and-leaf plot
- Check sheet
- Pareto chart
- Cause-and-effect diagram
- Defect concentration diagram
- Scatter diagram
- Control chart

The Shewhart control chart is probably the most technically sophisticated out of the seven tools.

1.2 Control charts

A control chart is one of the major techniques of Statistical Process Control (SPC). It was first introduced in 1924 by Walter A. Shewhart of the Bell Telephone Laboratories and it is a graphical display of a quality characteristic that has been measured or computed from a sample versus the sample number or time. It provides a basis for deciding whether the variation in the output of a process is due to common or special causes. If it is due to special causes, the process must be adjusted in order to bring the output back to acceptable quality levels. On the other hand, common causes are an expected part of the process and they are of much less concern to the manufacturer. The three fundamental uses of a control chart are the following:

- Reduction of process variability
- Monitoring and surveillance of a process
- Estimation of product or process parameters

A control chart consists of a center line and two other horizontal lines. The center line (CL) represents the average value of the quality characteristic, the upper line is the upper control limit (UCL) and the lower line is the lower

https://doi.org/10.26219/heal.aueb.5591

control limit (LCL). All these lines are specified from historical data as if the process is in control, which is also known as Phase I. Now the data gathered from Phase I can be used to estimate any unknown parameters. In Phase II, by using these limits and plotting current data to the chart, it is examined if the process is in control or out of control. If the points plot within the control limits, then the process is assumed to be in control and no further action is necessary. However, if at least one point plots outside of the control limits, then that is an evidence that the process is out of control and investigation and corrective action are required in order to find and eliminate the assignable cause or causes responsible for this behavior. If the process is in control, all the plotted points should have an essentially random pattern, which means that not only points plotted outside of the control limits are an evidence for an out of control process, but also points plotted inside the control limits, that behave in a systematic or non-random manner. A typical control chart is shown in Figure 2 (Douglas C. Montgomery, Introduction to Statistical Quality Control, 6th Edition).

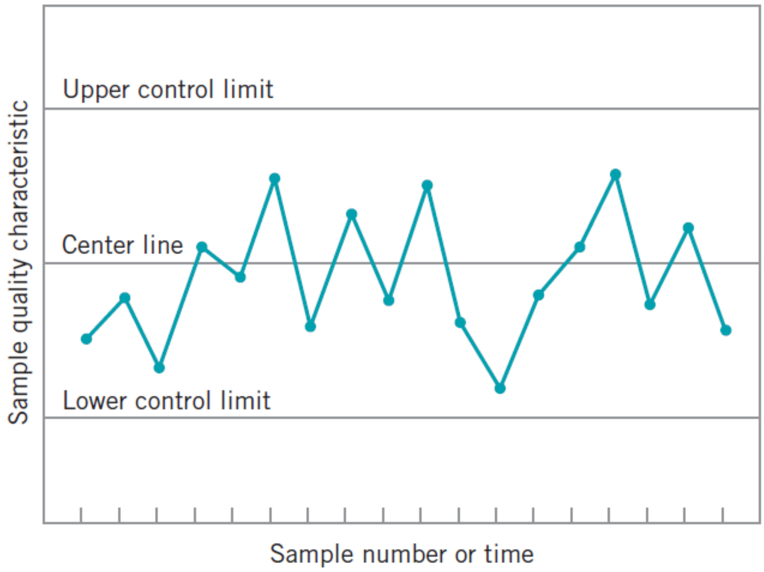


Figure 2. A typical control chart

A general model for a control chart is the Shewhart control chart. Then the center line, the upper and the lower control limits are given as follows:

$$UCL = \mu_w + L\sigma_w$$

$$CL = \mu_w$$

https://doi.org/10.26219/heal.aueb.5591

$$LCL = \mu_w - L\sigma_w$$

where w is a sample statistic that measures some quality characteristic of interest, μ_w is the mean of w, σ_w is the standard deviation of w and L is the "distance" of the control limits from the center line in terms of standard deviation units.

Control charts are classified into two general groups. These are the variables control charts and the control charts for attributes. When it comes to variables control charts, there are the following:

The X-bar chart is the most widely used chart for controlling central tendency. This chart is employed in situations where a sample mean is used to measure the quality of the output. The sample means are plotted on the chart, so that the mean value of a variable can be controlled. In this case, the control charts are the following:

$$UCL = \bar{\bar{X}} + A_2\bar{R}$$

$$CL = \bar{\bar{X}}$$

$$LCL = \bar{\bar{X}} - A_2\bar{R}$$

where $\bar{\bar{X}} = \frac{\bar{X}_1+\bar{X}_2+\cdots+\bar{X}_m}{m}$, m is the number of samples, $A_2 = \frac{3}{d_2\sqrt{n}}$, n is the sample size and $\bar{R} = \frac{R_1+R_2+\cdots+R_m}{m}$, where R_i , i=1,2,…,m are the ranges of the m samples.

The R-chart and the S-chart, in other words, charts based on either the sample range or the sample standard deviation, are used to control process variability. The center line and control limits for the R-chart are as follows:

$$UCL = D_4\bar{R}$$

$$CL = \bar{R}$$

$$LCL = D_3\bar{R}$$

where $D_3 = 1 - \frac{3d_3}{d_2}$, $D_4 = 1 + \frac{3d_3}{d_2}$ and $\bar{R} = \frac{R_1+R_2+\cdots+R_m}{m}$, where R_i , i=1,2,…,m are the ranges of the m samples.

For the development of the equations for the control limits of the X-bar and R charts we have used the random variable W=R/σ, which is called the relative range and it shows a relationship between the range of a sample of a normal distribution and the standard deviation of that distribution. The mean of W is d_2. Consequently, an estimator of σ is $\hat{\sigma} = R/d_2$. Therefore, if $\bar{R}$ is the average

https://doi.org/10.26219/heal.aueb.5591

range of m samples, then $\hat{\sigma} = \overline{R}/d_2$, which is an unbiased estimator of σ. Also, another unbiased estimator of σ is $\hat{\sigma} = \bar{s}/c_4$, where s is the sample standard deviation. This last one estimator will be used for the computation of the equations for the control limits of the S-chart.

The center line and control limits of the S-chart are given as follows:

$$UCL = B_4\bar{s}$$

$$CL = \bar{s}$$

$$LCL = B_3\bar{s}$$

where $B_3 = 1 - \frac{3}{c_4}\sqrt{1 - c_4^2}$ and $B_4 = 1 + \frac{3}{c_4}\sqrt{1 - c_4^2}$.

When it comes to control charts for attributes, one widely used chart is the control chart for fraction nonconforming (p-chart) or the control chart for number nonconforming (np-chart). The center line and the control limits for the p-chart are

$$UCL = p + 3\sqrt{\frac{p(1-p)}{n}}$$

$$CL = p$$

$$LCL = p - 3\sqrt{\frac{p(1-p)}{n}}$$

whereas the center line and the control limits for the np-chart are

$$UCL = np + 3\sqrt{np(1-p)}$$

$$CL = np$$

$$LCL = np - 3\sqrt{np(1-p)}$$

where n is the sample size and p is the process fraction nonconforming, which, when unknown, is estimated by the statistic $\bar{p}$.

In order to select the most appropriate control chart in every situation, practitioners are interested in how effective it is. The average run length (ARL) is the most common criterion used to assess a control chart's performance. It represents the average number of points of the monitoring statistic it takes the control chart to signal an out-of-control situation. When the process is in-control, we denote as ARL_0 the average run length before a false alarm, whereas, when the process is out-of-control, ARL_1 is the average run length to detect a real change in process and represents the detection speed of the chart. Generally, an efficient control chart must have large values of ARL_0 and small values of ARL_1. Another measure of performance of a control chart is the average time to signal (ATS), which refers to the number of time periods that occur until a signal is generated on the control chart. If samples are taken at fixed intervals of time that are h hours apart, then ATS = ARL x h.

Two very effective alternatives to the Shewhart control chart are the cumulative sum (CUSUM) control chart and the exponentially weighted moving average (EWMA) control chart, which offer important performance improvement relative to the Shewhart one. These charts are especially used when small process shifts are of interest and for Phase II process monitoring.

A CUSUM's most common implementation is in the tabular form. It may be constructed both for individual observations and for the averages of rational subgroups. For the construction of a tabular CUSUM for monitoring the mean of a process, let μ_0 be the target value for the quality characteristic x and C^+, C^- be two statistics. C^+ is called one-sided upper CUSUM and accumulates derivations from μ_0 that are above target, whereas C^- is called one-sided lower CUSUM and accumulates derivations from μ_0 that are below target. These statistics are computed as follows:

$$C_i^+ = \max\,[0, x_i - (\mu_0 + K) + C_{i-1}^+]$$

$$C_i^- = \max\,[0, (\mu_0 + K) - x_i + C_{i-1}^-]$$

where the starting values are $C_0^+ = C_0^- = 0$ and K is called the reference value. If either one of the above statistics exceeds the decision interval H, the process is considered to be out of control. The selection of the two parameters, K and H, is of great importance, as it can influence the performance of the CUSUM. By defining H=hσ and K=kσ, where σ is the standard deviation of the sample variable used in forming the CUSUM, then using h=4 or h=5 and k=1/2 will lead to a CUSUM that has good ARL performance against a shift of about 1σ in the process mean.

The EWMA chart has approximately equivalent performance to that of the CUSUM chart and it is typically used with individual observations as well, but in some ways it is easier to set up and operate. The exponentially weighted moving average is defined as follows:

$$z_i = \lambda x_i + (1-\lambda)z_{i-1}$$

where $0<\lambda\leq 1$ is a constant and the starting value is the process target, so that $z_0=\mu_0$.

With the following computations and the help of the above equation it is shown that the EWMA z_i is a weighted average of all previous sample means:

$$z_i = \lambda x_i + (1-\lambda)[\lambda x_{i-1} + (1-\lambda)\, z_{i-2}] = \lambda x_i + \lambda(1-\lambda)x_{i-1} + (1-\lambda)^2\, z_{i-2}$$

Continuing to substitute recursively for z_{i-j} , j=2,3,…,t, we obtain

$$z_i = \lambda\sum_{j=0}^{i-1}(1-\lambda)^j\, x_{i-j} + (1-\lambda)^i\, z_0$$

Since the EWMA can be viewed as a weighted average of all past and current observations, it is very insensitive to the normality assumption and is therefore, ideal for individual observations. In order to compute the control limits, it is needed to see that if the observations x_i are independent random variables with variance σ^2, then the variance of z_i is

$$\sigma_{z_i}^2 = \sigma^2\left(\frac{\lambda}{2-\lambda}\right)[1-(1-\lambda)^{2i}]$$

The EWMA chart is constructed by plotting z_i versus the sample number i or time and its center line and control limits are the following:

$$UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}[1-(1-\lambda)^{2i}]}$$

$$CL = \mu_0$$

$$UCL = \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}[1-(1-\lambda)^{2i}]}$$

where L is the width of the control limits.

The selection of the parameters L and λ is possible to give ARL performance for the EWMA control chart that closely approximates CUSUM ARL performance for detecting small shifts. For example, an equivalent design for the CUSUM with h=5 and k=1/2 is the EWMA with λ=0.1 and L=2.7. Very important is also to note that smaller values of λ are more appropriate for detecting smaller shifts.

1.3 The Weibull distribution

The Weibull distribution is named after the Swedish mathematician Waloddi Weibull, who described it in detail in 1951 and proposed it as a

https://doi.org/10.26219/heal.aueb.5591

distribution with various applications. However, the distribution was first identified in 1927 and first used in 1933 by Rosin and Rammler in description of "Laws governing the fineness of powered coal".

This distribution is the most widely used lifetime distribution and is also used in reliability engineering. Its great popularity and success is due to its versatility in modelling either increasing, constant or decreasing hazard rates. Specifically, it is used in lifetimes of many types of manufacturing item, reliability analysis of semi-conductors, ball-bearings, engines, soft weldings among others, reaction time analysis, biomedical applications, laboratory animals. Some other applications of the Weibull distribution are the following: modelling the inter-occurrence times of some natural phenomena such as earthquakes or outbreaks of contagious diseases, modelling returns on financial assets in an irregular market, where price bubbles are expected to occur. It has also been used to describe product properties such as strength, elongation and resistance.

Another reason for the great popularity of the Weibull distribution is that it has a large variety of shapes. This makes it extremely flexible in fitting data and it empirically fits many kinds of data. It can model right-skewed, left-skewed or symmetric data. That flexibility is the reason why engineers use the Weibull distribution to evaluate reliability and material strengths. The Weibull distribution is also a positive stable distribution, which means that the sum of independent variables from the distribution suitably normalized follows the same distribution.

Moreover, the Weibull distribution can be used as a limit distribution for the smallest of a large number of independent and identically distributed (iid) non-negative random variables. If X has a Weibull distribution, then ln(X) has an extreme value distribution. The relationship between the Weibull and the extreme value distribution is similar to the relationship between the normal and the log-normal distributions. Extreme value distributions play an important role in reliability analysis, too.

A distribution is mathematically defined by its probability density function (pdf) equation. The pdf of a three-parameter Weibull distribution, which is the most general expression of the Weibull pdf, is given by

https://doi.org/10.26219/heal.aueb.5591

$$f(X) = \frac{\beta}{\eta}(\frac{X-\gamma}{\eta})^{\beta-1}e^{-(\frac{X-\gamma}{\eta})^{\beta}}$$

where β is the shape parameter or slope, η is the scale parameter and γ is the location parameter. The median, the mean and the variance are, respectively, given by

$$m_e = \gamma + \eta(\ln 2)^{1/\beta}$$

$$E(X) = \gamma + \eta\Gamma(\frac{1}{\beta} + 1)$$

$$Var(X) = \eta^2[\Gamma\left(\frac{2}{\beta} + 1\right) - \Gamma^2\left(\frac{1}{\beta} + 1\right)]$$

where Γ() is the Gamma function, $\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}dx$.

When the location parameter is not used, in other words, when γ=0, then the pdf equation reduces to that of the two-parameter Weibull distribution. Thus, the pdf of a two-parameter Weibull distribution is given by

$$f(X) = \frac{\beta}{\eta}(\frac{X}{\eta})^{\beta-1}e^{-(\frac{X}{\eta})^{\beta}}$$

Moreover, if γ=0 and the shape parameter β is assumed to be a known constant β=c, then the pdf equation reduces to that of the one-parameter Weibull distribution, where only the scale parameter η needs to be estimated. The pdf of the one-parameter Weibull distribution is given by

$$f(X) = \frac{c}{\eta}(\frac{X}{\eta})^{c-1}e^{-(\frac{X}{\eta})^{c}}$$

The median, the mean and the variance for the last two cases, is derived from the most general forms of the three-parameter Weibull distribution by setting γ=0 and γ=0, β=c respectively.

The distribution equations may reduce to other distributions by varying its parameters. For example, if β=1 in the case of the three-parameter Weibull distribution, then that pdf reduces to the one of the two-parameter exponential distribution. Also, if β=1 and γ=0, it is the usual exponential distribution. When γ=0 and β=2, the pdf reduces to that of the Rayleigh distribution.

The following figures show the influence of the Weibull parameters on the shape of the Weibull pdf and the failure rate [53]. Firstly, Figure 3 presents the effect of different values of the shape parameter β on the shape of the pdf curve.

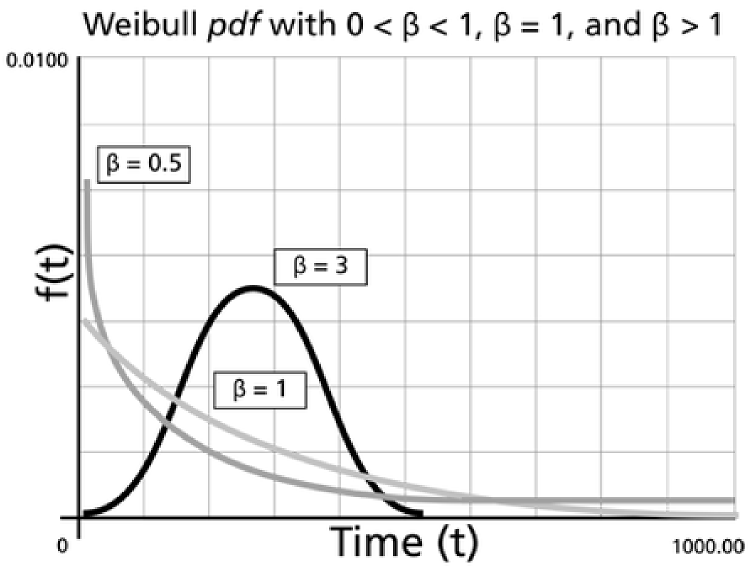


Figure 3. Effects of the shape parameter β

It is obvious that for β=3, and more general for 2.6 < β < 3.7, the Weibull pdf approximates the Normal pdf and the coefficient of skewness approaches zero. Additionally, for β < 2.6, the Weibull pdf is positively skewed and for β > 3.7, it is negatively skewed.

Secondly, Figure 4 presents the effect of different values of the scale parameter η on the shape of the pdf curve.

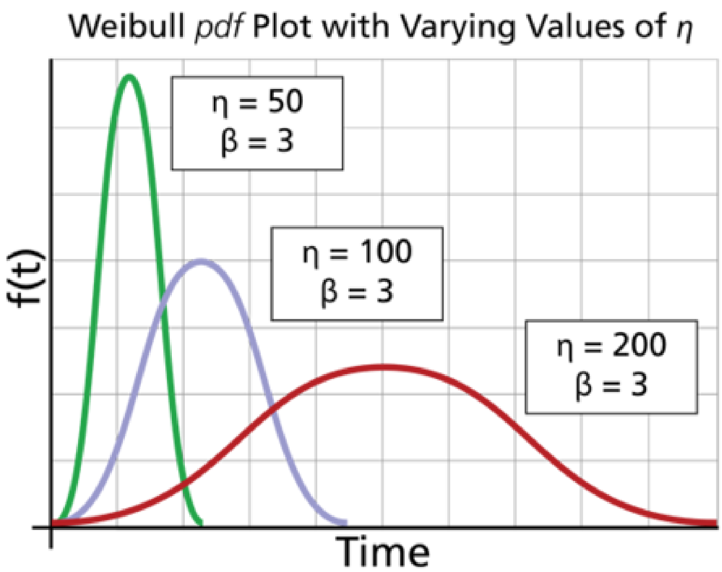


Figure 4. Effects of the scale parameter η

https://doi.org/10.26219/heal.aueb.5591

It is shown that for a constant shape parameter β, the pdf curve stretches out to the right and its height decreases as the scale parameter η increases.

Thirdly, Figure 5 presents the effect of different values of the location parameter γ on the shape of the pdf curve.

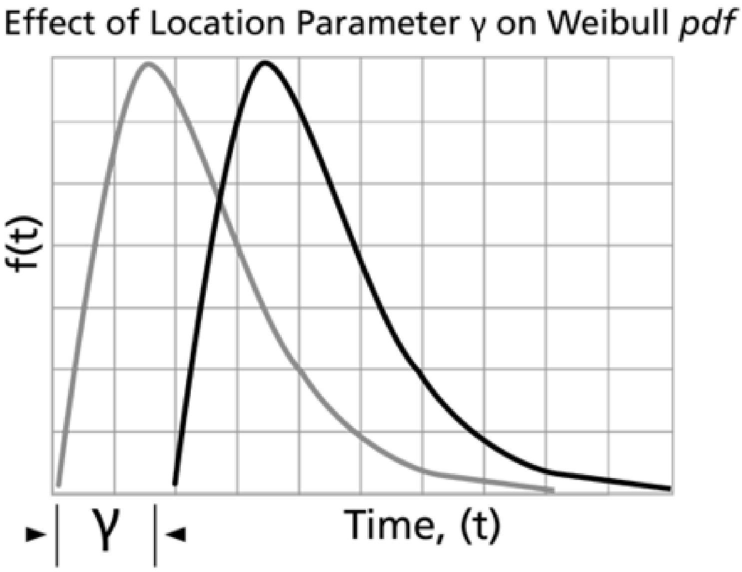


Figure 5. Effects of the location parameter γ

If $\gamma > 0$, the location parameter locates the distribution to the right, whereas if $\gamma < 0$, the distribution is located to the left.

The shape parameter β can also affect the Weibull failure rate, as shown in Figure 6.

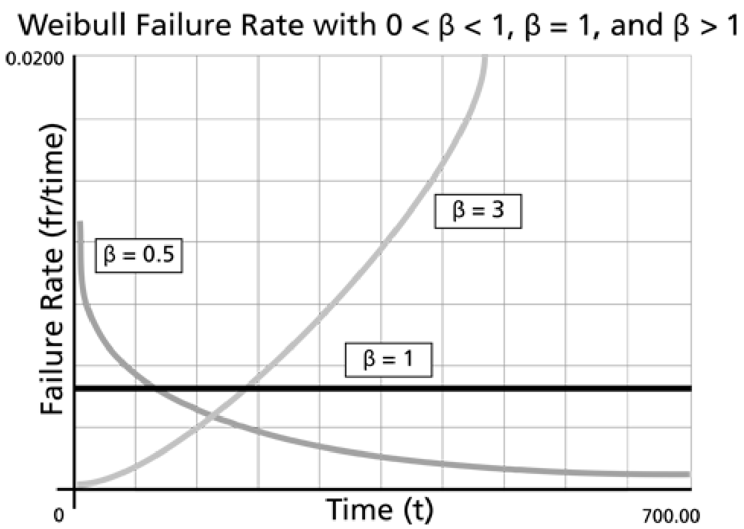


Figure 6. The effect of β on the failure rate

 https://doi.org/10.26219/heal.aueb.5591

The failure rate of the Weibull distribution decreases with time if $\beta<1$, it increases with time if $\beta>1$ and it remains constant if $\beta=1$. By combining the three lines, the plot looks like a bathtub curve.

In the industrial world, as seen before, the Weibull distribution is widely used and therefore, the output of a process is not always normally distributed, it can be skewed and closer to the Weibull distribution. So when the underlying distribution is the Weibull distribution, control charts, like the Shewhart one, that are designed based on the assumption that the process being monitored, produces a quality characteristic that can be approximated by a symmetrical normal distribution, can lead to false conclusions regarding the stability and the capability of the process. Wrong conclusions about a process, may cause great financial damage to a company. Thus, when the quality variable has a skewed distribution, the use of the normality assumption and as a result the use of the Shewhart X-bar and R or S-chart must be avoided. Because of the Weibull distribution's importance in real-life applications, the Weibull versions of the control charts are valuable tools for the practitioner in process monitoring.

CHAPTER 2

SHEWHART CONTROL CHARTS FOR THE WEIBULL DISTRIBUTION

2.1 Introduction

The use of the standard Shewhart X-bar and R or S control charts to monitor a process when the quality variable follows the Weibull distribution may be misleading because these charts are constructed under the normality assumptions. Also, control charts used in such situations cause an increase of Type I risk (i.e. probability of a false alarm) as the skewness of the distribution increases because of the variability in population. In this chapter, to deal with this problem, alternative methods which use asymmetric control limits are proposed. Moreover, control charts for monitoring the shape and scale parameter of the Weibull distribution are discussed and their performance measures are computed and compared in order to evaluate the chart's performance.

2.2 Methods to obtain control limits for the X-bar and R control charts

In this section, two methods (WV and SC method) will be described and used in order to obtain control limits for the X-bar and R control charts according to the article of Derya and Canan [2] and to that of Chan and Cui [3]. Also, a comparison between them gives the best one based on the Type I risk probabilities.

https://doi.org/10.26219/heal.aueb.5591

2.2.1 The WV method

The weighted variance (WV) method was first proposed by Choobineh and Ballard [6], which based on the semivariance approximation of Choobineh and Branting [7]. They obtained the asymmetric control limits for X-bar and R control charts for skewed distributions based on the standard deviation of sample means and range. Furthermore, by using this method, a simple technique was also proposed for constructing X-bar and R charts by Bai and Choi [8].

In the WV method there is no assumption about the distribution of the population and the control limits are adjusted according to the skewness of the population. This method decomposes the variance into two parts (upper variance σ_U^2 and lower variance σ_L^2, where $\sigma^2 = \sigma_U^2 + \sigma_L^2$) and according to Choobineh and Ballard [6], the basic idea is that a skewed distribution can be split into two segments at its mean, where each segment is used to create a new symmetric distribution. The two new distributions have the same mean but different variances (σ_U^2, σ_L^2). Let $P_X = P(X \leq \mu_X)$ be the probability that the quality variable X will be less than or equal to its mean μ_X. Then $\sigma_U^2 = P\sigma^2$ and $\sigma_L^2 = (1-P)\sigma^2$, where P, 1-P are the weights. When the process parameters are known, the control limits for the X-bar chart are given by

$$UCL_{\bar{x}} = \mu_x + 3\frac{\sigma_x}{\sqrt{n}}\sqrt{2P_x}$$

$$LCL_{\bar{x}} = \mu_x - 3\frac{\sigma_x}{\sqrt{n}}\sqrt{2(1-P_x)}$$

and the control limits of the R chart are given by

$$UCL_R = \mu_R + 3\sigma_R\sqrt{2P_x}$$

$$LCL_R = \max[0, \mu_R - 3\sigma_R\sqrt{2(1-P_x)}]$$

where μ_x and σ_x are the mean and the standard deviation of X, μ_R and σ_R are the mean and the standard deviation of the range of a sample size n (Bai and Choi [8]).

However, in practice, P_X and the process parameters are not known and must be estimated. μ_x is estimated by the grand mean of the subgroup means $\bar{\bar{X}}$, μ_R is estimated by the mean of the subgroup ranges $\bar{R}$ and P_X by using the

https://doi.org/10.26219/heal.aueb.5591

number of observations less than or equal to $\bar{\bar{X}}$: $\widehat{P_X} = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n}\delta(\bar{\bar{X}}-X_{ij})}{nk}$, where k and n are the number of samples and the number of observations in a subgroup respectively, whereas $\delta(X)=\begin{cases}1, X \geq 0\\0, \text{otherwise}\end{cases}$. In this case, in which the process parameters are unknown, the control limits of the X-bar chart are given by

$$UCL_{\bar{x}} = \bar{\bar{X}} + 3\frac{\bar{R}}{d_2^*\sqrt{n}}\sqrt{2\widehat{P_X}} = \bar{\bar{X}} + W_U\bar{R}$$

$$LCL_{\bar{x}} = \bar{\bar{X}} - 3\frac{\bar{R}}{d_2^*\sqrt{n}}\sqrt{2\left(1-\widehat{P_X}\right)} = \bar{\bar{X}} - W_L\bar{R}$$

and the control limits of the R control chart are given by

$$UCL_R = \bar{R}\left[1 + 3\frac{d_3^*}{d_2^*}\sqrt{2\widehat{P_X}}\right] = V_U\bar{R}$$

$$LCL_R = \bar{R}\left[1 - 3\frac{d_3^*}{d_2^*}\sqrt{2(1-\widehat{P_X})}\right] = V_L\bar{R}$$

where d_2^*, d_3^* are the control chart constants for the X-bar and R charts based on the WV method. These values are defined as the mean and standard deviation of the relative range R/σ and they have been obtained under the non-normality assumption. If the distribution is specified, they can be computed via numerical integration (Bai and Choi [8]).

2.2.2 The SC method

The skewness correction (SC) method was proposed by Chan and Cui [3] and it is used for constructing the X-bar and R control charts for skewed distributions. The degree of skewness is taken into consideration and no assumptions on the distribution are made in order to compute the asymmetric control limits. According to this method, the two control limits differ from the limits of the standardized Shewhart control chart by the amount of $\frac{\frac{4}{3}k_3}{(1+0.2k_3^2)}$, which is the skewness correction and k_3 is the skewness. If the process parameters are known, the control limits of the X-bar control chart are given by

https://doi.org/10.26219/heal.aueb.5591

$$UCL_{\bar{X}} = \mu_X + (3 + c_4^*)\frac{\sigma_X}{\sqrt{n}}$$

$$LCL_{\bar{X}} = \mu_X + (-3 + c_4^*)\frac{\sigma_X}{\sqrt{n}}$$

and the control charts of the R control chart are given by

$$UCL_R = \mu_R + (3 + d_4^*)\sigma_R$$

$$LCL_R = \mu_R + (-3 + d_4^*)\sigma_R$$

where c_4^* and d_4^* are the skewness corrections for the SC method. LCL_R is set to be zero if it is negative. When the process distribution is symmetric, $c_4^*= 0$ and the X-bar chart has the form of the conventional Shewhart chart. If the process parameters are unknown, the control limits of the X-bar chart are given by

$$UCL_{\bar{X}} = \bar{\bar{X}} + \left(3 + \frac{\frac{4k_3}{3\sqrt{n}}}{1 + \frac{0.2k_3^2}{n}}\right)\frac{\bar{R}}{d_2^*\sqrt{n}} \equiv \bar{\bar{X}} + A_U^*\bar{R}$$

$$LCL_{\bar{X}} = \bar{\bar{X}} + \left(-3 + \frac{\frac{4k_3}{3\sqrt{n}}}{1 + \frac{0.2k_3^2}{n}}\right)\frac{\bar{R}}{d_2^*\sqrt{n}} \equiv \bar{\bar{X}} - A_L^*\bar{R}$$

and the control limits of the R chart are given by

$$UCL_R = [1 + (3 + d_4^*)\frac{d_3^*}{d_2^*}]\bar{R} \equiv D_4^*\bar{R}$$

$$LCL_R = max[0, (1 + (-3 + d_4^*)\frac{d_3^*}{d_2^*})]\bar{R} \equiv D_3^*\bar{R}$$

When the distribution is symmetric, then the Type I risks of WV, SC and Shewhart charts are the same. However, according to Derya and Canan [2] and to Chan and Cui [3], when the underlying distribution is the Weibull distribution, the Shewhart chart has the worst performance both for the X-bar and the R chart, as the Type I risk increases too much when the skewness increases. Also, for the X-bar chart, the SC method gives the best results among all the previous methods. In particular, when the parameters are unknown, the SC method is very robust. Almost the same results are valid for the performance of the R chart based on the various methods. Specifically, for the R chart, the SC method gives the best results among the WV, Shewhart and SC methods.

https://doi.org/10.26219/heal.aueb.5591

As a conclusion, Type I risks of the SC X-bar and the SC R charts are closer to the conventional 0.27% than those of the WV charts, which means that the SC offers considerable improvement over the WV method when it is desirable for the Type I risk to be close to 0.27%.

2.3 Robust X-bar control chart using modified methods

When the parameters of a quality characteristic of the process are unknown, control charts are applied in a two-stages procedure according to the article of Derya [10]. In Phase I, control charts are used to study a historical dataset and determine the samples that are out of control. The process parameters are also estimated and the control limits are calculated for Phase II. In Phase II, based on these control charts, samples from the process are prospectively monitored for departures from the in-control state. Derya [10] describes three methods to construct the limits of the X-bar control chart for monitoring a Weibull distributed process. These are the modified Shewhart (MS), the modified weighted variance (MWV) and the modified skewness correction (MSC) methods. In these methods, the overall mean is replaced by a trimmed mean and the estimator of the standard deviation based on the ranges by the interquartile ranges.

2.3.1 Robust estimators

The main disadvantage of the mean of the sample means as an estimator of the process parameter μ is that it is not robust against outliers. If the distribution is skewed and has extreme outliers, trimming can result in improved efficiency. Therefore, the mean of the sample trimmed means is considered.
A trimmed mean is a method of averaging that removes a small designated percentage of the largest and smallest values before calculating the mean. Firstly, the values X_{ij} are ordered from lowest to highest for each one of the i samples, the desired amount of trimming is determined and then the mean is

https://doi.org/10.26219/heal.aueb.5591

calculated for all observations of a sample except the g=nα/2 smallest and largest observations, where n is the sample size and α is the percentage of samples to be trimmed. The trimmed mean is given by $T\overline{M}_{\alpha} = \frac{1}{k}\sum_{i=1}^{k} T\overline{M}_{\upsilon i}$, where $TM_{(\upsilon i)}$ is the υ-th ordered value of the sample trimmed means given by $T\overline{M}_{\upsilon i} = \frac{1}{n-2\lceil n\alpha \rceil}\sum_{j=\lceil n\alpha \rceil+1}^{n-\lceil n\alpha \rceil} X_{(ij)}$, where $\lceil n\alpha \rceil$ is the ceiling function, i.e. the smallest integer not less than nα.

Moreover, the mean of the sample range is also not robust against outliers and therefore, the mean of the sample interquartile ranges will be used instead of it. The mean of the sample interquartile ranges (IQR_s) is given by $I\bar{Q}R = \frac{1}{k}\sum_{i=1}^{k} IQR_i$, where IQR_i is the interquartile range of the sample i: $IQR_i = Q_{75,i} - Q_{25,i}$ and $Q_{r,i}$ is the r-th percentile of the values in sample i. The IQR for the Weibull distribution is given by $IQR_W = \left[-\frac{1}{\beta}\ln(0.25)\right]^{\frac{1}{\eta}} - \left[-\frac{1}{\beta}\ln(0.75)\right]^{\frac{1}{\eta}} = \frac{1}{\beta^{\frac{1}{\eta}}}\ln 4^{\frac{1}{\eta}} - \ln(\frac{4}{3})^{\frac{1}{\eta}}$, where β and η are the shape and scale parameters of the Weibull distribution respectively.

2.3.2 Modified methods for the X-bar control chart

In order to obtain the control limits for the X-bar chart by using robust estimators because the process parameters are assumed to be unknown, μ_X is estimated by using the trimmed mean of the subgroup trimmed means $T\overline{M}_{\alpha}$ and μ_R is estimated using the mean of the subgroup interquartile ranges $I\bar{Q}R$. The control limits of the X-bar chart based on the modified Shewhart (MS) method are the following:

$$UCL_{\bar{X}_{MS}} = T\overline{M}_{\alpha} + \frac{3}{d_2^Q\sqrt{n}} I\bar{Q}R$$

$$LCL_{\bar{X}_{MS}} = T\overline{M}_{\alpha} - \frac{3}{d_2^Q\sqrt{n}} I\bar{Q}R$$

where d_2^Q is a constant that depends on the subgroup size n.

https://doi.org/10.26219/heal.aueb.5591

The control limits of the X-bar chart based on the modified weighted variance (MWV) method are given as follows:

$$UCL_{\bar{x}_{MWV}} = T\overline{M}_{\alpha} + 3\frac{I\bar{Q}R}{d_2^Q\sqrt{n}}\sqrt{2\hat{P}_x^R}$$

$$LCL_{\bar{x}_{MWV}} = T\overline{M}_{\alpha} - 3\frac{I\bar{Q}R}{d_2^Q\sqrt{n}}\sqrt{2(1-\hat{P}_x^R)}$$

where d_2^Q is the control constant for the X-bar chart based on the MWV method and is defined as the mean of the interquartile range, $d_2^Q = E(\frac{IQR}{\sigma})$. The probability P_X is estimated from $\widehat{P}_x^R = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n}\delta(T\overline{M}_{\alpha}X - X_{ij})}{nk}$, where k and n are the number of samples and number of observations in a subgroup respectively, whereas $\delta(X)=\begin{cases} 1, \text{if } X \geq 0 \\ 0, \text{otherwise} \end{cases}$.

The control limits of the X-bar chart based on the modified skewness corresction (MSC) method are given as follows:

$$UCL_{\bar{X}_{MSC}} = T\overline{M}_{\alpha} + (3 + c_4^Q)\frac{I\bar{Q}R}{d_2^Q\sqrt{n}}$$

$$LCL_{\bar{X}_{MSC}} = T\overline{M}_{\alpha} + (-3 + c_4^Q)\frac{I\bar{Q}R}{d_2^Q\sqrt{n}}$$

where c_4^Q is the control chart constant for the MSC method and is computed by $c_4^Q = \frac{\frac{4}{3}k_3(TM_{\alpha})}{1+0.2k_3^2(TM_{\alpha})}$, where $k_3(TM_{\alpha})$ is the skewness of the subgroup trimmed means TM_{α}.

The performances of these three control charts based on the modified methods are compared in terms of the Type I risk probabilities p and the in-control ARL values. According to Derya [10], when the underlying distribution is Weibull, the MSC method has the best performance as it has the lowest values for p and the highest value for the in-control ARL in comparison to the MS and the MWV methods. All three methods can be used but the MSC is more appropriate and gives the desirable results when there are large outliers and the sample size is n=10. The MWV method can be used as an alternative.

 https://doi.org/10.26219/heal.aueb.5591

2.4 Methods to obtain control limits for the S control chart

In order to obtain control limits for the S control chart when the underlying distribution is Weibull and to avoid misleading conclusions due to non-normality, the weighted variance S control chart (WV-S) and the scaled weighted variance S control chart (SWV-S) are discussed and compared to the Shewhart S control chart according to the article of Atta et al. [11].

2.4.1 WV-S control chart

The two new symmetric distributions, $f_L(x)$ and $f_U(x)$, created by the WV method, have the same mean μ_X but different variances σ_L^2 and σ_U^2 (Choobineh and Ballard [6]) and are used to compute the limits of the WV control chart. According to Khoo et al. [12], the limits for the WV-S control chart when the process parameters are known, are given by

$$UCL_{WV-S} = \mu_S + 3\sigma_S\sqrt{2P_X}$$

$$LCL_{WV-S} = \mu_S - 3\sigma_S\sqrt{2(1-P_X)}$$

where μ_S and σ_S are the mean and standard deviation of S, respectively and $P_X = P(X \le \mu_X)$. When $P_X = \frac{1}{2}$, the WV-S control chart reduces to the standard S control chart.

When the process parameters are unknown, the control limits for the WV-S control chart are given by

$$UCL_{WV-S} = \bar{S}\left[1 + \frac{3\sqrt{1-(C'_4)^2}}{C'_4}\sqrt{2\hat{P}_X}\right] = B_U\bar{S}$$

$$LCL_{WV-S} = \bar{S}\left[1 - \frac{3\sqrt{1-(C'_4)^2}}{C'_4}\sqrt{2(1-\hat{P}_X)}\right] = B_L\bar{S}$$

where $C'_4 = \frac{E(S)}{\sigma_x}$ and $\hat{P}_X = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} I(\bar{\bar{X}} - X_{ij})}{mn}$, where m and n are the number of samples and the sample size, respectively and $I(x)=\begin{cases} 1, \text{if } x \ge 0 \\ 0, \text{otherwise} \end{cases}$.

https://doi.org/10.26219/heal.aueb.5591

2.4.2 SWV-S control chart

In the WV method, $f_L(x)$ and $f_U(x)$ are replaced by two normal distributions, having the same mean μ_X and variances σ_L^2, σ_U^2. The scaled weighted variance (SWV) method aims to improve the performance of the WV. According to Castagliola [13], the two functions $f_L(x)$ and $f_U(x)$ are not simply replaced by two normal distributions, but by two "bell-shaped" functions, $\varphi(x, \mu_X, \sigma_L, 2P_X)$ and $\varphi(x, \mu_X, \sigma_U, 2(1-P_X))$, with mean μ_X, σ_L^2 and σ_U^2 for second central moments, $2P_X$ and $2(1-P_X)$ for areas and $\varphi(x, \mu_X, t, k) = \frac{k^{3/2}}{t}\varphi(\frac{(x-\mu_X)\sqrt{k}}{t})$. It is called SWV because the function φ is scaled by the factor $\frac{k^{3/2}}{t}$.

The control limits for the SWV-S control chart when the process parameters are known are given by

$$UCL_{SWV-S} = \mu_S + \Phi^{-1}\left(1 - \frac{\alpha}{4(1-P_X)}\right)\sqrt{\frac{P_X}{(1-P_X)}}\sigma_S$$

$$LCL_{SWV-S} = \mu_S - \Phi^{-1}\left(1 - \frac{\alpha}{4P_X}\right)\sqrt{\frac{(1-P_X)}{P_X}}\sigma_S$$

where μ_S and σ_S are the mean and standard deviation of S respectively and α is the Type I error rate. When $P_X = \frac{1}{2}$, the SWV-S control chart reduces to the standard S control chart.

When the process parameters are unknown, the control limits for the SWV-S control chart are given by

$$UCL_{SWV-S} = \bar{S}\left[1 + \Phi^{-1}\left(1 - \frac{\alpha}{4(1-\hat{P}_X)}\right)\frac{\sqrt{1-(C'_4)^2}}{C'_4}\sqrt{\frac{\hat{P}_X}{(1-\hat{P}_X)}}\right]$$

$$LCL_{SWV-S} = \bar{S}\left[1 - \Phi^{-1}\left(1 - \frac{\alpha}{4(1-\hat{P}_X)}\right)\frac{\sqrt{1-(C'_4)^2}}{C'_4}\sqrt{\frac{(1-\hat{P}_X)}{\hat{P}_X}}\right]$$

where $C'_4 = \frac{E(S)}{\sigma_x}$ is a constant.

According to the conclusions of Atta et al. [11], the proposed SWV-S control chart has the best performance in comparison with the WV-S control chart and the standard S control chart. The comparisons are based on the false

https://doi.org/10.26219/heal.aueb.5591

alarm rate and the probability of out-of-control detection. The false alarm rate refers to the proportion of subgroup points plotting beyond the limits of a control chart, given that the process is actually in-control. On the other hand, the probability of out-of-control detection measures the ability of a chart to respond to a shift in the process and it is defined as the proportion of subgroup points plotting beyond the chart's limits when the process has shifted.

When the underlying distribution is Weibull, the SWV-S chart has the lowest false alarm rate and the highest probability of out-of-control detections for almost all levels of skewness and sample sizes, among the 3 charts. These results show its robustness to non-normality. Thus, the SWV-S control chart offers considerable improvement over the WV-S and standard S control charts when it is desirable for the false alarm rate to be close to the conventional 0.0027.

2.5 Shewhart control charts for monitoring reliability with Weibull lifetimes

The Weibull distribution is characterized by two parameters, shape and scale. These parameters can describe different aspects of lifetime variables in reliability and survival studies, in which, the Weibull distribution is of high importance. In the article of Faraz et al. [14], a new approach for monitoring a Weibull process is proposed, which is adaptable to any asymmetric distribution. According to this approach, the shape and scale parameters are assumed to be unknown and they can be estimated by initial in-control samples. Also, the in-control Weibull distribution is transformed to a standard normal distribution and then the process is monitored by Shewhart-type control charts.

After the Weibull observations are transformed to observations having the standard normal distribution (Faraz et al. [14]), then $\mu_0 = 0$ and $\sigma_0^2 = 1$ when the Weibull process is in-control. Furthermore, the relationship between the two Weibull parameters and the mean and variance of the standard normal distribution is examined. The shift size in the mean and variance of the

https://doi.org/10.26219/heal.aueb.5591

standard normal distribution can be measured as a result of any shift in the Weibull parameters and can be calculated by $\mu_1 = \mathrm{E}(\mathrm{Z}, \eta_1, \beta_1)$ and $\sigma_1^2 = \mathrm{Var}(\mathrm{Z}, \eta_1, \beta_1)$, where η_1, β_1 are the shifted scale and shape parameters, respectively. Thus, it is obvious that a shift in the scale or shape parameter leads to a shift in the mean and variance of the normal process and by controlling the normal process parameters, shifts in the Weibull parameters can also be detected.

Based on the transformed data (Z), Shewhart-type $\bar{\mathrm{Z}}$ and S^2 control charts will be used for monitoring shifts in the shape and scale parameters of a Weibull distributed process. The mean and variance of the random samples from the standard normal distribution N(0,1) are computed by using η_0, β_0, which are the estimated scale and shape parameters. The values of the mean and the variance are then plotted on the Shewhart $\bar{\mathrm{Z}}$ and S^2 control charts for process monitoring. The control limits and the center line of these charts are given by

$$(UCL_Z, CL_Z, LCL_Z) = \left(\frac{Z_{\gamma/2}}{\sqrt{n}}, 0, \frac{Z_{1-\gamma/2}}{\sqrt{n}}\right)$$

$$(UCL_S, CL_S, LCL_S) = \left(\frac{\chi^2_{\gamma/2}}{n-1}, 1, \frac{\chi^2_{1-\gamma/2}}{n-1}\right)$$

where $\mathrm{Z_p}$ and χ^2_{p} are the p-percentile of the standard normal and the chi-square distributions, respectively and γ is the type I error rate.

According to the conclusions of Faraz et al. [14], joint $\bar{\mathrm{Z}}$ and S^2 perform well in detecting different shifts in the Weibull parameters in terms of the ARL. A shift in these parameters leads to changes in the mean and variance of the transformed normal variables. Specifically, if only the shape parameter changes (η_0, β_1), then positive shifts in β cause small changes in the normal process mean and a decrease in standard deviation, whereas negative shifts in β do not have serious effects on the mean and cause an increase in standard deviation. If only the scale parameter changes (η_1, β_0), then negative shifts in η cause greater shifts in the normal parameters than that caused by a positive shift in η. Finally, if both Weibull parameters change (η_1, β_1), then a shift in β does not affect the normal process mean as long as the shift in η is small. Thus, an increase or decrease in the normal process parameters

suggests changes in the Weibull parameters, so $\bar{Z}$ and S^2 charts have good performance in detecting shifts in the shape and scale parameters.
This method can be considered as an asymmetric alternative to Shewhart control charts and its importance lies on its ease of use and flexibility for the case where the process distribution is Weibull. However, it can be applied to any distribution.

2.6 Normal individual and moving range charts

In this section, a transformation is discussed under which, the Weibull data follow an approximate normal distribution and the Shewhart control chart can be used. In order to monitor a process by using samples that consist of individual observations, the individuals Y and the moving range (MR) charts are considered (Pascual [15]). $MR_i = |Y_i - Y_{i-1}|$ is the moving range.

If the random variable X follows W(η,β), it has been shown that the transformed variable $Y=cX^{\alpha}$, α>0, c>0, follows also the Weibull distribution with parameters $W(c\eta^{\alpha},\beta/\alpha)$ (McCool and Joyner-Motley [16]). Moreover, according to Pascual [15], if W is an exponential random variable, then $W^{0.2777}$ is approximately normal with similar symmetry and kurtosis as these of the normal distribution. Based on these facts, it is obvious that, if X~W(η,β) then $X^{\beta}\sim W(\eta^{\beta},1)$ which is the exponential distribution with mean η^{β}. Thus, $Y=X^{0.2777\beta}$ is approximately normal.

In order to obtain the control limits of the two normal-based control charts, it is assumed that $X_i\sim W(\eta,\beta)$, λ>0 and hence, $Y_i=X_i^{\lambda\beta}\sim W(\eta^{\lambda\beta},1/\lambda)$. Then, the center line and the control limits of the individuals Y chart are given by (Pascual [15])

$$UCL_Y = \eta_s^{\lambda\beta_s}\lambda\Gamma(\lambda)\left[1 - 5.319149\left(1 - \frac{1}{2^{\lambda}}\right)\right]$$

$$CL_Y = E(Y_i) = \eta_s^{\lambda\beta_s}\lambda\Gamma(\lambda)$$

$$LCL_Y = \alpha^{\lambda\beta_s}\lambda\Gamma(\lambda)\left[1 + 5.319149\left(1 - \frac{1}{2^{\lambda}}\right)\right]$$

https://doi.org/10.26219/heal.aueb.5591

where η_s and β_s are the stable process parameters and for λ=0.2777, which was used for the normal transformation of the Weibull data, the center line and the control limits are the following

$$UCL_Y = 1.740382\eta_s^{0.2777\beta_s}$$

$$CL_Y = 0.901119\eta_s^{0.2777\beta_s}$$

$$LCL_Y = 0.061856\eta_s^{0.2777\beta_s}$$

The center line and the control limits of the MR chart are given by

$$UCL_{MR} = 6.534\eta_s^{\lambda\beta_s}\lambda\Gamma(\lambda)\left(1-\frac{1}{2^\lambda}\right)$$

$$CL_{MR} = 2\eta_s^{\lambda\beta_s}\lambda\Gamma(\lambda)\left(1-\frac{1}{2^\lambda}\right)$$

where η_s and β_s are the stable process parameters. There is no LCL for the MR chart because the lower 3-sigma limit is negative for λ values that yield close normal approximations. For λ=0.2777 the center line and the UCL are the following

$$UCL_{MR} = 1.030944\eta_s^{0.2777\beta_s}$$

$$CL_{MR} = 0.315563\eta_s^{0.2777\beta_s}$$

According to the conclusions of Pascual [15] and based on the ARL, the Y chart is in general effective in detecting increases in the mean and the variance but ineffective in detecting decreases. When it comes to detecting increases in the mean and the variance, the same remarks with the Y chart are also true for the MR chart. However, the behavior of the two charts' ARLs suggests that there are differences between them and the use of the MR chart may end up being more problematic than the individuals Y chart.

It should be noted that, the run length L is a geometric random variable with ARL=1/p, where p is the success probability. This is valid for the Y chart but not for the MR chart as the moving range contains two sampling periods and because of this, two consecutive moving ranges are not independent. For this reason, the ARL of the MR chart is computed with the help of a Fredholm integral equation (Pascual [15]).

https://doi.org/10.26219/heal.aueb.5591

2.7 Monitoring the shape parameter of a Weibull renewal process

It makes good sense to monitor changes in the shape parameter β of a Weibull distribution because, as stated in the article of Zhang et al. [17], it is particularly indicative and informative and it tends to be closely related to the failure mechanisms of a product. It is also an indication for the behavior of the hazard function in lifetime distributions. β>1 indicates an increasing hazard rate, β<1 indicates a decreasing hazard rate and β=1 indicates that the hazard rate is constant.

The Weibull renewal process is an extension of the Weibull distribution in terms of renewal theory. According to Zhang et al. [17], if $\{X_i, i=1,2,\ldots\}$ are independent and identically distributed Weibull random variables, then $\{X_i, i=1,2,\ldots\}$ is a Weibull renewal process. It should be clear that, if β=1, the Weibull renewal process reduces to the Poisson process, where the Time-Between-Events (TBE) are exponentially distributed.

In order to monitor the Weibull shape parameter, a Shewhart-type control chart is presented by Zhang et al. [17], which is called a Beta chart. This chart is based on the statistic $\hat{S} = \left(B_n\hat{\beta}\right)^{\lambda}$, which is appropriate for monitoring β. The process that lead to this statistic is the following: Let $\hat{\beta}$ be the maximum likelihood estimate (MLE) of β and $\frac{\hat{\beta}}{\beta}$ is a pivotal quantity whose distribution does not depend on the unknown parameter β. Bain [18] suggested the use of $B_n\hat{\beta}$, where $B_n = \frac{1}{E(\frac{\hat{\beta}}{\beta})}$ and is obtained using Monte Carlo simulation. Moreover, $B_n\hat{\beta}$ is an unbiased estimator of β and is asymptotically normal. However, its distribution is skewed when the sample size is small but in control charts, a normally distributed statistic is mostly wanted. For this reason, a Box-Cox transformation is applied on $B_n\hat{\beta}$ and gives the statistic $\hat{S} = \left(B_n\hat{\beta}\right)^{\lambda}$. This statistic has an approximately normal distribution and the value of λ=-0.5 is chosen as appropriate based on simulations, so $\hat{S} = \left(B_n\hat{\beta}\right)^{-0.5}$. Its mean and variance are the following:

$$E\left(\hat{S}\right) = B_n^{\lambda}\beta^{\lambda}E_n \text{ and } Var\left(\hat{S}\right) = B_n^{2\lambda}\beta^{2\lambda}V_n$$

where the values of E_n and V_n are obtained by Monte Carlo simulation.

https://doi.org/10.26219/heal.aueb.5591

The central line and the control limits of the Beta chart for monitoring the parameter β are given by (Zhang et al. [17])

$$UCL = (B_n\beta_{ic})^{-0.5}E_n + k(B_n\beta_{ic})^{-0.5}\sqrt{V_n}$$

$$CL = (B_n\beta_{ic})^{-0.5}E_n$$

$$LCL = (B_n\beta_{ic})^{-0.5}E_n - k(B_n\beta_{ic})^{-0.5}\sqrt{V_n}$$

where β_{ic} is the in-control value for β and k represents the distance of the control limits from the central line in standard deviation units. When it comes to out-of-control signals, it should be noted that $\hat{S}$ is a monotonically decreasing function of $\hat{\beta}$, so a point plotted above the UCL represents a decrease in β, whereas a point plotted below the LCL represents an increase in β.

The control limits in the above case are based on a known value of β_{ic}. When β_{ic} is unknown and must be estimated, the monitoring statistic is $\hat{S}_r = \left(B_r\hat{\beta}_r\right)^{-0.5}$, where r=3,4,…,20 for 20 in-control Weibull TBEs denotes the cumulative number of Weibull TBEs observed during Phase I and $\hat{\beta}_r$ is the MLE of β by using r Weibull TBEs. Then, the central line and the control limits of the Beta chart according to Zhang et al. [17] are given by

$$UCL_r = \left(B_r\hat{\beta}_r\right)^{-0.5}E_r + k\left(B_r\hat{\beta}_r\right)^{-0.5}\sqrt{V_r}$$

$$CL_r = \left(B_r\hat{\beta}_r\right)^{-0.5}E_r$$

$$LCL_r = \left(B_r\hat{\beta}_r\right)^{-0.5}E_r - k\left(B_r\hat{\beta}_r\right)^{-0.5}\sqrt{V_r}$$

For the construction of the Beta chart, values for β_{ic}, sample size n and k are appropriately chosen so that a desirable in-control ARL (ARL_0) value is achieved.

According to Zhang et al. [17], the performance of the Beta chart will be evaluated through the ARL. The ARL will be computed through Monte Carlo simulation because consecutive samples of a Weibull renewal process are correlated. It is known that the charts' performance is dependent on the shape parameter β and the sample size n but independent of the scale parameter η. When the shift in the shape parameter β is small, a large sample size n is required in order to obtain a small out-of-control ARL (ARL_1), which means that the shift will be detected faster. On the other hand, a large shift in β requires a small value for n, in order to detect the shift faster. The

 https://doi.org/10.26219/heal.aueb.5591

Beta chart is also ARL-unbiased, which means that the out-of-control ARL values are not higher than the in-control values. Through applications in reliability testing in the article of Zhang et al. [17] it was found that, the Beta chart is effective and makes good sense in real practice applications. The datasets used allow a comparison of this chart with an EWMA chart in order to find the one with better ARL performance. This comparison and its results will be discussed in Chapter 3: Exponentially weighted moving average control charts, in section 3.2.2.

2.8 Control charts for the scale parameter with fixed and variable sampling intervals

Shewhart control charts with fixed (FSI) and variable (VSI) sampling intervals are presented in the article of Ramalhoto and Morais [19], for the scale parameter of a three-parameter Weibull distribution. A shift in the scale parameter is important as it leads to a shift in the mean and the variance of the control variable. Two-sided and one-sided Shewhart control charts are presented and their performance is evaluated by using the function average time to signal (ATS) in order to control the mean and the variance of the Weibull control variable. The 'comparability criterion' and the 'primordial criterion' are taken into account in designing the charts. According to the 'comparability criterion', introduced by Reynolds et al. (1988), two or more control charts are comparable under control if, and only if, their average times to false alarm are equal. In 'primordial criterion' ANSS(θ)<ANSS(1), where θ is the shift in the scale parameter and ANSS(θ) is the average number of samples to signal. ANSS is also called average run length (ARL), which is the most commonly used measure of performance for FSI. The chart statistic and the 'continuation region' is the same both for the FSI and the VSI versions. In the VSI version, the 'continuation region' is partitioned into two intervals and the constant which defined that partition is obtained through the 'comparability criterion' under control.

The function ATS of all the control charts does not depend on the value of the location parameter or of the in-control scale parameter and under

https://doi.org/10.26219/heal.aueb.5591

control, does not depend on the shape parameter, too, when the probability of a false alarm and the sample size are fixed. The ‘primordial criterion’ is satisfied for all control charts, which means it is satisfied for both the two-sided and the lower and upper one-sided control charts and their FSI and VSI versions. Moreover, for fixed θ, all of them are quicker in detecting a shift as the shape parameter increases, except for the two-sided VSI control chart. For the two-sided control charts, based on the ATS criterion, the VSI versions are quicker in detecting a shift than are the FSI versions, which means that the VSI versions are always preferable to the FSI versions. The FSI version increases the speed of detection of a shift, when the shift is getting larger. For the one-sided control charts, under the ‘comparability criterion’, the VSI version is always preferable to the FSI version. In general, the advantage of the VSI version is even more obvious in the one-sided case than in the two-sided control chart case. Finally, one-sided control charts are preferable to the two-sided, as they perform better and have more powerful statistical properties.

https://doi.org/10.26219/heal.aueb.5591

 https://doi.org/10.26219/heal.aueb.5591

CHAPTER 3

EXPONENTIALLY WEIGHTED MOVING AVERAGE CONTROL CHARTS

3.1 Introduction

The exponentially weighted moving average (EWMA) control chart is especially used when the detection of small process shifts is of interest. It is a very effective alternative to the Shewhart control chart and it offers important performance improvement relative to Shewhart. Since the EWMA can be viewed as a weighted average of all past and current observations, it is very insensitive to the normality assumption and is therefore, ideal for individual observations. Smaller values of the smoothing parameter λ, which is used in EWMA charts, are more appropriate for detecting smaller shifts. In this chapter, various EWMA control charts are presented to monitor Weibull distributed processes. The main issues are monitoring the shape parameter of the Weibull distribution or the lifetimes' mean and also transforming the Weibull or exponential data to normal in order to apply the chart. Finally, performance measures are computed and compared in order to evaluate the chart's performance.

3.2 Moving range EWMA control charts for monitoring the shape parameter

According to the article of Akhundjanov and Pascual [20], the shape parameter β of the Weibull distribution plays a big role in practical applications and thus, the importance of its monitoring is obvious. However, previous literature relies mostly on the assumption that β is known and stable.

In this section, the design of control charts for monitoring shifts in the shape parameter will be discussed, when only a single observation per sampling period is used in order to represent cases, in which there are limitations in time and resources. For this reason, Pascual and Nguyen [21] proposed the moving range control charts, but it was found that they perform well when the shift in β is large rather than when it is small to moderate. So, here the suggested chart is the EWMA control chart based on the moving range statistic.

3.2.1 Monitoring statistic

The statistic that will be used for monitoring the shape parameter is the moving range, R_i. This statistic is derived from the following process: Let Y_i be a single observation in period i, for i=1,2,... and it also follows the Weibull distribution with scale parameter α and shape parameter β, $Y_i \sim W(\alpha, \beta)$. Then $X_i = \log(Y_i)$ follows the smallest extreme value distribution (SEV) with location parameter μ=log(α) and scale parameter σ=1/β, $X_i \sim SEV(\mu, \sigma)$. As $R_i = |X_i - X_{i-1}|$, i=2,3,... is denoted the SEV moving range between the periods i-1 and i. The distributional properties of R_i depend only on σ and not on μ. Hence, R_i is an appropriate statistic for monitoring shifts in the SEV scale parameter σ, or equivalently, in the Weibull shape parameter β (since σ=1/β). This means that the monitoring of β will be achieved through the process of monitoring σ.

3.2.2 Control limits and ARL performance

An out-of-control signal on the EWMA control charts occurs when

$$E_i > h_U = \mu_R + \sigma_R h_U^* \quad \text{or} \quad E_i < h_L = \mu_R + \sigma_R h_L^*$$

where $E_i = \lambda R_i + (1-\lambda)E_{i-1}$ is the EWMA value for the i period, $E_1 = E(R) = \mu_R$ and h_U, h_L are the control limits for the EWMA. A central line can be drawn to represent typical values of E_i under stable process, for example CL=μ_R and the control limits are defined with the help of h_U^*, h_L^*, which are the standardized control limits and are derived as follows: Let $R_i^* = \frac{(R_i - \mu_R)}{\sigma_R}$ be the standardized moving range, then h_U^*, h_L^* are the standardized control limits based on R_i^* and they are derived from simulations. Specifically, for the design of the EWMA control chart, values for the weighting parameter λ and for the standardized control limits h_U^*, h_L^* are chosen, so that, when the process is stable, a specific in-control ARL value is achieved. Simulations to determine the values of h_U^*, h_L^* are performed for various combinations of λ and in-control ARL (ARL_0). The value of the Weibull scale parameter does not need to be specified. Also, both two-sided and one-sided (lower and upper) EWMA control charts are discussed. The upper-sided chart is recommended for detecting increases in σ (decreases in β) and the lower-sided chart is proposed for detecting decreases in σ (increases in β), whereas the two-sided control chart is more appropriate when the detection of any changes in σ or β are of interest.

According to the findings of Akhundjanov and Pascual [20], one-sided charts are unique and unbiased for fixed λ and ARL_0, whereas for two-sided charts, the combinations of h_U^* and h_L^* are infinitely many, but only one of them corresponds to an unbiased chart and that is the one that will be chosen. Comparisons also indicate that, for small or moderate shifts in σ based on the ARL, the proposed moving range EWMA control chart detects the process shift faster than the moving range chart proposed by Pascual and Nguyen [21]. However, Zhang et al. [17] proposed in their article the Beta chart, as seen in Chapter 2, in section 2.7, and their comparisons showed that the Beta chart detects a shift in β faster than the moving range EWMA chart proposed by Akhundjanov and Pascual [20]. In other words, the moving range EWMA chart is outperformed by the Beta chart based on ARL values.

3.2.3 Estimators of stable-process σ

Of course, in order to obtain the control limits, the stable-process value of σ needs to be known. When the parameter is unknown, this can be achieved

https://doi.org/10.26219/heal.aueb.5591

by two proposed approaches. The first one is the standards-given case, in which the stable-process values of parameters are chosen according to prior knowledge or experience and the control limits are constructed using these fixed values of the parameters. The second one is the retrospective case, in which the stable-process values of parameters are estimated by a sample generated from a stable process and then, the control limits are computed based on these estimates. In the last case, either the unbiased estimator of σ ($\widehat{\sigma}_{UE}$) or the maximum likelihood estimator of σ ($\widehat{\sigma}_{MLE}$) can be used. Moreover, the control limits are treated as random variables when the retrospective approach is used, whereas they are fixed quantities when the standards-given case is used.

Finally, the sample size requirements for Phase I estimations of the retrospective approach were examined through simulation by Akhundjanov and Pascual [20] in order to achieve similar run length distributions for both approaches. According to their conclusions, the MLE estimator, $\widehat{\sigma}_{MLE}$, requires smaller sample size for Phase I than the $\widehat{\sigma}_{UE}$ and thus, it is preferred.

3.3 EWMA charts for monitoring the mean of censored Weibull lifetimes

In the article of Zhang and Chen [22] lower and upper-sided EWMA charts are proposed to detect changes in the process mean when the underlying distribution is Weibull and censoring occurs at a fixed level (Type I censoring: the experiment stops at a predetermined time, at which point any subjects remaining are right censored). According to Steiner and MacKay [23], the use of the traditional monitoring methods is not directly applicable on censored data. They also say that, changes in the mean are changes in the scale parameter when the shape parameter is fixed and they propose to replace the censored data with conditional expected values (CEV), CEV(X)=E[X|X > $\left(\frac{c}{\alpha_o}\right)^{\beta_o}$], where c is the fixed level at which censoring occurs. Based on that idea are constructed the one-sided EWMA charts suggested by Zhang and Chen [22]. The selection of the smoothing parameter λ, as well as, of K_L and K_u must be considered. K_L and K_u are multipliers for the lower and upper-

https://doi.org/10.26219/heal.aueb.5591

sided EWMA charts respectively, that together with λ control the chart's performance. A combination of (λ, K_L) or (λ, K_u) must be found, that for a given censoring rate, a specified in-control ARL value is achieved and the chart signals quickly when the process is out-of-control.

Then, the EWMA CEV charts are compared with the Shewhart-type charts proposed by Steiner and MacKay [24]. The comparison is based on ARL and standard deviations of run lengths (SDRL) values, which are derived through simulations. According to the results of Zhang and Chen [22], the lower-sided EWMA CEV chart has a better performance than the Shewhart-type for detecting process mean decreases. Also, the upper-sided EWMA CEV chart performs well in monitoring mean increases, whereas the Shewhart-type chart is not appropriate for monitoring mean increases when the censoring rate is high. Finally, a value of λ=0.1 or λ=0.2 is recommended in order to detect small to moderate shifts in the process mean and at the same time to achieve a good performance based on ARL and SDRL.

3.4 Monitoring Weibull quantiles by EWMA charts

In the article of Pascual et al. [25], EWMA control charts are used to monitor the p-quantile of the Weibull distribution with Type II censored samples. In Type II censoring, the experiment stops when a predetermined number are observed to have failed and the remaining subjects are right-censored. Monitoring the quantiles is of high importance, as a shift in the quantiles of the strength or lifetime distribution is possible to indicate a deterioration in product quality. Also, the ARL of the charts are computed and used to compare the performance of these charts with other.

3.4.1 Distribution assumptions and monitoring statistic

Let T~W(η, β), where η and β are the scale and shape parameters of the Weibull distribution, respectively. If Y=log(T), then it is known that Y~SEV(μ, σ), where μ and σ are the location and scale parameters of the

https://doi.org/10.26219/heal.aueb.5591

smallest extreme value distribution, respectively. The p-quantile of the SEV distribution is given by $y_p = \mu + \sigma s_p$, where $s_p = \log[-\log(1-p)]$ is the p-quantile of the standard SEV distribution SEV(0,1). The p-quantile of the Weibull distribution is $w_p = \exp(y_p)$ and thus, monitoring y_p is equivalent to monitoring w_p.

EWMA control charts for monitoring the Weibull quantiles were constructed by Haghighi et al. [26] based on the distribution of a pivotal quantity conditioned on ancillary statistics. Because of the intractable form of the monitoring statistic, an ancillary statistic was used to help derive analytical forms of the pdf and cdf. In this way, simulation can also be avoided when it comes to ARL if the EWMA control limits are fixed over time. This happens because in that case, the ARL can be computed using a Fredholm integral equation based on the distribution of the monitoring statistic conditioned on ancillary statistics. The ancillary statistics for a SEV distribution is given by $a = \left(a_1 = \frac{y_1 - \hat{\mu}}{\hat{\sigma}}, \dots, a_n = \frac{y_n - \hat{\mu}}{\hat{\sigma}}\right)'$, where $\hat{\mu}, \hat{\sigma}$ are the ML estimates of the SEV parameters μ, σ. Moreover, the control charts that will be discussed, are based on the following statistic, $Q_0 = \frac{\hat{y}_p - y_p^0}{\hat{\sigma}}$, where y_p^0 is the stable-process p-quantile value. The distribution of Q_0 can be approximated by the distribution of $Q_0|a_0$.

3.4.2 ARL and control limits for EWMA charts

The EWMA statistic needed for the construction of the EWMA chart based on the statistic Q_0 is given by $E_i = \lambda Q_{0i} + (1-\lambda)E_{i-1}$, where $0<\lambda\leq 1$ is the smoothing parameter, E_0 is a specified starting value and Q_{0i} is a statistic value in period i, i=1,2,… . Pascual et al. [25] present two approaches to derive the control limits. It should be noted that, the control limits are determined so that, a specified in-control ARL value is achieved. According to the first approach, the control limits are given by

$$UCL = \mu(a_0) + K\sigma(a_0)\sqrt{\frac{\lambda}{2-\lambda}[1-(1-\lambda)^{2i}]}$$

$$CL = \mu(a_0)$$

$$LCL = \mu(a_0) - K\sigma(a_0)\sqrt{\frac{\lambda}{2-\lambda}[1-(1-\lambda)^{2i}]}$$

since $\mathrm{E(E_i)} = \mu(\mathrm{a_0})$ and $\mathrm{Var(E_i)} = \sigma^2(\mathrm{a_0})\sqrt{\frac{\lambda}{2-\lambda}[1-(1-\lambda)^{2i}]}$. Here, it is obvious that, control limits vary with the sampling period I, so the numerical calculation of the ARL is difficult and that leads to simulations for the computation of the control limits of this chart, which will be referred to, as $\mathrm{EWMA_1}$.

The second approach is about choosing fixed values for the control limits. Control limits are determined so that ARL is L_0 when the process is stable with respect to the quantile of interest. In that case the chart will be called $\mathrm{EWMA_2}$. The numerical calculation of the ARL is possible here, by using the Fredholm integral equation. Then, a combination of the fixed control limits is searched, which leads to an ARL-unbiased chart. That combination of the control limits is the one with the lowest maximized ARL value. If that combination exists, then the ARL value that was found, should be the already specified in-control ARL value (L_0).

Whereas the $\mathrm{EWMA_2}$ is ARL-unbiased, $\mathrm{EWMA_1}$ might not be unbiased for relatively small shifts. However, Pascual et al. [25] considered also charts based on Western Electric alarm rules and their comparisons based on ARL performance, show that $\mathrm{EWMA_1}$ and $\mathrm{EWMA_2}$ are recommended to monitor shifts in the quantiles of Weibull or SEV distribution.

3.5 Impact of model misspecification on the exponential EWMA charts

In the article of Pehlivan and Testik [27], the exponential EWMA control chart for the time-between-events (TBE) is proposed for the early detection of process deterioration. The chart's sensitivity in detecting mean shifts is investigated and it is known that this chart is designed under the assumption that the TBE follows the exponential distribution. The TBE mean is assumed to be known, but the exponential assumption is not correct and the

 https://doi.org/10.26219/heal.aueb.5591

case, where the true underlying distribution of TBE is the Weibull distribution is examined.

The lower-sided exponential EWMA chart, which is designed to detect decreases in the mean, is used by Pehlivan and Testik [27] to monitor the process. The chart is obtained by plotting the statistic $Z_t = \min\{B, (1-\lambda) Z_{t-1} + \lambda X_t\}$ against the sample number t, where $\{X_t, i = 1,2,\ldots\}$ is the sequence of the i.i.d. TBE, $0<\lambda\leq 1$, B is a positive boundary and Z_0 is selected to be the in-control mean value (θ_0). The process is out-of-control if $Z_t\leq h$, where h is the control limit, such that $h\leq \theta_0\leq B$. B ensures that Z_t is at most a certain distance away from the control limit and can lead to improvement of the chart's sensitivity for the early detection of mean shifts, whereas a small value of λ can cause the chart to be more sensitive to small mean shifts. The influence of B and λ on the chart is obvious and for this reason, these parameters must be defined, so as, to lead to a robust ARL performance of the chart. In order to evaluate the performance of the chart, measures such as the ARL, the standard deviation of run length and percentiles of the run length distribution are computed under departures from the exponential distribution with the use of the Markov chain method.

According to the findings of the article studied in this section, the exponential EWMA control charts are extremely robust to departures from the assumed exponential distribution of the TBE observations. When the true underlying distribution is Weibull, it was shown that a very small smoothing parameter value such as λ=0.01 or 0.05 provides a very robust performance and B=2 performs well in many cases, too. A study of the ARL values shows that such a chart will not signal a false alarm often under deviations from the assumed distribution (large in-control ARL value) and process deteriorations will be detected very fast when a shift occurs (small out-of-control ARL value). Finally, despite, the fact that departures from the assumed exponential distribution are a concern, the exponential EWMA control chart is suggested for use in practical application because of its robustness and its good performance in the early detection of process deteriorations.

https://doi.org/10.26219/heal.aueb.5591

3.6 A study of EWMA chart with transformed exponential data

In the article of Ji Ying Liu et al. [28], a new EWMA chart is proposed, in order to monitor exponentially distributed time-between-events (TBE) data. Before the chart is constructed, the TBE data are transformed to approximate the normal distribution using the double square root (SQRT) transformation. The exponential data X_t are transformed to approximately normal data Y_t by using the transformation $y=x^{0.25}$, $x\geq 0$ and the control limits of the EWMA chart based on the transformed data are given by

$$UCL = \mu_0 + L\sigma_0\sqrt{\frac{\lambda}{(2-\lambda)}[1-(1-\lambda)^{2t}]}$$

$$CL = \mu_0$$

$$LCL = \mu_0 - L\sigma_0\sqrt{\frac{\lambda}{(2-\lambda)}[1-(1-\lambda)^{2t}]}$$

where $0<\lambda\leq 1$ is the smoothing factor, L is a design parameter, t is the sample number (or time) and μ_0, σ_0, when they are unknown, can be estimated by $\hat{\mu}_0 = \bar{y} = \sum_{t=1}^{n} y_t$ and $\hat{\sigma}_0 = \sqrt{\frac{1}{n-1}[\sum_{t=1}^{n}(y_t-\bar{y})^2]}$, respectively.

The performance of the EWMA chart with transformed exponential data is evaluated by the ARL, which is computed with the use of the Markov chain approach. The in-control ARL (ARL_0) of this chart depends on the values of L and λ and does not depend on the exponential mean, whereas the out-of-control ARL (ARL_1) depends on the shift level and on L and λ. The study of Ji Ying Liu et al. [28] shows that the optimal value of λ, when the amount of the shift is fixed, decreases with the increase of ARL_0.

The EWMA chart with transformed exponential data is compared to the X-MR chart and to the exponential EWMA chart in terms of ARL values and its robustness is examined when the TBE distribution is Weibull. The proposed chart in the article of Ji Ying Liu et al. [28] has a good performance in monitoring exponentially distributed TBE data and it is also more sensitive in detecting both process improvements and process deteriorations than the X-MR chart. Moreover, in terms of ARL, the exponential EWMA chart has similar performance with the EWMA chart with transformed exponential data.

 https://doi.org/10.26219/heal.aueb.5591

However, sometimes the exponential assumption of the TBE distribution is not valid and the Weibull distribution is a better alternative. Due to the transformation of the TBE data to normal, the proposed chart here is also robust for the case when the underlying distribution is Weibull. It is shown that the smaller the smoothing factor λ, the better the robustness of the EWMA chart. Especially when λ=0.05, the chart is very robust to Weibull distribution for the ARL_0 and it is more sensitive for Weibull data when the shape parameter is η>1 than it is for the exponential data (η=1) both for upward and downward shifts. Furthermore, its performance is better when the shape parameter is larger. Finally, it is observed that the EWMA chart with transformed exponential data is not very sensitive to the parameters L, λ and thus, the procedure of selecting these parameters is easier than the one of the exponential EWMA.

3.7 EWMA control charts based on the moving average statistic and $\ln S^2$

In the article of Wang and Cheng [29], an EWMA control chart based on the moving average statistic and one based on $\ln S^2$ are proposed, in order to monitor the process mean and variability when the underlying distribution is Weibull. These charts are used after the transformation of the Weibull data to standard normal and their performance is evaluated through the ARL values, which are computed using the Markov chain approach. Finally, some comparisons are made between them and the other control charts used for monitoring the mean and process variability.

3.7.1 Weibull data transformation to standard normal data

In order to introduce the new EWMA control charts, the idea of Wang and Cheng [29] was firstly to transform the Weibull data to standard normal data and then to monitor the process by applying the conventional design methods, used on the EWMA chart, for these normal data. For the transformation, the suggestion of Faraz et al. [14] was used, according to which, the error function can be applied to transform normal to Weibull data and the inverse error function to transform Weibull to standard normal data. Thus, here are needed the following: $x = \sqrt{2}\mathrm{erf}^{-1}\left[1 - 2e^{-(\frac{y}{\theta})^{\beta}}\right]$ and $x \sim N(0,1)$, where $\mathrm{erf}(z) = \frac{1}{\sqrt{\pi}}\int_{-z}^{z} e^{-u^2}du$, and β and θ are the shape and scale parameters, respectively, of the Weibull distribution.

3.7.2 EWMA-MA and EWMA-lnS² control charts

The moving average (MA) statistic of span size w at time i is computed by including sequentially a new subgroup in the averaging and removing the oldest subgroup in the span. It is given by

$$\begin{cases} MA_i = \dfrac{\bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_{i-1}}{i}, i < w \\ MA_i = \dfrac{\bar{x}_i + \bar{x}_{i-1} + \cdots + \bar{x}_{i-w+1}}{i}, i \geq w \end{cases}$$

The mean and the variance of the MA statistic are given by $E(MA_i) = 0$ for $i \geq 1$ and $Var(MA_i) = \frac{1}{nw}$ for $i \geq w$.

The EWMA statistic for monitoring the process mean based on the MA is given by $Z_i = \lambda MA_i + (1-\lambda)Z_{i-1}$, where $Z_0 = 0$ and the central line and control limits are given by

$$UCL_{MA} = L\sqrt{\frac{\lambda}{nw(2-\lambda)}}$$

$$CL_{MA} = 0$$

$$LCL_{MA} = -L\sqrt{\frac{\lambda}{nw(2-\lambda)}}$$

where L is a control chart constant under a desired ARL value and $Var(Z_i) = \frac{\lambda}{nw(2-\lambda)}$ for $i \geq w$.

The EWMA statistic for monitoring the process variance based on $\ln S^2$ is given by $EWMA_i = \max\{\lambda \ln S_i^2 + (1-\lambda)EWMA_{i-1}, 0\}$, where $EWMA_0 = 0$ and the upper control limit is given by

 https://doi.org/10.26219/heal.aueb.5591

$$UCL_{S^2} = K\sqrt{\frac{\lambda}{2-\lambda}\left(\frac{2}{n-1}+\frac{2}{(n-1)^2}+\frac{4}{3(n-1)^3}-\frac{16}{15(n-1)^5}\right)}$$

where K is a control chart constant under a desired ARL value and the asymptotic variance $\mathrm{Var}(\mathrm{EWMA_i}) = \frac{\lambda}{2-\lambda}\left(\frac{2}{n-1}+\frac{2}{(n-1)^2}+\frac{4}{3(n-1)^3}-\frac{16}{15(n-1)^5}\right)$.

The values of the design parameters L and λ are chosen so that they give the shortest out-of-control ARL value based on an already specified fixed in-control ARL value. The ARL is calculated using the Markov chain method, since the points plotted on the EWMA-MA control chart are not independent.

3.7.3 Comparisons

The performances of the two new proposed control charts are compared to other charts that are also used to detect such shifts. The EWMA-MA chart is compared to the $\overline{X}$ and EWMA-$\overline{X}$ charts for monitoring the process mean, whereas the EWMA-lnS^2 is compared to the S^2 and S^2-MA charts for monitoring the process variability. The comparisons are made in terms of the ARL values and the results of Wang and Cheng [29] show that the EWMA-MA outperforms the $\overline{X}$ and EWMA-$\overline{X}$ control charts for detecting shifts in the process mean. Also, the EWMA-lnS^2 outperforms the S^2 and S^2-MA control charts for detecting shifts in the process variability.

3.8 The impact of Weibull data and autocorrelation on Shewhart and EWMA charts

According to the article of Black et al. [36], it is common in real-world situations for autocorrelated and Weibull data to be found. In that case, the assumption of independent and identically distributed normal data is not valid and thus, the robustness and performance of Shewhart and EWMA control charts could be affected. For this reason, their robustness and performance are examined and compared in the case where, the process generates Weibull

 https://doi.org/10.26219/heal.aueb.5591

and/or autocorrelated data. Also, the experimentation consists of cases with various levels of process shift, various autocorrelation coefficients and Weibull shape parameters.

A comparison between the two charts has already been made but only with autocorrelated data and with the most widely used techniques to be: the use of autoregressive models, ignoring autocorrelation, plotting residuals and adjusting control limits. However, the study of Black et al. [36] was about both violations of the normality assumption in the light of Weibull data and autocorrelation.

Each case of the experimentation consists firstly of the retrospective stage, during which, control limits are computed based on generated data. Secondly, in the prospective stage, more data are generated until the occurrence of an out-of-control point. Then, the control chart's performance is evaluated based on ARL values. They showed that, in general, the EWMA is preferable to the Shewhart chart when data are Weibull distributed and autocorrelated. More specifically, the EWMA outperforms the Shewhart in 62% of the cases and especially when autocorrelation effects are low to moderate. On the other hand, the Shewhart outperforms the EWMA in 35% of the cases and especially when autocorrelation is high and the process shift is either zero or high. The rest 3% of the cases are inconclusive.

3.9 An attribute sampling plan using EWMA for Weibull distribution

Acceptance sampling is a method of statistical quality control, according to which, a lot or batch will be accepted or rejected based on the inspection of a sample from this batch. In the article of Aslam et al. [37] is stated that there are two basic categories of acceptance sampling, attribute and variable sampling plans. In attribute sampling plans, a product is just classified as conforming or nonconforming, whereas in variable sampling plans, the decision to accept or reject a product is based on measurements of the quality characteristics. There are also many kinds of sampling plans, such as single or double sampling plan and group, repetitive group or skip-lot sampling plans. In every case it is of high importance to satisfy both producer's and customer's risk. The producer's risk is the probability to reject a good lot, whereas the customer's risk is the probability to accept a bad lot. Aslam et al. [37] also proposed a new attribute acceptance sampling plan based on the EWMA statistic when the product lifetime follows the Weibull distribution and a time-truncated test is applied to obtain observations.

The EWMA statistic that is used, is given by $\mathrm{EWMA_t} = \lambda S + (1 - \lambda)\mathrm{EWMA_{t-1}}$, where S is the observed number of failures until time t_0, which is the specified time of the end of the time-truncated life test. If $\mathrm{EWMA_t} > \mathrm{c}$, then the lot is rejected, whereas, if $\mathrm{EWMA_t} \leq \mathrm{c}$, then the lot is accepted, and c is the acceptance number of the sampling plan. The probability of accepting a lot is computed and the parameters of the plan (sample size and acceptance number of the sampling plan) are chosen so that, the sample size is minimized and the consumer's and producer's risk are satisfied. Then, the assumption of the Weibull distribution is considered and applied to the proposed plan.

Finally, a comparison was made between the efficiency of the proposed plan and the conventional single sampling plan (SSP) without using the EWMA statistic. The SSP is a special case of the proposed plan when λ=1. The results show that the proposed plan requires a smaller sample size and acceptance number than the SSP. That is its main advantage and it is also what makes a sampling plan preferable. Thus, the proposed plan is much better than the SSP according to the conclusions of Aslam et al. [37] and for this reason, it is recommended for products' inspection in industry.

https://doi.org/10.26219/heal.aueb.5591

CHAPTER 4

CUMULATIVE SUM CONTROL CHARTS

4.1 Introduction

The cumulative sum (CUSUM) control chart is especially used when the detection of small process shifts is of interest. It is a very effective alternative to the Shewhart control chart and it offers important performance improvement relative to Shewhart. A CUSUM's most common implementation is in the tabular form. It may be constructed both for individual observations and for the averages of rational subgroups. In this chapter, various CUSUM control charts are presented to monitor Weibull distributed processes. The main issues are monitoring Weibull time-between-events (TBE) and CUSUM designs for transformed data. Finally, performance measures are computed and compared in order to evaluate the chart's performance and also a mixed EWMA-CUSUM control chart is discussed, where the EWMA and the CUSUM statistics are combined for the Weibull distribution, as it can lead to better performance

4.2 A CUSUM design for transformed exponential data

In the article of Alwan [38] a CUSUM chart is proposed for transformed exponential data. If the random variable X follows the Weibull distribution with shape parameter β and scale parameter η, then for β=1, the distribution reduces to exponential. So, it is obvious that the exponential is a special case of the Weibull distribution. The proposed chart is based on a transformation applied to the exponential data in order to achieve approximate normality. Then, the standard control charts, which are based on the normality assumption, can be applied for the approximate normal data.

The process, which is followed for the transformation of the data, is the following: If $X \sim W(\eta, \beta)$, then for the transformed variable $Y = X^p$, $Y \sim W(\eta^p, \frac{\beta}{p})$. So, a variable that follows the exponential distribution, i.e. $W(\eta,1)$, after the transformation it follows the $W(\eta^p, \frac{1}{p})$. According to Nelson [39], $W(\alpha,3.6)$ has zero skewness and kurtosis value close to the one of the normal distribution. For p=0.2777, 1/p=1/0.2777 is nearly equal to 3.6. So, the power transformation that will be used is p=0.2777. Thus, if X follows the exponential distribution, then, $Y = X^{0.2777}$ follows an approximate normal distribution. The mean and standard deviation of Y are given by $\mu_Y = 0.901119\beta^{0.2777}$ and $\sigma_Y = 0.277956\beta^{0.2777}$, respectively.

The CUSUM chart is usually implemented in the tabular form. The lower one-sided cumulative statistic is used to detect a downward shift in the process mean and for the transformed exponential data, it is given by

$$C_i^- = \min[0, x_i^{0.2777} - (\mu_Y^0 - k^- \sigma_Y^0) + C_{i-1}^-]$$

whereas, the upper one-sided cumulative statistic is used to detect an upward shift in the process mean and for the transformed exponential data, it is given by

$$C_i^+ = \min[0, x_i^{0.2777} - (\mu_Y^0 + k^+ \sigma_Y^0) + C_{i-1}^+]$$

where $\mu_Y^0 = 0.901119\beta_0^{0.2777}$, $\sigma_Y^0 = 0.277956\beta_0^{0.2777}$,$\beta_0$ is the in-control value of the process mean and k^-, k^+ are reference values.

For the lower one-sided CUSUM, an out-of-control signal happens if $C_i^- \leq -h^-\sigma$ and for the upper one-sided CUSUM, an out-of-control signal happens if $C_i^+ \geq h^+\sigma$, where σ is the process standard deviation. The value of h^- or h^+ is selected given a value of k^- or k^+, so that a specified in-control ARL value is achieved. ARL is estimated through Monte Carlo simulation and its values for the proposed CUSUM chart show that it performs well in detecting mean shifts.

https://doi.org/10.26219/heal.aueb.5591

4.3 CUSUM control charts for monitoring Weibull distributed time between events

In the article of Shafae et al. [40], the development of three CUSUM charts is described to monitor Weibull distributed time between events (TBE) observations and their performance is evaluated and compared. These charts are the Weibull CUSUM (WCUSUM) chart, the exponential CUSUM (ECUSUM) chart and an adjusted ECUSUM (AECUSUM) chart. The main interest is about detecting deteriorations in the process, which is indicated by a decrease in the TBE mean. So, the charts are constructed to monitor small decreases in the TBE mean. The performance comparison is based on the ARL and average time to signal (ATS) values.

4.3.1 The Weibull CUSUM chart

The WCUSUM is constructed based on the assumption that the underlying TBE distribution is Weibull. Let Y be a random variable that represents the time duration between two successive events. It is assumed that Y follows a Weibull distribution with shape parameter β and scale parameter η. Monitoring for a decrease in η is equivalent to monitoring for a decrease in the TBE mean of the Weibull distribution if β is fixed. The WCUSUM statistic at time i in order to detect that decrease is given by

$$C_{W,i} = \max(0, C_{W,i-1} - y_i^{\beta} + k_W)$$

where $C_{W,0} = 0$, $i = 1,2,\dots$ and k_W is a reference parameter given by $k_W = \frac{\beta \ln(\frac{\eta_1}{\eta_0})}{(\eta_0^{-\beta} - \eta_1^{-\beta})}$, η_0, η_1 are the in-control and out-of-control values, respectively. When $C_{W,i} \geq h_W$, there is an out-of-control signal, where for a given value of k_W, h_W is selected so as, to achieve a desired in-control ARL value.

When the in-control values of the Weibull parameters are unknown, then they are estimated by maximum likelihood estimation.

https://doi.org/10.26219/heal.aueb.5591

4.3.2 The exponential CUSUM chart

The ECUSUM chart is constructed based on the assumption that the underlying TBE distribution is exponential and its robustness will be examined in the section 4.3.4 for the case when the true underlying distribution is Weibull with shape parameters varying between 1 and 4. Let X be a random variable that represents the time duration between two successive events. It is assumed that X follows an exponential distribution with parameter μ>0. μ is also the TBE mean value in which we are interested in detecting decreases. The ECUSUM statistic at time i in order to detect the decreases is given by

$$C_{e,i} = \max(0, C_{e,i-1} - x_i + k_e)$$

where $C_{e,0} = 0$, $i = 1,2, \ldots$ and k_e is a reference parameter given by $k_e = \frac{\ln(\frac{\mu_1}{\mu_0})}{(\mu_0^{-1} - \mu_1^{-1})}$, μ_0, μ_1 are the in-control and out-of-control mean values, respectively.

When $C_{e,i} \geq h_e$, there is an out-of-control signal and h_e is selected based on given k_e and in-control ARL values.

4.3.3 The adjusted exponential CUSUM chart

The ECUSUM chart does not account for the true underlying distribution, which is Weibull. So, an adjustment is applied in order to account for it. This adjustment allows for the AECUSUM to be directly comparable to the WCUSUM chart. For the design of the AECUSUM chart, the control limit obtained for the ECUSUM chart is adjusted so that the desired in-control ARL value is achieved and the charts become comparable. So, h_{ae}, which is the AECUSUM control limit is obtained through a simulation-based technique.

When $C_{e,i} \geq h_{ae}$, there is an out-of-control signal and h_{ae} is selected based on the reference parameter k_e, on the in-control ARL value and on the simulated TBE observations, which follow the true Weibull distribution.

https://doi.org/10.26219/heal.aueb.5591

4.3.4 Comparisons

For the value of the shape parameter β=1, all the three charts are equivalent. However, for β≠1, the in-control ARL values of the ECUSUM chart vary significantly from the specified value. The in-control ARL and ATS values vary from very small to extremely large values depending on the value of the shape parameter, which is not reasonable. So, the ECUSUM is not robust to departures from the exponential distribution and it cannot be compared to the other two charts, as it can lead to misleading conclusions because of the different in-control ARL values of the three charts when monitoring Weibull distributed data.

Comparisons between the WCUSUM and AECUSUM charts based on the ARL and ATS, showed that the WCUSUM's performance is slightly better or very similar to the AECUSUM's. Thus, for Weibull distributed TBE data, using the WCUSUM chart is recommended.

4.4 The effect of estimation error on RAST CUSUM chart performance

According to the article of Zhang et al. [41], in industrial processes, all the units are assumed to have homogeneous characteristics. However, when it comes to healthcare settings, it is obvious that patients are heterogeneous according to their physical conditions. So, in order to take the patient heterogeneity into account, risk-adjusted control charts were proposed. Risk-adjusted CUSUM charts are either Bernoulli or survival time CUSUM charts based on the monitored variables (binary outcome or survival times). In Phase I of the control chart, the unknown parameters must be estimated based on historical data from the in-control process. The error of this parameter estimation influences the chart performance and the effect of this error on risk-adjusted survival time (RAST) CUSUM chart in continuous time is studied in the article of Zhang et al. [41].

 https://doi.org/10.26219/heal.aueb.5591

The statistic used on the RAST CUSUM chart is $S(t) = R(t) - \min_{s \le t} R(s)$, where R(t) is the log-likelihood ratio test statistic for in-control versus out-of-control. The risk adjustment is achieved using an accelerated failure time (AFT) regression model to account for patient heterogeneity. The anticipated survival time distribution is assumed to be Weibull. It also suggested that the median run length (medRL) is used as a performance metric because it is easy to compute through simulation. The ARL cannot be used, since the CUSUM's run length distribution in the case studied, does not follow the Markovian assumptions and so, the ARL would be hard to approximate. Also, the standard deviation of the medRL will be used as a performance metric in order to see the variation in the chart performance.

The effect of estimation error on RAST CUSUM charts is studied for different sample sizes, specified in-control medRL, adverse event rate and patient variability. The results of the study of Zhang et al. [41], show that the sample size affects significantly the variability of the CUSUM chart performance. Specifically, the variation in the medRLs increases as the sample size increases. Also, an increase in the specified in-control medRL ($medRL_0$) causes an increase of the same amount to the standard deviation of the true $medRL_0$. The adverse event rate has the most significant effect on estimation error. When it doubles, the standard deviation of the true $medRL_0$ decreases by a factor of about $\sqrt{2}$ and this decrease is more significant for the true out-of-control medRL ($medRL_1$). The influence of patient variability on the CUSUM chart performance is, in general, not significant. Low variation has little effect on the standard deviation of medRLs and high variation has a slightly bigger impact.

4.5 A mixed EWMA-CUSUM control chart

It is known that the EWMA and CUSUM control charts are appropriate for detecting small shifts in the manufacturing process. Abbas et al. [42] suggested using a combination of the two charts to monitor the process, the mixed EWMA-CUSUM chart. In the article of Aslam [43] this chart is applied

https://doi.org/10.26219/heal.aueb.5591

to Weibull data, which means that the EWMA and CUSUM statistics are combined for the Weibull distribution. The mixed chart's performance is evaluated through the ARL.

The mixed chart detects small shifts in the time between events (TBE) mean. A change in the TBE mean corresponds to a shift in the scale parameter of the Weibull distribution when the shape parameter is known. A decrease in the TBE means deterioration of the process. According to Aslam [43], the EWMA-CUSUM statistic is derived through the following process: Let Y be a Weibull distributed quality characteristic with shape parameter β and scale parameter η, then, it is known that, Y_i^{β} follows the exponential distribution with mean η^{β}. The EWMA statistic at time i, is given by

$$Q_i = \lambda Y_i^{\beta} + (1-\lambda)Q_{i-1}$$

where 0<λ≤1 is the smoothing constant. Then, the CUSUM-EWMA statistic at time i is given by

$$C_{W,i} = max[0, C_{W,i-1} - Q_i + k_W]$$

where $C_{W,0} = 0$, $i = 1,2, \dots$ and the reference parameter $k_W = \frac{\beta \ln(\frac{\eta_1}{\eta_0})}{(\eta_0^{-\beta} - \eta_1^{-\beta})}$, where $\eta_0, \eta_1$are the in-control and out-of-control scale parameters. The control limit h_W of the mixed EWMA-CUSUM chart is given by Lucas [44] as $h_W = h_t \eta_0^{\beta}$ and the process is out-of-control if $C_{W,i} > h_W$.

The computation of the ARL, in order to evaluate the chart's performance, is achieved through Monte Carlo simulation. According to the results of Aslam [43] and based on the ARL, the proposed mixed control chart performs well in detecting the shift in the process and it also performs better compared to already existing control charts that could be used in such cases. The mixed EWMA-CUSUM control chart can be used in health department for monitoring infections and also to monitor external shocks to system and failure rate in mechanical systems.

https://doi.org/10.26219/heal.aueb.5591

https://doi.org/10.26219/heal.aueb.5591

CHAPTER 5

ILLUSTRATIVE EXAMPLES

5.1 Introduction

In chapter 2, the WV and the SC methods, which use asymmetric control limits, were proposed so that the standard Shewhart $\overline{X}$ and R control charts can be used without giving misleading results, even when the quality variable follows the Weibull distribution. In chapters 3 and 4, it was seen that a specific transformation of exponential or Weibull data allows for the EWMA and CUSUM control charts to be able to be used effectively. In this chapter, three examples will be presented that contain $\overline{X}$**,** R, EWMA and CUSUM control charts based on the above methods of the previous chapters, the data come from simulation and it will be demonstrated how these charts detect the different shifts in the process mean. All the computations have been conducted using the R programming language and for the charts' presentation the library 'qcc' has been used.

5.2 $\overline{X}$ and R control charts

Let us assume that we intend to control the quality variable X, which follows the Weibull distribution with shape parameter β and scale parameter η=1. A scale parameter of η=1 is chosen for convenience since the skewness depends on the shape parameter only. The quality variable will be controlled using the WV and the SC methods, which will be compared based on the Type I risks. The two methods will be compared for subgroups of size n=2, 3, 5 and for values of the shape parameter β=2.15, 1.20, 0.86, which correspond to the values k=0.5, 1.5, 2.5 of skewness, respectively.

https://doi.org/10.26219/heal.aueb.5591

In order to construct the $\overline{X}$ and R control charts, the control charts constants for each method had to be defined. According to Derya and Canan [2], for the values of β, k that we chose, the constants for the computation of the control charts' limits are given in the following tables:

WV Method					
β	**k**	W_U	W_L	V_U	V_L
2.15	0.50	1.97	1.83	3.43	0.00
1.20	1.50	2.45	1.87	4.06	0.00
0.86	2.50	2.74	1.96	5.23	0.00
SC Method					
β	**k**	A_U^*	A_L^*	D_4^*	D_3^*
2.15	0.50	2.20	1.62	4.26	0.00
1.20	1.50	2.78	1.25	4.95	0.00
0.86	2.50	3.22	1.23	5.66	0.00

Table 1. $\overline{X}$ and R charts constants for n=2

WV Method					
β	**k**	W_U	W_L	V_U	V_L
2.15	0.50	1.08	0.99	2.72	0.00
1.20	1.50	1.32	1.01	3.36	0.00
0.86	2.50	1.49	1.08	4.16	0.00
SC Method					
β	**k**	A_U^*	A_L^*	D_4^*	D_3^*
2.15	0.50	1.16	0.90	3.12	0.00
1.20	1.50	1.46	0.73	3.82	0.00
0.86	2.50	1.71	0.65	4.53	0.00

Table 2. $\overline{X}$ and R charts constants for n=3

https://doi.org/10.26219/heal.aueb.5591

WV Method					
β	**k**	**W_U**	**W_L**	**V_U**	**V_L**
2.15	0.50	0.61	0.56	2.25	0.00
1.20	1.50	0.74	0.57	2.82	0.00
0.86	2.50	0.81	0.58	3.52	0.00
SC Method					
β	**k**	**A_U^***	**A_L^***	**D_4^***	**D_3^***
2.15	0.50	0.65	0.53	2.45	0.15
1.20	1.50	0.78	0.45	3.10	0.15
0.86	2.50	0.92	0.40	3.75	0.06

Table 3. $\overline{X}$ and R charts constants for n=5

Firstly, 30 subgroups of size n from the Weibull distribution with shape parameter β and scale parameter η=1 were generated to compute the control limits. Then 100 subgroups of size n from the same distribution with the same parameters are generated, to check if their $\overline{X}$ and R, fall outside the control limits of the corresponding charts. This procedure was repeated 10000 times and finally, the Type I risk is estimated as the percentage of the $\overline{X}$ and R that fell outside the control limits. The results of this procedure are presented on Table 4 below.

			n=2		**n=3**		**n=5**	
β	**k**	**Methods**	$\overline{X}$ chart	R chart	$\overline{X}$ chart	R chart	$\overline{X}$ chart	R chart
2.15	**0.50**	WV	0.0054	0.0109	0.0038	0.0053	0.0032	0.0030
		SC	0.0043	0.0021	0.0034	0.0013	0.0027	0.0017
1.20	**1.50**	WV	0.0071	0.0142	0.0051	0.0083	0.0033	0.0057
		SC	0.0061	0.0052	0.0042	0.0036	0.0035	0.0052
0.86	**2.50**	WV	0.0121	0.0132	0.0094	0.0110	0.0068	0.0081
		SC	0.0072	0.0099	0.0062	0.0075	0.0049	0.0064

Table 4. Type I risks based on two methods when the process distribution is Weibull with shape parameter β and corresponding skewness k.

It is obvious that the SC method has a smaller Type I risk than the WV method for both the $\overline{X}$ and the R chart and for every value of skewness. This conclusion is also confirmed on Figures 7, 8 and 9 below.

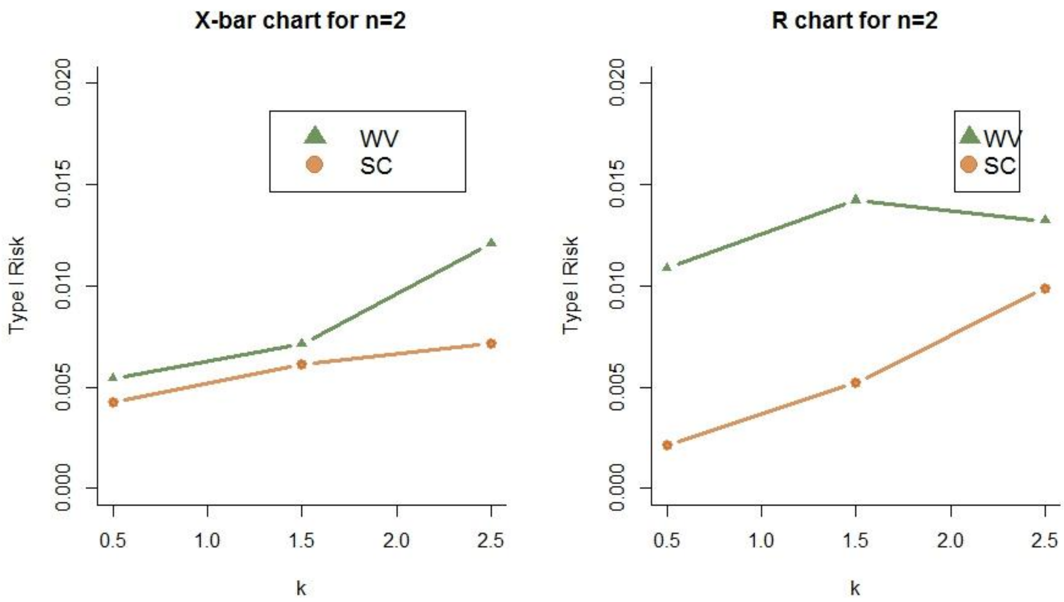


Figure 7. Type I risks of $\overline{X}$ and R charts for n=2

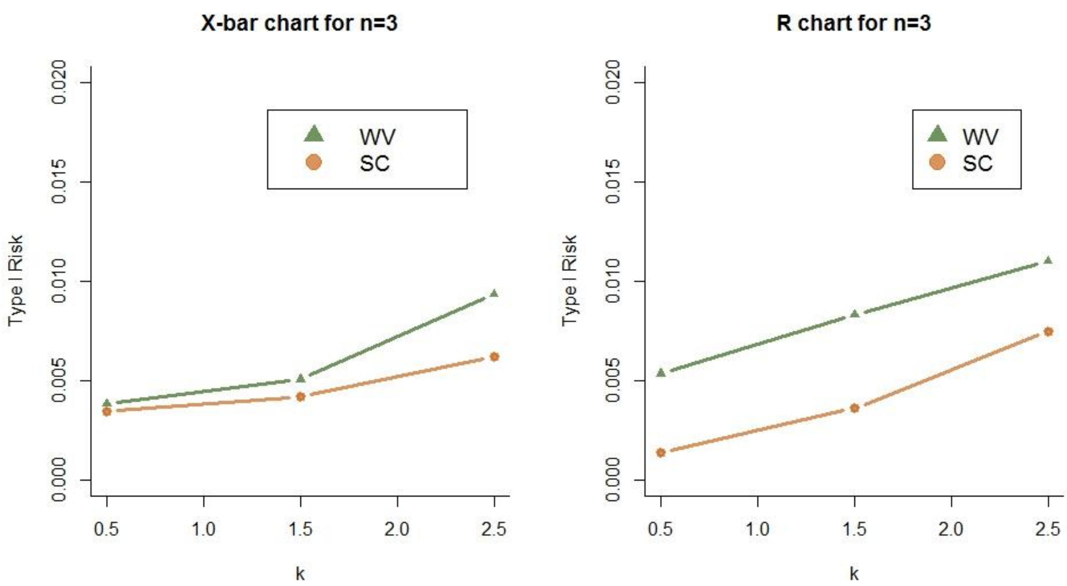


Figure 8. Type I risks of $\overline{X}$ and R charts for n=3

https://doi.org/10.26219/heal.aueb.5591

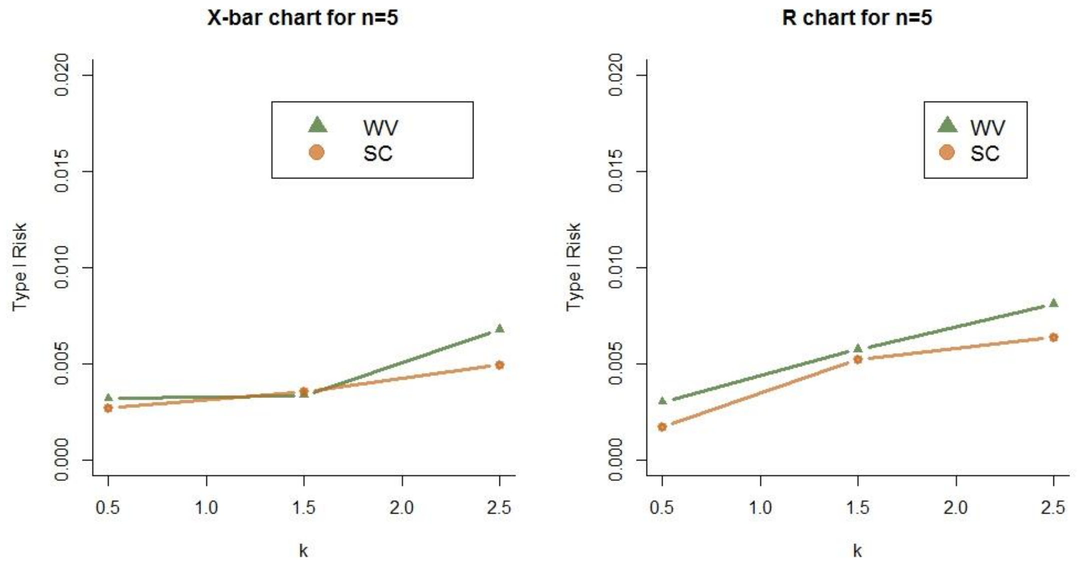


Figure 9. Type I risks of $\overline{\mathrm{X}}$ and R charts for n=5

According the results of both Table 4 and the above figures, the difference in Type I risks between the two methods is bigger in the R chart than in the $\overline{\mathrm{X}}$ chart. Moreover, the Type I risks of the SC method are closer to the conventional 0.27% than those of the WV method and in most cases, that is more pronounced when the skewness increases.

5.3 EWMA control charts

As seen in Chapter 3, Section 3.6 and according to the article of Ji Ying Liu et al. [28], the conventional EWMA control chart can be used to monitor exponentially distributed TBE data, as long as these data approximate the normal distribution. This is achieved by using the double square root (SQRT) transformation.

5.3.1 The SQRT transformation for exponential TBE data

Let us assume that the process we want to monitor, when it is in-control, it follows the exponential distribution with mean equal to 1 and after

https://doi.org/10.26219/heal.aueb.5591

the SQRT transformation, the data approximate the normal distribution. Firstly, a table of out-of-control ARL (ARL_1) values is presented for different combinations of the λ, L parameters and for different shift levels by using simulation. The combinations of λ, L are chosen so that according to Ji Ying Liu et al. [28], an in-control ARL (ARL_0) value of 500 is achieved.

Shift	**L**	**2.279**	**2.611**	**2.799**	**2.921**	**2.953**	**2.938**
(β_1/β_0)	**λ**	**0.02**	**0.05**	**0.1**	**0.2**	**0.4**	**0.5**
0.2		10.62	9.73	8.97	8.95	36.28	43.49
0.5		37.41	33.22	53.42	90.38	127.06	141.10
0.7		85.84	106.71	131.75	163.54	176.86	186.88
1.2		122.92	138.40	159.08	197.76	235.08	248.84
1.5		58.64	56.34	62.83	78.58	93.04	110.92
2.0		22.95	21.43	20.69	25.95	37.59	41.27
3.5		9.19	8.81	7.62	6.52	6.48	6.82
5.0		7.18	6.16	4.92	4.5	4.02	3.80

Table 5. The out-of-control ARLs of some selective EWMA charts with transformed exponential data (in-control ARL=500)

Let us now assume that 20 TBE data are generated from an exponential distribution with mean equal to 1 when the process is in-control. Then, 50 points are generated from an out-of-control process which follows an exponential distribution with mean equal to 0.2 so the shift level is 0.2. The ARL_0 value is set to be 500, so the parameters of the EWMA chart are λ=0.20 and L=2.921. After transforming the data to approximate normality, it is obvious that the EWMA chart performs well as it detects the shift at the 26th point, as can be seen in Figure 10 and confirmed by Table 5.

https://doi.org/10.26219/heal.aueb.5591

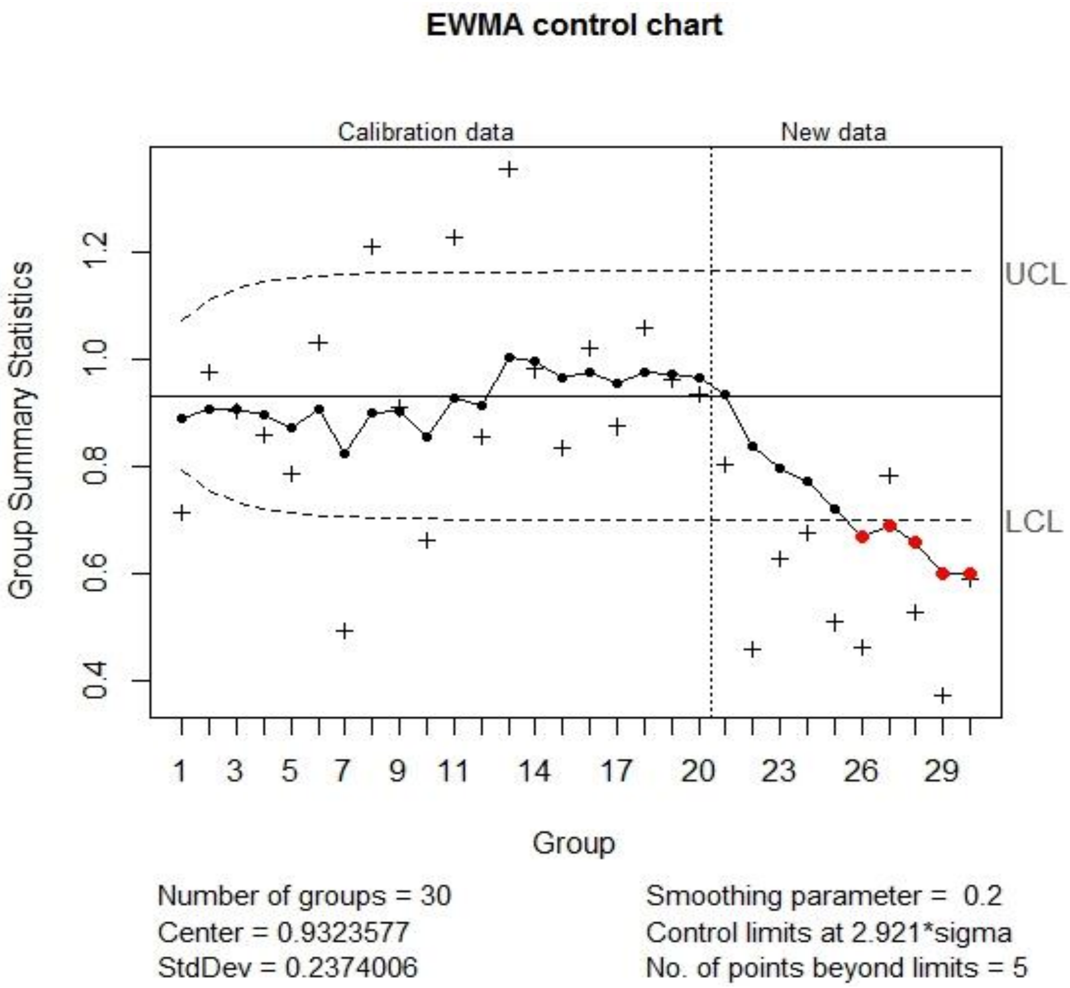


Figure 10. The EWMA chart with transformed exponential data

It would be interesting to see how the standard Shewhart control chart would perform in this example. Of course, it can be used as the data are approximately normal after the transformation. It can be seen in Figure 11 below, that the performance of the Shewhart control chart is not good as it takes 55 points to detect the shift. That was expected due to the small size of the shift.

https://doi.org/10.26219/heal.aueb.5591

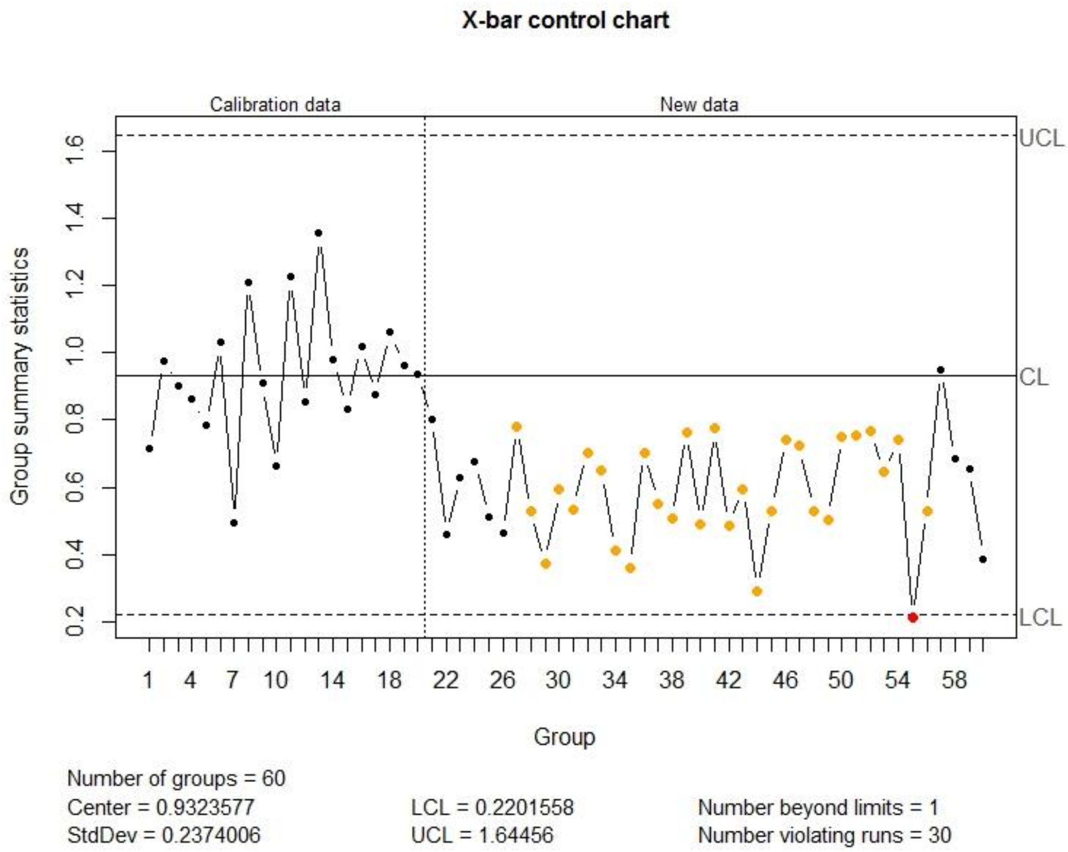


Figure 11. The Shewhart $\overline{X}$ chart with transformed exponential data

5.3.2 The EWMA control charts with Weibull transformed data

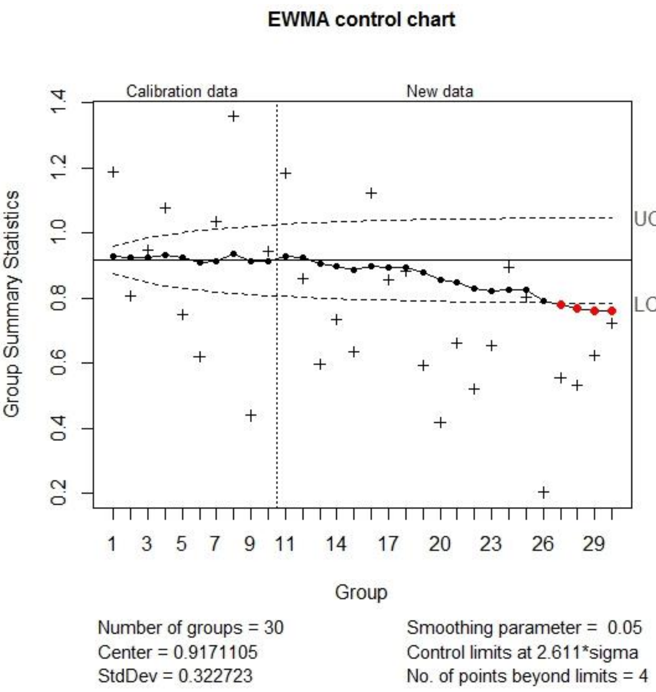
The Weibull distribution is more widely used to model TBE data compared to the exponential distribution, which is a special case of the Weibull. However, a Weibull distribution with shape parameter β and scale parameter η, remains a Weibull distribution with parameters $\beta^{0.25}$ and 4η, respectively, after the double SQRT transformation. So, it is interesting to see how the EWMA chart with the double SQRT transformation would react when the actual TBE data follow the Weibull distribution.

In the following example, the shape parameter β will be held fixed, whereas the scale parameter η will change. This is reasonable since in practical applications the scale parameter is more likely to change due to assignable causes, while the shape parameter is more related to the natural properties of the system and is rather stable. So, the shift in the process mean

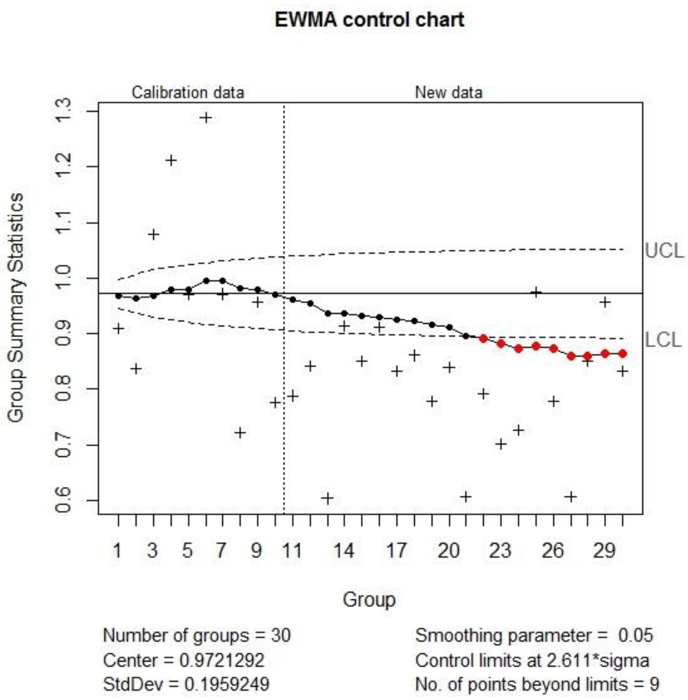
https://doi.org/10.26219/heal.aueb.5591

depends on the shift of the scale parameter $(\frac{\mu_1}{\mu_0} = \frac{\eta_1}{\eta_0})$. In the example both upward and downward shifts will be examined. According to Ji Ying Liu et al. [28], the chart is more sensitive for Weibull distributed data with β>1 than for exponential data (β=1), for both upward and downward shifts and the following results, confirm their conclusions.

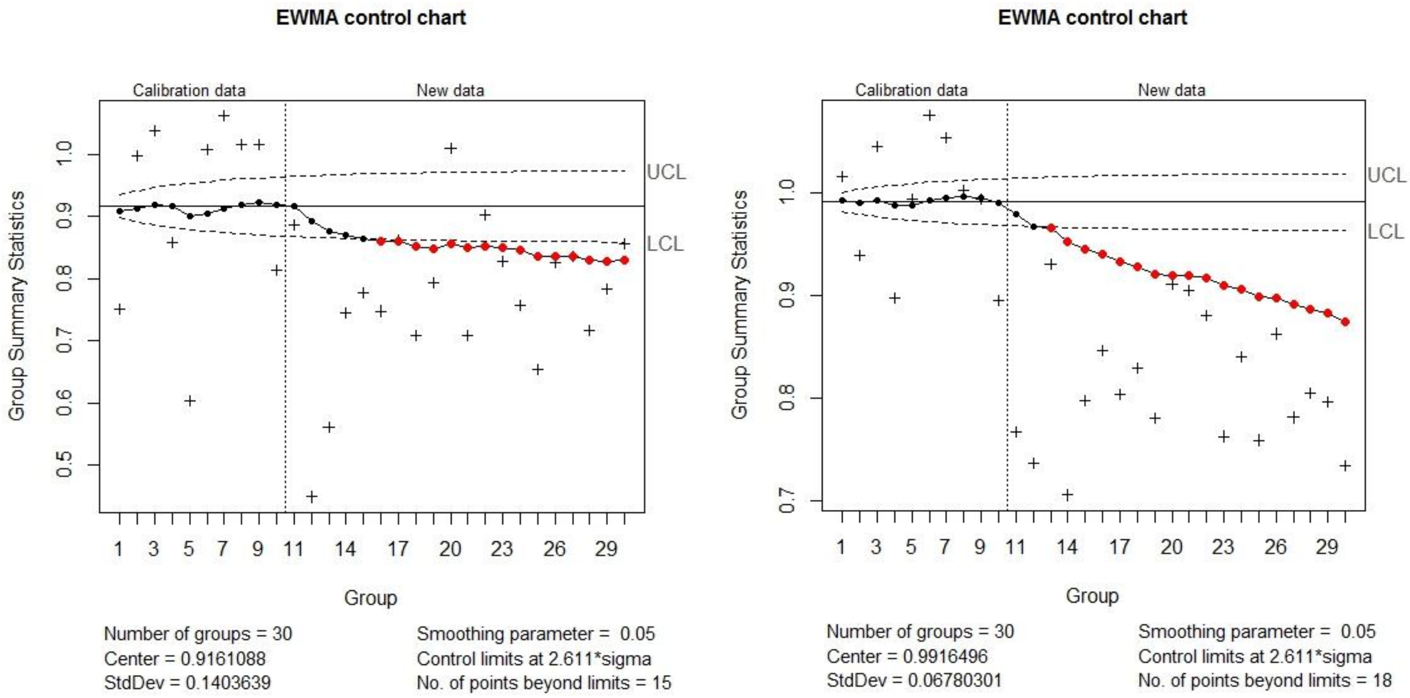
Let us now assume that 10 TBE data are generated from the Weibull(β,η_0) distribution, where η_0=1 when the process is in-control. Then, 20 points are generated from an out-of-control process which follows the Weibull(β,η_1) distribution, where η_1=0.5 so the level of the downward shift is 0.5. The parameters of the EWMA chart are set to be λ=0.05 and L=2.611.



a) β=1



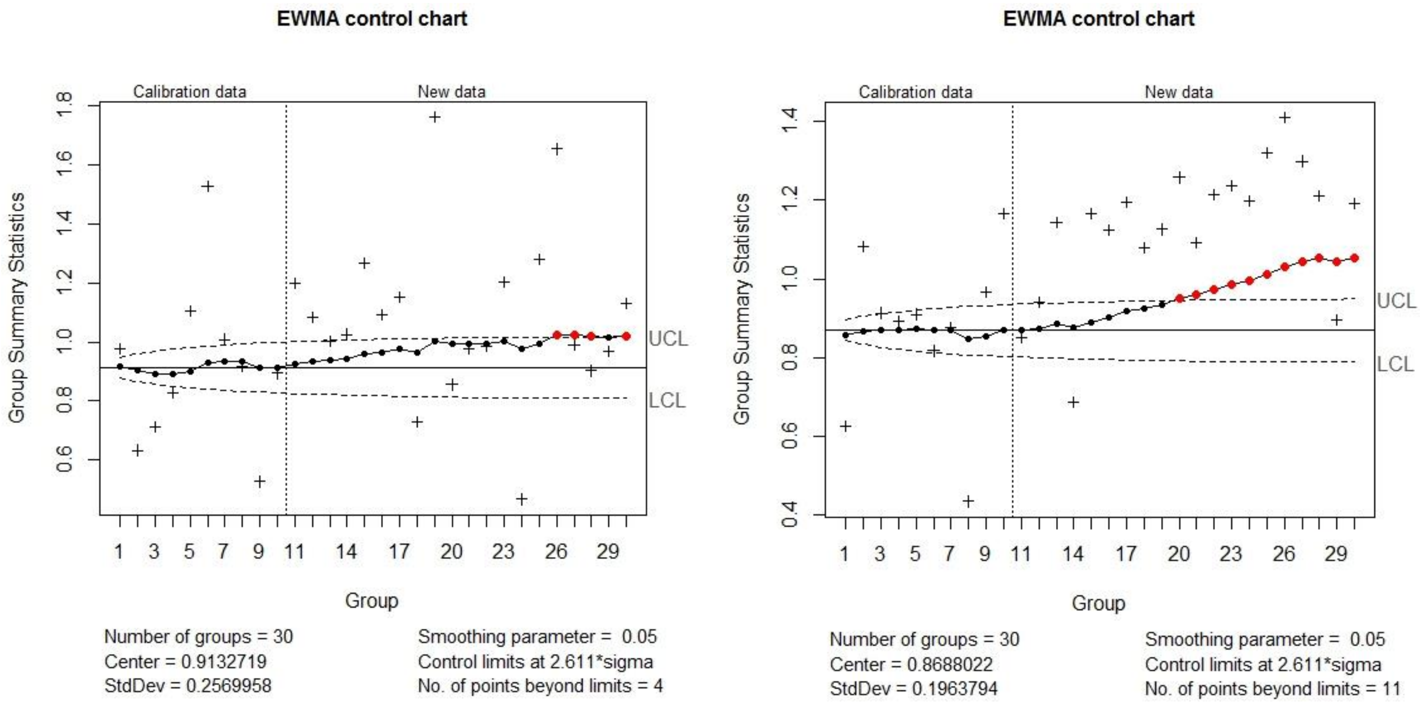
b) β=2



c) β=3 d) β=4

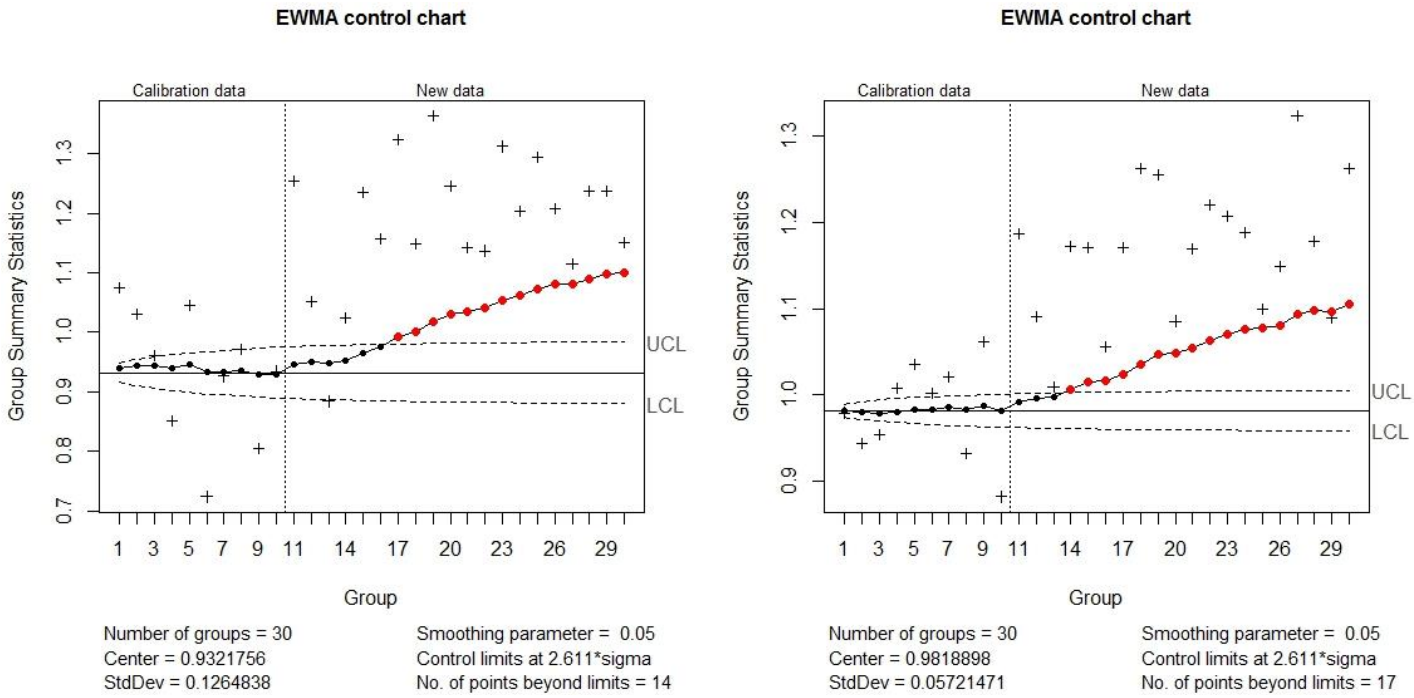
Figure 12. Downward shifts on EWMA charts with transformed Weibull data

For the upward shift, the parameters are the same as in the above case, except for the value of η_1, which is now η_1=2, so the level of the downward shift is 2.



a) β=1 b) β=2

https://doi.org/10.26219/heal.aueb.5591



c) β=3 d) β=4

Figure 13. Upward shifts on EWMA charts with transformed Weibull data

From Figures 12 and 13, it can be seen that, the larger the shape parameter is, the better the performance of the EWMA chart, as it detects the shift more quickly both for upward and downward shifts. This confirms the conclusions of Ji Ying Liu et al. [28] that the chart is more sensitive for Weibull distribution with β>1, since the chart with the transformed exponential data (β=1) has the worst performance compared to the charts where β>1. However, the chart does not perform well when β<1, as it does not detect the shift quickly, so it is not recommended for Weibull data under the transformation discussed here.

5.4 CUSUM control charts

As seen in Chapter 4, Section 4.2 and according to the article of Alwan [38], the standard CUSUM control chart, which is based on the normality assumption, can be applied for approximate normal data. He also proposes to transform exponential data, which are a special case of the Weibull distribution, using the same transformation that was presented in Section 6.3

 https://doi.org/10.26219/heal.aueb.5591

for the application of the EWMA control chart. Of course, after the transformation, the exponential data approximate the normal distribution and the CUSUM chart can be used.

Finally, before going on to the examples, it is useful to remember that the CUSUM chart is usually implemented in the tabular form. The lower one-sided cumulative statistic is used to detect a downward shift in the process mean, whereas the upper one-sided cumulative statistic is used to detect an upward shift in the process mean

5.4.1 CUSUM control chart for transformed exponential data

Let us assume, like in the previous example, that 20 TBE data are generated from an exponential distribution with mean equal to 1 when the process is in-control. Then, 50 points are generated from an out-of-control process which follows an exponential distribution with mean equal to 0.2 so the shift is downward and the lower one-sided cumulative statistic is expected to detect the shift. After transforming the data to approximate normality, it is obvious that the CUSUM chart performs well, like the EWMA in Section 6.3.1 does, as it detects the shift at the 27th point, as can be seen in Figure 14.

 https://doi.org/10.26219/heal.aueb.5591

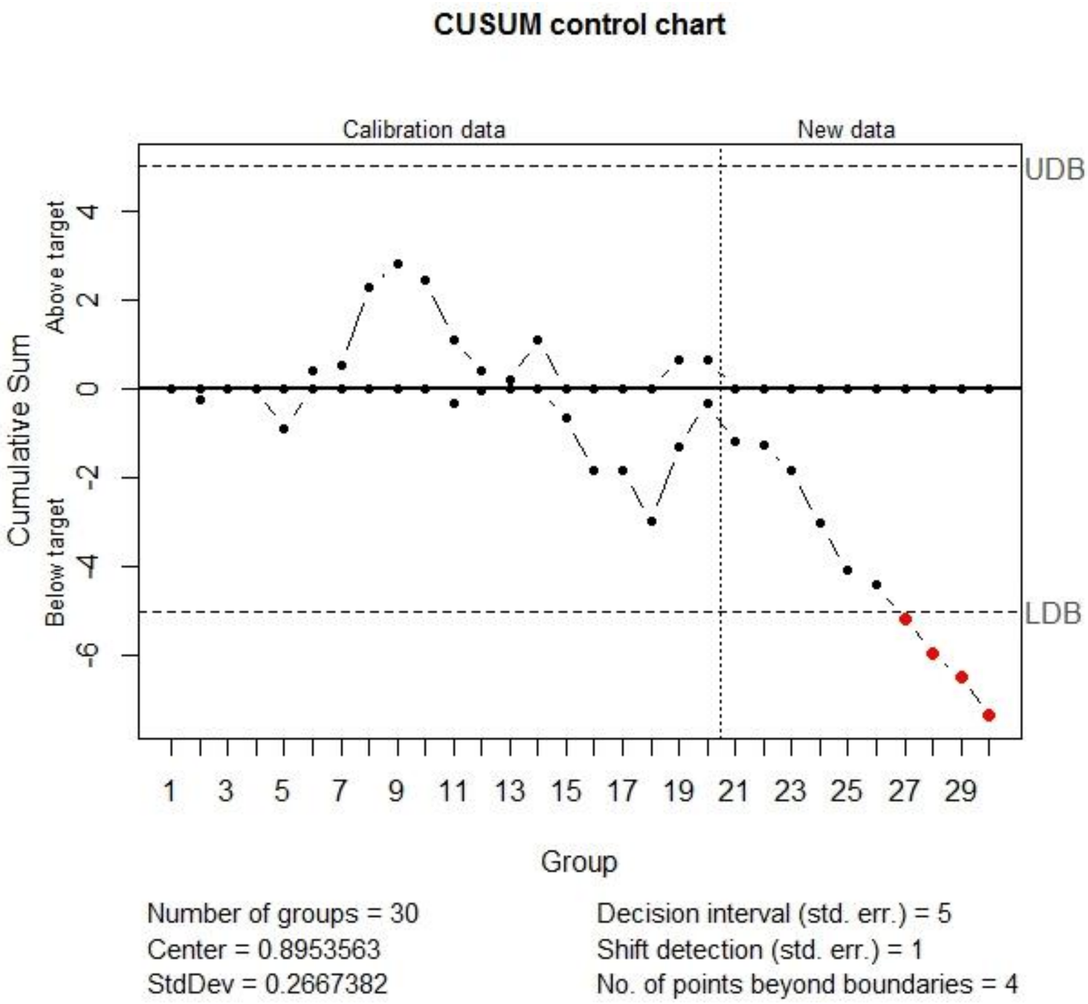


Figure 14. The CUSUM chart with transformed exponential data

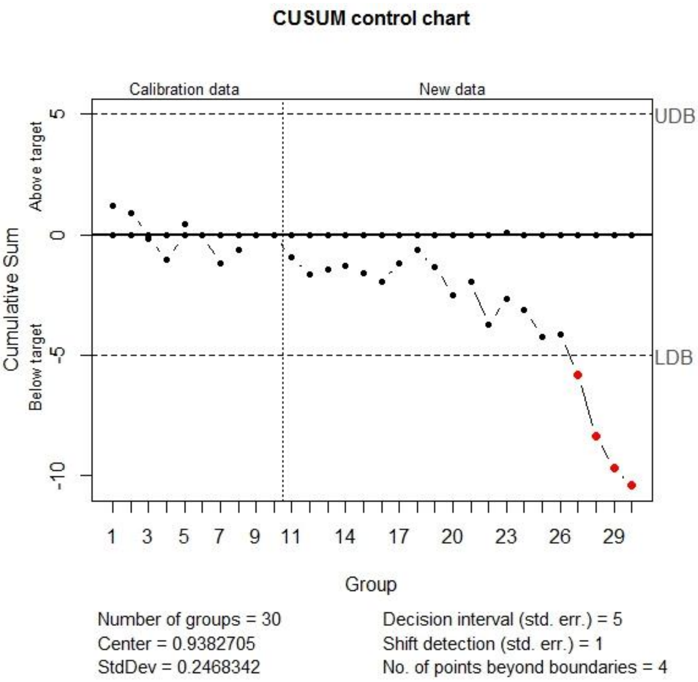
5.4.2 The CUSUM control charts with Weibull transformed data

As stated in Section 6.3.2, the Weibull distribution is more widely used to model TBE data compared to the exponential distribution, which is a special case of the Weibull. It was seen that the EWMA chart is robust for transformed Weibull data and the results of Ji Ying Liu et al. [28] were confirmed. Now, it would be interesting to check through a similar example if these are also valid for the CUSUM chart. It should be reminded that what is wanted to be checked is if the chart is more sensitive for Weibull distributed data with β>1 than for exponential data (β=1), for both upward and downward shifts.

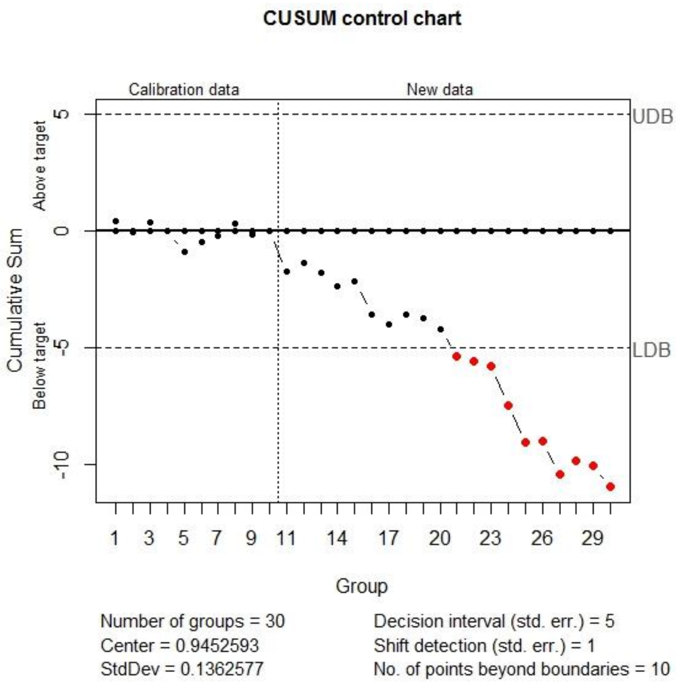
https://doi.org/10.26219/heal.aueb.5591

In the following example, the shape parameter β will be held fixed, whereas the scale parameter η will change for the same reasons that were analyzed in Section 6.3.2. In the example both upward and downward shifts will be examined.

Let us now assume that 10 TBE data are generated from the Weibull(β,η_0) distribution, where η_0=1 when the process is in-control. Then, 20 points are generated from an out-of-control process which follows the Weibull(β,η_1) distribution, where η_1=0.5 so the level of the downward shift is 0.5.

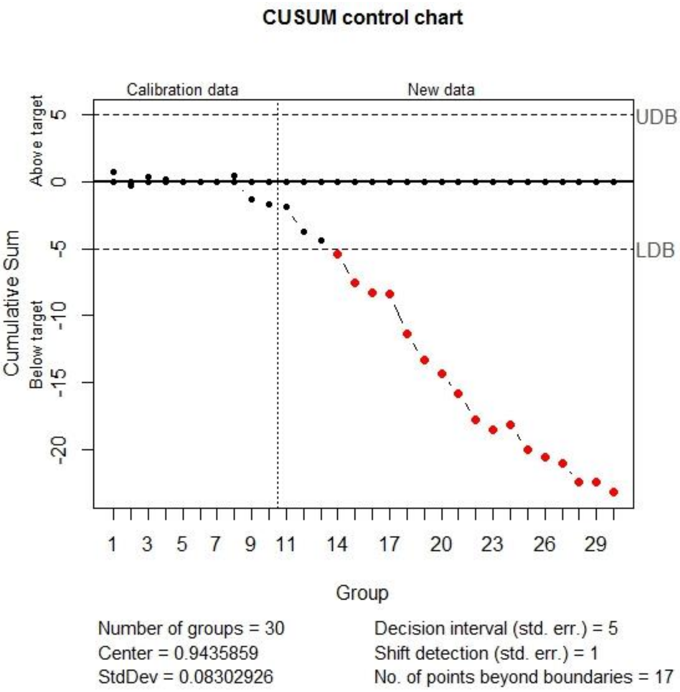


a) β=1

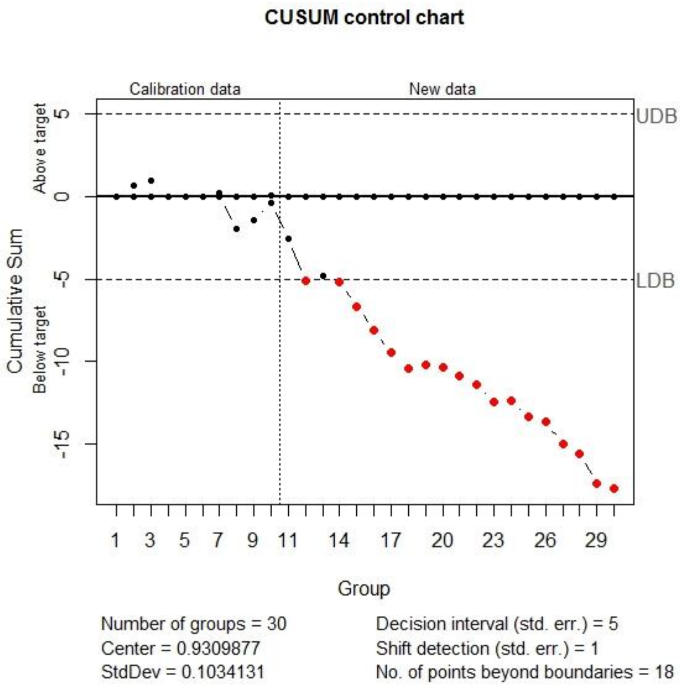


b) β=2

 https://doi.org/10.26219/heal.aueb.5591



c) β=3

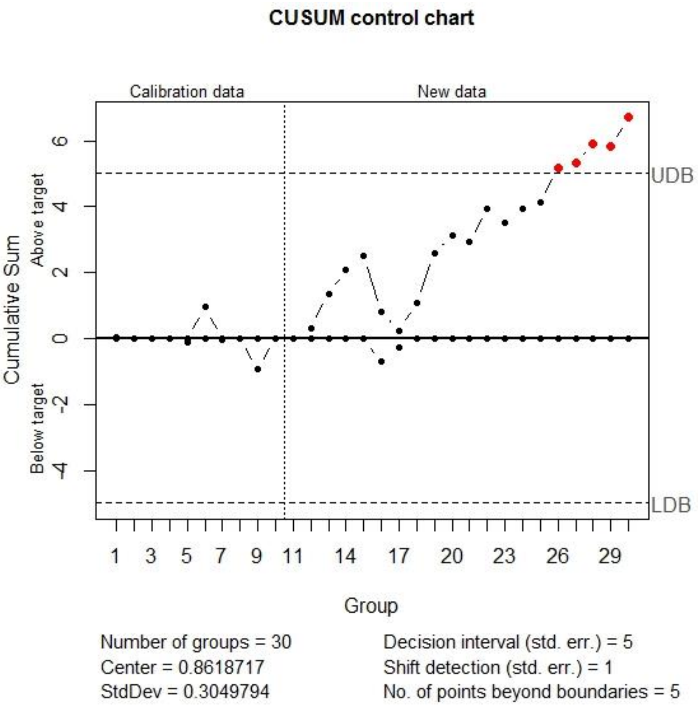


d) β=4

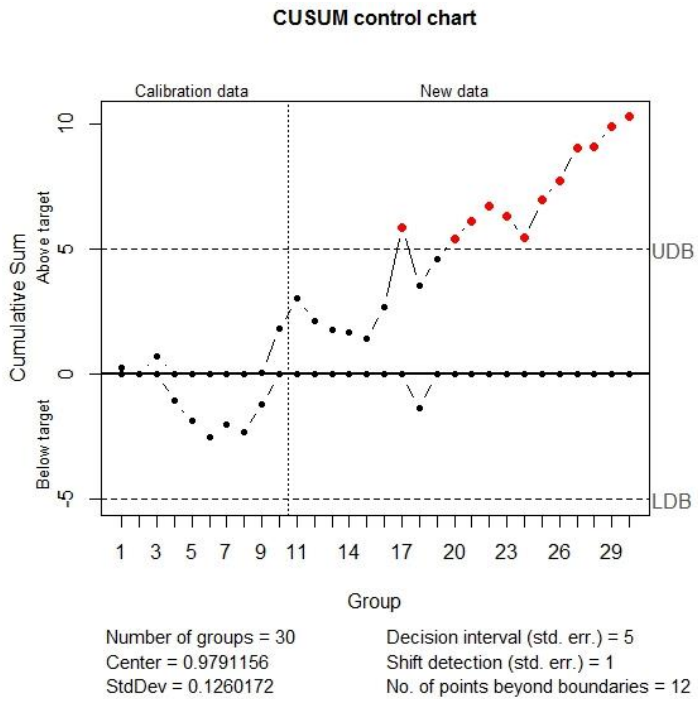
Figure 15. Downward shifts on CUSUM charts with transformed Weibull data

For the upward shift, the parameters are the same as in the above case, except for the value of η_1, which is now η_1=2, so the level of the downward shift is 2.

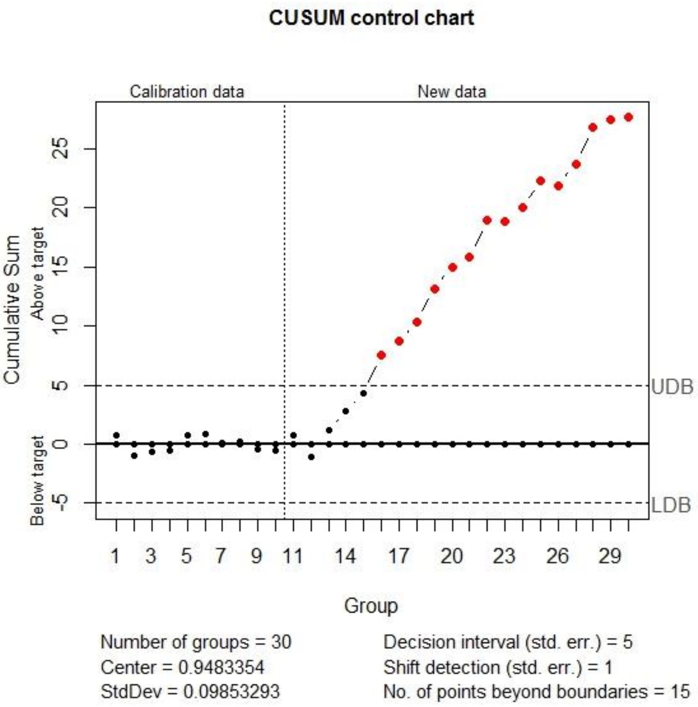
https://doi.org/10.26219/heal.aueb.5591



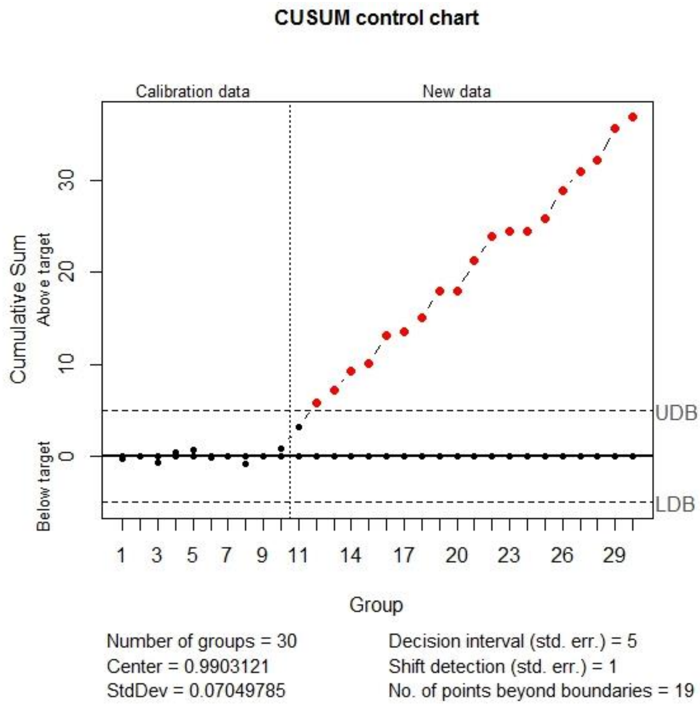
a) β=1



b) β=2



c) β=3



d) β=4

Figure 16. Upward shifts on CUSUM charts with transformed Weibull data

From Figures 15 and 16, it can be seen that, the larger the shape parameter is, the better the performance of the CUSUM chart, as it detects the shift more quickly both for upward and downward shifts. This shows that the

 https://doi.org/10.26219/heal.aueb.5591

conclusions of Ji Ying Liu et al. [28] that the chart is more sensitive for Weibull distribution with $\beta>1$ are not only valid for the EWMA chart, but for the CUSUM chart, too. The chart with the transformed exponential data ($\beta=1$) has the worst performance compared to the charts where $\beta>1$. However, the CUSUM chart, like the EWMA chart, does not perform well when $\beta<1$, as it does not detect the shift quickly, so it is not recommended for Weibull data under the transformation discussed here.

https://doi.org/10.26219/heal.aueb.5591

https://doi.org/10.26219/heal.aueb.5591

CHAPTER 6

GENERAL CONCLUSIONS AND DISCUSSION

In this thesis, a review of control charts when the process distribution is Weibull was presented. More specifically, the standard Shewhart, EWMA and CUSUM control charts could lead to misleading results if used for not normally distributed, but heavy tailed and skewed data. However, the use of the Weibull distribution is wide, so, the methods weighted variance (WV) and skewness correction (SC) were proposed for constructing the Shewhart X-bar, R and S control charts. Based on Type I risks, the SC method was seen to be the best one. Also, Shewhart control charts for monitoring the shape and scale parameter of the Weibull distribution were discussed, as a shift in them may be connected to many other variables' shifts. For example, the Beta chart, which is a Shewhart-type control chart, was used to monitor the Weibull shape parameter. The case when a sample consists of individual observations was also examined by using the individuals Y and the moving range (MR) control charts and it was concluded that the MR chart may end up being more problematic than the individuals Y chart. Afterwards, EWMA control charts were proposed for detecting small process shifts and for monitoring the shape parameter, the lifetimes' mean or the quantiles of the Weibull distribution, as well as, EWMA charts based on the moving average statistic and on $\ln S^2$. Additionally, it was seen that, in case of autocorrelated data, independence and normality assumptions of the Shewhart and EWMA control charts are invalid and their performance is affected. So, both charts were compared under the violation of that assumptions and the EWMA was found to outperform the Shewhart chart when the data were Weibull distributed and autocorrelated. Moreover, CUSUM control charts are also used mostly for detecting small process shifts and in order to monitor Weibull distributed processes, various CUSUM control charts were presented, such as the Weibull CUSUM, the exponential CUSUM and the adjusted exponential CUSUM chart. Based on ARL and ATS values, the Weibull CUSUM was the one

https://doi.org/10.26219/heal.aueb.5591

having the best performance. For both EWMA and CUSUM charts, a transformation of exponential data to normal is suggested so that the standard control charts can be applied. Then, the charts have a good performance in monitoring exponentially distributed TBE data based on the ARL values and the EWMA is also robust for the case when the underlying distribution is Weibull. A combination of the last two control charts, in other words, a mixed EWMA-CUSUM chart was presented for detecting small process shifts and based on the ARL, it performs better than the already existing control charts that could be used in such cases. Finally, on the last section, the performance of X-bar and R control charts based on the WV and SC methods was demonstrated using simulation data, as well as, the good performance of the EWMA and CUSUM control charts for the transformed exponential data. Also, it was seen that in that case, the CUSUM chart when the underlying distribution is Weibull, is robust, as well.

As a future research, many interesting issues could be pursued. For instance, the effect of parameter estimation on the performance of the Beta chart or finding some analytical methods for computing its ARL values or even finding a theoretical treatment of the correlation between consecutive samples of the monitoring statistic used on this chart. Moreover, the EWMA control charts, which were based on the moving average statistic and on $\ln S^2$ could be extended and used for other non-normal data but also, these two statistics could be combined to create one mixed new control chart. Furthermore, when it comes to the case of the autocorrelated data, it should be noted that it is common for the process mean of an autocorrelated process to exhibit a trend. So, it would be of interest to analyze that trend in the process shift. Finally, it could be examined if the mixed EWMA-CUSUM control chart that was described could be used for other distributions and under which circumstances.

https://doi.org/10.26219/heal.aueb.5591

REFERENCES

[1] Montgomery, D. C. (2009). Introduction to statistical quality control. John Wiley & Sons (New York).

[2] Derya, K., & Canan, H. (2012). Control charts for skewed distributions: Weibull, gamma, and lognormal. Metodoloski zvezki, 9(2), 95.

[3] Chan, L. K., & Cui, H. J. (2003). Skewness correction X and R charts for skewed distributions. Naval Research Logistics (NRL), 50(6), 555-573.

[6] Choobineh, F., & Ballard, J. L. (1987). Control-limits of QC Charts for skewed distributions using weighted-variance. IEEE transactions on reliability, 36(4), 473-477.

[7] Choobineh, F., & Branting, D. (1986). A simple approximation for semivariance. European Journal of Operational Research, 27(3), 364-370.

[8] Bai, D. S., & Choi, I. S. (1995). X and R control charts for skewed populations. Journal of Quality Technology, 27(2), 120-131.

[9] Chang, Y. S., & Bai, D. S. (2001). Control charts for positively-skewed populations with weighted standard deviations. Quality and Reliability Engineering International, 17(5), 397-406.

[10] Karagöz, D. (2016). Robust X control chart for monitoring the skewed and contaminated process. Hacettepe J Math Stat. doi, 10.

[11] Atta, A. M., Shoraim, M. H., Yahaya, S. S., Zain, Z., & Ali, H. (2016). A SCALED WEIGHTED VARIANCE S CONTROL CHART FOR SKEWED POPULATIONS. Journal of Theoretical and Applied Information Technology, 91(1), 61.

[12] Khoo, M. B.C., Atta, A.M. A. and Chen, C-H. (2009a). Proposed X-bar and S control charts for skewed distributions, Proceedings of the International Conference on Industrial Engineering and Engineering Management (IEEM 2009), Dec. pp. 389-393, Hong Kong.

[13] Castagliola, P. (2000). control chart for skewed populations using a scaled weighted variance method. International Journal of

 https://doi.org/10.26219/heal.aueb.5591

Reliability, Quality and Safety Engineering, 7(03), 237-252.

[14] Faraz, A., Saniga, E. M., & Heuchenne, C. (2015). Shewhart control charts for monitoring reliability with Weibull lifetimes. Quality and Reliability Engineering International, 31(8), 1565-1574.

[15] Pascual, F. (2013). Individual and moving ratio charts for Weibull processes. In Stochastic Orders in Reliability and Risk (pp. 331-350). Springer, New York, NY.

[16] McCool, J. I., & Joyner-Motley, T. (1998). Control charts applicable when the fraction nonconforming is small. Journal of Quality Technology, 30(3), 240.

[17] Zhang, C. W., Ye, Z., & Xie, M. (2017). Monitoring the shape parameter of a Weibull renewal process. IISE Transactions, 49(8), 800-813.

[18] Bain, L.J. (1978) Statistical Analysis of Reliability and Life-Testing Models: Theory and Methods, Marcel Dekker, New York, NY.

[19] Ramalhoto, M. F., & Morais, M. (1999). Shewhart control charts for the scale parameter of a Weibull control variable with fixed and variable sampling intervals. Journal of Applied Statistics, 26(1), 129-160.

[20] Akhundjanov, S. B., & Pascual, F. (2015). Moving range EWMA control charts for monitoring the Weibull shape parameter. Journal of Statistical Computation and Simulation, 85(9), 1864-1882.

[21] Pascual, F., & Nguyen, D. (2011). Moving range charts for monitoring the Weibull shape parameter with single observation samples. Quality and Reliability Engineering International, 27(7), 905-919.

[22] Zhang, L., & Chen, G. (2004). EWMA charts for monitoring the mean of censored Weibull lifetimes. Journal of Quality Technology, 36(3), 321.

[23] Steiner, S., & MacKay, R. J. (2001). Detecting changes in the mean from censored lifetime data. In Frontiers in Statistical Quality Control 6 (pp. 275-289). Physica, Heidelberg.

[24] Steiner, S. H., & Mackay, R. J. (2000). Monitoring processes with

https://doi.org/10.26219/heal.aueb.5591

highly censored data. Journal of Quality Technology, 32(3), 199.

[25] Pascual, F., Yang, S., & Ye, M. (2017). Monitoring Weibull quantiles by EWMA charts based on a pivotal quantity conditioned on ancillary statistics. Quality and Reliability Engineering International, 33(1), 103-119.

[26] Haghighi, F., Pascual, F., & Castagliola, P. (2015). Conditional control charts for Weibull quantiles under type II censoring. Quality and Reliability Engineering International, 31(8), 1649-1664.

[27] Pehlivan, C., & Testik, M. C. (2010). Impact of model misspecification on the exponential EWMA charts: a robustness study when the time-between-events are not exponential. Quality and Reliability Engineering International, 26(2), 177-190.

[28] Liu, J. Y., Xie, M., Goh, T. N., & Chan, L. Y. (2007). A study of EWMA chart with transformed exponential data. International Journal of Production Research, 45(3), 743-763.

[29] Wang, F. K., & Cheng, X. B. (2017). Exponentially weighted moving average control charts based on the moving average statistic and lnS2 for monitoring a Weibull process with subgroups. Quality and Reliability Engineering International, 33(8), 1901-1913.

[30] Lowry, C. A., Woodall, W. H., Champ, C. W., & Rigdon, S. E. (1992). A multivariate exponentially weighted moving average control chart. Technometrics, 34(1), 46-53.

[31] Linderman, K. and Love, T. E. (2000a). Economic and economic-statistical design of MEWMA control charts. Journal of Quality Technology, 32, 410-417.

[32] Linderman, K. and Love, T. E. (2000b). Implementing the economic and economic-statistical design of MEWMA control charts. Journal of Quality Technology, 32, 457-463.

[33] Love, T., & Linderman, K. (2003). A Weibull process failure mechanism for the economic design of MEWMA control charts. Journal of Statistical computation and Simulation, 73(3), 195-202.

[34] McWilliams, T. P. (1989). Economic control chart designs and the in-control time distribution: A sensitivity study. Journal of Quality

https://doi.org/10.26219/heal.aueb.5591

Technology, 21(2), 103-110.

[35] Borror, C. M., Montgomery, D. C., & Runger, G. C. (1999). Robustness of the EWMA control chart to non-normality. Journal of quality Technology, 31(3), 309-316.

[36] Black, G., Smith, J., & Wells, S. (2011). The impact of Weibull data and autocorrelation on the performance of the Shewhart and exponentially weighted moving average control charts. International Journal of Industrial Engineering Computations, 2(3), 575-582.

[37] Aslam, M., Balamurali, S., Jun, C. H., & Meer, A. (2017). Time-truncated attribute sampling plans using EWMA for Weibull and Burr type X distributions. Communications in Statistics-Simulation and Computation, 46(6), 4173-4184.

[38] Alwan, L. C. (2000). Designing an effective exponential CUSUM chart without the use of nomographs. Communications in Statistics-Theory and Methods, 29(12), 2879-2893.

[39] Nelson, L. S. (1994). A control chart for parts-per million nonconforming items. J. Qual. Technol., 26(3), 239-240.

[40] Shafae, M. S., Dickinson, R. M., Woodall, W. H., & Camelio, J. A. (2015). Cumulative sum control charts for monitoring Weibull-distributed time between events. Quality and Reliability Engineering International, 31(5), 839-849.

[41] Zhang, M., Xu, Y., He, Z., & Hou, X. (2016). The Effect of Estimation Error on Risk-Adjusted Survival Time CUSUM Chart Performance. Quality and Reliability Engineering International, 32(4), 1445-1452.

[42] Abbas, N., Riaz, M., & Does, R. J. (2013). Mixed exponentially weighted moving average–cumulative sum charts for process monitoring. Quality and Reliability Engineering International, 29(3), 345-356.

[43] Aslam, M. (2016). A mixed EWMA–CUSUM control chart for Weibull-distributed quality characteristics. Quality and Reliability Engineering International, 32(8), 2987-2994.

[44] Lucas, J. M. (1982). Combined Shewhart-CUSUM quality control

schemes. Journal of quality technology, 14(2), 51-59.

[45] Borror, C. M., Keats, J. B., & Montgomery, D. C. (2003). Robustness of the time between events CUSUM. International Journal of Production Research, 41(15), 3435-3444.

[46] Pascual, F. (2010). EWMA charts for the Weibull shape parameter. Journal of Quality Technology, 42(4), 400-416.

[47] Oliveira, J. W., Valença, D. M., Medeiros, P. G., & Marçula, M. (2016). Risk-adjusted monitoring of time to event in the presence of long-term survivors. Biometrical Journal, 58(6), 1485-1505.

[48] Hardoon, S. L., Lewsey, J. D., & van der Meulen, J. H. (2007). Continuous monitoring of long-term outcomes with application to hip prostheses. Statistics in medicine, 26(28), 5081-5099.

[49] Charongrattanasakul, P., & Pongpullponsak, A. (2017). Optimizing the Cost of Integrated Model for Fuzzy Failure Weibull Distribution Using Genetic Algorithm. Engineering Journal (Eng. J.), 21(2), 253-268.

[50] Chang, Y. S. (2007). Multivariate CUSUM and EWMA control charts for skewed populations using weighted standard deviations. Communications in Statistics—Simulation and Computation®, 36(4), 921-936.

[51] Khoo, M. B., Teh, S. Y., & Eng, M. Y. (2010). A comparison of multivariate control charts for skewed distributions using weighted standard deviations. Malaysian Journal of Fundamental and Applied Sciences, 6(1).

[52] Chang, Y. S., & Bai, D. S. (2001). Control charts for positively-skewed populations with weighted standard deviations. Quality and Reliability Engineering International, 17(5), 397-406.

[53] Wikipedia, Reliawiki, The Weibull distribution, Chapter 8: The Weibull distribution,http://reliawiki.org/index.php/The_Weibull_Distribution

https://doi.org/10.26219/heal.aueb.5591