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Dynamic relationships between latent yield curve factors, economic growth, and monetary policy: a cointegration analysis.

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1 Abstract

The literature suggests that the slope factor of the yield curve reflects long-term economic activity, while the curvature factor captures shorter-term changes in monetary policy stance (Argyropoulos & Tzavalis 2016). This paper provides clear-cut evidence that there exists a cointegrating long-term equilibrium between the yield curve's latent factors, economic activity, and the monetary policy instrument. We analyse this dynamic relationship using US data. We employ the Dynamic Nelson-Siegel model within a state-space framework to extract the latent factors through the Kalman filter. We construct the implied vector error correction model to quantify both the long-term cointegrated dynamics and short-term adjustments. Impulse response analysis and variance decomposition are conducted to evaluate how shocks to the yield curve factors propagate through the economy and derive policy implications based on the findings.

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2 Acknowledgments

I would like to express my deepest gratitude towards my Professors at the MSc Programme who taught me economic theory, and particularly, Professor Tzavalis from whom I learned time-series econometrics, and whose extensive work in the field served as the foundation of this dissertation.

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3 Introduction

The yield curve, which represents the relationship between interest rates (bond yields) and bond maturities, has long been regarded as a crucial tool for predicting economic activity, indicating an underlying relationship between the two variables. Traditionally, the term spread - defined as the difference between long- and short-term interest rates - has been used to forecast future economic growth. An upward-sloping yield curve indicates expansion while an inverted yield curve signals a potential recession. However, recent literature has suggested that proceeding with a more nuanced fashion by decomposing the term structure of the yield curve into its latent factors - namely, slope and curvature - provides a superior depiction of future macroeconomic conditions, enhancing the model's predictive ability (Argyropoulos & Tzavalis 2016).

More specifically, the literature (see, e.g., Argyropoulos & Tzavalis 2016) has showed that the slope factor contains information about longer-term economic activity. Conversely, the curvature factor is shown to maintain predictive power on shorter business cycles, which is associated with current developments in the monetary policy stance. Therefore, there is potentially an implied vector of underlying relationships between the slope factor, the curvature factor, economic activity, and the monetary policy instrument. This paper studies the dynamic relationships between the yield curve factors and the aforementioned macroeconomic variables (economic activity, and monetary instrument). There has been work in the literature addressing those relationships (see, e.g., Diebold & Li 2006; Diebold, Ji & Li 2006; Diebold & Li 2006; Diebold et al. 2003; Ang & Piazzesi 2003; Ang et al. 2006; Argyropoulos & Tzavalis 2015; Estrella & Hardouvelis 1991; Stock & Watson 2003). Most of these studies employ a vector autoregressive process in order to capture the short-term dynamics between the latent components of the yield curve and macroeconomic indicators. They find strong evidence of the effects of macroeconomic variables on the future movements that the slope and curvature factors of the term structure of the yield curve have. There is also evidence for a reverse causal direction as well, albeit weaker (Diebold, Rudebusch & Aruoba 2006).

In this paper we address the following questions, after first relying on recent literature decomposing the yield curve into the latent slope and curvature factors (see, e.g., Diebold, Rudebusch & Aruoba 2006; Argyropoulos & Tzavalis 2016): First, if there exists a cointegrating relationship between these two latent factors of the yield curve and the above two macroeconomic indicators (growth and monetary instrument). Second, if there is, what are the adjustments mechanisms that lead to a mean-reversion towards the long-run equilibrium. Regarding the first question, there are two studies that have tested and proved the existence of a cointegrating vector between Treasury rates for different maturities of the term structure, with the corresponding error correction model outperforming the augmented vector autoregressive model (see, Bradley & Lumpkin 1992; Zhang 1993). However, in this paper we will explore the cointegration space between the latent components of the term structure (slope and curvature) with the two key

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macroeconomic variables discussed above. That is, if there a strong long-term equilibrium between those latent factors and those two macroeconomic variables, that despite temporary disequilibria, the forces of the economy will eventually be leading them to a mean-reversion, back to their equilibrium. This is important for both policymakers and bond market participants to better comprehend the underlying mechanisms of the bond market, affected by economic growth and the current course of action of the central banks.

Regarding the second question, upon confirming the existence of a strong cointegrating relationship between the aforementioned, we will construct the implied vector error correction model to study how and at what speed shorter disequilibria are getting adjusted (or corrected) back to long-term equilibria. To commence this analysis, we will employ the dynamic Nelson-Siegel model for the term structure (DNS model, hereafter), whose convenient state-space representation allows for a comprehensive estimation of the latent factors of the yield curve; namely, the level (or shift), the slope, and curvature, as is performed in the recent literature (see, e.g., Diebold & Li 2006; Diebold et al. 2003). The DNS model of the term structure, as proposed by Nelson & Siegel (1987), assumes in alignment with the current literature, that the level factor of the yield curve captures only parallel shifts of interest rates, not a functional of of maturity intervals, and therefore it is cancelled out in term spreads upon considering separate maturity bond yields (interest rates) (see, e.g., Argyropoulos & Tzavalis 2015; Bliss 1997; Litterman & Scheinkman 1991).

The paper is structured as follows. In Section 2, we provide the relevant literature review for the term structure of the yield curve. In Section 3, we discuss the data, and analyse the theoretical framework that we will follow, starting with the DNS model, followed by cointegration theory, the error correction model, and finally impulse responses and variance decomposition. In Section 4, we estimate the DNS model, retrieve the slope and curvature factors of the yield curve, and whereupon proceed with the empirical cointegration analysis. Section 4 discusses the findings and concludes the paper.

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4 Literature Review

4.1 The Yield Curve as a Predictor of Economic Activity

The term structure of interest rates, known as the yield curve, is defined as the spread between long- and short-term interest rates, and it depicts how the returns (yields) of bonds (or debt instruments, in general), vary across different maturities (Estrella & Trubin 2006). Normally, the interest rates increase alongside maturities, due to the increased risk implied by the extra duration (Parker & Schularick 2021). The theory suggests that short-term interest rates are representative of the central banks' monetary policy decisions, whilst the long-term interest rates reflect bond market participants' expectations about future short-term interest rates, inflation, and risk premia (Argyropoulos & Tzavalis 2016). The latter is known as the Rational Expectations Hypothesis of the Term Structure. The expectations hypothesis of the term structure (REHTS) states that movements in long-term interest rates are due to movements in expected future short-term interest rates (Diebold, Rudebusch & Aruoba 2006; Tzavalis & Wickens 1998). Additionally, the slope of the yield curve is defined as the term spread between long- and short-term interest rates (Argyropoulos & Tzavalis 2016). According to the theory, a contractionary monetary policy that increases the short-term interest rate and drives future recessions, produces a zero or negative term spread (Argyropoulos & Tzavalis 2016). This course of action alters expectations for future short-term interest rates, and thereby decreases long-term rates because the market's expectation for future recessionary periods increases savings demand. As a result, this potentially flattens, or even inverts, the yield curve. In contrast, an expansionary monetary policy in which the central bank decreases short-term interest rates leads to a positive term spread and thus to a positive slope of the yield curve (Argyropoulos & Tzavalis 2016).

There is a large literature examining the predictive ability of the term structure of interest rates (the yield curve) about future economic activity (see, e.g., Estrella & Hardouvelis 1991; Argyropoulos & Tzavalis 2015; Ang & Piazzesi 2003; Ang et al. 2006). The most popular exercise performed to capture the information contained on interest rates about forthcoming economic activity involves a regression where the term spread between long- and short-term interest rates (slope of the yield curve) acts as a regressor, and GDP growth as a regressand. The outcome of this exercise illustrates a positive relationship between the spread of the term and the real output. More specifically, they illustrate that when long-term interest rates move positively further away than short-term interest rates, real economic activity increases for numerous quarters ahead (see, e.g., Estrella & Mishkin 1997).

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4.2 Decomposing the Yield Curve: Slope and Curvature Factors

There is work in the literature on extracting latent factors from the term structure of the yield curve by decomposing the term spread into the slope and curvature factors (Diebold & Li 2006; Diebold, Rudebusch & Aruoba 2006). The most widely used model in the literature that offers a parsimonious representation of the yield curve is the Dynamic Nelson-Siegel model (DNSM) (Nelson & Siegel 1987). To proceed with the extraction of the above latent factors, Diebold, Rudebusch & Aruoba (2006) suggested the application of the Kalman filter that has been claimed to facilitate real-time estimation, rendering it responsive to market developments.

The slope factor represents the difference in bond yield between long-term and short-term maturities. The literature suggests that the slope factor of the yield curve reflects future real business cycle conditions (Bekaert et al. 2010). Conversely, the curvature factor captures the yield curve's concavity[1], and contains information about short-term interest rate volatility and central bank interventions (Dewachter & Lyrio 2006). It is found that the curvature factor captures short or medium-term adjustments in monetary policy (Dewachter et al. 2006). For instance, if the central bank deems economic growth to be rather high, and proceeds to a contractionary monetary policy upon the term spread, the curvature factor will reflect the impact of such course of action (Mönch 2012).

4.3 Cointegration and Yield Curve - Macroeconomic Interactions

Cointegration analysis examines the potential existence of a long-term equilibrium relationship between non-stationary and integrated of order one time-series. Engle & Granger (1987) and Johansen (1988) proposed the very concept of cointegration and subsequently provided methodologies upon which we can identify and estimate such relationships. This analysis is pertinent for studying long-term potential equilibrium relationships between the yield curve factors and fundamental macroeconomic variables such as economic activity and monetary stance.

The dynamic interactions between the macroeconomy and the yield curve are present in the literature. There is evidence to suggest the effects of macroeconomic variables on future fluctuations in the yield curve, as well as evidence of an, albeit weaker, inverse causal direction (Diebold, Rudebusch & Aruoba 2006). It has been proven that the slope factor has significant predictive ability over economic activity on longer horizons, whilst the curvature factor has predictive ability on shorter-term fluctuations of output that are potentially correlated with changes in monetary policy (Argyropoulos & Tzavalis 2016). Moreover, Stock & Watson (1993) demonstrated that liquidity fluctuations, illustrated by M2 money supply, can impact both the yield curve and economic activity. Those three crucial results suggest that

[1]See appendix B for a formal mathematical definition of concavity.

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potentially there is a long-term equilibrium relationship between the aforementioned variables, that has to be examined through full cointegrating analysis.

4.4 Forecasting Economic Activity Using Yield Curve Factors

In recent forecasting literature, there have been numerous studies surrounding the predictive accuracy of the yield curve (see, e.g., Ang & Piazzesi 2003; Ang et al. 2006, Argyropoulos & Tzavalis 2015). Mönch 2012 concluded that incorporating the slope and curvature factors improves forecasting accuracy of output than by running traditional regressions. These results were confirmed with a clear-cut fashion by Argyropoulos & Tzavalis 2016, who clearly illustrated that the latent slope and curvature factors of the yield curve contain more information about future fluctuations in real output than the term spread itself, which was primarily used in the literature hitherto. This happens because these two factors enter the term spread with opposite signs and reflect distinct information about future economic activity (Argyropoulos & Tzavalis 2016). These results remain robust to the augmentation of the interest rate level and growth rate level as well in the regression model, with decreased out-of-sample mean squared errors compared to traditional regressions involving solely the term spread (Argyropoulos & Tzavalis 2016).

4.5 Contribution

There has been extensive research upon the yield curve's and its factors' forecasting prowess, as well as its relationship with macroeconomic variables (Argyropoulos & Tzavalis 2016; Ang & Piazzesi 2003; Diebold et al. 2003). However, the goal of this paper is to seek any long-term equilibrium relationships between the yield curve's factors and the relevant macroeconomic variables on output growth and the monetary policy stance, and subsequently study the long- and short-term dynamics as well as adjustment mechanisms that dictate those relationships. The ultimate goal of this paper is to provide useful insights for policymakers and market participants into those intricate relationships.

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5 Data and Methodology

This section presents the data and empirical methodology used to examine the dynamic relationship between the slope and curvature factors of the yield curve and key macroeconomic variables, namely GDP growth and monetary policy stance. First, we describe the data sources, including yield curve estimates, macroeconomic indicators, and their transformations. Next, we outline the estimation of latent yield curve factors using the Dynamic Nelson-Siegel (DNS) model within a state-space framework. The analysis proceeds with cointegration testing to determine long-term equilibrium relationships, followed by the specification of a Vector Error Correction Model (VECM) to capture both long-run and short-run dynamics. Finally, we discuss the forecasting techniques and econometric tests employed to evaluate predictive accuracy.

5.1 Data

The yield curve factors—slope and curvature—are extracted using the Dynamic Nelson-Siegel (DNS) model. The slope factor is constructed as the difference between long-term and short-term interest rates, reflecting expectations about economic growth. The curvature factor captures medium-term yield movements, which are sensitive to shifts in monetary policy. The macroeconomic variables include the Industrial Productivity Index (ipi_t), and to reflect the monetary policy stance we use the M2 money supply ($m2_t$).

This study's data set consists of 515 monthly observations from 1982:1 to 2024:11 for the United States. This period encompasses multiple economic cycles, including recessions, expansions, and shifts in monetary policy regimes. The monthly frequency allows for a detailed analysis of both short-term dynamics and long-term relationships between yield curve factors and macroeconomic variables. All data were sourced from the Federal Reserve Economic Data (FRED) database. To estimate the yield curve factors (slope and curvature) this study utilises US Treasury yields with maturities ranging from 3 months to 30 years. Specifically, to fit the dynamic term-structure Nelson-Siegel model (Nelson & Siegel 1987), the following 19 maturities are included $\tau = (3, 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108, 120, 240, 360)$. Enough maturities are needed to capture the shape of the yield curve, which typically includes short, medium, and long-term maturities. Usually, the literature uses until the first 17 of the above maturities (see, e.g., Argyropoulos & Tzavalis 2016), but here we include the 20- and 30-year maturities to better capture long-term dynamics, as this is the very nature of the cointegration analysis which seeks a long-term equilibrium relationship. These yields are reported as annualized percentages and represent risk-free interest rates for their respective maturities. Incorporating this broad range of maturities allows for an accurate estimation of the DNS model.

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Short-term yields (e.g., 3-month, 6-month, 1-year) help capture the immediate term structure of interest rates and respond to short-term monetary policy changes. Medium-term yields (e.g., 2-year, 3-year, 5-year) capture the expected trajectory of the economy in the foreseeable future and provide a balance between short-term and long-term expectations. Long-term yields (e.g., 10-year, 20-year, 30-year) reflect longer-term expectations about inflation, growth, and risk premia, and are most commonly used for long-term financial planning, such as for pension funds or investment portfolios (Tzavalis & Wickens 1998).

As is suggested in the literature, we are using monthly data for fitting the yield curve model as it allows for a better capture of the yield curve's dynamic movements and hence a more accurate calibration of the model's parameters (Argyropoulos & Tzavalis 2016).

Two key macroeconomic indicators are included to investigate the relationship between the yield curve factors and broader economic dynamics. First, to approximate economic activity we the Industrial Production Index (IPI). The Industrial Production Index is employed as a proxy for economic activity. While real GDP is a traditional measure of economic output, its quarterly frequency is incompatible with the monthly estimation of yield curve factors. The IPI, being a monthly series, provides a more suitable alternative and effectively captures fluctuations in industrial output, which runs parallel with economic growth. Second, to capture the current monetary policy stance, we use the the M2 Money Supply, that reflects the liquidity available in the economy and is influenced by monetary policy decisions. It includes cash, checking deposits, and easily convertible near money such as savings deposits and money market securities. The growth rate of M2 is used to approximate changes in monetary policy stance, as fluctuations in money supply are often tied to the Federal Reserve's policy adjustments. Before estimation, the data undergoes several preprocessing steps to ensure accuracy and consistency. All time-series are subjected to unit root tests, including the Augmented Dickey-Fuller, Phillips-Perron, and KPSS tests, to assess stationarity. Non-stationary series are differenced as required, particularly in preparation for the cointegration analysis. The Industrial Production Index (IPI) is transformed into annualized percentage changes to approximate the growth rate of economic activity, and the M2 Money Supply is converted into monthly growth rates to reflect changes in liquidity which is affected by the central bank and therefore serves as an indicator of the monetary policy stance.

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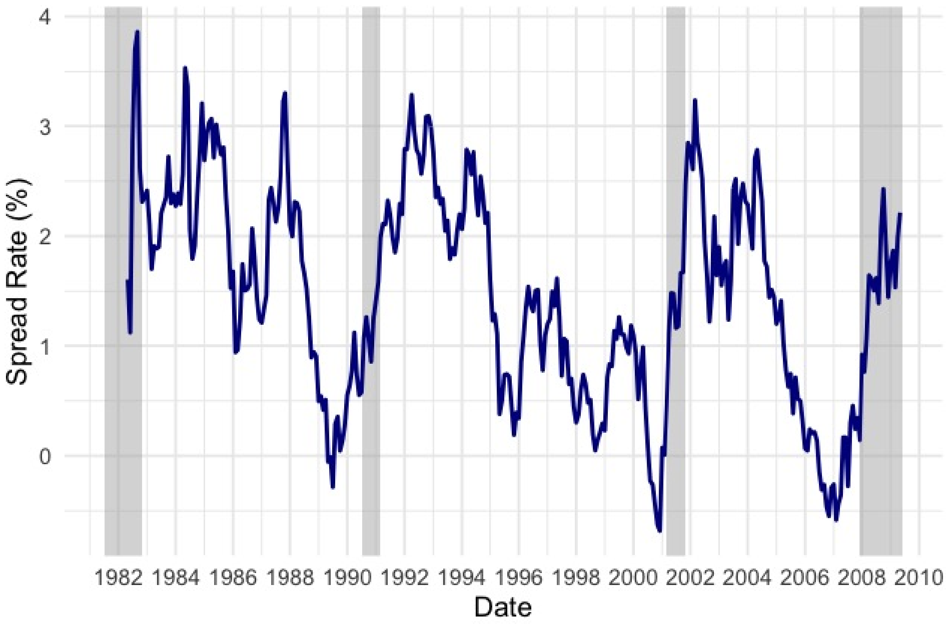

Figure 1: United States Term spread $5y - 3m$ and NBER recessionary periods (shaded areas).

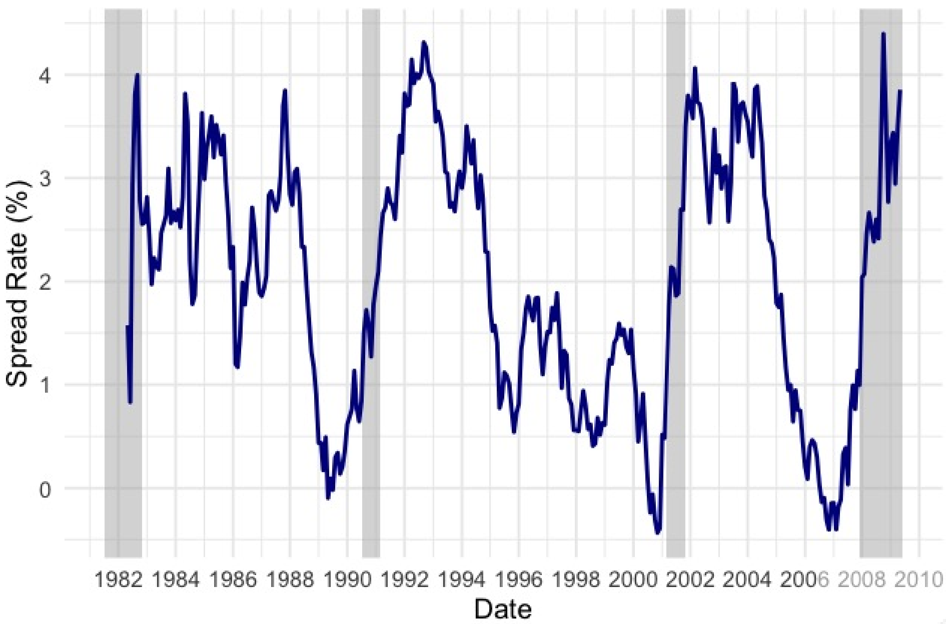

Figure 2: United States Term spread $10y - 3m$ and NBER recessionary periods (shaded areas).

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5.2 Methodology

5.2.1 State Space Models

Before delving into fitting the yield curve with latent factor models, we need to introduce a dynamic model that will be used to build and estimate this endeavour. In many economic variables, we cannot directly measure the relationship of independent and dependent variables that might by underlying or unobserved. The most common way to deal with this problem, is to apply state space models and to express our dynamic system into *spate-space representation* (Hamilton 1994).

A state space model is a dynamic system that, first, includes the representation of the observed time series y_t by the unobserved state vector ξ_t. Second, the dynamic representation of ξ_t is usually a Markov process, or an *AR(1)*, where ξ_t only depends on its history ξ_{t-1} as some innovation. Therefore, the *spate-space representation* of the dynamics of y_t includes the following system of two equations.

The measurement (observation) equation

$$y_t = \Gamma \xi_t + \epsilon_t$$

The state transition equation

$$\xi_t = \Phi \xi_{t-1} + C\mu_{t-1} + \eta_t$$

The vector μ is the mean vector while both the measurement error ϵ_t and the state error η_t are *i.i.d.* white noises with

$$\epsilon_t \sim N(0, H), \quad \text{and} \quad \eta_t \sim N(0, Q).$$

The *measurement equation* illustrates the linear relationship between the observed time series y_t and the unobserved state ξ_t. The *state transition equation* demonstrates the evolution of the state vector from $t-1$ to t. It is clear that there are several latent variables that have to be estimated as part of this dynamic model. Therefore, we rely upon a widely used and very effective method called the Kalman filter.

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5.2.2 Kalman Filter

The *Kalman Filter* is a method used to estimate the latent variables of a linear state space representation. To this end, this filter constructs the corresponding log-likelihood function and proceeds to estimate those latent variables using the maximum log-likelihood method.

The *Kalman Filter* starts from estimating ξ_t given the initial condition of ξ_0, the observed time series of the measurement vector y_t, as well as the information contained on the matrices Γ, Φ, C, H, Q. The *Kalman Filter* includes two steps: *prediction*, and *update* (Kim & Bang 2018). Suppose we are in period $t-1$ and consider the state vector ξ_{t-1} and its covariance matrix Σ_{t-1}, and we update in period t. Next, we present the two steps of the Kalman Filter algorithm.

Prediction

$$\xi_{t|t-1} = \Phi\xi_{t-1} + C\mu_{t-1} \quad \text{predicted state estimate}$$

$$\Sigma_{t|t-1} = \Phi\Sigma_{t-1}\Phi' + Q \quad \text{predicted state error covariance}$$

Update

$$\xi_{t|t} = \xi_{t|t-1} + K_t u_t \quad \text{updated state estimate}$$

$$\Sigma_{t|t} = (I - K_t\Gamma)\Sigma_{t|t-1} \quad \text{updated state error covariance}$$

where $u_t = y_y - \Gamma\xi_{t|t-1}$ is the measurement residual, and $K_t = \Sigma_{t|t-1}\Gamma'(H + \Gamma\Sigma_{t|t-1}\Gamma')^{-1}$ is called the *Kalman gain*. Due to the Kalman filter's recursive nature, we need to set initial guesses for the state vector and error covariance matrix.

Kalman filter provides all parameters needed to set up the log-likelihood function. Due to its complexity, however, we use the prediction error decomposition format of the log-likelihood function to simplify the estimation. Therefore, the log-likelihood function using this decomposition is given by

$$\begin{aligned}\mathcal{L}(Y_t|\omega) &= \sum_{t=1}^{T} \mathcal{L}(y_t|Y_{t-1}, \omega) \\ &= -\sum_{t=1}^{T}\left(\frac{N}{2}\log 2\pi + \frac{1}{2}\log|\Omega_{t|t-1}| + \frac{1}{2}u_t'\Omega_{t|t-1}^{-1}u_t\right)\end{aligned}$$

where N is the sample size of the time series, ω is the vector containing the unknown parameters, $Y_t = (y_1, \ldots, y_N)'$, $u_t = y_t - y_{t-1} = y_t - \Gamma\xi_{t|t-1}$, and $\Omega_{t|t-1} = H + \Gamma\Sigma_{t|t-1}\Gamma'^{-1}$. We perform maximum likelihood estimation by optimising the above function to estimate the unknown parameters.

We have now presented the general method of modelling and estimating latent factor models. Subsequently, we proceed with the application for the specific dynamic model for yields.

5.2.3 Dynamic Nelson-Siegel Model

We first need to retrieve estimates of the slope and curvature factors of the yield curve from the data. To achieve this, we fit the dynamic term structure Nelson and Siegel (1987) model (DNSM) into our zero-coupon interest rates data set, as is done in the literature (Argyropoulos & Tzavalis 2016). We fit the model separately for each country we analyse (in our case, solely the US). Through this model, we decompose the term spreads into the slope and curvature factors of the yield curve, by writing interest rates $r_{it}(\tau)$ in state-space form as follows:

$$r_{it}(\tau) = l_{it} + s_{it}(\frac{1-e^{-\lambda_i \tau}}{\lambda_i \tau}) + c_{it}(\frac{1-e^{-\lambda_i \tau}}{\lambda_i \tau} - e^{-\lambda_i \tau}), \quad \forall i. \tag{1}$$

where $\tau = \tau_1, \tau_2, \ldots, \tau_n$ denote different maturity intervals, and l_{it}, s_{it} and c_{it} are latent variables which denote the three factors spanning the term structure of interest rates r_{it}, for all τ. Specifically, l_{it} represents the level factor of the yield curve that captures parallel shifts to $r_{it}(\tau)$, for all τ, which is usually tied to changes in long-run expectations about inflation (Argyropoulos & Tzavalis 2016). Next, the slope factor of the yield curve, s_{it}, that indicates the impact of changes in future business cycles upon r_{it}, converges to unity, as $\tau \to 0$, and to zero, as $\tau \to \infty$ (Argyropoulos & Tzavalis, 2016). Last, c_{it} represents the curvature factor of the yield curve. Conversely with the slope, the curvature converges to zero as $\tau \to 0$ and $\tau \to \infty$, which renders it concave in τ (Argyropoulos & Tzavalis 2016). The impact of the curvature upon $r_{it}(\tau)$ is claimed to be stronger in short- and medium-term maturity interest rates (Argyropoulos & Tzavalis 2016; Christensen et al. 2010). The structural parameter of the DNSM λ_i determines the exponentially decaying effects of factors s_{it} and c_{it} on $r_{it}(\tau)$.

It is worthy to note here, that the dynamic Nelson and Siegel model is used to better capture the progression of the bond market time by time, and thus is an advancement of the *classical Nelson and Siegel model* by adding a dynamic behaviour to the latent factors, and is formed as follows:

$$r_{it}(\tau) = l_i + s_i(\frac{1-e^{-\lambda_i \tau}}{\lambda_i \tau}) + c_i(\frac{1-e^{-\lambda_i \tau}}{\lambda_i \tau} - e^{-\lambda_i \tau}), \quad \forall i.$$

We observe that the only difference with its dynamic version is that in the classical model, the latent term structure factors are not permitted to change over time. The dynamic version allows for the yield curve factors to be *time-varying parameters* and not constant throughout time, as this modelling has been empirically observed to better capture the dynamics of the yield curve (Diebold & Li 2006).

Following, we take the spread between two interest rates, $r_{it}(\tau_l)$ and $r_{it}(\tau_s)$, with long- and short-end maturity intervals respectively, say τ_l and τ_s. Equation 1 implies that the term spread $spr_{it}(\tau_l, \tau_s)$ is:

$$spr_{it}(\tau_l, \tau_s) = r_{it}(\tau_l) - r_{it}(\tau_s) = \gamma_{si}^{(\tau_l,\tau_s)} s_{it} - \gamma_{ci}^{(\tau_l,\tau_s)} c_{it} \tag{2}$$

for all i, where

$$\gamma_{si}^{(\tau_l,\tau_s)} = [(\frac{1-e^{-\lambda_i \tau_l}}{\lambda_i \tau_l}) - (\frac{1-e^{-\lambda_i \tau_s}}{\lambda_i \tau_s})]$$

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and

$$\gamma_{ci}^{(\tau_l,\tau_s)} = [(\frac{1-e^{-\lambda_i \tau_l}}{\lambda_i \tau_l} - e^{-\lambda_i \tau_l}) - (\frac{1-e^{-\lambda_i \tau_s}}{\lambda_i \tau_s} - e^{-\lambda_i \tau_s})].$$

We observe that the term spread is solely determined by the slope and curvature factors, as the level factor is cancelled out. The slope and curvature coefficients of (4) depend on maturity intervals τ_s and τ_l, and parameter λ_i. After estimating λ_i from the data, we proceed extracting γ_{si} and γ_{ci}. These two illustrate the speed with which a change in the slope and curvature factors impacts $spr_{it}(\tau_l, \tau_s)$ can deteriorate (Argyropoulos & Tzavalis 2016).

To retrieve estimates of factors, s_{it} and c_{it}, we fit the DNSM into the interest rate data set, by applying the Kalman filter. To this end, we write the measurement (3) as follows:

$$r_{it} = \Gamma_i(\lambda_i)x_{it} + \epsilon_{it} \tag{3}$$

where $r_{it} = (r_{it}(\tau_1), r_{it}(\tau_2), \ldots, r_{it}(\tau_N))'$, and N denotes the number of different maturity intervals used in the estimation. In our case N = 19, with the following maturities included: $\tau = (3, 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108, 120, 240, 360)$, indicated in months. $\Gamma()$ is an $(N \times 3)$-dimension matrix of loading coefficients, defined as

$$\Gamma_i(\lambda_i) = \begin{bmatrix} 1 & (\frac{1-e^{-\lambda_i \tau_1}}{\lambda_i \tau_1}) & (\frac{1-e^{-\lambda_i \tau_1}}{\lambda_i \tau_1} - e^{-\lambda_i \tau_1}) \\ 1 & (\frac{1-e^{-\lambda_i \tau_2}}{\lambda_i \tau_2}) & (\frac{1-e^{-\lambda_i \tau_2}}{\lambda_i \tau_2} - e^{-\lambda_i \tau_2}) \\ \vdots & \vdots & \vdots \\ 1 & (\frac{1-e^{-\lambda_i \tau_N}}{\lambda_i \tau_N}) & (\frac{1-e^{-\lambda_i \tau_N}}{\lambda_i \tau_N} - e^{-\lambda_i \tau_N}) \end{bmatrix},$$

where $\epsilon_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \Sigma_\epsilon)$, and $x_{it} = (l_{it}, s_{it}, c_{it})'$ is the vector of state variables. Vector x_{it} is assumed that follows a vector autoregressive process of lag order one, i.e.,

$$\begin{bmatrix} l_{it} \\ s_{it} \\ c_{it} \end{bmatrix} = \begin{bmatrix} \mu_l \\ \mu_s \\ \mu_c \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix} \begin{bmatrix} l_{it-1} \\ s_{it-1} \\ c_{it-1} \end{bmatrix} + \begin{bmatrix} u_{it}^l \\ u_{it}^s \\ u_{it}^c \end{bmatrix}, \tag{4}$$

or

$$x_{it} = \mu + \Phi x_{it-1} + u_{it} \tag{5}$$

where $u_{it} = (u_{it}^l, u_{it}^s, u_{it}^c)'$, with $u_{it} \overset{i.i.d.}{\sim} \mathcal{N}(0, \Sigma_u)$. Then, (3) and (5) constitute a state space system of equations whose error term vector is given as

$$\begin{bmatrix} \epsilon_{it} \\ u_{it} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_\epsilon & 0 \\ 0 & \Sigma_u \end{bmatrix}\right),$$

where Σ_ϵ and Σ_u are the variance–covariance matrices of error terms ϵ_{it} and u_{it}, respectively. The error ϵ_{it} and u_{it} will be assumed to be uncorrelated, as is done in the literature (Argyropoulos & Tzavalis 2016; Diebold & Li 2006).

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5.2.4 Cointegration Theory

We now turn to the theory of cointegration, and whereupon, the theory of the error correction model which we will use in the empirical analysis in the next chapter. The theoretical framework used follows the classic textbook of Hamilton (1994).

We start our analysis with two economic series y_{1t} and y_{2t}, which are first-difference stationary or $I(0)$, and we state the following definition:

Cointegration

Cointegration means that there is one linear combination between y_{1t} and y_{2t} (i.e., a structural relationship) which yields a stationary process $I(0)$, i.e.,

$$\beta_1 y_{1t} + \beta_2 y_{2t} = u_t \tag{6}$$

where u_t is stationary ($I(0)$), or by normalising at y_{1t},

$$y_{1t} - \beta y_{2t} = u_{1t}$$

where $\beta = -\frac{\beta_2}{\beta_1}$ and $u_{1t} = \frac{1}{\beta_1} u_t$,

$$y_{1t} = \beta y_{2t} + u_{1t} \tag{7}$$

This definition can be extended to include more than one regressors and cointegrating regressions:

Definition of r cointegrated vectors

Let a vector of M $I(0)$ economic variables $y_t = (y_{1t}, y_{2t}, \ldots, y_{Mt})'$. If there are r cointegrated relationships, with $r < M$, then the errors

$$u_t = \beta' y_t \quad \text{or} \quad \begin{bmatrix} u_{1t} \\ u_{2t} \\ \vdots \\ u_{rt} \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} & \ldots & \beta_{1M} \\ \beta_{21} & \beta_{22} & \ldots & \beta_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{r1} & \beta_{r2} & \ldots & \beta_{rM} \end{bmatrix} \begin{bmatrix} y_{1t} \\ y_{2t} \\ \vdots \\ y_{Mt} \end{bmatrix}, \quad \text{are } I(0).$$

The matrix β is $r \times M$ and its row-vectors denoted as $\beta_i' = (\beta_{i1}, \beta_{i2}, \ldots, \beta_{iM})'$, $\quad i = 1, 2, \ldots, r$ are called cointegrating vectors.

Testing for Cointegration

There are two widespread used methods of testing whether a regression is a cointegrating one. The first, known as the Engle-Granger Two-Step Test, is a very simple process that involves carrying out an ADF

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test on the residuals of the static regression. More specifically, consider two variables y_t and x_t, where $y_t, x_t \sim I(1)$, and the following regression:

$$y_t = \alpha + \beta x_t + u_t.$$

Step 1: We carry out OLS to estimate $y_t = \hat{\alpha} + \hat{\beta} x_t + \hat{u}_t$ and capture the residuals $\hat{u}_t$.

Step 2: We run an ADF test on the auxiliary regression of $\hat{u}_t$,

$$\hat{u_t} = \phi \hat{u_{t-1}} + \epsilon_t,$$

under the null hypothesis $H_0 : \phi = 1$ against the alternative $H_a : \phi < 1$. If there is a unit root, then we have no evidence of a cointegrating relationship. In other words, $\hat{u}_t$ is a measure of disequilibrium, and testing for cointegration is equivalent to testing whether $\hat{u}_t$ is stationary.

However, in the presence of issues, such as correlation between u_t and ϵ_t, the Engle-Granger test might provide false results or might not capture a cointegrating relationship. Therefore, in the literature a much stronger method for testing for cointegration is used, that of Johansen [2].

Estimation of Cointegrated Relationships

Much like testing for cointegration, there are numerous methods of estimating a cointegration relationship. We shall, once again, first state the most straight-forward method, proposed by Engle-Granger, and subsequently the more accurate method of Johansen.

The static OLS framework

The simplest method for estimating cointegration relationships uses the static OLS framework. Consider two variables y_t and x_t, where $y_t, x_t \sim I(1)$, and the following cointegrating regression:

$$y_t = \alpha + \beta x_t + u_t$$

$$\Delta x_t = e_t.$$

Here, Δ denotes the difference operator. The cointegrating vector is $CV = (1 \quad -\beta)'$ and $u_t \sim I(0)$. Under the null hypothesis of existence of a cointegrating relationship, i.e., $H_0 : u_t \sim I(0)$, we can estimate the slope coefficients α and β using OLS. We make two remarks:

1: Both OLS estimators $\hat{\alpha}_{OLS}$ and $\hat{\beta}_{OLS}$ do not converge to a normal distribution, but rather a non-standard one.

2: The LS estimator of slope coefficient $\hat{\beta}_{OLS}$ is said to be super-consistent.

[2]See Appendix A for a full presentation of the Granger Representation Theorem.

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Obviously, the non-normality of the distribution of the OLS estimators, renders making inference about the estimates of cointegrating vector coefficients problematic. Furthermore, under this framework there is correlation between e_t and u_t, which reflects the bias of the LS estimator $\hat{\beta}_{OLS}$ in small samples.

The dynamic OLS framework

To reduce this bias in the limiting distribution of $\hat{\beta}_{OLS}$ that arises from the correlation between e_t and u_t, Stock & Watson (1993) suggested augmenting the static regression model by extra stationary differenced-terms. Specifically, they proposed including dynamic lead and lagged terms of Δx_t in the RHS to capture the small sample bias effects in estimating the cointegrating relationships, i.e.,

$$y_t = \alpha + \beta x_t + \sum_{s=-p}^{p} \gamma_s \Delta x_{t-s} + u_t \tag{8}$$

For instance, for $p = 2$, equation (8) implies

$$y_t = \alpha + \beta x_t + \gamma_{-2}\Delta x_{t+2} + \gamma_{-1}\Delta x_{t+1} + \gamma_0 \Delta x_t + \gamma_1 \Delta x_{t-1} + \gamma_2 \Delta x_{t-2} + u_t$$

The above framework is known as dynamic OLS, or *DOLS*.

Multivariate models and cointegration: the Johansen's procedure

Now, we consider the generalisation of the aforementioned analysis, for multivariate models, with more than two variables considered. Consider the following VAR(1) model in reduced form with n variables:

$$y_t = \delta + Ay_{t-1} + e_t$$

where A is the coefficient matrix, e_t is the innovation vector and δ is the drift term. The stationarity condition requires all eigenvalues of A to be within the unit root circle. The eigenvalues of A are the roots of the n_{th} order characteristic polynomial $|A - \lambda I_n|$ obtained by solving the characteristic equation $|A - \lambda I_n| = 0$. For stability, λ_1 and λ_2 must be less than one in modulus.

The above VAR(1) can be transformed in error correction form (more on this on the next section) as such:

$$\begin{aligned} y_t &= \delta + Ay_{t-1} + e_t \\ &\Leftrightarrow y_t - y_{t-1} = \delta + Ay_{t-1} - y_{t-1} + e_t \\ &\Leftrightarrow \Delta y_t = \delta - (I_n - A)y_{t-1} + e_t \\ &\Leftrightarrow \Delta y_t = \delta + \Pi y_{t-1} + e_t \end{aligned}$$

where $\Pi = -(I_n - A)$ is an $n \times n$ matrix.

Similarly, consider the VAR(2) model, that can be presented in error correction form as follows:

$$\begin{aligned} y_t &= \delta + A_1 y_{t-1} + A_2 y_{t-2} + e_t \\ &\Leftrightarrow y_t - y_{t-1} = \delta + A_1 y_{t-1} - y_{t-1} + +A_2 y_{t-1} - A_2 y_{t-1} + A_2 y_{t-2} + e_t \\ &\Leftrightarrow \Delta y_t = \delta - (I_n - A_1 - A_2) y_{t-1} - A_2 \Delta y_{t-1} + e_t \\ &\Leftrightarrow \Delta y_t = \delta + \Pi y_{t-1} + \Gamma \Delta y_{t-1} + e_t \end{aligned}$$

where $\Pi = -(I_n - A_1 - A_2)$ and $\Gamma = -A_2$.

This can be generalised for the VAR(p) as follows:

$$\begin{aligned} y_t &= \delta + A_1 y_{t-1} + A_2 y_{t-2} + \cdots + A_p y_{t-p} + e_t \\ \Leftrightarrow \Delta y_t &= \delta + \Pi y_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta y_{t-j} + e_t \end{aligned}$$

where $\Pi = -(I_n - \sum_{j=1}^{p} A_i)$ and $\Gamma_j = -A_{j+1}, \quad j = 1, 2, \ldots, p-1$.

Testing for cointegration with Johansen's method

The above representation is important because the rank of the matrix $\Pi_{n \times n}$ tells us how many independent linear combinations there are amongst the variables, where

$$0 \leq rank(\Pi_{n \times n}) \leq n.$$

The rank of Π can only be 0 if Π is a null matrix, where in this case we just have a VAR in first differences, without further level terms. Hence, variables are indeed $I(1)$ but not cointegrated. Conversely, if Π has full rank, then this means that there are n linearly independent combinations of the variables that are stationary, which implies that we can solve for each of the n variables in a unique fashion and obtain stationary relationships. In other words, all variables are $I(0)$ and have a VAR in levels.

However, if there are $I(1)$ variables, then the rank of Π must be strictly

$$0 < rank(\Pi_{n \times n}) < n.$$

Suppose $rank(\Pi_{n \times n}) = r$. This states that there are r linearly independent combinations among the n variables, which are $I(0)$, and thus those are the cointegrating relationships. Additionally, in order for $rank(\Pi_{n \times n}) = r$, Π must have r non-zero eigenvalues. Therefore, to examine the number of cointegrating relationships or vectors, is equivalent to examining the number of non-zero eigenvalues of the Π matrix.

The critical values for testing the rank of the Π matrix have been tabulated by Johansen (1988) and Johansen & Juselius (1990). There are two tests that have been proposed to conduct this examination, as will be discussed below.

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Let the theoretical eigenvalues of the Π matrix in decreasing order be

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$$

The idea put forward by Johansen (1988) is that if $rank(\Pi_{n\times n}) = r$, then $log(1-\lambda_j) = 0, \quad j = r+1, r+2, \ldots, n$ for the smallest $n-r$ eigenvalues of Π.

Let $\hat{\lambda_1} \geq \hat{\lambda_2} \geq \ldots \geq \hat{\lambda_n}$ be the estimated eigenvalues. The following two methods are used to test for the number of cointegrated relationships.

a) The $\hat{\lambda_{max}}$ *test* is used to test the null hypothesis of h cointegrating relations against the alternative of $h+1$ cointegrating relations. The maximum eigenvalue statistic is defined as:

$$\hat{\lambda_{max}} = -Tlog(1-\hat{\lambda_{r+1}}), \quad r = 0, 1, \ldots, n-1$$

with

$$H_0 : rank(\Pi) \leq r \quad vs \quad H_a : rank(\Pi) = r+1$$

and tests whether $\hat{\lambda_{r+1}}$ is significantly different from zero.

b) The *Trace test* is used to test the null hypothesis of r cointegrating relationships against the alternative of n cointegrating relationships (as many as the variables in vector y_t). The Trace statistic is defined as:

$$\lambda_{trace} = -T \sum_{i=r+1}^{n} log(1-\hat{\lambda}_i), \quad r = 0, 1, \ldots, n-1$$

with

$$H_0 : rank(\Pi) \leq r \quad vs \quad H_a : rank(\Pi) \geq r+1$$

and tests whether the smallest $n-r$ eigenvalues of Π are significantly different from zero.

For both tests, the asymptotic distribution of the above test statistics are not normal and their critical values are tabulated by Johansen (1988); see also Hamilton (1994).

Both of the above tests are carried out in a sequential manner starting from the smaller eigenvalues and working towards the larger, $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$. The null hypothesis is rejected should the test statistic value is large. Additionally, the eigenvalues that correspond to the non-stationary, $I(1)$, relationships are zero, or $\lambda_i = 0, \quad i = r+1, \ldots, n$. This implies that only the first eigenvalues are non-zero, or $\lambda_i \neq 0, \quad i = 1, \ldots, r$, which suggests the following equivalency of the null hypothesis:

$$H_0 : \text{There are at most } r \text{ cointegrating relationships}$$

is equivalent to

$$H_0 : \lambda_i = 0, \quad i = r+1, \ldots, n.$$

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5.2.5 Vector Error Correction Model (VECM)

Next, we present the theory of the Vector Error Correction Model for which again we follow the textbook of Hamilton (1994).

It is proven that cointegrated series cannot be represented in a VAR framework in first differences[3]. Notwithstanding, cointegrated series can be represented in a Vector Error Correction Model, which includes the disturbance terms of the cointegrating relationships as an error correction term. Assume that y_{2t} is a random walk with

$$y_{2t} = y_{2t-1} + u_{2t}, \quad \Delta y_{2t} = u_{2t} \tag{9}$$

and taking the first difference of (7),

$$\Delta y_{1t} = \beta \Delta y_{2t} + \Delta u_{1t},$$

we have

$$\Delta y_{1t} = \beta u_{2t} + u_{1t} - u_{1t-1}, \quad \text{using} \quad (8)$$

$$\Delta y_{1t} = -u_{1t-1} + \beta u_{2t} + u_{1t}, \quad \text{since} \quad u_{1t-1} = y_{1t-1} - \beta y_{2t-1}$$

we obtain

$$\Delta y_{1t} = -(y_{1t-1} - \beta y_{2t-1}) + \beta u_{2t} + u_{1t} \tag{10}$$

We can write equations (8) and (9) as a system of equations, as follows:

$$\begin{aligned} \Delta y_{1t} &= -(y_{1t-1} - \beta y_{2t-1}) + \epsilon_{1t}, &\quad \text{with} \quad \epsilon_{1t} = \beta u_{2t} + u_{1t} \\ \Delta y_{2t} &= \quad 0 \quad + \epsilon_{2t}, &\quad \text{with} \quad \epsilon_{2t} = u_{2t} \end{aligned}$$

or

$$\begin{aligned} \Delta y_t = \begin{bmatrix} \Delta y_{1t} \\ \Delta y_{2t} \end{bmatrix} &= \begin{bmatrix} -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & -\beta \end{bmatrix} \begin{bmatrix} y_{1t-1} \\ y_{2t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} \\ &= -\alpha \beta' y_{t-1} + \epsilon_t, \quad \alpha = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \beta' = (1, \quad -\beta) \\ &= \Pi y_{t-1} + \epsilon_t, \quad \Pi = -\alpha\beta' \end{aligned}$$

Therefore, we can represent cointegrated series as follows:

Vector Error Correction Model (VECM)

$$\Delta y_t = \Pi y_{t-1} + \epsilon_t, \quad \textit{where} \quad \Pi = \begin{bmatrix} -1 & \beta \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} = \begin{bmatrix} \beta u_{2t} + u_{1t} \\ u_{2t} \end{bmatrix} \tag{11}$$

[3] See appendix for a full proof.

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The term $y_{1t} - \beta y_{2t} = u_{1t}$ is the cointegrating relationship and is referred to as an error correction term. The negative sign of u_{1t-1} in the equation $\Delta y_{1t} = -(y_{1t-1} - \beta y_{2t-1}) + \epsilon_{1t}$ implies that, at time $t-1$, a positive value of $u_{1t-1} = y_{1t-1} - \beta y_{2t-1}$ means that y_{1t-1} is larger than its long-run equilibrium level illustrated by βy_{2t-1}. The structure of the VECM representation, therefore, implies that this will drive a correction of y_{1t} towards its long-run equilibrium level which is βy_{2t}. This is also referred to in the literature as mean-reversion.

VAR in levels and VECM representation under cointegration

Consider a VAR(p) process:

$$y_t = \mu + \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + ... + \Phi_p y_{t-p} + \epsilon_t,$$

or using the lag operator,

$$(I_m - \Phi_1 L - \Phi_2 L^2 - ... - \Phi_p L^p) y_t = \mu + \epsilon_t$$

Using the following lag-polynomial relationship [4],

$$\Phi(L) \equiv I_m - \Phi_1 L - \Phi_2 L^2 - ... - \Phi_p L^p = (I_m - \phi L) - (J_1 L + J_2 L^2 + ... + J_{p-1} L^{p-1})(1 - L),$$

where $\phi = \Phi_1 + \Phi_2 + ... + \Phi_p$ and $J_s = -(\Phi_{s+1} + \Phi_{s+2} + ... + \Phi_p)$, for $s + 1, 2, ..., p-1$, the VAR(p) can be rewritten as

$$(I_m - \phi L) y_t = (J_1 L + J_2 L^2 + ... + J_{p-1} L^{p-1}) \Delta y_t + \mu + \epsilon_t$$

or

$$y_t = J_1 \Delta y_{t-1} + J_2 \Delta y_{t-2} + ... + J_{p-1} \Delta y_{t-p+1} + \phi y_{t-1} + \mu + \epsilon_t$$

Subtracting y_{t-1} from both sides, yields

$$\Delta y_t = J_1 \Delta y_{t-1} + J_2 \Delta y_{t-2} + ... + J_{p-1} \Delta y_{t-p+1} + (\phi - I_n) y_{t-1} + \mu + \epsilon_t$$

or

$$\Delta y_t = J_1 \Delta y_{t-1} + J_2 \Delta y_{t-2} + ... + J_{p-1} \Delta y_{t-p+1} - (I_n - \Phi_1 - \Phi_2 - \ldots - \Phi_p) y_{t-1} + \mu + \epsilon_t$$

or

$$\Delta y_t = J_1 \Delta y_{t-1} + J_2 \Delta y_{t-2} + ... + J_{p-1} \Delta y_{t-p+1} - \Phi(1) y_{t-1} + \mu + \epsilon_t \qquad (12)$$

Cointegration implies that $u_t = \beta' y_t$ are $I(0)$. Furthermore, if we treat L as a scalar and set $L = 1$, then the polynomial $\Phi(L)$ has at least one unit root, with its determinant

$$|I_n - \Phi_1 L - \Phi_2 L^2 - \ldots - \Phi_p L^p| = 0$$

for $L = 1$, and $\Phi(1)$ is a singular matrix. If there are r cointegrating relationships, then $rank[\Phi(1)] = r < n$. Importantly, this implies that $-\Phi(1)$ can be factorised as:

$$-\Phi(1) = -\alpha\beta' = \Pi$$

[4] For a full derivation of this polynomial, see (Hamilton 1994, p. 517)

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where Π is an $n \times n$ matrix, α is an $n \times r$ vector, and β' is an $r \times n$ vector.
Therefore, (12) becomes

$$\Delta y_t = J_1 \Delta y_{t-1} + J_2 \Delta y_{t-2} + ... + J_{p-1} \Delta y_{t-p+1} - \alpha\beta' y_{t-1} + \mu + \epsilon_t \quad (13)$$

where $-\alpha\beta' = \Pi$, or, using the definition of cointegration $u_{t-1} = \beta' y_{t-1}$, we get

$$\Delta y_t = J_1 \Delta y_{t-1} + J_2 \Delta y_{t-2} + ... + J_{p-1} \Delta y_{t-p+1} - \alpha u_{t-1} + \mu + \epsilon_t{}^5.$$

Estimation of cointegrated systems - The Johansen's method

As has been discussed previously, the above VECM representation of cointegrated variables

$$\Delta y_t = J_1 \Delta y_{t-1} + J_2 \Delta y_{t-2} + ... + J_{p-1} \Delta y_{t-p+1} - \Pi y_{t-1} + \mu + \epsilon_t, \quad \Pi = -\alpha\beta'$$

enables testing for the correct number of cointegrated relationships r. But, more than that, it helps us estimate the slope coefficients of these regressions, because for more than one cointegrating relationship, $r > 1$, OLS is inappropriate. To this end, Johansen & Juselius (1990) have developed maximum likelihood estimators. To see this, we write the above VECM representation as

$$\Delta y_t = (\mu, \Pi, J_1, \ldots, J_{p-1}) \begin{bmatrix} 1 \\ y_{t-1} \\ \vdots \\ \Delta y_{t-p+1} \end{bmatrix} + \epsilon_t = \Pi^* x_t + \epsilon_t$$

and using this, we set up the log-likelihood of the VECM as

$$\mathcal{L}(\Omega, \Pi) = -\frac{nT}{2} log(2\pi) - \frac{T}{2} log(|\Omega|) - \frac{1}{2T} \sum_{t=1}^{T} (\Delta y_t - \Pi^* x_t)' \Omega^{-1} (\Delta y_t - \Pi^* x_t)$$

subject to the constraint:

$$\Pi = -\alpha\beta'$$

where β will contain the cointegrating vectors and α will contain the adjustment coefficients.

Maximising the log-likelihood function with respect to $\mu, J_1, \ldots, J_{p-1}$ yields the estimates of $\mu(\Pi), J_1(\Pi), \ldots, J_{p-1}(\Pi)$ by running the following SURE-type regressions:

$$\Delta y_t = \zeta_0 + \zeta_1 \Delta y_{t-1} + \zeta_2 \Delta y_{t-2} + \ldots + \zeta_p \Delta y_{t-p+1} + u_t$$

and

$$y_t = \xi_0 + \xi_1 \Delta y_{t-1} + \xi_2 \Delta y_{t-2} + \ldots + \xi_p \Delta y_{t-p+1} + v_t$$

Crucially, Johansen (1990) has proved that the ML estimate of the cointegrating vectors β is equal to the matrix $\hat{V}$ containing the r eigenvectors $(\hat{v_1}, \ldots, \hat{v_r})$ that correspond to the r largest eigenvalues

[5] See also appendix for the statement of the Granger Representation Theorem.

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$\hat{\lambda_1}, \ldots, \hat{\lambda_r}$. That is,

$$\hat{\beta_{ML}} = (\hat{\beta}_1, \ldots, \hat{\beta}_r) = (\hat{v}_1, \ldots, \hat{v}_r),$$

where $\hat{v}_i$ is the corresponding eigenvector of the eigenvalue $\hat{\lambda_i}$, $i = 1, \ldots, r$. More specifically, the cointegrating vectors β can be computed by calculating the eigenvectors of

$$\hat{\Sigma_{vv}^{-1}} \hat{\Sigma_{vu}} \hat{\Sigma_{uu}^{-1}} \hat{\Sigma_{uv}}$$

where

$$\hat{\Sigma_{vv}} = \frac{1}{T} \sum_{t=1}^{T} \hat{v}_t \hat{v}_t{}' \quad \hat{\Sigma_{uu}} = \frac{1}{T} \sum_{t=1}^{T} \hat{u}_t \hat{u}_t{}' \quad \hat{\Sigma_{uv}} = \frac{1}{T} \sum_{t=1}^{T} \hat{u}_t \hat{v}_t{}' = \hat{\Sigma_{vu}}$$

MA representation of cointegrated variables in 1^{st} differences

Cointegrated variables have a moving average (MA) representation of their 1^{st} differences. To see this, consider (7) and (9),

$$y_{1t} = \beta y_{2t} + u_{1t}$$

and

$$y_{2t} = y_{2t-1} + u_{2t}, \quad \Delta y_{2t} = u_{2t}$$

Taking the first difference of (7)

$$\begin{aligned}
\Delta y_{1t} &= \beta \Delta y_{2t} + \Delta u_{1t} \\
&= \beta u_{2t} + u_{1t} - u_{1t-1}, \quad \text{using (9)} \\
&= \epsilon_{1t} - (\epsilon_{1t-1} - \beta \epsilon_{2t-1}), \quad \text{by defining} \quad u_{2t} = \epsilon_{2t} \quad \text{and} \quad u_{1t} = \epsilon_{1t} - \beta u_{2t} \\
&= (1 - L)\epsilon_{1t} + \beta L \epsilon_{2t}
\end{aligned}$$

where L is the lag operator. Writing them in matrix form,

$$\Delta y_t = \begin{bmatrix} \Delta y_{1t} \\ \Delta y_{2t} \end{bmatrix} = \begin{bmatrix} (1 - L)\epsilon_{1t} + \beta L \epsilon_{2t} \\ \epsilon_{2t} \end{bmatrix} = \begin{bmatrix} (1 - L) & \beta L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} = \Psi(L)\epsilon_t$$

or, in more compact form, the MA(∞) representation in first-differences is

$$\Delta y_t = \Psi(L)\epsilon_t \tag{14}$$

where

$$\Psi(L) = \begin{bmatrix} (1 - L) & \beta L \\ 0 & 1 \end{bmatrix}$$

and

$$\epsilon_t = \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix}$$

This illustrates that if two variables are cointegrated then their first-differences have a vector MA representation, which can be generalised for n variables. Notwithstanding, if we treat L as a scalar variable,

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and evaluate $\Psi(L)$ at $L = 1$, then

$$\begin{bmatrix} (1-1) & \beta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \beta \\ 0 & 1 \end{bmatrix},$$

which has determinant $|\Psi(1)| = 0$, which implies that $\Psi(L)$ has a unit root. Consequently, the MA representation is not invertible, and we, therefore, cannot obtain a VAR in first-differences. That is,

$$\Psi(L)^{-1}\Delta y_t \neq \epsilon_t.$$

This is important as we will use this MA(∞) representation to obtain the impulse responses later.

5.2.6 Impulse Response Function and Forecast Error Variance Decomposition

Following Diebold et. al (2004), we subsequently describe the theoretical framework that will be used in our models for the computation of Impulse Response Functions ($IRFs$) and Variance Decompositions (VDs).

All models in the paper can be presented in a VAR(p) form in levels,

$$y_t = \mu + A_1 y_{t-1} + A_2 y_{t-2} + \ldots + A_p y_{t-p} + \epsilon_t,$$

where y_t is an $n \times 1$ vector of endogenous variables, A_i is the coefficient matrix of lag i, and μ is the constant vector. The residuals ϵ_t follow

$$\epsilon_t \sim N(0, \quad \Omega),$$

where Ω is the variance-covariance matrix, that is not necessarily diagonal. To proceed with the computation of the $IRFs$ and VDs, we must express the VAR(p) model in moving average Wold MA(∞) form as

$$y_t = \mu + \Psi_0 \epsilon_t + \Psi_1 \epsilon_{t-1} + \Psi_2 \epsilon_{t-2} + \ldots = \sum_{i=0}^{\infty} \Psi_i \epsilon_{t-i}, \quad (15)$$

or

$$y_t = \mu + \Psi(L)\epsilon_t,$$

where $\Psi_0 = I_n$, $\Psi_i = J^i$, for $i = 1, 2, \ldots$ and to link the matrices A^h for $h = 1, 2, \ldots, p$ with the MA matrices Ψ_h for $h = 1, 2, \ldots$, we use the following recursive formula as illustrated by Hamilton (1994, p. 303), given as

$$\Psi_h = A_1 \Psi_{h-1} + A_2 \Psi_{h-2} + \ldots + A_p \Psi_{h-p}$$

with $\Psi_0 = 1$ and $\Psi_h = 0$ for $h < 0$.

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Impulse Response Functions

Following Diebold et. al (2006), we define the IRF of the system as the responses of the endogenous variables to one unit shocks in the innovation terms. Intuitively, given that all the variables in our system are in percentage terms, we present impulse responses as a one percentage point shocks to the innovation terms.

Usually, the variance-covariance matrix of the residuals is not diagonal, that is the innovation terms ϵ_{jt} are not uncorrelated with each other and not independent, or *iid*. Therefore, those shocks cannot be viewed as fundamental (or structural) economic shocks, and thus we would have to proceed with some factorisation to orthogonalise the innovation terms ϵ_{jt} to render them uncorrelated, and subsequently view them as fundamental shocks. As is often used in practice (Diebold et al. 2003), we will employ the Cholesky decomposition, which is a special triangular factorisation, that not only orthogonalises the vector of innovations ϵ_t to obtain a vector u_t with independent (orthogonal) innovations, but also normalises the structural shocks to have unit-variance. This way, we can measure the effects of a unit increase in the structural shocks.

Therefore, we consider two cases:

1. Diagonal Ω

   The MA(∞) representation enables us to to compute the response of a variable y_{jt+h} for $h = 0, 1, 2, \ldots$ of vector y_{t+h} with respect to innovation ϵ_{jt} of vector ϵ_t as the change at time t projection of y_{jt+h} as follows

   $$IRF(h)_{ji} \equiv \frac{\partial \hat{y_{jt+h}}}{\partial \epsilon_{it}} = [\Psi_h]_{ji}, \quad \text{for} \quad h = 1, 2, \ldots,$$

   where $[\Psi_h]_{ji}$ is the (j, i) element of the corresponding matrix. Alternatively, if we collect all possible combinations (j, i) in matrix form, we can present the above as

   $$IRF(h) \equiv \frac{\partial \hat{y_{t+h}}}{\partial \epsilon_t'} = \Psi_h, \quad \text{for} \quad h = 1, 2, \ldots$$

2. Non-Diagonal Ω

   As mentioned above, when Ω is not diagonal, we cannot compute the IRFs using the original innovation terms ϵ_{jt} as these are not structural economic shocks. Therefore, we use the Cholesky decomposition to obtain a lower-triangular matrix P that satisfies:

   $$\Omega = \Sigma_\epsilon = PP'$$

   Then the normalised structural errors, that we denote as u_t, are given as

   $$u_t = P^{-1}\epsilon_t$$

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which implies that

$$Pu_t = \epsilon_t$$

To see the existence of unit-variance,

$$\begin{aligned}\Sigma_u &\equiv Var(u_t)\\ &= \mathbb{E}(u_t u_t')\\ &= \mathbb{E}(P^{-1}\epsilon_t\epsilon_t' P'^{-1})\\ &= P^{-1}\mathbb{E}(\epsilon_t\epsilon_t')P'^{-1}\\ &= P^{-1}\Sigma_\epsilon P'^{-1}\\ &= P^{-1}PP'P'^{-1}\\ &= I_n\end{aligned}$$

The IRFs, defined as the response of y_{t+h} to a one unit shock to u_t, are given as

$$IRF(h) \equiv \frac{\partial \hat{y_{t+h}}}{\partial u_t'} = \Psi_h P$$

Variance Decompositions

Following Hamilton (1994, pp. 323-324) and Diebold (2006) we present the framework for deriving the variance decomposition. First, to identify the forecast error s periods ahead into the future, we use

$$y_{t+s} - \hat{y_{t+s|t}} = \Psi_0\epsilon_{t+s} + \Psi_1\epsilon_{t+s-1} + \ldots + \Psi_{s-1}\epsilon_{t+1}$$

The mean squared error of this s-period-ahead forecast is

$$\begin{aligned}MSE(\hat{y_{t+s|t}}) &= \mathbb{E}[(y_{t+s} - \hat{y_{t+s|t}})(y_{t+s} - \hat{y_{t+s|t}})']\\ &= \Omega + \Psi_1\Omega\Psi_1' + \Psi_2\Omega\Psi_2' + \ldots + +\Psi_{s-1}\Omega\Psi_{s-1}'\end{aligned}$$

where

$$\Omega = \mathbb{E}(\epsilon_t\epsilon_t')$$

The contribution of the j^{th} variable to the MSE of the s-period-ahead forecast, for the case of diagonal variance-covariance matrix Ω is

$$MSE(s)_j = Var(\epsilon_{jt})[I_n I_n' + \Psi_1 I_n I_n'\Psi_1' + \ldots + \Psi_{s-1} I_n I_n'\Psi_{s-1}'].$$

Alternatively, for the case of the non-diagonal Ω employing the Cholesky decomposition and using the P factor, the MSE is given as

$$MSE(s)_j = [P_j P_j' + \Psi_1 P_j P_j'\Psi_1' + \ldots + \Psi_{s-1} P_j P_j'\Psi_{s-1}'].$$

The MSE of the s-period-ahead forecast can be written as the sum of n terms, and is given by

$$MSE(s) = \sum_{j=1}^{n} MSE(s)_j$$

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where both $MSE(s)$ and $MSE(s)_j$ are $n \times n$ matrices. We define the VD at horizon s as the fraction of the MSE of the i^{th} variable due to shocks to the j^{th} as

$$VDC(s)_{ji} = \frac{MSE(s)_{ji}}{MSE(s)_i} \tag{16}$$

where $MSE(s)_{ji}$ and $MSE(s)_i$ denote the (i, i) elements of the corresponding matrices.

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6 Results

In this chapter we present the results of the empirical analysis. First, we start with the estimation of the factors of the yield curve from the DNS model. Next, we proceed with the results of the cointegration analysis, before presenting the vector error correction model (VECM) analysis and estimation. Finally, the impulse responses (IRFs) and variance decompositions (VDs) are presented.

6.1 DNSM Analysis

In this section, we present the estimation results of the Dynamic Nelson-Siegel Model (DNSM), which extracts the level, slope, and curvature factors from the yield curve. These factors will subsequently serve as indicators of macroeconomic conditions and monetary policy stance. Our analysis follows the methodology outlined in Argyropoulos & Tzavalis (2016), applying a state-space framework with Kalman filtering to dynamically estimate the latent factors.

In Figures 3A-C, we graphically present estimates of factors l_{it}, s_{it}, and c_{it}. In Table 1, we present some descriptive statistics of these estimates of l_{it}, s_{it}, and c_{it}, and in Table 2, the cross-correlation coefficients for those factors. In Table 3, we report estimates for the parameters of the DNS model.

More specifically, Table 1 presents the summary statistics of the three estimated yield curve factors: level (l_{it}), slope (s_{it}), and curvature (c_{it}). The mean values suggest that the level factor exhibits the highest persistence, which aligns with its role in capturing the long-term average yield. The slope factor, which reflects expectations about future economic growth, displays significant variation, with a standard deviation of 2.69. The curvature factor, associated with short- to medium-term yield movements and monetary policy adjustments, shows the highest volatility, with a standard deviation of 6.38. The autocorrelation coefficients at various lags confirm the persistence of these factors. The level factor retains high autocorrelation over time, reflecting its long-run stability. The slope factor demonstrates strong persistence at one-year ($\rho(12) = 0.82$) and two-year ($\rho(24) = 0.71$) lags, indicating that its impact on economic conditions extends beyond short horizons. The curvature factor, on the other hand, shows a somewhat more rapid decay in autocorrelation ($\rho(12) = 0.78$, $\rho(24) = 0.46$), reinforcing its role as a short-term adjustment mechanism.

The cross-correlation matrix presented in Table 2, provides insights into the relationships between the three factors. The level and slope factors exhibit almost no correlation (-0.03), suggesting that long-term yield shifts do not necessarily translate into changes in the slope. The slope and curvature factors are positively correlated (0.64), indicating that shifts in business cycle expectations tend to coincide with

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short-term policy-driven fluctuations in the yield curve. The negative correlation between the level and curvature factors (-0.62) suggests that periods of higher long-term yields are typically associated with reduced short-term yield volatility.

The Kalman filter estimates illustrated in Table 3, provide key insights into the dynamic behaviour of the extracted yield curve factors. The estimated transition matrix Φ suggests that all three factors exhibit strong persistence, with diagonal elements close to unity. The slope and curvature factors display minor feedback effects, with the slope factor influencing future curvature changes ($\phi_{23} = 0.0079$). The estimated drift terms (μ) reveal that the level factor tends to increase slightly over time ($\mu = 0.0567^*$), while the curvature factor shows a negative trend (-0.0709), suggesting a gradual reduction in short-term yield curve adjustments.

Figure 3 presents the estimated trajectories of the level, slope, and curvature factors over time. Consistent with theoretical expectations, the slope factor fluctuates in response to business cycle dynamics, while the curvature factor displays sharper, shorter-lived adjustments. Figure 4 and Figure 5 further illustrate the factor loading coefficients across different maturities. The loading coefficients for the slope factor (s_{it}) decrease as maturity increases, confirming that short-term rates respond more significantly to changes in economic conditions. The loading coefficients for the curvature factor (c_{it}) peak at intermediate maturities, reinforcing its role in capturing medium-term fluctuations.

The results align with the findings of Argyropoulos & Tzavalis (2016) and Diebold & Li (2006), which suggest that the slope factor serves as a leading indicator of economic growth, with higher values associated with expansions and lower values predicting recessions. The curvature factor primarily reflects shifts in monetary policy stance, with increases indicating easing measures and decreases suggesting tightening. These findings confirm that the slope and curvature factors contain richer predictive information about macroeconomic conditions than the simple term spread, supporting the argument that decomposing the yield curve into its latent components enhances forecasting accuracy (Argyropoulos & Tzavalis 2016).

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Table 1: Descriptive statistics of the term structure (yield curve) factors.

Factor	Mean	St. dev	$\rho(1)$	$\rho(12)$	$\rho(24)$
Level	1.55	1.84	0.98	0.69	0.37
Slope	2.30	2.69	0.99	0.82	0.71
Curvature	10.18	6.38	0.99	0.78	0.46

Notes: The table presents descriptive statistics of the estimates of term structure (yield curve) factors l_{it}, s_{it}, and c_{it}, namely their mean, standard deviation, and autocorrelation coefficients of one month, one and two years.

Table 2: Cross-correlation coefficients of the estimated factors.

	Level	Slope	Curvature
Level	1.00	-0.03	-0.62
Slope		1.00	0.64
Curvature			1.00

Notes: The table presents values of the cross-correlation coefficients of level, slope and yield curve factors (l_{it}, s_{it}, and c_{it}), for the case of the US considered in our analysis.

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Table 3: Kalman Filter Estimates of (5).

	Φ	
1.0011***	-0.0055	-0.0061
(0.0060)	(0.0057)	(0.0112)
-0.00866^{*}	0.9930***	0.0079
(0.0041)	(0.0040)	(0.0077)
-0.0030	0.0020	1.0016***
(0.0022)	(0.0021)	(0.0042)
	μ	
0.0567*	-0.0218	-0.0709
(0.0246)	(0.0237)	(0.0464)
	Σ_η	
0.0232	0.0081	-0.0032
(0.001)	(0.001)	(0.001)
	0.0215	-0.0159
	(0.001)	(0.001)
		0.0823
		(0.002)
	Adjusted $\overline{\mathrm{R}^2}$	
0.9931	0.997	0.998

Notes: The table presents estimates of model (5), for the United States (US). Our sample consists of 515 monthly observations from 1982:01 to 2024:11. Standard errors are reported in parentheses. $^{***}p < 0.001$, $^{*}p < 0.05$.

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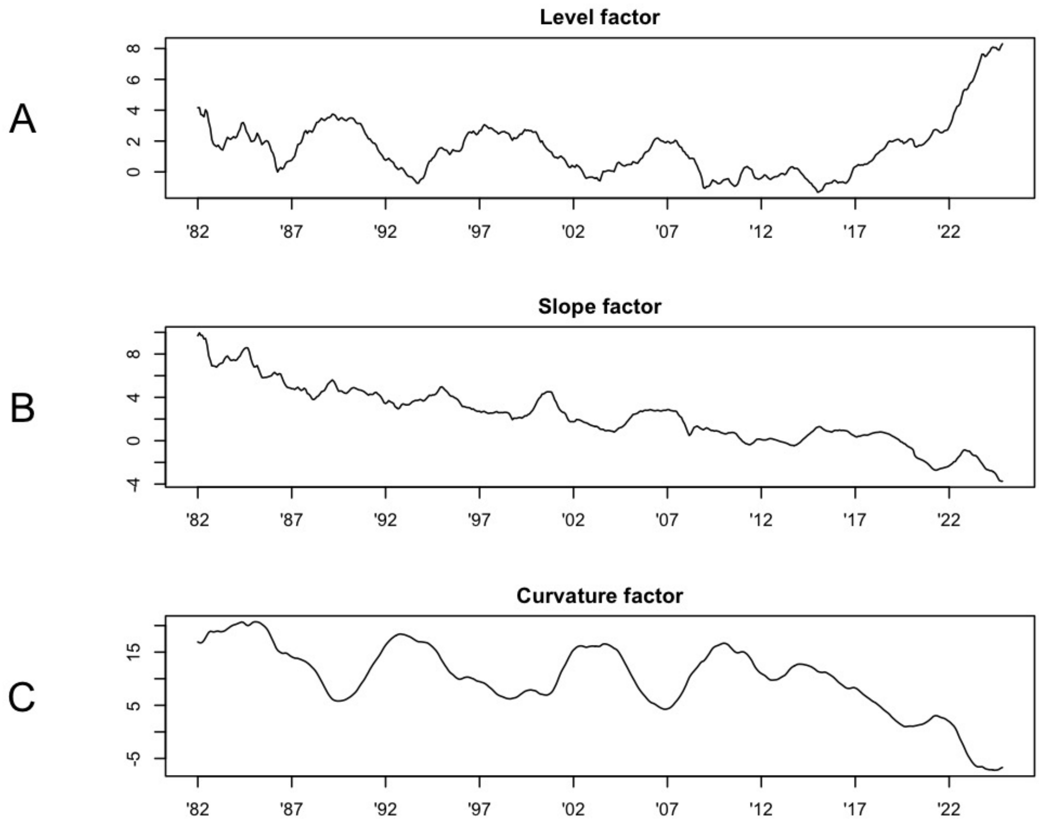

Figure 3: A: Estimates of level factors l_{it}. B: Estimates of slope factors s_{it}. C: Estimates of curvature factors c_{it}.

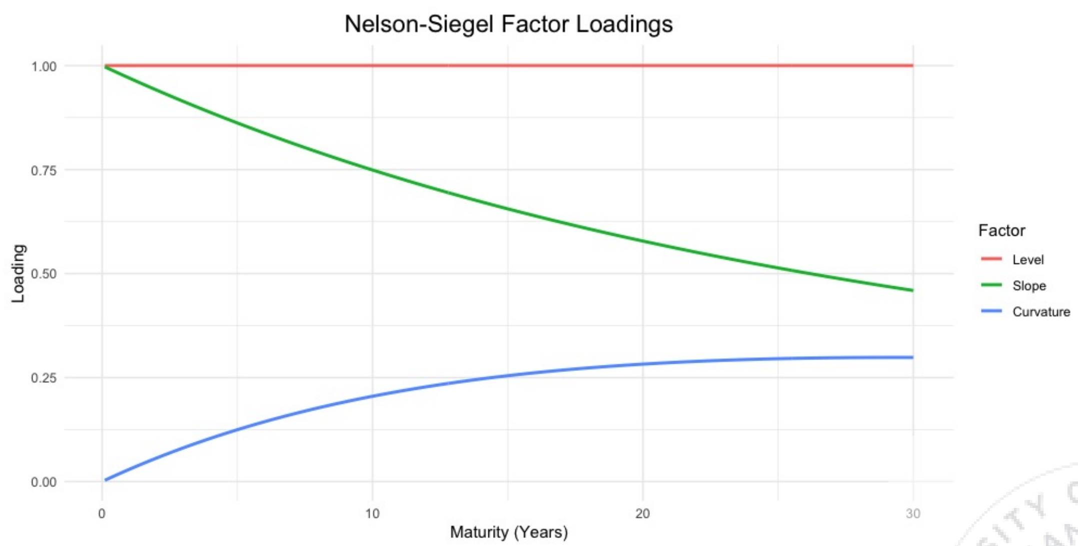

Figure 4: Factor loading coefficients

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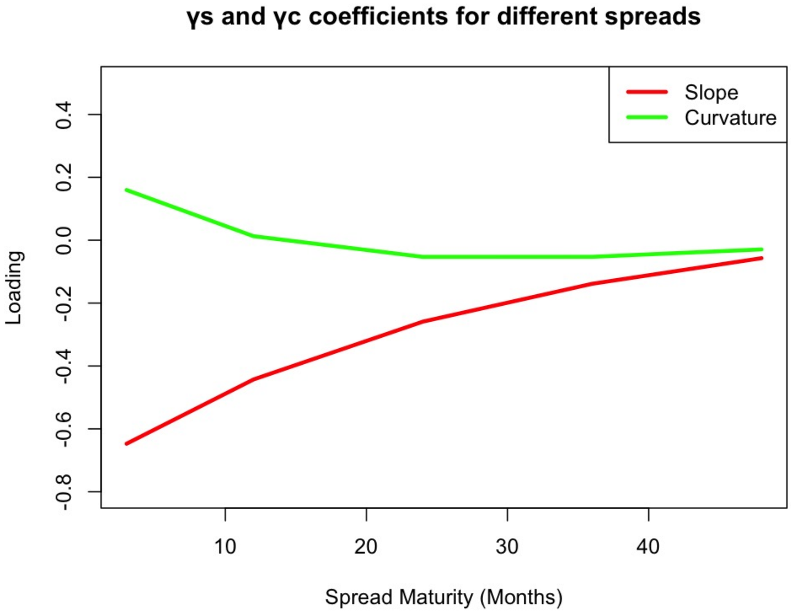

Figure 5: Factor loading coefficients $\gamma_{si}^{(\tau_l,\tau_s)}$ and $\gamma_{ci}^{(\tau_l,\tau_s)}$.

6.2 Cointegration Analysis

Before delving into the cointegration analysis, we first proceed with standard stationarity tests, namely Augmented Dickey-Fuller, Phillips–Perron, and Kwiatkowski–Phillips–Schmidt–Shin, for ipi_t, $m2_t$, s_t, and c_t. As illustrated in Table 4, all four variables are non-stationary in levels. Therefore, we proceed with taking the growth rates of ipi_t and $m2_t$, and the first-differences of s_t, and c_t. The reason we take the growth rates for the former two variables is because it is more meaningful from an economic standpoint. Subsequently, we run the same three tests for unit roots for ipi_t^{growth}, $m2_t^{growth}$, Δs_t[6], and Δc_t. We find that now all variables have become stationary. Therefore, all four variables considered are integrated of order 1, or $I(1)$, thereby fulfilling cointegration's requirement.

Subsequently, we proceed with setting up a VAR(p) model of the four variables in levels, and we use it to determine the correct lag structure p. The information criteria results are reported in Table 5. We conclude that the correct lag structure is 5. We will use this lag structure for the Johansen's procedure below, because we need to form the Π matrix and conduct the Trace and $\hat{\lambda_{max}}$ tests to find the number of cointegrating relationships r. The Johansen cointegration test results are reported in Table 6.

[6] Δ denotes the difference operator.

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Table 4: Unit root tests

Variable	ADF p-value	PP p-value	KPSS p-value
ipi_t	0.7412	0.8502	0.0100
$m2_t$	0.9619	0.9900	0.0100
s_t	0.0439	0.0346	0.0100
c_t	0.0927	0.8949	0.0100
ipi_t^{growth}	0.0100	0.0100	0.0636
$m2_t^{growth}$	0.0100	0.0100	0.1000
Δs_t	0.0100	0.0100	0.1000
Δc_t	0.0100	0.0100	0.1000

Table 5: VAR Lag Selection Criteria

Lag	AIC	BIC	HQIC	FPE
1	1.4777	1.5434	1.6451	4.3831
2	-2.9676	-2.8495	-2.6665	0.0514
3	-3.4082	-3.2376	-2.9732	0.0331
4	-3.5188	-3.2957	-2.9500	0.0296
5	-3.5879	-3.3123	-2.8852	0.0277
6	-3.5632	-3.2351	-2.7266	0.0284
7	-3.5370	-3.1564	-2.5667	0.0291
8	-3.5125	-3.0794	-2.4083	0.0298
9	-3.5066	-3.0210	-2.2686	0.0300
10	-3.4773	-2.9392	-2.1054	0.0309

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We define a Vector Autoregressive (VAR) model of order p for our four variables in reduced form as follows:

$$\begin{bmatrix} ipi_t \\ m2_t \\ s_t \\ c_t \end{bmatrix} = \begin{bmatrix} \mu_{ipi} \\ \mu_{m2} \\ \mu_s \\ \mu_c \end{bmatrix} + \begin{bmatrix} \alpha_{11}^1 & \alpha_{12}^1 & \alpha_{13}^1 & \alpha_{14}^1 \\ \alpha_{21}^1 & \alpha_{22}^1 & \alpha_{23}^1 & \alpha_{24}^1 \\ \alpha_{31}^1 & \alpha_{32}^1 & \alpha_{33}^1 & \alpha_{34}^1 \\ \alpha_{41}^1 & \alpha_{42}^1 & \alpha_{43}^1 & \alpha_{44}^1 \end{bmatrix} \begin{bmatrix} ipi_{t-1} \\ m2_{t-1} \\ s_{t-1} \\ c_{t-1} \end{bmatrix} + \ldots + \begin{bmatrix} \alpha_{11}^p & \alpha_{12}^p & \alpha_{13}^p & \alpha_{14}^p \\ \alpha_{21}^p & \alpha_{22}^p & \alpha_{23}^p & \alpha_{24}^p \\ \alpha_{31}^p & \alpha_{32}^p & \alpha_{33}^p & \alpha_{34}^p \\ \alpha_{41}^p & \alpha_{42}^p & \alpha_{43}^p & \alpha_{44}^p \end{bmatrix} \begin{bmatrix} ipi_{t-p} \\ m2_{t-p} \\ s_{t-p} \\ c_{t-p} \end{bmatrix} + \begin{bmatrix} \epsilon_t^{ipi} \\ \epsilon_t^{m2} \\ \epsilon_t^{s} \\ \epsilon_t^{c} \end{bmatrix}$$

or, in compact form,

$$y_t = \mu + A_1 y_{t-1} + \ldots + A_p y_{t-p} + \epsilon_t$$

where y_t is a 4×1 vector, μ is a 4×1 constant vector, A_i the 4×4 coefficient matrix for lag i, and ϵ_t is the 4×1 vector of white noise innovations, and we assume that

$$\epsilon_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \Sigma_\epsilon)$$

Table 6: Johansen Cointegration Test Results (Trace & $\hat{\lambda_{max}}$ Eigenvalue)

Rank Hypothesis	λ_{trace}	λ_{trace} Critical Value (5%)	$\hat{\lambda_{max}}$ Statistic	$\hat{\lambda_{max}}$ Critical Value (5%)
$r \leq 0$	10.7830	9.2400	10.7830	9.2400
$r \leq 1$	23.3820	19.9600	12.5990	15.6700
$r \leq 2$	42.6477	34.9100	19.2657	22.0000
$r \leq 3$	71.5257	53.1200	28.8780	28.1400

The role of the yield curve in forecasting economic activity has been extensively studied in the literature, with particular attention given to the slope and curvature factors of the term structure (Argyropoulos & Tzavalis 2016). The slope factor is commonly associated with expectations regarding future economic growth, while the curvature factor captures short-term adjustments in monetary policy stance (Argyropoulos & Tzavalis 2016). These findings have provided a strong theoretical and empirical basis for the current study, which investigates the dynamic relationships between economic activity (illustrated by ipi_t), monetary policy stance (illustrated by $m2_t$), and yield curve factors (s_t and c_t).

The Johansen cointegration test was conducted to determine whether a long-term equilibrium relationship exists among the four variables ipi_t, $m2_t$, s_t, and c_t. The results reveal strong evidence of cointegration, suggesting that despite potential short-term fluctuations, these variables exhibit a stable, long-run relationship.

The trace test results indicate the presence of at least three cointegrating vectors, while the maximum eigenvalue test identifies only one significant cointegrating relationship at the 5% significance level. Given that the trace test is generally considered more robust in the literature (Cheung & Lai 1993), these findings suggest a stronger interdependence between economic activity, monetary policy stance, and the yield curve structure than what the $\hat{\lambda_{max}}$ test alone would suggest. This is consistent with

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previous empirical studies, such as Diebold et al. (2006) and Christensen et al. (2010), which also find significant cointegrating relationships between yield curve factors and macroeconomic indicators.

These results provide empirical support for the theoretical predictions of the yield curve hypothesis. More specifically, the slope factor (defined as the difference between long-term and short-term yields) is confirmed as a leading indicator of business cycle conditions. This aligns with Tzavalis' findings (2016), which show that the slope component retains significant predictive power for economic activity up to five years ahead. Additionally, the curvature factor, which enters the term spread with the opposite sign to the slope, exhibits cointegration with both economic activity and monetary policy stance. This supports the idea that curvature captures short-term fluctuations in policy adjustments, as suggested by Dewachter & Lyrio (2006). The presence of cointegration between money supply $m2_t$ and the yield curve suggests that monetary expansionary/contractionary policies have a long-term effect on both economic activity and yield curve dynamics. This is consistent with the expectations hypothesis of the term structure, where central bank interventions influence yield movements through expectations of future short-term rates (Tzavalis & Wickens 1998).

The Johansen cointegration analysis strongly supports the hypothesis that economic growth, monetary policy stance, and the yield curve dynamics are interlinked over the long run. The confirmation of these long-run relationships not only aligns with existing theoretical literature but also reinforces the necessity of employing a VECM framework to disentangle short-run from long-run effects. The next section will proceed with estimating the VECM, allowing for a deeper understanding of the transmission mechanisms between monetary policy, interest rate expectations, and economic activity.

6.3 VECM Analysis

The existence of cointegration among these variables suggests that a Vector Error Correction Model (VECM) is the appropriate econometric framework for further analysis. Unlike a standard VAR model, which would only capture short-run dynamics, the VECM accounts for both long-run equilibrium relationships and short-run deviations, thereby providing a richer understanding of how monetary policy and yield curve factors influence economic activity over time.

The VECM representation can be written as:

$$\Delta y_t = \Pi y_{t-1} + \sum_{i-1}^{p-1} \Gamma_i \Delta y_{t-i} + \mu + \epsilon_t \tag{17}$$

where:

- $y_t = (ipi_t, m2_t, s_t, c_t)'$ is the vector of endogenous variables.

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- Δy_t denotes first differences.
- Πy_{t-1} captures the long-run equilibrium relationships through the error correction term:

$$\Pi = \alpha\beta'$$

  where:
  - α is the adjustment matrix, indicating how quickly each variable adjusts to disequilibrium.
  - β is the cointegration matrix, representing the cointegrating relationships.
- Γ_i are short-run adjustment matrices.
- μ is a constant term.
- ϵ_t is a vector of error terms.

Since the Johansen test suggests at least one cointegration relationship, the estimated VECM will include an error correction term (ECT) to adjust for deviations from long-run equilibrium.

The cointegrating equation estimated by maximum likelihood is given as:

$$ipi_t = -0.00345m2_t + 9.03307s_t - 5.16428c_t + \epsilon_t. \tag{18}$$

The negative coefficient of $m2_t$ (-0.00345) suggests that an increase in money supply leads to a marginal long-run contraction in industrial production. This aligns with theoretical expectations where excessive liquidity may contribute to inflationary pressures that dampen real economic activity. The s_t coefficient (9.03307) is strongly positive, reinforcing the well-established link between yield curve steepness and future economic growth. A steeper yield curve reflects expectations of higher future growth, consistent with the findings of Tzavalis (2016). The c_t coefficient (-5.16428) indicates that a rise in medium-term rates, relative to short- and long-term rates, is associated with a decline in economic activity, suggesting that short-term policy shocks impact business cycles and investment decisions. These results confirm cointegration among the four variables, meaning they share a long-run equilibrium despite short-term fluctuations.

The ECT measures how quickly deviations from long-run equilibrium are corrected. The ECT coefficient for IPI (-0.0018) suggests that economic activity adjusts very slowly to deviations from the long-run equilibrium. M2's ECT coefficient (0.0499) suggests monetary policy also exhibits weak mean-reversion. Conversely, curvature's ECT coefficient (0.0006***) is significant, indicating that yield curve curvature quickly corrects deviations. This slow adjustment process is consistent with structural economic theories, where interest rate expectations and liquidity conditions adjust gradually over time (Brayton et al. 1997).

The VECM representation also illustrates short-run dynamics. First, there is significant negative impact between $m2_{t-1}$ and ipi_t (-0.0053^{***}), implying that expansionary monetary policy (higher M2) leads

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Table 7: Vector Error Correction Model (VECM) Results

Variable	ipi_t	$m2_t$	s_t	c_t
ECT	-0.0018 (0.0015)	0.0499 (0.0903)	-5.9e-05 (0.0002)	0.0006 (0.0001)***
μ	0.0640 (0.0608)	4.5435 (3.6891)	-0.0141 (0.0081).	-0.0168 (0.0044)***
Δipi_{t-1}	0.0590 (0.0518)	-18.0026 (3.1467)***	0.0081 (0.0069)	-0.0057 (0.0038)
$\Delta m2_{t-1}$	-0.0053 (0.0008)***	0.7804 (0.0511)***	-50.9184 (20.4941)*	-25.1747 (37.0374)
Δs_{t-1}	1.1881 (0.3376)***	-0.2516 (0.0816)**	0.6961 (0.0451)***	-0.1468 (0.0247)***
Δc_{t-1}	-0.2604 (0.6102)	-25.1747 (37.0374)	-0.2516 (0.0816)**	1.5364 (0.0446)***
Δipi_{t-2}	-0.0412 (0.0523)	5.8026 (3.1739).	0.0036 (0.0070)	0.0006 (0.0038)
$\Delta m2_{t-2}$	0.0080 (0.0010)***	-0.2725 (0.0623)***	-3.9e-05 (0.0001)	2.1e-05 (7.5e-05)
Δs_{t-2}	-0.3610 (0.4109)	0.2804 (24.9385)	-0.1590 (0.0549)**	0.1914 (0.0300)***
Δc_{t-2}	0.2023 (1.0973)	50.2816 (66.6024)	-0.0037 (0.1466)	-0.8795 (0.0801)***
Δipi_{t-3}	0.1904 (0.0487)***	-1.7921 (2.9560)	0.0019 (0.0065)	0.0004 (0.0036)
$\Delta m2_{t-3}$	0.0007 (0.0011)	0.2301 (0.0658)***	3.4e-08 (0.0001)	-2.7e-05 (7.9e-05)
Δs_{t-3}	-0.2607 (0.4160)	8.4836 (25.2520)	0.0687 (0.0556)	-0.0876 (0.0304)**
Δc_{t-3}	1.0184 (1.0879)	-80.2456 (66.0341)	0.2412 (0.1454).	0.3549 (0.0794)***
Δipi_{t-4}	0.0707 (0.0453)	14.3365 (2.7502)***	0.0004 (0.0061)	-0.0045 (0.0033)
$\Delta m2_{t-4}$	-0.0019 (0.0008)*	0.0866 (0.0503).	-7.6980 (20.5350)	56.8979 (36.4714)
Δs_{t-4}	-0.1113 (0.3383)	-7.6980 (20.5350)	0.0158 (0.0452)	0.0531 (0.0247)*
Δc_{t-4}	-1.1868 (0.6009)*	56.8979 (36.4714)	-0.0171 (0.0803)	-0.0453 (0.0439)

Notes: The table presents estimates of the VECM model (17), for the United States (US). Standard errors are reported in parentheses. $^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$.

to lower economic growth in the short run. But, lagged expansionary policy has a positive impact, suggesting that monetary transmission mechanisms operate with delay, as seen from the coefficient of $m2_{t-2}$ and ipi_t (0.0080***). This pattern supports theoretical predictions that monetary policy has short-term contractionary effects (due to liquidity traps or inflationary pressures) but turns expansionary in the medium term (Werning 2011). Next, a steeper yield curve predicts higher economic growth in the short run, as illustrated from the coefficient of s_{t-1} and ipi_t (1.1881***). These results support the findings of Tzavalis (2016), where it is argued that the yield curve slope remains a robust predictor of business cycles.

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6.4 Impulse Response Analysis

The impulse response functions demonstrate the dynamic effects of shocks to the endogenous variables. The primary goal of the IRF analysis in this paper is to examine how a shock to yield curve factors s_t and c_t propagates through the economy upon economic activity (illustrated by ipi_t) and monetary policy stance (represented by $m2_t$), as well as the opposite direction. The impulse responses are computed *in levels* and by employing the *Cholesky decomposition*, a special case of triangular factorisation, that not only ensures innovations remain orthogonal, but also provides unit variance to capture the unit shocks.

From Figure 6 we observe that a positive shock to the slope factor s_t initially increases economic activity (represented by ipi_t). This implies that a steepening of the yield curve, usually associated with expansionary monetary policy, has also expansionary effects upon real economic activity. This impact is statistically significant for approximately 8-12 months (at least 3 quarters), after which it begins to wear off. This impulse response aligns with prior findings in the literature, that have suggested that the slope of the yield curve retains significant predictive power over forthcoming economic activity (Argyropoulos & Tzavalis 2016).

From Figure 7 we observe that a positive shock to the curvature factor c_t initially decreases money supply $m2_t$ for a little over than 5 months. However, later on the response reverts and turns to positive illustrating eventual monetary accommodation following an initial contractionary short-term. This is aligned with the theory suggesting that the curvature factor represents short-term monetary adjustments. From Figure 7 we can also see that a shock in the the slope factor s_t leads to a significant drop in $m2_t$, suggesting that an increase in the term spread leads to a drop in money supply. Again, this is in accordance with the theory, as a heightened term spread indicates a lower short-term interest rates relative to long-term interest rates, which increases today's demand for money (following an expansionary monetary policy), thereby decreasing supply of money.

From the opposite direction, in Figure 8 we see that a positive ipi_t shock on s_t translates to a concave-shaped increase of the slope factor lasting multiple quarters. Crucially, Figure 9 demonstrates that an increase in ipi_t leads to a convex-shaped[7] fall on curvature lasting several quarters. This is aligned with the theory because an increase in economic activity will potentially lead the central bank to a contractionary course of action that will increase recessionary expectations, thus decreasing long-term interest rates, and thereby reducing the yield curve's concavity (curvature). Conversely, a positive shock on money supply $m2_t$ leads to a slight decrease in curvature c_t lasting approximately 6-8 months, before reversing trajectory and having a positive impact the holds for several quarters. Again, this result is parallel to theory suggesting that the curvature factor captures short-term monetary policy adjustments.

[7] See appendix B for a formal mathematical definition of convexity.

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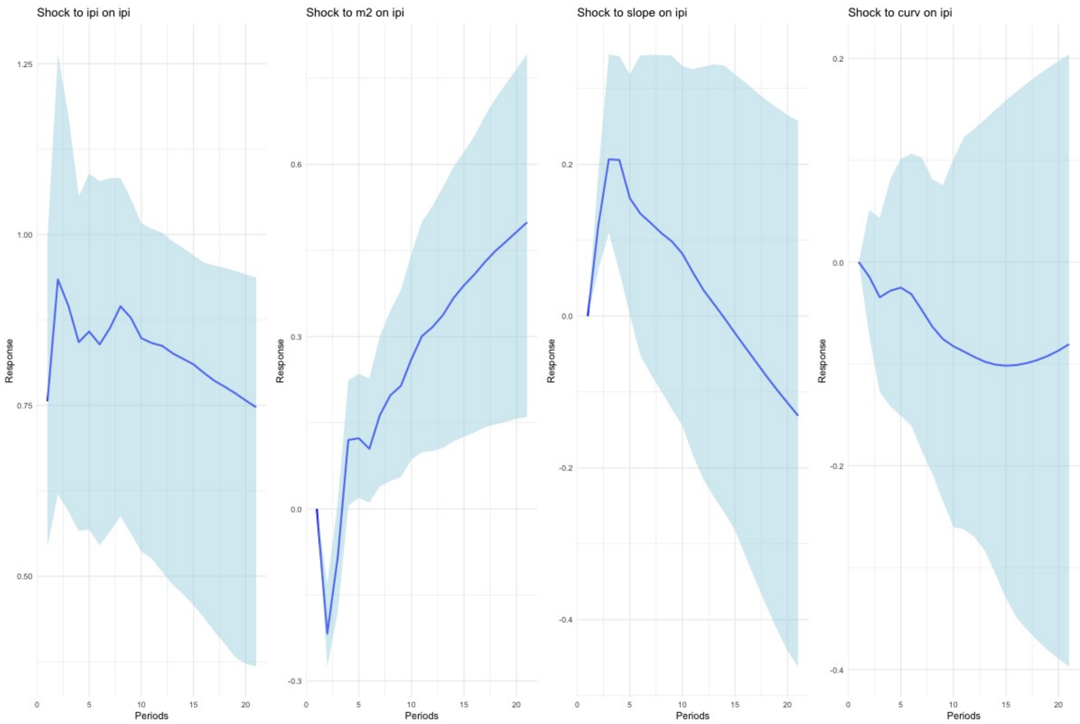

Figure 6: IPI's Responses to Shocks. The plot shows the impulse responses to a Cholesky one unit innovation to each variable. Number of months shown on the x axis. IRFs are computed in levels.

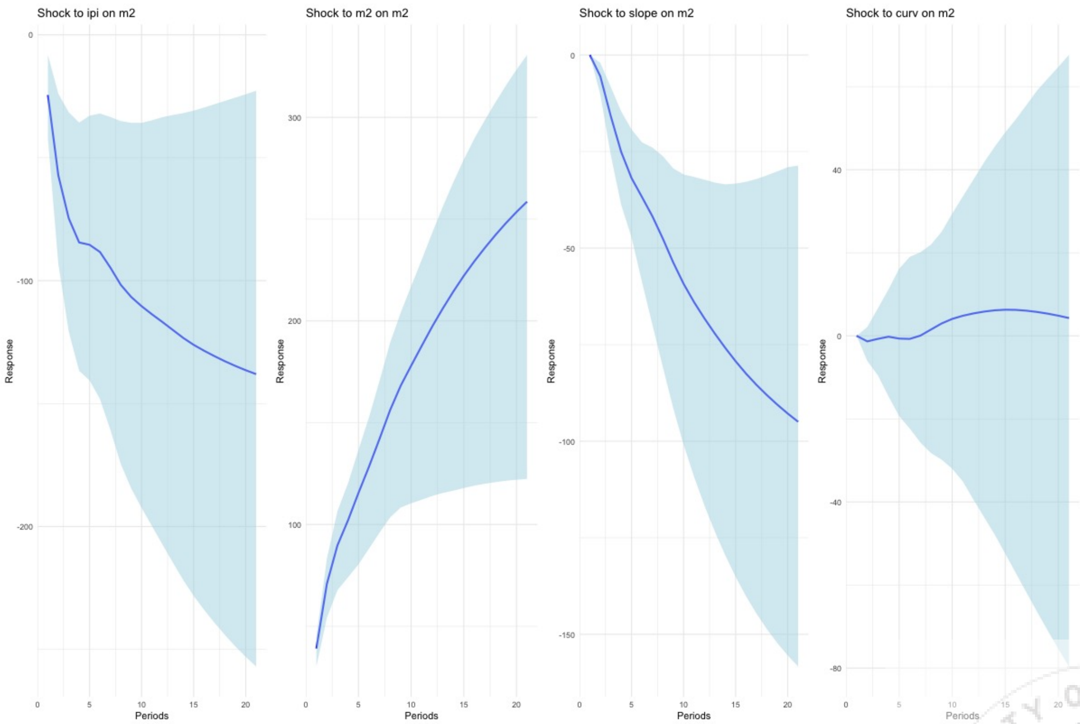

Figure 7: M2's Responses to Shocks. The plot shows the impulse responses to a Cholesky one unit innovation to each variable. Number of months shown on the x axis. IRFs are computed in levels.

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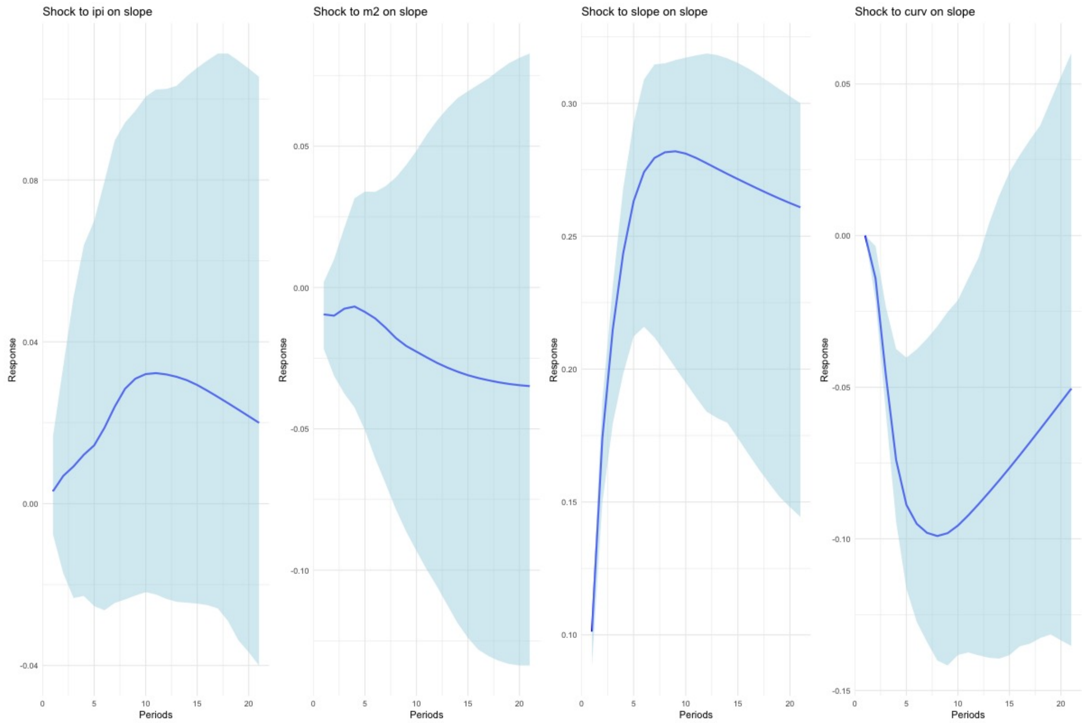

Figure 8: Slope factor's Responses to Shocks. The plot shows the impulse responses to a Cholesky one unit innovation to each variable. Number of months shown on the x axis. IRFs are computed in levels.

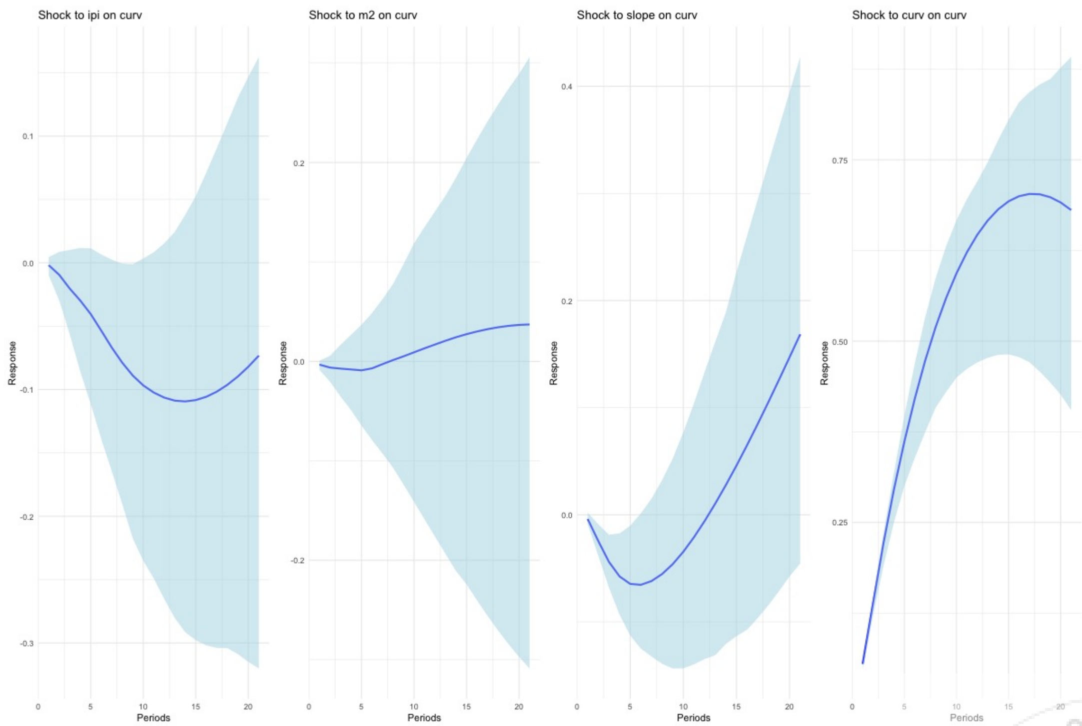

Figure 9: Curvature factor's Responses to Shocks. The plot shows the impulse responses to a Cholesky one unit innovation to each variable. Number of months shown on the x axis. IRFs are computed in levels.

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6.5 Variance Decomposition

Variance Decomposition illustrates the proportion of the forecast error variance of an endogenous variables that is attributable to itself as well as the rest of the endogenous variables in the system. In other words, by conducting this decomposition, we assess the relative contributions of all variables in the system $(ipi_t, m2_t, s_t, c_t)$ to a specific variable over different horizons.

The results in Table 8 and Figure 10, illustrate both numerically and graphically how shocks to each variable propagates throughout the system over time via all four variables. Specifically, the variance of economic activity is largely driven by its own past values in the short- to medium-term, accounting for more than 90% of its total variance. However, in the long-term - 5 years ahead - only about half of its variance depends on its past values, and 38% is contributed by money supply $m2_t$ illustrating that future output is depended upon monetary policy. Crucially, we also observe that whilst the contribution of the slope factor is zero in the first month, after 60 months it has risen to 8%. This once again verifies that the slope factor contains information about economic activity in the long-term (Argyropoulos & Tzavalis 2016).

Regarding the variance of money supply $m2_t$, we observe that in the short-term about two-thirds of its variance are attributed to its own past values while the rest one-third is due to ipi_t, with the two yield curve factors not contributing at all. However, in the long-term the contribution of the slope factor to the variance of the money supply has gradually spiked from zero to 10%, indicating that the slope factor also contains information about future movements of the money supply.

Turning to the yield curve factors, we first observe that the variance of the slope in the short-term is almost entirely attributed to its own past values with very little from the money supply. In the long-term, we also observe some slight variation originating from the curvature factor too. Similarly to the slope, the variance of the curvature is also almost entirely attributed to its own past values in the short-run. However, and in contrast to the slope's variance, in the long-term we surprisingly observe that only half of its variance is attributed to its own past values, while more than 40% originates from the slope, a significant 8% comes from economic activity, and very little from the money supply. Consequently, we see that future values of the curvature are heavily relied upon the slope, but also there is information in economic activity, which is explained by the fact that today's output fluctuations influence tomorrow's monetary policy which in turn affects the concavity of the yield curve which is a process taking months (or years) in total.

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Table 8: Forecast Error Variance Decomposition

ipi_t					s_t				
Horizon	ipi_t	$m2_t$	s_t	c_t	Horizon	ipi_t	$m2_t$	s_t	c_t
1	1.00	0.00	0.00	0.00	1	0.00	0.01	0.99	0.00
2	0.96	0.03	0.01	0.00	2	0.00	0.00	0.99	0.00
3	0.95	0.02	0.02	0.00	3	0.00	0.00	0.97	0.03
4	0.95	0.02	0.03	0.00	4	0.00	0.00	0.95	0.05
5	0.95	0.02	0.03	0.00	5	0.00	0.00	0.93	0.07
6	0.95	0.02	0.03	0.00	6	0.00	0.00	0.92	0.08
7	0.95	0.02	0.03	0.00	7	0.00	0.00	0.91	0.08
8	0.95	0.03	0.03	0.00	8	0.00	0.00	0.90	0.09
9	0.94	0.03	0.03	0.00	9	0.01	0.00	0.90	0.09
10	0.94	0.03	0.02	0.00	10	0.01	0.00	0.90	0.09
24	0.82	0.15	0.02	0.01	24	0.01	0.01	0.91	0.07
60	0.52	0.38	0.08	0.02	60	0.00	0.01	0.95	0.03
$m2_t$					c_t				
Horizon	ipi_t	$m2_t$	s_t	c_t	Horizon	ipi_t	$m2_t$	s_t	c_t
1	0.28	0.72	0.00	0.00	1	0.00	0.00	0.01	0.99
2	0.37	0.63	0.00	0.00	2	0.00	0.00	0.03	0.97
3	0.39	0.60	0.01	0.00	3	0.01	0.00	0.03	0.96
4	0.39	0.59	0.02	0.00	4	0.01	0.00	0.04	0.96
5	0.37	0.60	0.03	0.00	5	0.01	0.00	0.03	0.96
6	0.35	0.61	0.04	0.00	6	0.01	0.00	0.03	0.96
7	0.34	0.62	0.04	0.00	7	0.01	0.00	0.03	0.96
8	0.32	0.63	0.05	0.00	8	0.02	0.00	0.02	0.96
9	0.31	0.64	0.05	0.00	9	0.02	0.00	0.02	0.96
10	0.30	0.64	0.06	0.00	10	0.02	0.00	0.01	0.97
24	0.23	0.69	0.08	0.00	24	0.02	0.00	0.03	0.95
60	0.19	0.71	0.10	0.00	60	0.08	0.01	0.41	0.50

Note: Each entry gives the proportion of the forecast variance (at the specified forecast horizon).

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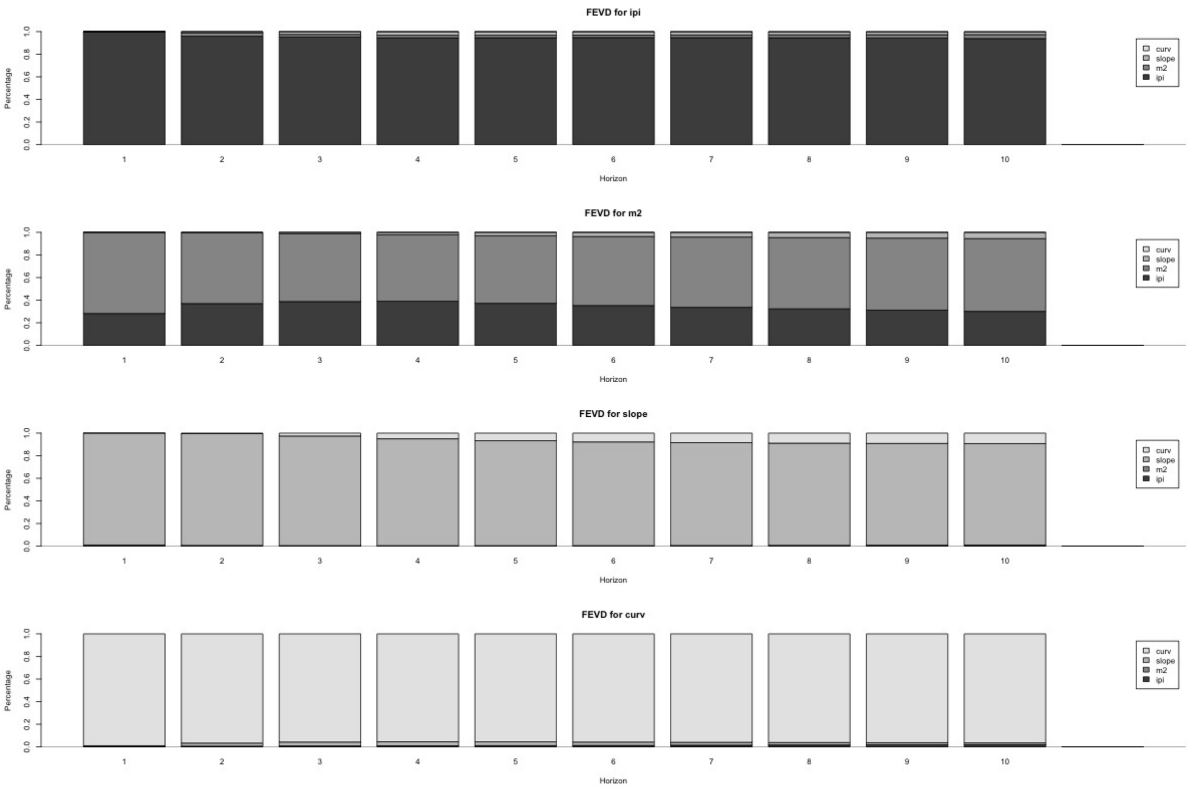

Figure 10: Variance Decomposition

7 Discussion

It is illustrated in the literature that the slope factor of the yield curve is interconnected with long-term business conditions, and the curvature factor with shorter-term economic activity which is, in turn, associated with current monetary stance (Argyropoulos & Tzavalis 2016). This paper has presented an examination of the dynamic relationships between the latent factors of the term structure of the yield curve (slope and curvature) and two macroeconomic variables that represent the above two implied relationships, namely the industrial production (economic activity) and money supply (monetary stance). To do so, first, the paper utilises the well-known dynamic Nelson-Siegel term structure model in a state space representation in order to extract the yield curve's latent factors by applying the Kalman filter (Nelson & Siegel 1987).

In alignment with our expectations, this paper proves that there is a long-term equilibrium cointegrating relationship between the yield curve's factors, economic activity, and money supply. This finding confirms the theoretical framework of the rational expectations hypothesis of the term structure (Fama 1984; Tzavalis & Wickens 1998) that suggests that the term structure of interest rates encapsulates market expectations about future economic conditions, and even more so its latent factors, as has been illustrated by Argyropoulos & Tzavalis (2016). Crucially, the presence of cointegration implies that deviations from this long-term equilibrium are temporary, and there are adjustment forces in the economy that push

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towards a mean-reversion.

The short-run dynamics between those variables have been studied in the literature (see, e.g., Diebold, Rudebusch & Aruoba 2006; Diebold et al. 2003). However, the implementation of a VECM sheds light on both the short-run and long-run dynamics of the four endogenous variables in question. The error correction term's significance demonstrates the speed at which disequilibria adjust back to their long-term cointegrating relationship, which can be considered a measure of the yield curve's responsiveness to macroeconomic shifts.

Impulse response analysis illustrated that a positive shock to the term spread, increases economic activity for several quarters, before reversing to a negative effect. Conversely, a positive shock to slope has a negative effect on the money supply, which implies that a decrease of the short-term interest rates relative to the long-term ones, increases today's demand for money, thereby decreasing money supply. Interestingly, a positive shock to the curvature factor, has initially a negative impact on the money supply but soon it becomes positive, indicating the functioning of contractionary monetary policy that is being accommodated in the medium-term, aligning with the relevant literature (Argyropoulos & Tzavalis 2015). Furthermore, as was expected from the literature (see, e.g., Argyropoulos & Tzavalis 2016), an increase in economic activity has a positive effect on the slope, but an increase in the money supply leads to a fall in the slope which is explained by a contractionary monetary policy.

From the variance decomposition analysis, we observed that the slope factor does not contribute to the variance of economic activity in the short-run, but gradually increases its contribution reaching a peak in the long-run, which is aligned with the relevant literature that suggests that the slope factors contains information about future economic activity (Argyropoulos & Tzavalis 2016). The same occurs with the case of money supply, where output constantly contributes to its variance, but the slope factor does not at all initially, but gradually increases its contribution reaching a significant amount of variance in the long-run. Another interesting result, is that the variance of the curvature, albeit initially being almost solely dependent upon its own past values, eventually in the long-run is largely dependent upon the variance of the slope factor, and significantly dependent upon business cycles as well. This result is also consistent with the relevant findings in the literature (see, e.g., Diebold et al. 2003).

The findings of the present study are largely aligned with the existing literature. Evidence by Estrella & Hardouvelis (1991) that the yield curve is a highly efficient predictor of economic activity, is corroborated by the present paper by proving a cointegrating relationship between the yield curve factors and key macroeconomic variables. Accordingly, Diebold, Ji & Li (2006) employed the dynamic Nelson-Siegel model for the term structure to forecast the yield curve from macroeconomic variables, resulting in a robust model, which this paper confirms as well.

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8 Conclusion

This study has examined the dynamic relationship between the slope and curvature factors of the yield curve and two macroeconomic indicators - economic activity and the monetary policy instrument - which are shown in the literature to be interconnected. We have extracted the latent factors of the term structure of the yield curve by employing the dynamic Nelson-Siegel model for the term structure within a state-space representation. The cointegration analysis proved that there exists a long-term cointegrating equilibrium relationship between the slope and curvature factors, economic activity, and the money supply. We subsequently constructed a vector error correction model to study both the short-run adjustments as well as the long-run interactions.

Our findings reinforce the hypothesis in the literature that the slope factor serves as a reliable predictor of real business cycle conditions over the long-run, while the curvature factor depicts the short-run adjustments on monetary policy that the central bank produces. We have provided clear-cut evidence that there is a cointegrating equilibrium between the yield curve factors, business conditions, and the course of action followed by the central banks. This is significant for policymakers and bond-market participants suggesting that despite short-run deviations, there are market forces in play that will lead to a mean-reversion towards the long-run equilibrium relationship found. These findings also strengthen the hypothesis found in the literature which states that decomposing the yield curve into its latent components can provide superior forecasting results than relying solely on the term spread. Future research could expand on these findings by incorporating additional macroeconomic indicators to test whether there is a more complex cointegrating relationship active.

Further research is proposed upon this analysis, but instead of viewing latent yield curve factors as $I(1)$ processes, view them as *fractionally-integrated* series and subsequently perform fractional-cointegration analysis to better capture the inherent *long-memory* nature of the term structure of interest rates. Appendix C provides an introduction to fractionally - integrated series, and its consequential *fractional - cointegration* analysis, as illustrated in Appendix D. This is significant because interest rates have *long memory*, either in mean or in variance. This means that shocks upon those series take a very long time to disappear, thereby rendering the fractionally - integrated representation of those series potentially better as the simpler $I(1)$ representation might lose significant information of this long memory.

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A Granger Representation Theorem

Following Engle & Granger (1987), we present the Granger Representation Theorem below as well as an exercise.

Consider an $n \times 1$ vector y_t where Δy_t has the Wold representation

$$\Delta y_t \equiv (1-L)y_t = \delta + \epsilon_t + \Psi_1 \epsilon_{t-1} + \Psi_2 \epsilon_{t-2} + \ldots = \delta + \Psi(L)\epsilon_t$$

where $\epsilon_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \Sigma_\epsilon)$ and,

$$\sum_{s=0}^{\infty} \Psi_s < \infty, \quad \text{absolutely summable.}$$

Suppose that there are exactly r cointegrating relationships amongst the elements of y_t. Then, there exists a $(r \times n)$ matrix β' whose rows are linearly independent such that the $(r \times 1)$ vector u_t, defined as,

$$u_t = \beta' y_t \;\sim\; I(0),$$

i.e., it is stationary. Additionally, the matrix β' has the following property:

$$\beta' \Psi(1) = 0 \quad \text{and} \quad \beta' \delta = 0.$$

Furthermore, if the process can be represented as a VAR(p) in levels,

$$y_t = c + \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \ldots + \Phi_p y_{t-p} + \epsilon_t$$

or

$$\Phi(L) y_t = c + \epsilon_t$$

where

$$\Phi(L) = \mathbb{I}_{\ltimes} - \Phi_1 L - \Phi_2 L^2 - \ldots - \Phi_p L^p,$$

then, $\exists$ a matrix α of dimension $(n \times r)$, such as,

$$\Phi(1) = -\alpha\beta'$$

and Δy_t has the following VECM representation:

$$\Delta y_t = J_1 \Delta y_{t-1} + J_2 \Delta y_{t-2} + \ldots + J_{p-1} \Delta y_{t-p+1} + c + \Phi(1) y_{t-1} + \epsilon_t$$

or

$$\Delta y_t = J_1 \Delta y_{t-1} + J_2 \Delta y_{t-2} + \ldots + J_{p-1} \Delta y_{t-p+1} + c + -\alpha\beta' u_{t-1} + \epsilon_t$$

where

$$J_s = -[\Phi_{s+1} + \Phi_{s+2} + \ldots + \Phi_p], \quad \text{for} \quad s = 1, 2, \ldots, p-1.$$

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The following exercise is from Professor Tzavalis.

Consider the following cointegrating relationship

$$y_{1t} = \beta_2 y_{2t} + \beta_3 y_{3t} + u_{1t},$$

with

$$\Delta y_{2t} = u_{2t}, \quad \text{and} \quad \Delta y_{3t} = u_{3t}.$$

Give the MA(∞) representation of the vector $\begin{bmatrix} \Delta y_{1t} \\ \Delta y_{2t} \\ \Delta y_{3t} \end{bmatrix}$ and obtain $\Psi(1)$. Next, verify the following condition for the existence of cointegration, implied by the Engle-Granger representation theorem:

$$\beta' \Psi(1) = 0,$$

where $\beta' = (1, -\beta_2, -\beta_3)$.

Solution

Take the first difference of y_{1t} and obtain

$$\begin{aligned} \Delta y_{1t} &= \beta_2 \Delta y_{2t} + \beta_3 \Delta y_{3t} + \Delta u_{1t} \\ &= \beta_2 u_{2t} + \beta_3 u_{3t} + \Delta u_{1t} \\ &= \beta_2 u_{2t} + \beta_3 u_{3t} + (1-L) u_{1t} \\ &= (1-L) u_{1t} + \beta_2 u_{2t} + \beta_3 u_{3t}. \end{aligned}$$

Now, we can obtain the Wold MA(∞) representation of the first differences as follows,

$$\begin{bmatrix} \Delta y_{1t} \\ \Delta y_{2t} \\ \Delta y_{3t} \end{bmatrix} = \begin{bmatrix} (1-L) & \beta_2 & \beta_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \end{bmatrix}$$

or

$$\Delta y_t = \Psi(L) u_t.$$

Observe, that if we treat L as a scalar and we compute $\Psi(L)$ at $L = 1$ we obtain,

$$\begin{aligned} \beta' \Psi(1) &= (1, -\beta_2, -\beta_3) \begin{bmatrix} 0 & \beta_2 & \beta_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 0 - \beta_2 \cdot 0 - \beta_3 \cdot 0 & 1 \cdot \beta_2 - \beta_2 \cdot 1 - \beta_3 \cdot 0 & 1 \cdot \beta_3 - \beta_2 \cdot 0 - \beta_3 \cdot 1 \end{bmatrix} = (0,0,0)' = 0 \end{aligned}$$

or

$$\beta' \Psi(1) = 0,$$

thereby verifying the condition for the existence of cointegration implied by the Engle-Granger representation theorem.

B Convexity and Concavity

Following Mas-Colell et al. (1995), we first define convex sets.

The set $A \subset \mathbb{R}^n$ is convex if $\alpha x + (1-\alpha)x' \in A$ whenever $x, x' \in A$ and $\alpha \in [0,1]$. In words, this means that if two vectors x, x' that belong in a convex set $A \subset \mathbb{R}^n$, then any line segment that connects those two vectors will also lie in A.

Next, we define a concave and convex function.

The function $f : A \to \mathbb{R}^n$, defined in a convex set $A \subset \mathbb{R}^n$, is concave if

$$f(\alpha x + (1-\alpha)x') \geq \alpha f(x) + (1-\alpha) f(x')$$

for all $x, x' \in A$ and all $\alpha \in [0,1]$.

Conversely, the function $f : A \to \mathbb{R}^n$, defined in a convex set $A \subset \mathbb{R}^n$, is convex if

$$f(\alpha x + (1-\alpha)x') \leq \alpha f(x) + (1-\alpha) f(x')$$

for all $x, x' \in A$ and all $\alpha \in [0,1]$.

C Fractionally Integrated Series

Following mainly the representation of Hamilton (1994), we present the theory of *fractionally - integrated* series. A property of various economic time series, especially financial series, is that they show *long memory* either in their mean or variance, thereby rendering any shocks upon them to disappear rather slowly. To model such behaviour, the concept of *fractionally - integrated* series has been proposed, which lies between $I(0)$ and $I(1)$ processes. In other words, the number of differences the process receives to become stationary, d, is not necessarily an integer, but rather it takes real values.

Consider the following auto-regressive time-series model,

$$\Delta^d y_t = (1-L)^d y_t = c + \epsilon_t.$$

It can be proved that if $-0.5 < d < 0.5$ then the above model is *stationary*. However, if $d \geq 0.5$ then the series is *non-stationary*, that is, its variance is divergent as $t \to \infty$. Furthermore, it also proven that if $0 \leq d < 0.5$ then the autocorrelation function of the time-series model, $\rho(.)$, is proportional to k^{2d-1}, as $k \to \infty$. This explains the aforementioned *long memory* as the autocorrelations of the model decay

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hyperbolically to zero, i.e., slower, in contrast to the exponentially decaying ones of a stationary process. This is also important because we use this rough approximation to estimate d, as we will see.

The Binomial Theorem

We use Newton's *Binomial Theorem* which allows the expansion of $(1-L)^d y_t$ as an infinite series for any $d > -1$. The expansion is given by:

$$\Delta^d y_t = (1-L)^d y_t = \sum_{k=0}^{\infty} \binom{d}{k} (-L)^k$$

where the binomial coefficients are defined by

$$\binom{d}{k} = \frac{d(d-1)(d-2)\dots(d-k+1)}{k!}.$$

Using this formula, our initial expansion can be written as:

$$(1-L)^d = 1 - dL + \frac{d(d-1)}{2!}L^2 - \frac{d(d-1)(d-2)}{3!}L^3 + \dots$$

It can be shown, that if $d < 0.5$, the process is *weakly stationary* and has an $MA(\infty)$ Wold representation given by:

$$y_t = (1-L)^d u_t = \sum_{i=0}^{\infty} \theta_i u_{t-i}$$

where

$$\theta_k = \frac{d(1+d)(2+d)\cdots(k-1+d)}{k!}.$$

Additionally, if $d > -0.5$, then the MA process is *invertible* and y_t has an alternative $AR(\infty)$ representation given by:

$$y_t = \sum_{i=1}^{\infty} \phi_i y_{t-i} + u_t$$

where

$$\phi_k = \frac{-d(1-d)(2-d)\cdots(k-1-d)}{k!}.$$

According to the study of Campbell et al. (1997), comparing the autocorrelation function of a stationary fractionally integrated model and that of a simple short-memory *AR(1)* process demonstrates that the former stationary *FI* series has a *hyperbolic decay*, whilst the latter stationary *AR(1)* has a *geometric decay*. After 100 periods, the *AR(1)* model, as expected, exhibits negligible to no autocorrelation, but the *FI* continues to have large autocorrelation.

Next, we present the generalisation of *ARIMA* models to incorporate real values of d.

ARFIMA Models

The above AR process can be transformed to an *autoregressive fractionally integrated moving average process*, $ARFIMA(p, d, q)$, given as:

$$\Phi(L)(1-L)^d y_t = c + \Theta(L)\epsilon_t.$$

where

$$\Phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p$$

and

$$\Theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q.$$

The model can be rewritten as:

$$\Phi(L)\Delta^d y_t = c + \Theta(L)\epsilon_t$$

or

$$\Phi(L)\Delta^{1+(d-1)} y_t = c + \Theta(L)\epsilon_t$$

or

$$\Delta y_t = \Phi(L)^{-1}\Delta^{1-d} c + \Phi(L)^{-1}\Delta^{1-d}\Theta(L)\epsilon_t$$

or

$$\Delta y_t = \mu + A(L)\epsilon_t$$

where

$$\mu = \Phi(L)^{-1}\Delta^{1-d} c \quad \text{and} \quad A(L) = \Phi(L)^{-1}\Delta^{1-d}\Theta(L).$$

We also have to note that if $d < 1$, then the $I(d)$ process is *mean-reverting* where shocks tend to disappear in the long-run, in contrast to when $d \geq 1$ where shocks tend to have permanent effect.

Estimating d

If $0 \leq d < 0.5$, then it can be shown that the autocorrelation function of *ARFIMA* models is proportional to and can be approximated by

$$\rho(k) \simeq k^{2d-1}, \quad \text{as} \quad k \to \infty.$$

This is very useful because using it, we can provide an estimate for the fractional differencing parameter d. Taking logarithms of the above approximation yields

$$\log|\rho(k)| \simeq (2d-1)\log k.$$

Therefore, we can estimate $(2d-1)$ as the β coefficient in the following regression:

$$\log|\rho(\hat{i})| = \alpha + \beta \log k + u_i$$

where,

$$\hat{\beta} = 2\hat{d} - 1$$

so that,

$$\hat{d} = (\hat{\beta} + 1)/2.$$

D Fractional Cointegration

Traditional cointegration analysis imposes that all variables included - in our case, interest rates - are $I(1)$ unit root processes which cannot diverge from a long-run equilibrium relationship (Abbritti et al. 2023). However, as it has been suggested, this creates a problematic non-mean reversion in rates (Abbritti et al. 2023). As explained in the literature, treating interest rates as $I(1)$ processes, implies that shocks upon interest rates have permanent effect, even though this is not illustrated by historical data (see, e.g., Diebold & Rudebusch 2013; Campbell et al. 1997). Therefore, we need a framework that is based upon the long memory behaviour of interest rates, where shocks are long-lasting, in contrast to $I(0)$ processes, but are not permanent, in contrast to $I(1)$ processes.

To this end, we can employ the *fractional cointegration vector autoregressive (FCVAR) model*, as suggested by Johansen & Nielsen (2012). The FCVAR is used to determine the long-run equilibrium relationship between fractionally integrated series. Given two *real numbers* d, b, the components of the vector y_t are said to be cointegrated of order *d, b*, denoted $y_t \sim CI(d, b)$ if all variables of y_t are $I(d)$ and there exists a vector $\beta \neq 0$ such that $u_t = \beta' y_t \sim I(\lambda) = I(d - b)$, $b > 0$. The FCVAR model introduced by Johansen & Nielsen (2012) is a generalisation of Johansen's (1995) cointegrated vector autoregressive (CVAR) model. The addition is that FCVAR allows to conduct the analysis with fractional processes of order d and cointegration order $d - b$, $b > 0$. Before presenting the FCVAR model, we present the non-fractional version CVAR (Abbritti et al. 2023).

Let X_t with $t = 1, \ldots, T$ be a $p-$dimensional $I(1)$ time series vector. The CVAR model is

$$\Delta X_t = \alpha^* \beta^{*'} X_{t-1} + \sum_{i=1}^{k} \Gamma_i^* \Delta X_{t-i} + \epsilon_t = \alpha^* \beta^{*'} L X_t + \sum_{i=1}^{k} \Gamma_i^* \Delta L^i X_t + \epsilon_t, \tag{19}$$

where Δ is the first difference operator, that is $\Delta = (1 - L)$, k is the lag length of the VAR, α^* is the matrix of adjustment parameters, β^* is the matrix of cointegrating vectors, and Γ_i^* are the coefficient matrices that govern the short-run $I(0)$ VAR dynamics. To derive FCVAR, we replace the difference and lag operators Δ and L in (19) by their fractional counterparts Δ^b and L_b, respectively. This yields

$$\Delta^b X_t = \alpha \beta' L_b X_t + \sum_{i=1}^{k} \Gamma_i^* \Delta^b L_b^i X_t + \epsilon_t \tag{20}$$

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The vectors α and β have the same interpretation with the previous α^* and β^*, but they only coincide in the case where $d = b = 1$. We, then, apply $X_t = \Delta^{d-b}(Y_t - \mu)$, where Y_t is the $p \times 1$ vector of the time series under analysis, and μ is the mean vector containing level parameters that serve as a non-zero starting point of our time series. Consequently, we have

$$\Delta^d(Y_t - \mu) = \alpha\beta' L_b \Delta^{d-b}(Y_t - \mu) + \sum_{i=1}^{k} \Gamma_i^* \Delta^d L_b^i (Y_t - \mu) + \epsilon_t, \tag{21}$$

where $\epsilon_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \Omega)$, α and β are $p \times r$ matrices, with $0 \le r \le p$, and r represents the number of cointegrating relationships.

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References

Abbritti, M., Carcel, H., Gil-Alana, L., Moreno, A. et al. (2023), 'Term premium in a fractionally cointegrated yield curve', *Journal of Banking & Finance* **149**, 106777.

Ang, A. & Piazzesi, M. (2003), 'A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables', *Journal of Monetary economics* **50**(4), 745–787.

Ang, A., Piazzesi, M. & Wei, M. (2006), 'What does the yield curve tell us about gdp growth?', *Journal of econometrics* **131**(1-2), 359–403.

Argyropoulos, E. & Tzavalis, E. (2015), 'Term spread regressions of the rational expectations hypothesis of the term structure allowing for risk premium effects', *Studies in Nonlinear Dynamics & Econometrics* **19**(1), 49–70.

Argyropoulos, E. & Tzavalis, E. (2016), 'Forecasting economic activity from yield curve factors', *The North American Journal of Economics and Finance* **36**, 293–311.

Bekaert, G., Cho, S. & Moreno, A. (2010), 'New keynesian macroeconomics and the term structure', *Journal of Money, Credit and Banking* **42**(1), 33–62.

Bliss, R. R. (1997), 'Movements in the term structure of interest rates', *Economic Review-Federal Reserve Bank of Atlanta* **82**(4), 16.

Bradley, M. G. & Lumpkin, S. A. (1992), 'The treasury yield curve as a cointegrated system', *Journal of Financial and Quantitative Analysis* **27**(3), 449–463.

Brayton, F., Mauskopf, E., Reifschneider, D. & Tinsley, P. (1997), 'The role of expectations in the frb/us macroeconomic model', *Fed. Res. Bull.* **83**, 227.

Campbell, J. Y., Lo, A. W., MacKinlay, A. C. & Whitelaw, R. F. (1997), 'The econometrics of financial markets', *Macroeconomic Dynamics* **2**(4), 559–562.

Cheung, Y. & Lai, K. (1993), 'Finite sample sizes of johnansen's likelihood ratio tests for cointegration. oxford bulletin of economics and statistics, nº 55', *Oxford Bulletin* .

Christensen, J. H., Lopez, J. A. & Rudebusch, G. D. (2010), 'Inflation expectations and risk premiums in an arbitrage-free model of nominal and real bond yields', *Journal of Money, Credit and Banking* **42**, 143–178.

Dewachter, H. & Lyrio, M. (2006), 'Macro factors and the term structure of interest rates', *Journal of Money, Credit and Banking* pp. 119–140.

Dewachter, H., Lyrio, M. & Maes, K. (2006), 'A joint model for the term structure of interest rates and the macroeconomy', *Journal of Applied Econometrics* **21**(4), 439–462.

https://doi.org/10.26219/heal.aueb.8459

Diebold, F. X., Ji, L. & Li, C. (2006), 'A three-factor yield curve model: non-affine structure, systematic risk sources, and generalized duration', *Long-Run Growth and Short-Run Stabilization: Essays in Memory of Albert Ando. Cheltenham, UK: Edward Elgar* pp. 240–274.

Diebold, F. X. & Li, C. (2006), 'Forecasting the term structure of government bond yields', *Journal of econometrics* **130**(2), 337–364.

Diebold, F. X. & Rudebusch, G. D. (2013), *Yield curve modeling and forecasting: the dynamic Nelson-Siegel approach*, Princeton University Press.

Diebold, F. X., Rudebusch, G. D. & Aruoba, S. B. (2003), The macroeconomy and the yield curve: A nonstructural analysis, Technical report, CFS Working Paper.

Diebold, F. X., Rudebusch, G. D. & Aruoba, S. B. (2006), 'The macroeconomy and the yield curve: a dynamic latent factor approach', *Journal of econometrics* **131**(1-2), 309–338.

Engle, R. F. & Granger, C. W. (1987), 'Co-integration and error correction: representation, estimation, and testing', *Econometrica: journal of the Econometric Society* pp. 251–276.

Estrella, A. & Hardouvelis, G. A. (1991), 'The term structure as a predictor of real economic activity', *The journal of Finance* **46**(2), 555–576.

Estrella, A. & Mishkin, F. S. (1997), 'The predictive power of the term structure of interest rates in europe and the united states: Implications for the european central bank', *European economic review* **41**(7), 1375–1401.

Estrella, A. & Trubin, M. (2006), 'The yield curve as a leading indicator: Some practical issues', *Current issues in Economics and Finance* **12**(5).

Fama, E. F. (1984), 'The information in the term structure', *Journal of financial economics* **13**(4), 509–528.

Hamilton, J. D. (1994), *Time Series Analysis*, Princeton University Press.

Johansen, S. (1988), 'Statistical analysis of cointegration vectors', *Journal of economic dynamics and control* **12**(2-3), 231–254.

Johansen, S. (1995), *Likelihood-based inference in cointegrated vector autoregressive models*, OUP Oxford.

Johansen, S. & Juselius, K. (1990), 'Maximum likelihood estimation and inference on cointegration—with appucations to the demand for money', *Oxford Bulletin of Economics and statistics* **52**(2), 169–210.

Johansen, S. & Nielsen, M. Ø. (2012), 'Likelihood inference for a fractionally cointegrated vector autoregressive model', *Econometrica* **80**(6), 2667–2732.

Kim, Y. & Bang, H. (2018), Introduction to kalman filter and its applications, *in* 'Introduction and implementations of the Kalman filter', IntechOpen.

https://doi.org/10.26219/heal.aueb.8459

Litterman, R. & Scheinkman, J. (1991), 'Common factors affecting bond returns', *Journal of fixed income* pp. 54–61.

Mas-Colell, A., Whinston, M. D. & Green, J. R. (1995), *Microeconomic theory*, Vol. 1, Oxford university press New York.

Mönch, E. (2012), 'Term structure surprises: the predictive content of curvature, level, and slope', *Journal of Applied Econometrics* **27**(4), 574–602.

Nelson, C. R. & Siegel, A. F. (1987), 'Parsimonious modeling of yield curves', *Journal of business* pp. 473–489.

Parker, D. & Schularick, M. (2021), 'The term spread as a predictor of financial instability', *Federal Reserve Bank of New York* .

Stock, J. H. & Watson, M. W. (1993), 'A simple estimator of cointegrating vectors in higher order integrated systems', *Econometrica: journal of the Econometric Society* pp. 783–820.

Stock, J. H. & Watson, M. W. (2003), 'Forecasting output and inflation: The role of asset prices', *Journal of economic literature* **41**(3), 788–829.

Tzavalis, E. & Wickens, M. (1998), 'A re-examination of the rational expectations hypothesis of the term structure: reconciling the evidence from long-run and short-run tests', *International Journal of Finance & Economics* **3**(3), 229–239.

Werning, I. (2011), Managing a liquidity trap: Monetary and fiscal policy, Technical report, National Bureau of Economic Research.

Zhang, H. (1993), 'Treasury yield curves and cointegration', *Applied Economics* **25**(3), 361–367.