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A Bayesian Kalman Filter approach in correcting near surface temperature forecasts

By

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ΤΜΗΜΑ ΣΤΑΤΙΣΤΙΚΗΣ

Μία Μπεϋζιανή Kalman Filter προσέγγιση για τη διόρθωση των προβλέψεων της θερμοκρασίας κοντά στην επιφάνεια

Βασιλεία Ν. Ραφαήλ

ΔΙΑΤΡΙΒΗ

Που υποβλήθηκε στο Τμήμα Στατιστικής

του Οικονομικού Πανεπιστημίου Αθηνών

ως μέρος των απαιτήσεων για την απόκτηση

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DEDICATION

There are a few distinguished and admirable people
who managed
to keep their morality and dignity
and gain the love and respect from all the people who met
during their life…

In memory to my beloved father
(06/09/1953-01/05/2009)

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VITA

I was born in Halandri, in 1978. I graduated from the 1st High School of Halandri in 1996. In 1997 I entered the Department of Mathematics in the University of Athens and I obtained my degree in 2003. In October 2006, parallelly to my work, I started attending the course of studies of the Master Program in Statistics, in Athens University of Economics and Business. After its completion I have been occupied with my Master Thesis.

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ABSTRACT

Vasilia Raphael

A Bayesian Kalman Filter approach in correcting near surface temperature forecasts

March 2010

In Meteorology, Numerical Weather Prediction (NWP) models are used to provide forecasts for various weather related parameters. One such parameter, with high interest to the general public, is the near surface (2m) temperature. It is well known that the NWP based forecasts are inaccurate and various post processing methods exist, for improving the predictions.

In this thesis a new method is proposed for the correction of near surface (2m) temperature forecasts provided by a NWP model. This procedure constitutes a Bayesian approach, based on a Kalman filter model. We provide an algorithm that results improved temperature predictions for future time-steps, by combining past observations and their corresponding NWP forecasts.

The developed methodology is illustrated through the application of the proposed algorithm to a real data set, consisting of the observed and the forecasted temperatures of 700 days at a particular meteorological station (Thessaloniki).

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ΠΕΡΙΛΗΨΗ

Βασιλεία Ραφαήλ

Μία Μπεϋζιανή Kalman Filter προσέγγιση για τη διόρθωση των προβλέψεων της θερμοκρασίας κοντά στην επιφάνεια

Μάρτιος 2010

Στην μετεωρολογία, τα αριθμητικά μοντέλα πρόγνωσης καιρού χρησιμοποιούνται για προβλέψεις σε ποικίλες καιρικές παραμέτρους. Μία τέτοια παράμετρος, με υψηλό ενδιαφέρον για το κοινό, είναι η θερμοκρασία κοντά στην επιφάνεια. Είναι γνωστό ότι οι προγνώσεις αυτές είναι ανακριβείς και διάφορες μέθοδοι αναπτύσσονται με σκοπό τη βελτίωσή τους.

Σε αυτή τη διατριβή προτείνεται μία καινούργια μεθοδολογία βελτίωσης των προβλέψεων της θερμοκρασίας (κοντά στην επιφάνεια) που προέρχονται από ένα αριθμητικό μοντέλο πρόγνωσης καιρού. Η μέθοδος αυτή αποτελεί μία Μπεϋζιανή προσέγγιση και βασίζεται σε ένα Kalman Filter μοντέλο. Συνδυάζοντας τις προηγούμενες παρατηρήσεις και τις αντίστοιχες προβλέψεις, δημιουργούμε έναν αλγόριθμο ο οποίος μας δίνει βελτιωμένες προγνώσεις της θερμοκρασίας για μελλοντικές χρονικές στιγμές.

Η μεθοδολογία που αναπτύσσεται απεικονίζεται μέσω της εφαρμογής του προτεινόμενου αλγορίθμου σε ένα πραγματικό σύνολο δεδομένων, το οποίο αποτελείται από τις πραγματικές καθώς και τις αντίστοιχες προβλεπόμενες τιμές της θερμοκρασίας 700 ημερών σε ένα συγκεκριμένο μετεωρολογικό σταθμό (Θεσσαλονίκη).

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Chapter 1

Introduction

People always made efforts in order to understand the complicated laws that govern the universe. Ancient civilizations created myths in order to explain complex phenomena. Later, some primitive methods have been created to predict some periodical phenomena. Gradually, divinities were replaced by cycles, shapes, numbers and mathematics. Nowadays, while one theory reverses the other and one standard replaces the other, something remains stable: the laws that rule nature are mathematical.

In 1587, Newton claimed that the nature laws could be described by mathematical methods. In 1922, the mathematician Lewis Frei Richardson, in his work "Weather prediction with numerical methods" described the mathematical methods which one could use in order to predict the weather evolution. A few years later, Lorenz-one of the greatest pioneers in numerical based forecast-after continual attempts, he managed with his primitive computer to create a simple model of the atmosphere that constituted the ancestor of current atmospheric models. In the following years, there were several attempts until the weather numerical forecast had been an undeniable reality.

Nowadays, as the progress that has been achieved is enormous, the results that we receive from the Numerical Weather Prediction (NWP) models are more reliable and more analytic (Hellenic National Meteorological Service).

We will focus our attention to the SKIRON modeling system (Kallos et al., 1997) which was developed by the Atmospheric Modeling and Weather Forecasting Group (AM&WFG) of the University of Athens and it is based on the Eta/NCEP model (Janjic, 1994). SKIRON runs in operational and research mode in a large number of meteorological centers and institutes such as the University of Athens, the Hellenic National Meteorological Service, the Hellenic Centre for Marine Research, the Aristotle University of Thessaloniki, the Atmospheric Sciences Research Centre of the University of Albany and others. SKIRON is a full-physics atmospheric model appropriate for regional simulations in areas with varying physiographic

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characteristics. It has the unique capability to use either the "step-mountain" Eta vertical coordinate (Mesinger, 1984) or the customary sigma coordinate. On the horizontal, the variables are represented on the staggered Arakawa E-grid. The model includes the option to use either the hydrostatic assumption or non-hydrostatic dynamics (Janjic et al., 2001). The hydrostatic version of the system was successfully used operationally in the Atmospheric Modeling and Weather Forecasting Group (AM&WFG) of the University of Athens since 1997 while the non-hydrostatic SKIRON is in operational use since January 2003 (http://forecast.uoa.gr). The model has been successfully applied to a large number of different regions (Kallos and Pytharoulis, 2005; Papadopoulos et al., 2002; Papadopoulos, 2001). SKIRON is also used operationally in the Department Meteorology and Climatology of the Aristotle University of Thessaloniki being initialized 12 UTC (15 Greek Time during summer-time and 14 Greek Time during winter-time) daily and producing 3-day forecasts (http://meteo.geo.auth.gr/skiron). It is integrated separately in two domains. The one domain, called SKIRON20, covers the entire Mediterranean region and most of Europe with a horizontal grid-spacing of $0.20° \times 0.20°$ latitude-longitude (approximately $20\ \text{km} \times 20\ \text{km}$). The other domain, called SKIRON5, covers Greece with a horizontal grid-spacing of $0.05° \times 0.05°$ latitude-longitude (approximately $5\ \text{km} \times 5\ \text{km}$).

However, it is well known that NWP models exhibit systematic errors in the forecasts of weather parameters, such as temperature, especially near the surface. Various techniques and methods have been proposed in the literature to tackle the systematic bias problems. The Kalman Filter theory (Kalman, 1960; Meinhold and Singpurwalla, 1983) provides an efficient tool to correct systematic errors, combining observations with model forecasts (Homleid, 1995; Anadranistakis et al., 2002; Galanis et al., 2006). Especially, a one-dimensional Kalman Filter based technique is proposed for the correction of maximum and minimum near surface (2m) temperature forecasts obtained by a Numerical Weather Prediction model (Galanis and Anadranistakis, 2002).

In our study, using a one-dimensional Kalman Filter model, we present a modified procedure to estimate this systematic error between observed and forecasted 2m-temperatures. This estimation, as is added to the NWP model prediction for the next time-step, gives an improved forecast for the parameter of interest.

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The presentation of our procedure is starting in the second section of chapter 2, as we define the Kalman Filter model. Thereafter, in section 2.4 we provide a recursive formula of the required systematic error. The calculation of the error of any time t, that will improve the temperature forecast for the next time-step $t+1$, is based on the previous differences between temperature observations and forecasts. Additionally, the specification of the systematic error depends on various parameters, that we discuss in chapter 3. In chapter 4 the new proposed procedure is applied for the correction of 2m-temperature forecasts, obtained by the NWP model SKIRON5, to the station of Thessaloniki and for the time period from 01 June 2007 to 30 April 2009. The worth noting results of this application are presented in section 4.3. Finally, the concluding remarks along with the directions to future work are provided at the end of this chapter.

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Chapter 2

The Kalman Filter based approach for correcting near surface temperature forecasts

2.1 Introduction

This chapter is organized into 4 sections. Section 2.2 provides the background theory of the Kalman Filter to help the reader familiarize with the concept. In section 2.3 we present a short description of a one-dimensional Kalman Filter based technique, which has already been proposed for the correction of maximum and minimum near surface (2m) temperature forecasts obtained by a NWP model. Finally, in the last section of this chapter, we provide the extensive proof of the new proposed recursive formula for the estimation of the required systematic error between observed and forecasted temperatures.

2.2 Kalman Filter

The Kalman Filter commonly employed by control engineers and other physical scientists has been successfully used in such diverse areas as the processing of signals in aerospace tracking, the underwater sonar and the statistical control of quality. More recently, it has also been used in some non-engineering applications such as short-term forecasting and the analysis of life lengths from dose-response experiments. Due to the fact that much of the first published literature on the Kalman Filter was in the engineering journals, using a language, notation and style alien to statisticians, there was a delay of several years before statisticians become familiar with this useful methodology. Nowadays, the model, the notions and the techniques of Kalman Filtering are potentially of great interest to statisticians owing to their similarity to

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linear models of regression and time series analysis and because of their great utility in applications.

The Kalman Filter theory provides equations for recursively updating estimates of an unknown process, combining observations related to the process and knowledge about the parameter evolution in time.

In what follows, we provide the equations that compose a Kalman Filter model and a short description of the recursive estimation procedure.

A complete description of the Kalman Filter theory can be found in (Kalman, 1960) and a Bayesian variant of this in (Meinhold and Singpurwalla, 1983).

2.2.1 The Kalman Filter model

A Kalman Filter model is composed of two basic equations. The first one refers to the relationship between the observed values of a variable of interest Y_t and the unobservable parameter θ_t, at times *1,…, t-1, t*, which is linear and is specified by the observation equation

$$Y_t = F_t \theta_t + v_t \tag{2.2.1}$$

where F_t is a known quantity and the observation error v_t is assumed to be normally distributed with mean zero and known variance V_t, denoted as $v_t \sim N(0, V_t)$.

The second equation that describes how the unknown parameter of interest θ_t changes over time is the system equation

$$\theta_t = G_t \theta_{t-1} + w_t \tag{2.2.2}$$

where as before G_t is a known quantity and for the system equation error we assume that $w_t \sim N(0, W_t)$, with W_t being known.

2.2.2 The Kalman Filter estimation procedure

In this section, Kalman Filter will be provided in the general multivariate form.

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The term "Kalman Filter" refers to a recursive procedure for inference about the state of nature θ_t. The key idea here is that given the data $\boldsymbol{Y}_t = (Y_1, Y_2, \ldots, Y_t)$, inference about θ_t can be carried out through a direct application of Bayes's theorem which can be written as:

$$f(\theta_t \mid Y_t) \propto f(Y_t \mid \theta_t, Y_{t-1}) \times f(\theta_t \mid Y_{t-1}) \tag{2.2.3}$$

where the posterior distribution of θ_t given the data $f(\theta_t \mid Y_t)$ is proportional to the likelihood $f(Y_t \mid \theta_t, Y_{t-1})$ and the prior distribution $f(\theta_t \mid Y_{t-1})$.

The recursive procedure starts at time 0, where we assume that

$$\theta_0 \sim N\left(\hat{\theta}_0, \Sigma_0\right) \tag{2.2.4}$$

where $\hat{\theta}_0$ and Σ_0 assumed to be known.

At time *t-1* we have that the posterior distribution of θ_{t-1} given the data $\boldsymbol{Y}_{t-1} = (Y_1, Y_2, ..., Y_{t-1})$ is given by

$$(\theta_{t-1} \mid Y_{t-1}) \sim N\left(\hat{\theta}_{t-1}, \Sigma_{t-1}\right) \tag{2.2.5}$$

where $\hat{\theta}_{t-1}$ and Σ_{t-1} are the mean and the variance of $(\theta_{t-1} \mid Y_{t-1})$, respectively.

We now look forward to time *t*, but in two stages:

A. Prior to observing Y_t and

B. After observing Y_t

Stage A.

At time *t*, before the data $\boldsymbol{Y}_t$ is available, we can use the system equation to update the information we had about θ from stage *t-1* to *t* :

Since $\theta_t = G_t \theta_{t-1} + w_t$ and $(\theta_{t-1} \mid Y_{t-1}) \sim N\left(\hat{\theta}_{t-1}, \Sigma_{t-1}\right)$ we have that the prior distribution of θ_t is given by

$$(\theta_t \mid Y_{t-1}) \sim N\left(G_t \hat{\theta}_{t-1}, R_t\right) \tag{2.2.6}$$

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where $R_t = G_t \Sigma_{t-1} G_t^{'} + W_t$.

Stage B.

On observing Y_t, our goal is to compute the posterior distribution of θ_t.

However, to do this, we need to know the likelihood $f\left(Y_t \mid \theta_t, Y_{t-1}\right)$.

Let $\hat{Y}_t$ denotes the prediction of Y_t. Combining the equations (2.2.1) and (2.2.2), it follows that $\hat{Y}_t = F_t G_t \hat{\theta}_{t-1}$.

We define e_t to be the error from predicting Y_t, from time *t-1*. Thus

$$e_t = Y_t - \hat{Y}_t = Y_t - F_t G_t \hat{\theta}_{t-1} \tag{2.2.7}$$

Since F_t, G_t and $\hat{\theta}_{t-1}$ are all known, observing Y_t is equivalent to observing e_t. Thus (2.2.3) can be rewritten as:

$$f\left(\theta_t \mid Y_t\right) \propto f\left(Y_t \mid \theta_t, Y_{t-1}\right) \times f\left(\theta_t \mid Y_{t-1}\right) \Leftrightarrow$$

$$\Leftrightarrow f\left(\theta_t \mid Y_t, Y_{t-1}\right) = f\left(\theta_t \mid e_t, Y_{t-1}\right) \propto f\left(e_t \mid \theta_t, Y_{t-1}\right) \times f\left(\theta_t \mid Y_{t-1}\right) \tag{2.2.8}$$

with $f\left(e_t \mid \theta_t, Y_{t-1}\right)$ being the likelihood.

Using the fact that $Y_t = F_t \theta_t + v_t$, we get:

$$e_t = Y_t - F_t G_t \hat{\theta}_{t-1} = F_t \theta_t + v_t - F_t G_t \hat{\theta}_{t-1} = F_t\left(\theta_t - G_t \hat{\theta}_{t-1}\right) + v_t \tag{2.2.9}$$

Thus $E\left[e_t \mid \theta_t, Y_{t-1}\right] = F_t\left(\theta_t - G_t \hat{\theta}_{t-1}\right)$ and $Var\left(e_t \mid \theta_t, Y_{t-1}\right) = V_t$

So the likelihood of $\left(e_t \mid \theta_t, Y_{t-1}\right)$ is described by

$$\left(e_t \mid \theta_t, Y_{t-1}\right) \sim N\left(F_t\left(\theta_t - G_t \hat{\theta}_{t-1}\right), V_t\right) \tag{2.2.10}$$

Based on Bayes theorem we obtain the posterior distribution of θ_t as:

$$f\left(\theta_t \mid Y_t, Y_{t-1}\right) = \frac{f\left(e_t \mid \theta_t, Y_{t-1}\right) \times f\left(\theta_t \mid Y_{t-1}\right)}{\int f\left(e_t, \theta_t \mid Y_{t-1}\right) d\theta_t} \tag{2.2.11}$$

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Finally, it is proved (Meinhold and Singpurwalla, 1983) that the posterior distribution of θ_t is given by:

$$\left(\theta_t \mid e_t, Y_{t-1}\right) \sim N\left(\hat{\theta}_t, \Sigma_t\right) \tag{2.2.12}$$

where

$$\hat{\theta}_t = G_t \hat{\theta}_{t-1} + R_t F_t' \left(V_t + F_t R_t F_t'\right)^{-1} e_t \tag{2.2.13}$$

$$\Sigma_t = R_t - R_t F_t' \left(V_t + F_t R_t F_t'\right)^{-1} F_t R_t \tag{2.2.14}$$

where $R_t = G_t \Sigma_{t-1} G_t' + W_t$

2.3 A one-dimensional Kalman Filter for correcting temperature values

As it has already been mentioned, a Kalman Filter model has been used to correct systematic errors, combining observations and forecasts of temperature.

In this application, the unobservable parameter θ_t is the systematic deviation between observed temperature value and 2m-temperature model forecast, at time *t*.

The data Y_t are defined as the differences between forecasts and observations at time *t*.

Below, we describe a one-dimensional Kalman Filter based technique, which has already been proposed for the correction of maximum and minimum near surface (2m) temperature forecasts obtained by a Numerical Weather Prediction model (Galanis and Anadranistakis, 2002).

According to that procedure, the Kalman Filter system equation has the form, which is described as

$$\theta_t = \theta_{t-1} + w_t$$

$$Y_t = \theta_t + v_t$$

where θ_t is describing the systematic deviation between observed and forecasted temperature and Y_t is the difference between forecast and observation at time *t*; w_t, v_t are normal zero-mean variables with variances W_t, V_t, respectively.

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Based on the Kalman Filter estimation procedure and the relationship (2.2.13), applied to the univariate case and for $G_t = F_t = 1$, the estimation $\hat{\theta}_t$ is given by the equation

$$\hat{\theta}_t = \hat{\theta}_{t-1} + \frac{R_t}{V_t + R_t} e_t$$

where $R_t = \Sigma_{t-1} + W_t$, $e_t = Y_t - \hat{\theta}_{t-1}$, $\Sigma_t = R_t - \frac{R_t}{V_t + R_t} R_t$ and $\hat{\theta}_0$, Σ_0 :known.

The estimation of the variances W_t, V_t constitutes one of the most serious difficulties in that procedure. The proposed estimation of W_t and V_t, is based on the sample of the last 7 values of $w_t = \hat{\theta}_t - \hat{\theta}_{t-1}$ and $v_t = Y_t - \hat{\theta}_t$. Precisely, W_t and V_t are assumed equal to $\frac{1}{6}\sum_{i=0}^{6}\left(w_{t-i} - \frac{\sum_{i=0}^{6} w_{t-i}}{7}\right)^2$ and $\frac{1}{6}\sum_{i=0}^{6}\left(v_{t-i} - \frac{\sum_{i=0}^{6} v_{t-i}}{7}\right)^2$,

where $w_{t-i} = \hat{\theta}_{t-i} - \hat{\theta}_{t-i-1}$ and $v_{t-i} = Y_{t-i} - \hat{\theta}_{t-i}$, respectively.

Finally, the estimation $\hat{\theta}_t$ is added to the prediction of the Numerical Weather Prediction model for the next time-step to provide the final improved forecast for the time *t+1*.

2.4 The new proposed estimation procedure

In our study, we present a modified procedure to improve the near surface temperature forecast, obtained by a NWP model. In this section we will derive the new proposed recursive formula for the estimation of the error $\hat{\theta}_t$, using the previous differences $Y_1, Y_2, \ldots, Y_{t-1}$ between observed and forecasted temperatures.

In our procedure a one-dimensional Kalman Filter model is also used. For the variance V_t of the difference $v_t = Y_t - \theta_t$ we assume that it does not change over time and being unknown. We model it by a continuous random variable, which follows the

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inverted gamma distribution with parameters known positive constants α, β ($IG(\alpha,\beta)$). We start with the model description.

2.4.1 The description of the model

We define θ_t as the systematic error between observed and forecasted temperature at time t.

The state at time t, θ_t, is related to the state at time t-1, θ_{t-1}, through the equation

$$\theta_t = \theta_{t-1} + w_t \text{ , for all } t \qquad (2.4.1)$$

where w_t has a normal distribution with mean zero and variance W_t, i.e.,

$$w_t \sim N(0, W_t)$$

Let Y_t, the data, denote the difference between observed and forecasted temperature at time t.

We assume that the relation between Y_t and θ_t is linear and is given by the equation

$$Y_t = \theta_t + v_t \text{ , for all } t \qquad (2.4.2)$$

where v_t has a normal distribution with zero mean and variance V_t, i.e.,

$$v_t \sim N(0, V_t)$$

For the variances W_t, V_t of the error terms w_t, v_t, respectively, we further assume the following:

- W_t, V_t do not change over time, thus $W_t = W$, $V_t = V$, for all t.
- The variance W is a multiple of the variance V; concretely, W, V are connected by the equation $W = \kappa V$, where κ is a known positive constant.

The unknown variance V is a continuous random variable assumed to follow the inverted gamma distribution with parameters α, β, i.e.,

$$V \sim IG(\alpha, \beta) \qquad (2.4.3)$$

where α, β are known positive constants.

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According to the above assumption and the relationships (2.4.1), (2.4.2) and (2.4.3) we come to the conclusion that, at time t, the distribution of $(\theta_t|\theta_{t-1},V)$ is given by

$$(\theta_t|\theta_{t-1},V) \sim N(\theta_{t-1},\kappa V) \tag{2.4.4}$$

and the distribution of $(Y_t|\theta_t,V)$ is given by

$$(Y_t|\theta_t,V) \sim N(\theta_t,V) \tag{2.4.5}$$

where $V \sim IG(\alpha,\beta)$ and κ, α, β : known positive constants.

2.4.2 Preliminaries

As it has already been mentioned, we are interested in estimating the systematic error between observed and forecasted temperature at time t, defined as θ_t, given the data $\boldsymbol{Y}_t=(Y_1, Y_2,...,Y_t)$.

Equivalently, our aim is to obtain a point estimate of θ_t. In Bayesian inference, the complete description of the unknown parameter θ_t can be obtained only through the posterior distribution of $(\theta_t|\boldsymbol{Y}_t)$. In order to obtain a point estimate of θ_t, we simply need to select a summary feature of the posterior distribution $f\left(\theta_t | Y_t\right)$, such as its mean, median or mode. It is known that, under squared error loss function, the Bayes rule of θ_t is the expected value of the posterior distribution of $(\theta_t|\boldsymbol{Y}_t)$, described by $\hat{\theta}_t=E[\theta_t|\boldsymbol{Y}_t]$ (Carlin and Louis, 1996; Casella and Berger, 2001).

The key idea of the estimation procedure is that the posterior distribution of $(\theta_t|\boldsymbol{Y}_t)$ can be obtained from the joint posterior distribution of $(\theta_t,V|\boldsymbol{Y}_t)$, where the distribution of $(\theta_t,V|\boldsymbol{Y}_t)$ arises through a direct application of Bayes theorem, which can be written as:

$$f(\theta_t,V|\boldsymbol{Y}_t) \propto f(\theta_t,V|\boldsymbol{Y}_{t-1}) \times f(Y_t|\theta_t,V,\boldsymbol{Y}_{t-1})$$

where the likelihood of $(Y_t|\theta_t,V,\boldsymbol{Y}_{t-1})$ is described by (2.4.5) :

$$(Y_t|\theta_t,V,\boldsymbol{Y}_{t-1}) \sim N(\theta_t,V)$$

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The joint prior distribution of $(\theta_t, V|\boldsymbol{Y}_{t-1})$ will be computed in the upcoming theorem, using (2.4.4), (2.4.3) :

$$(\theta_t|\theta_{t-1}, V) \sim N(\theta_{t-1}, \kappa V)$$

$$V \sim IG(\alpha,\beta)$$

where κ, α, β : known positive constants.

In addition, we assume that at time $t=0$ it holds that:

$$(\theta_0|V) \sim N\left(\hat{\theta}_0, B_0 V\right)$$

where $B_0 = \kappa$: known positive constant and $\hat{\theta}_0$: known

2.4.3 The recursive estimation procedure

Theorem 2.1

Let V, θ_t, Y_t be random variables such that, for all $t \geq 1$,

$V \sim IG(\alpha,\beta)$

$(\theta_t|\theta_{t-1}, V) \sim N(\theta_{t-1}, \kappa V)$

$(Y_t|\theta_t, V, \boldsymbol{Y}_{t-1}) \sim N(\theta_t, V)$

where κ, α, β : known positive constants.

We assume that $(\theta_0|V) \sim N\left(\hat{\theta}_0, B_0 V\right)$, where $B_0 = \kappa$: known positive constant and $\hat{\theta}_0$:known.

Then, for all $t \geq 1$,

a) The prior distribution of $(\theta_t|V, \boldsymbol{Y}_{t-1})$ is given by

$$(\theta_t|V, \boldsymbol{Y}_{t-1}) \sim N\left(\hat{\theta}_{t-1}, A_t V\right)$$

with pdf

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$$f(\theta_t | V, \boldsymbol{Y}_{t-1}) = \frac{1}{\sqrt{2\pi A_t v}} \exp\left(-\frac{\left(\theta_t - \hat{\theta}_{t-1}\right)^2}{2A_t v} \right) \tag{2.4.6}$$

where $A_t = B_{t-1} + \kappa$

b) The posterior distribution of $(\theta_t | V, \boldsymbol{Y}_t)$ is given by

$$(\theta_t | V, \boldsymbol{Y}_t) \sim N\left(\hat{\theta}_t, B_t V\right)$$

with pdf

$$f(\theta_t | V, \boldsymbol{Y}_t) = \frac{1}{\sqrt{2\pi B_t v}} \exp\left(-\frac{\left(\theta_t - \hat{\theta}_t\right)^2}{2B_t v} \right) \tag{2.4.7}$$

where $B_t = \dfrac{A_t}{A_t + 1}$ and $\hat{\theta}_t = B_t Y_t + (1 - B_t)\hat{\theta}_{t-1}$

c) The joint prior distribution of $(\theta_t, V | \boldsymbol{Y}_{t-1})$ is NIG$\left(\hat{\theta}_{t-1}, A_t V, a, \beta\right)$ (Normal Inverted Gamma), with pdf

$$f(\theta_t, V | \boldsymbol{Y}_{t-1}) = \frac{1}{\sqrt{2\pi A_t v}} \exp\left(-\frac{\left(\theta_t - \hat{\theta}_{t-1}\right)^2}{2A_t v} \right) \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \frac{1}{v^{\alpha+1}} \exp\left(-\frac{1}{\beta v} \right) \tag{2.4.8}$$

d) The joint posterior distribution of $(\theta_t, V | \boldsymbol{Y}_t)$ is NIG$\left(\hat{\theta}_t, B_t V, \dfrac{2a+1}{2}, N_t^{-1}\right)$, with pdf given by

$$f(\theta_t, V | \boldsymbol{Y}_t) = \frac{1}{\sqrt{2\pi B_t v}} \exp\left(-\frac{\left(\theta_t - \hat{\theta}_t\right)^2}{2B_t v} \right) \times$$

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$$\times \frac{1}{\Gamma\left(\frac{2a+1}{2}\right)\left(N_t^{-1}\right)^{\frac{2a+1}{2}}} \frac{1}{v^{\frac{2a+1}{2}+1}} \exp\left(-\frac{1}{N_t^{-1} v}\right) \tag{2.4.9}$$

where $N_t = \frac{\left(y_t - \hat{\theta}_{t-1}\right)^2}{2\left(A_t + 1\right)} + \frac{1}{\beta}$

e) $(\theta_t | \boldsymbol{Y}_t)$ has the following distribution

$$f(\theta_t | \boldsymbol{Y}_t) = \frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)} \frac{1}{\sqrt{\pi}} \left(2B_t N_t\right)^{\frac{2a+1}{2}} \frac{1}{\left(\left(\theta_t - \hat{\theta}_t\right)^2 + 2B_t N_t\right)^{\frac{(2a+1)+1}{2}}} \tag{2.4.10}$$

which is a transformation of the t-distribution with *2α+1* degrees of freedom.

$\hat{\theta}_t$ is the expected value of $(\theta_t | \boldsymbol{Y}_t)$, described by

$$E[\theta_t | \boldsymbol{Y}_t] = \hat{\theta}_t = B_t Y_t + \left(1 - B_t\right) \hat{\theta}_{t-1} \tag{2.4.11}$$

Proof :

The proof of this theorem is based on the mathematical induction process.

Concretely, the proof is composed of three induction steps:

A. We prove the relationships a) – e) for *t=1*

a) To prove that $(\theta_1 | V) \sim N\left(\hat{\theta}_0, A_1 V\right)$, where $A_1 = B_0 + \kappa$, it is sufficient to prove that the pdf of $(\theta_1 | V)$ is

$$f(\theta_1 | V) = \frac{1}{\sqrt{2\pi \cdot 2\kappa v}} \exp\left(-\frac{\left(\theta_1 - \hat{\theta}_0\right)^2}{2 \cdot 2\kappa v}\right) \tag{2.4.12}$$

since $B_0 = \kappa$ and $A_1 = 2\kappa$.

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To establish (2.4.11), we note that

$$f(\theta_1|V)=\int_{-\infty}^{+\infty} f(\theta_1 \mid \theta_0, V) f(\theta_0 \mid V) d\theta_0$$

According to the assumptions of the theorem,

$(\theta_0|V)\sim N\left(\hat{\theta}_0, B_0 V\right)$, where $B_0=\kappa$ and

$(\theta_1|\theta_0,V)\sim N(\theta_0,\kappa V)$

Then

$f(\theta_1|V)=$

$$=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi\kappa v}}\exp\left(-\frac{(\theta_1-\theta_0)^2}{2\kappa v}\right)\frac{1}{\sqrt{2\pi\kappa v}}\exp\left(-\frac{\left(\theta_0-\hat{\theta}_0\right)^2}{2\kappa v}\right)d\theta_0=$$

$$=\frac{1}{2\pi\kappa v}\int_{-\infty}^{+\infty}\exp\left(-\frac{\theta_1^2+\theta_0^2-2\theta_1\theta_0+\theta_0^2+\hat{\theta}_0^2-2\hat{\theta}_0\theta_0}{2\kappa v}\right)d\theta_0=$$

$$=\frac{1}{2\pi\kappa v}\int_{-\infty}^{+\infty}\exp\left(-\frac{2\theta_0^2-2\left(\theta_1+\hat{\theta}_0\right)\theta_0+\theta_1^2+\hat{\theta}_0^2}{2\kappa v}\right)d\theta_0=$$

$$=\frac{1}{2\pi\kappa v}\int_{-\infty}^{+\infty}\exp\left(-\frac{\theta_0^2-2\frac{\theta_1+\hat{\theta}_0}{2}\theta_0+\frac{\theta_1^2+\hat{\theta}_0^2}{2}}{\kappa v}\right)d\theta_0=$$

$$=\frac{1}{2\pi\kappa v}\int_{-\infty}^{+\infty}\exp\left(-\frac{\left(\theta_0-\frac{\theta_1+\hat{\theta}_0}{2}\right)^2+\frac{\theta_1^2+\hat{\theta}_0^2}{2}-\left(\frac{\theta_1+\hat{\theta}_0}{2}\right)^2}{2\frac{\kappa v}{2}}\right)d\theta_0=$$

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$$=\frac{1}{2\pi\kappa v}\sqrt{2\pi\frac{\kappa v}{2}}\exp\left(\frac{\left(\frac{\theta_1+\hat{\theta}_0}{2}\right)^2-\frac{\theta_1^2+\hat{\theta}_0^2}{2}}{\kappa v}\right)=$$

$$=\frac{1}{2\sqrt{\pi\kappa v}}\exp\left(\frac{\theta_1^2+\hat{\theta}_0^2+2\hat{\theta}_0\,\theta_1-2\theta_1^2-2\hat{\theta}_0^2}{4\kappa v}\right)=$$

$$=\frac{1}{\sqrt{2\pi\cdot 2\kappa v}}\exp\left(-\frac{\theta_1^2+\hat{\theta}_0^2-2\hat{\theta}_0\,\theta_1}{2\cdot 2\kappa v}\right)=$$

$$=\frac{1}{\sqrt{2\pi\cdot 2\kappa v}}\exp\left(-\frac{\left(\theta_1-\hat{\theta}_0\right)^2}{2\cdot 2\kappa v}\right)$$, which is equal to (2.4.12)

b) Next we need to show that the posterior distribution of $(\theta_1 | V, Y_1) \sim N\left(\hat{\theta}_1, B_1 V\right)$,

where $B_1 = \frac{A_1}{A_1+1}$ and $\hat{\theta}_1 = B_1 y_1 + (1-B_1)\hat{\theta}_0$.

Since $B_0 = \kappa$, $A_1 = 2\kappa$, $B_1 = \frac{2\kappa}{2\kappa+1}$ and $\hat{\theta}_1 = \frac{2\kappa}{2\kappa+1} y_1 + \frac{1}{2\kappa+1}\hat{\theta}_0$,

we will need to show that

$$f(\theta_1 | V, Y_1) = \frac{1}{\sqrt{2\pi\frac{2\kappa}{2\kappa+1}v}}\exp\left(-\frac{\left(\theta_1-\left(\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0\right)\right)^2}{2\frac{2\kappa}{2\kappa+1}v}\right) \qquad (2.4.13)$$

In fact,

$f(\theta_1 | V, Y_1) \propto f(Y_1 | \theta_1, V) f(\theta_1 | V)$

According to the assumptions of the theorem, the likelihood of $(Y_t | \theta_t, V, \boldsymbol{Y}_{t-1})$ for *t=1* is described by

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$$f(Y_1|\theta_1,V)=\frac{1}{\sqrt{2\pi v}}\exp\left(-\frac{(y_1-\theta_1)^2}{2v}\right)$$

and according to (2.4.12) of part a),

$$f(\theta_1|V)=\frac{1}{\sqrt{2\pi\cdot 2\kappa v}}\exp\left(-\frac{\left(\theta_1-\hat{\theta}_0\right)^2}{2\cdot 2\kappa v}\right)$$

Thus,

$$f(\theta_1|V,Y_1)\propto f(Y_1|\theta_1,V)f(\theta_1|V)\propto$$

$$\propto\frac{1}{\sqrt{2\pi v}}\exp\left(-\frac{(y_1-\theta_1)^2}{2v}\right)\frac{1}{\sqrt{2\pi\cdot 2\kappa v}}\exp\left(-\frac{\left(\theta_1-\hat{\theta}_0\right)^2}{2\cdot 2\kappa v}\right)\propto$$

$$\propto\exp\left(-\frac{2\kappa\left(y_1^2+\theta_1^2-2y_1\theta_1\right)+\theta_1^2+\hat{\theta}_0^2-2\hat{\theta}_0\theta_1}{4\kappa v}\right)\propto$$

$$\propto\exp\left(-\frac{(2\kappa+1)\theta_1^2-2\left(2\kappa y_1+\hat{\theta}_0\right)\theta_1}{4\kappa v}\right)\propto\exp\left(-\frac{\theta_1^2-2\frac{2\kappa y_1+\hat{\theta}_0}{2\kappa+1}\theta_1}{2\frac{2\kappa v}{2\kappa+1}}\right)\propto$$

$$\propto\exp\left(-\frac{\left(\theta_1-\frac{2\kappa y_1+\hat{\theta}_0}{2\kappa+1}\right)^2}{2\frac{2\kappa v}{2\kappa+1}}\right)\propto\exp\left(-\frac{\left(\theta_1-\left(\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0\right)\right)^2}{2\frac{2\kappa}{2\kappa+1}v}\right)$$

Thus, (2.4.13) holds.

c) The joint pdf of (θ_1,V) is represented by the product

$$f(\theta_1,V)=f(\theta_1|V)f(V)$$

where the pdf of $(\theta_1|V)$ has already been established in part a) and is given by

$$f(\theta_1|V)=\frac{1}{\sqrt{2\pi\cdot 2\kappa v}}\exp\left(-\frac{\left(\theta_1-\hat{\theta}_0\right)^2}{2\cdot 2\kappa v}\right)$$

Furthermore, $V\sim IG(\alpha,\beta)$ with pdf

$$f(V)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\frac{1}{v^{\alpha+1}}\exp\left(-\frac{1}{\beta v}\right)$$

Hence, the joint prior distribution of (θ_1,V) is NIG$\left(\hat{\theta}_0,2\kappa V,a,\beta\right)$, with pdf

$$f(\theta_1,V)=\frac{1}{\sqrt{2\pi\cdot 2\kappa v}}\exp\left(-\frac{\left(\theta_1-\hat{\theta}_0\right)^2}{2\cdot 2\kappa v}\right)\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\frac{1}{v^{\alpha+1}}\exp\left(-\frac{1}{\beta v}\right)$$

d) To prove that the joint posterior distribution of $(\theta_1,V|Y_1)$ is

NIG$\left(\hat{\theta}_1,B_1V,\frac{2a+1}{2},N_1^{-1}\right)$, with pdf given by

$$f(\theta_1,V|Y_1)=\frac{1}{\sqrt{2\pi B_1 v}}\exp\left(-\frac{\left(\theta_1-\hat{\theta}_0\right)^2}{2B_1 v}\right)\times$$

$$\times\frac{1}{\Gamma\left(\frac{2a+1}{2}\right)\left[N_1^{-1}\right]^{\frac{2a+1}{2}}}\frac{1}{v^{\frac{2a+1}{2}+1}}\exp\left(-\frac{1}{N_1^{-1}v}\right)$$

where $N_1=\frac{\left(y_1-\hat{\theta}_0\right)^2}{2\left(A_1+1\right)}+\frac{1}{\beta}$, $B_1=\frac{A_1}{A_1+1}$ and $\hat{\theta}_1=B_1y_1+\left(1-B_1\right)\hat{\theta}_0$,

it is sufficient to prove that the pdf of $(\theta_1,V|Y_1)$ is given by

$$f(\theta_1,V|Y_1)=\frac{1}{\sqrt{2\pi\frac{2\kappa}{2\kappa+1}v}}\exp\left(-\frac{\left(\theta_1-\left(\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0\right)\right)^2}{2\frac{2\kappa}{2\kappa+1}v}\right)\times$$

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$$\times \frac{1}{\Gamma\left(\frac{2a+1}{2}\right)\left[\left(\frac{\left(y_1-\hat{\theta}_0\right)^2}{2(2\kappa+1)}+\frac{1}{\beta}\right)^{-1}\right]^{\frac{2a+1}{2}}} \frac{1}{v^{\frac{2a+1}{2}+1}} \exp\left(-\frac{1}{\left(\frac{\left(y_1-\hat{\theta}_0\right)^2}{2(2\kappa+1)}+\frac{1}{\beta}\right)^{-1}\cdot v}\right) \quad (2.4.14)$$

since $B_0=\kappa$, $A_1=2\kappa$, $B_1=\frac{2\kappa}{2\kappa+1}$ and $\hat{\theta}_1=\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0$

To obtain (2.4.14), we first note that $f(\theta_1,V|Y_1) \propto f(Y_1|\theta_1,V)f(\theta_1,V)$

According to the assumptions of the theorem, the likelihood of $(Y_1|\theta_1,V)$ is described by

$$f(Y_1|\theta_1,V)=\frac{1}{\sqrt{2\pi v}}\exp\left(-\frac{(y_1-\theta_1)^2}{2v}\right)$$

Furthermore, part c) allows us to know that the joint prior distribution of (θ_1,V) is $\mathrm{NIG}\left(\hat{\theta}_0, 2\kappa V, a, \beta\right)$, with pdf

$$f(\theta_1,V)=\frac{1}{\sqrt{2\pi\cdot 2\kappa v}}\exp\left(-\frac{\left(\theta_1-\hat{\theta}_0\right)^2}{2\cdot 2\kappa v}\right)\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\frac{1}{v^{\alpha+1}}\exp\left(-\frac{1}{\beta v}\right)$$

Hence,

$f(\theta_1,V|Y_1) \propto$

$$\propto \frac{1}{\sqrt{2\pi v}}\exp\left(-\frac{(y_1-\theta_1)^2}{2v}\right)\frac{1}{\sqrt{2\pi\cdot 2\kappa v}}\exp\left(-\frac{\left(\theta_1-\hat{\theta}_0\right)^2}{2\cdot 2\kappa v}\right)\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\frac{1}{v^{\alpha+1}}\exp\left(-\frac{1}{\beta v}\right)$$

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$$\propto \frac{1}{\sqrt{v}} \exp\left(-\frac{2\kappa\left(y_1^2 + \theta_1^2 - 2y_1\theta_1\right) + \theta_1^2 + \hat{\theta}_0^2 - 2\hat{\theta}_0\,\theta_1}{2\cdot 2\kappa v} \right) \frac{1}{v^{\left(a+\frac{1}{2}\right)+1}} \exp\left(-\frac{1}{\beta v}\right) \propto$$

$$\propto \frac{1}{\sqrt{v}} \exp\left(-\frac{(2\kappa+1)\theta_1^2 - 2\left(2\kappa y_1 + \hat{\theta}_0\right)\theta_1 + 2\kappa y_1^2 + \hat{\theta}_0^2}{2\cdot 2\kappa v} \right) \frac{1}{v^{\left(a+\frac{1}{2}\right)+1}} \exp\left(-\frac{1}{\beta v}\right) \propto$$

$$\propto \frac{1}{\sqrt{v}} \exp\left(-\frac{\theta_1^2 - 2\frac{2\kappa y_1 + \hat{\theta}_0}{2\kappa+1}\theta_1 + \frac{2\kappa y_1^2 + \hat{\theta}_0^2}{2\kappa+1}}{2\cdot \frac{2\kappa}{2\kappa+1} v} \right) \frac{1}{v^{\left(a+\frac{1}{2}\right)+1}} \exp\left(-\frac{1}{\beta v}\right) \propto$$

$$\propto \frac{1}{\sqrt{v}} \exp\left(-\frac{\left(\theta_1 - \frac{2\kappa y_1 + \hat{\theta}_0}{2\kappa+1}\right)^2}{2\cdot \frac{2\kappa}{2\kappa+1} v} \right) \frac{1}{v^{\left(a+\frac{1}{2}\right)+1}} \times$$

$$\times \exp\left(-\frac{\frac{2\kappa y_1^2 + \hat{\theta}_0^2}{2\kappa+1} - \left(\frac{2\kappa y_1 + \hat{\theta}_0}{2\kappa+1}\right)^2}{2\cdot \frac{2\kappa}{2\kappa+1} v} \right) \exp\left(-\frac{1}{\beta v}\right) \propto$$

$$\propto \frac{1}{\sqrt{v}} \exp\left(-\frac{\left(\theta_1 - \left(\frac{2\kappa}{2\kappa+1} y_1 + \frac{1}{2\kappa+1}\hat{\theta}_0\right)\right)^2}{2\frac{2\kappa}{2\kappa+1} v} \right) \frac{1}{v^{\left(a+\frac{1}{2}\right)+1}} \times$$

$$\times \exp\left(-\frac{(2\kappa+1)\left(2\kappa y_1^2 + \hat{\theta}_0^2\right) - \left(2\kappa y_1 + \hat{\theta}_0\right)^2}{2\cdot 2\kappa(2\kappa+1)v} \right) \exp\left(-\frac{1}{\beta v}\right) \propto$$

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$$\propto \frac{1}{\sqrt{2\pi \frac{2\kappa}{2\kappa+1} v}} \exp\left(-\frac{\left(\theta_1 - \left(\frac{2\kappa}{2\kappa+1} y_1 + \frac{1}{2\kappa+1} \hat{\theta}_0 \right) \right)^2}{2 \frac{2\kappa}{2\kappa+1} v} \right) \frac{1}{v^{\left(a+\frac{1}{2}\right)+1}} \times$$

$$\times \exp\left(-\frac{4\kappa^2 y_1^2 + 2\kappa \hat{\theta}_0^2 + 2\kappa y_1^2 + \hat{\theta}_0^2 - 4\kappa^2 y_1^2 - \hat{\theta}_0^2 - 4\kappa y_1 \hat{\theta}_0}{2 \cdot 2\kappa (2\kappa+1) v} \right) \exp\left(-\frac{1}{\beta v} \right)$$

Let $C = \frac{1}{\sqrt{2\pi \frac{2\kappa}{2\kappa+1} v}} \exp\left(-\frac{\left(\theta_1 - \left(\frac{2\kappa}{2\kappa+1} y_1 + \frac{1}{2\kappa+1} \hat{\theta}_0 \right) \right)^2}{2 \frac{2\kappa}{2\kappa+1} v} \right) \frac{1}{v^{\left(a+\frac{1}{2}\right)+1}}$

Then

$f(\theta_1, V | Y_1) \propto$

$$\propto C \times \exp\left(-\frac{4\kappa^2 y_1^2 + 2\kappa \hat{\theta}_0^2 + 2\kappa y_1^2 + \hat{\theta}_0^2 - 4\kappa^2 y_1^2 - \hat{\theta}_0^2 - 4\kappa y_1 \hat{\theta}_0}{2 \cdot 2\kappa (2\kappa+1) v} \right) \exp\left(-\frac{1}{\beta v} \right) \propto$$

$$\propto C \times \exp\left(-\frac{2\kappa \hat{\theta}_0^2 + 2\kappa y_1^2 - 4\kappa y_1 \hat{\theta}_0}{2 \cdot 2\kappa (2\kappa+1) v} \right) \exp\left(-\frac{1}{\beta v} \right) \propto$$

$$\propto C \times \exp\left(-\frac{\hat{\theta}_0^2 + y_1^2 - 2 y_1 \hat{\theta}_0}{2(2\kappa+1) v} \right) \exp\left(-\frac{1}{\beta v} \right) \propto C \times \exp\left(-\frac{\left(y_1 - \hat{\theta}_0 \right)^2}{2(2\kappa+1) v} \right) \exp\left(-\frac{1}{\beta v} \right) \propto$$

$$\propto C \times \exp\left(-\left(\frac{\left(y_1 - \hat{\theta}_0 \right)^2}{2(2\kappa+1)} + \frac{1}{\beta} \right) \cdot \frac{1}{v} \right) \propto C \times \exp\left(-\frac{1}{\left[\frac{\left(y_1 - \hat{\theta}_0 \right)^2}{2(2\kappa+1)} + \frac{1}{\beta} \right]^{-1} \cdot v} \right) \propto$$

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$$\propto C \times \frac{1}{\Gamma\left(\frac{2a+1}{2}\right)\left[\left(\frac{\left(y_1-\hat{\theta}_0\right)^2}{2(2\kappa+1)}+\frac{1}{\beta}\right)^{-1}\right]^{\frac{2a+1}{2}}} \exp\left(-\frac{1}{\left(\frac{\left(y_1-\hat{\theta}_0\right)^2}{2(2\kappa+1)}+\frac{1}{\beta}\right)^{-1}\cdot v}\right)$$

Thus,

$$f(\theta_1, V|Y_1) = \frac{1}{\sqrt{2\pi\frac{2\kappa}{2\kappa+1}v}} \exp\left(-\frac{\left(\theta_1-\left(\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0\right)\right)^2}{2\frac{2\kappa}{2\kappa+1}v}\right) \times$$

$$\times \frac{1}{\Gamma\left(\frac{2a+1}{2}\right)\left[\left(\frac{\left(y_1-\hat{\theta}_0\right)^2}{2(2\kappa+1)}+\frac{1}{\beta}\right)^{-1}\right]^{\frac{2a+1}{2}}} \frac{1}{v^{\frac{2a+1}{2}+1}} \exp\left(-\frac{1}{\left(\frac{\left(y_1-\hat{\theta}_0\right)^2}{2(2\kappa+1)}+\frac{1}{\beta}\right)^{-1}\cdot v}\right)$$

e) Our aim is to prove that the pdf of the marginal $(\theta_1|Y_1)$ is given by

$$f(\theta_1|Y_1) = \frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)} \frac{1}{\sqrt{\pi}} \left(2B_1N_1\right)^{\frac{2a+1}{2}} \frac{1}{\left(\left(\theta_1-\hat{\theta}_1\right)^2+2B_1N_1\right)^{\frac{(2a+1)+1}{2}}}$$

where $N_1 = \dfrac{\left(y_1-\hat{\theta}_0\right)^2}{2\left(A_1+1\right)}+\dfrac{1}{\beta}$

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and the expected value of the distribution is

$$E[\theta_1|Y_1]=\hat{\theta}_1=B_1 y_1+\left(1-B_1\right)\hat{\theta}_0$$

or equivalently,

$$E[\theta_1|Y_1]=\hat{\theta}_1=\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0 \qquad (2.4.15)$$

since $A_1=2\kappa$ and $B_1=\frac{2\kappa}{2\kappa+1}$

The proof of this part of the theorem is based on the following equation

$$f(\theta_1|Y_1)=\int_0^{+\infty} f\left(\theta_1,V\mid Y_1\right)dv$$

We have already shown that the joint posterior pdf of $(\theta_1,V|Y_1)$ is given by (2.4.14) :

$$f(\theta_1,V|Y_1)=\frac{1}{\sqrt{2\pi\frac{2\kappa}{2\kappa+1}v}}\exp\left(-\frac{\left(\theta_1-\left(\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0\right)\right)^2}{2\frac{2\kappa}{2\kappa+1}v}\right)\times$$

$$\times\frac{1}{\Gamma\left(\frac{2a+1}{2}\right)\left[\left(\frac{\left(y_1-\hat{\theta}_0\right)^2}{2\left(2\kappa+1\right)}+\frac{1}{\beta}\right)^{-1}\right]^{\frac{2a+1}{2}}}\frac{1}{v^{\left(\frac{2a+1}{2}\right)+1}}\exp\left(-\frac{1}{\left(\frac{\left(y_1-\hat{\theta}_0\right)^2}{2\left(2\kappa+1\right)}+\frac{1}{\beta}\right)^{-1}\cdot v}\right)$$

Therefore,

$$f(\theta_1|Y_1)=\int_0^{+\infty} f\left(\theta_1,V\mid Y_1\right)dv=$$

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$$=\int_0^{+\infty}\frac{1}{\sqrt{2\pi\frac{2\kappa}{2\kappa+1}v}}\exp\left(-\frac{\left(\theta_1-\left(\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0\right)\right)^2}{2\frac{2\kappa}{2\kappa+1}v}\right)\frac{1}{\Gamma\left(\frac{2a+1}{2}\right)\left[\left(\frac{\left(y_1-\hat{\theta}_0\right)^2}{2(2\kappa+1)}+\frac{1}{\beta}\right)^{-1}\right]^{\frac{2a+1}{2}}}\times$$

$$\times\frac{1}{v^{\left(\frac{2a+1}{2}\right)+1}}\exp\left(-\frac{1}{\left(\frac{\left(y_1-\hat{\theta}_0\right)^2}{2(2\kappa+1)}+\frac{1}{\beta}\right)^{-1}\cdot v}\right)dv=$$

$$=\frac{1}{\sqrt{2\pi\frac{2\kappa}{2\kappa+1}}}\frac{1}{\Gamma\left(\frac{2a+1}{2}\right)\left[\left(\frac{\left(y_1-\hat{\theta}_0\right)^2}{2(2\kappa+1)}+\frac{1}{\beta}\right)^{-1}\right]^{\frac{2a+1}{2}}}\times$$

$$\times\int_0^{+\infty}\frac{1}{v^{(a+1)+1}}\exp\left(-\frac{\frac{\left(\theta_1-\left(\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0\right)\right)^2}{2\frac{2\kappa}{2\kappa+1}}}{v}\right)\exp\left(-\frac{\frac{\left(y_1-\hat{\theta}_0\right)^2}{2(2\kappa+1)}+\frac{1}{\beta}}{v}\right)dv=$$

$$=\frac{1}{\sqrt{2\pi\frac{2\kappa}{2\kappa+1}}}\frac{1}{\Gamma\left(\frac{2a+1}{2}\right)\left[\left(\frac{\left(y_1-\hat{\theta}_0\right)^2}{2(2\kappa+1)}+\frac{1}{\beta}\right)^{-1}\right]^{\frac{2a+1}{2}}}\times$$

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$$\times \int_0^{+\infty} \frac{1}{v^{(a+1)+1}} \exp\left(- \frac{1}{\left(\frac{\left(\theta_1 - \left(\frac{2\kappa}{2\kappa+1} y_1 + \frac{1}{2\kappa+1} \hat{\theta}_0 \right)\right)^2}{2\frac{2\kappa}{2\kappa+1}} + \frac{\left(y_1 - \hat{\theta}_0\right)^2}{2(2\kappa+1)} + \frac{1}{\beta} \right)^{-1} \cdot v} \right) dv =$$

$$= \frac{1}{\sqrt{2\pi \frac{2\kappa}{2\kappa+1}}} \frac{1}{\Gamma\left(\frac{2a+1}{2}\right) \left[\left(\frac{\left(y_1 - \hat{\theta}_0\right)^2}{2(2\kappa+1)} + \frac{1}{\beta} \right)^{-1} \right]^{\frac{2a+1}{2}}} \times$$

$$\times \Gamma(\alpha+1) \left(\left(\frac{\left(\theta_1 - \left(\frac{2\kappa}{2\kappa+1} y_1 + \frac{1}{2\kappa+1} \hat{\theta}_0 \right)\right)^2}{2\frac{2\kappa}{2\kappa+1}} + \frac{\left(y_1 - \hat{\theta}_0\right)^2}{2(2\kappa+1)} + \frac{1}{\beta} \right)^{-1} \right)^{\alpha+1} =$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma\left(\frac{2a+1}{2}\right)} \frac{1}{\sqrt{2\pi \frac{2\kappa}{2\kappa+1}}} \frac{\left(\left(\frac{\left(\theta_1 - \left(\frac{2\kappa}{2\kappa+1} y_1 + \frac{1}{2\kappa+1} \hat{\theta}_0 \right)\right)^2}{2\frac{2\kappa}{2\kappa+1}} + \frac{\left(y_1 - \hat{\theta}_0\right)^2}{2(2\kappa+1)} + \frac{1}{\beta} \right)^{-1} \right)^{\alpha+1}}{\left(\left(\frac{\left(y_1 - \hat{\theta}_0\right)^2}{2(2\kappa+1)} + \frac{1}{\beta} \right)^{-1} \right)^{\frac{2a+1}{2}}} =$$

$$= \frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)} \frac{1}{\sqrt{2\pi \frac{2\kappa}{2\kappa+1}}} \left(\frac{\left(y_1 - \hat{\theta}_0\right)^2}{2(2\kappa+1)} + \frac{1}{\beta} \right)^{\frac{2a+1}{2}} \times$$

$$\times \frac{1}{\left(\frac{\left(\theta_1-\left(\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0\right)\right)^2}{2\frac{2\kappa}{2\kappa+1}}+\frac{\left(y_1-\hat{\theta}_0\right)^2}{2(2\kappa+1)}+\frac{1}{\beta}\right)^{\alpha+1}}=$$

$$=\frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)}\frac{1}{\sqrt{2\pi\frac{2\kappa}{2\kappa+1}}}\left(\frac{\beta\left(y_1-\hat{\theta}_0\right)^2+2(2\kappa+1)}{2\beta(2\kappa+1)}\right)^{\frac{2a+1}{2}}\times$$

$$\times \frac{1}{\left(\frac{\beta(2\kappa+1)\left(\theta_1-\left(\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0\right)\right)^2+\beta\frac{2\kappa}{2\kappa+1}\left(y_1-\hat{\theta}_0\right)^2+2\frac{2\kappa}{2\kappa+1}(2\kappa+1)}{2\beta\frac{2\kappa}{2\kappa+1}(2\kappa+1)}\right)^{a+1}}=$$

$$=\frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)}\frac{1}{\sqrt{2\pi\frac{2\kappa}{2\kappa+1}}}(2\beta\cdot 2\kappa)^{a+1}\left(\frac{\beta\left(y_1-\hat{\theta}_0\right)^2+2(2\kappa+1)}{2\beta(2\kappa+1)}\right)^{\frac{2a+1}{2}}\times$$

$$\times \frac{1}{\left(\beta(2\kappa+1)\left(\theta_1-\left(\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0\right)\right)^2+\beta\frac{2\kappa}{2\kappa+1}\left(y_1-\hat{\theta}_0\right)^2+2\cdot 2\kappa\right)^{a+1}}=$$

$$=\frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)}\frac{1}{\sqrt{2\pi\frac{2\kappa}{2\kappa+1}}}(2\beta\cdot 2\kappa)^{a+1}\left(\frac{\beta\left(y_1-\hat{\theta}_0\right)^2+2(2\kappa+1)}{2\beta(2\kappa+1)}\right)^{\frac{2a+1}{2}}\times$$

$$\times\left(\frac{1}{\beta(2\kappa+1)}\right)^{\alpha+1}\frac{1}{\left(\left(\theta_1-\left(\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0\right)\right)^2+\frac{\beta\frac{2\kappa}{2\kappa+1}\left(y_1-\hat{\theta}_0\right)^2+2\cdot 2\kappa}{\beta(2\kappa+1)}\right)^{a+1}}=$$

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$$=\frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)}\frac{1}{\sqrt{2\pi\frac{2\kappa}{2\kappa+1}}}\left(\frac{2\cdot 2\kappa}{2\kappa+1}\right)^{a+1}\left(\frac{\beta\left(y_1-\hat{\theta}_0\right)^2+2\left(2\kappa+1\right)}{2\beta\left(2\kappa+1\right)}\right)^{\frac{2a+1}{2}}\times$$

$$\times\frac{1}{\left(\left(\theta_1-\left(\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0\right)\right)^2+\frac{\beta\frac{2\kappa}{2\kappa+1}\left(y_1-\hat{\theta}_0\right)^2+2\cdot 2\kappa}{\beta(2\kappa+1)}\right)^{\frac{(2a+1)+1}{2}}}=$$

$$=\frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)}\frac{1}{\sqrt{\pi}}\left(\frac{2\cdot 2\kappa}{2\kappa+1}\right)^{\frac{2a+1}{2}}\left(\frac{\beta\left(y_1-\hat{\theta}_0\right)^2+2\left(2\kappa+1\right)}{2\beta\left(2\kappa+1\right)}\right)^{\frac{2a+1}{2}}\times$$

$$\times\frac{1}{\left(\left(\theta_1-\left(\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0\right)\right)^2+\frac{\beta\frac{2\kappa}{2\kappa+1}\left(y_1-\hat{\theta}_0\right)^2+2\cdot 2\kappa}{\beta(2\kappa+1)}\right)^{\frac{(2a+1)+1}{2}}}=$$

$$=\frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)}\frac{1}{\sqrt{\pi}}\left(\frac{\beta\frac{2\kappa}{2\kappa+1}\left(y_1-\hat{\theta}_0\right)^2+2\cdot 2\kappa}{\beta\left(2\kappa+1\right)}\right)^{\frac{2a+1}{2}}\times$$

$$\times\frac{1}{\left(\left(\theta_1-\left(\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0\right)\right)^2+\frac{\beta\frac{2\kappa}{2\kappa+1}\left(y_1-\hat{\theta}_0\right)^2+2\cdot 2\kappa}{\beta(2\kappa+1)}\right)^{\frac{(2a+1)+1}{2}}}$$

$$=\frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)}\frac{1}{\sqrt{\pi}}\left(2\frac{2\kappa}{2\kappa+1}\frac{\left(y_1-\hat{\theta}_0\right)^2}{2\kappa+1}+2\frac{2\kappa}{2\kappa+1}\frac{1}{\beta}\right)^{\frac{2a+1}{2}}\times$$

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$$\times \frac{1}{\left(\left(\theta_1-\left(\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0\right)\right)^2+2\frac{2\kappa}{2\kappa+1}\left(\frac{\left(y_1-\hat{\theta}_0\right)^2}{2\left(2\kappa+1\right)}+\frac{1}{\beta}\right)\right)^{\frac{(2a+1)+1}{2}}}$$

Thus,

$f(\theta_1|Y_1)=$

$$=\frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)}\frac{1}{\sqrt{\pi}}\left(2B_1N_1\right)^{\frac{2a+1}{2}}\frac{1}{\left(\left(\theta_1-\left(\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0\right)\right)^2+2B_1N_1\right)^{\frac{(2a+1)+1}{2}}}$$

where $N_1=\frac{\left(y_1-\hat{\theta}_0\right)^2}{2\left(A_1+1\right)}+\frac{1}{\beta}$

We make the transformation

$$t_1=\frac{\theta_1-\left(\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0\right)}{\frac{\sqrt{2B_1N_1}}{\sqrt{2\alpha+1}}} \tag{2.4.16}$$

The Jacobian of the transformation is $\frac{\sqrt{2B_1N_1}}{\sqrt{2a+1}}$ and the marginal pdf of $(t_1|Y_1)$ is given by

$$f(t_1|Y_1)=\frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)}\frac{1}{\sqrt{\pi}}\left(2B_1N_1\right)^{\frac{2a+1}{2}}\frac{1}{\left(t_1^2\cdot\frac{2B_1N_1}{2a+1}+2B_1N_1\right)^{\frac{(2a+1)+1}{2}}}\frac{\sqrt{2B_1N_1}}{\sqrt{2a+1}}=$$

$$=\frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)}\frac{1}{\sqrt{\pi}}\left(2B_1N_1\right)^{\frac{2a+1}{2}}\frac{1}{\left(2B_1N_1\right)^{a+1}}\frac{1}{\left(\frac{t_1^2}{2a+1}+1\right)^{\frac{(2a+1)+1}{2}}}\frac{\sqrt{2B_1N_1}}{\sqrt{2a+1}}=$$

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$$=\frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)}\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{2a+1}}\frac{1}{\left(\frac{t_1^2}{2a+1}+1\right)^{\frac{(2a+1)+1}{2}}}\Leftrightarrow$$

$$\Leftrightarrow f(t_1|Y_1)=\frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)}\frac{1}{\sqrt{(2a+1)\pi}}\frac{1}{\left(\frac{t_1^2}{2a+1}+1\right)^{\frac{(2a+1)+1}{2}}},$$

which means that $(t_1|Y_1)$ has the t-distribution with $2\alpha+1$ degrees of freedom; hence, the expected value of the distribution is zero, i.e.

$$E[t_1|Y_1]=0 \tag{2.4.17}$$

From (2.4.16), (2.4.17), it follows that

$$E\left[\left.\frac{\theta_1-\left(\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0\right)}{\frac{\sqrt{2B_1\mathrm{N}_1}}{\sqrt{2\alpha+1}}}\right|\mathrm{Y}_1\right]=0\Leftrightarrow E[\theta_1|Y_1]=\frac{2\kappa}{2\kappa+1}y_1+\frac{1}{2\kappa+1}\hat{\theta}_0,$$

which is equal to (2.4.15)

B. We assume that the relationships a) – e) hold for $t\text{-}1$

Concretely,

a) The prior distribution of $(\theta_{t-1}|V,\boldsymbol{Y}_{t-2})$ is given by

$$(\theta_{t-1}|V,\boldsymbol{Y}_{t-2})\sim N\left(\hat{\theta}_{t-2},A_{t-1}V\right)$$

with pdf

$$f(\theta_{t-1}|V,\boldsymbol{Y}_{t-2})=\frac{1}{\sqrt{2\pi A_{t-1}v}}\exp\left(-\frac{\left(\theta_{t-1}-\hat{\theta}_{t-2}\right)^2}{2A_{t-1}v}\right)$$

where $A_{t-1}=B_{t-2}+\kappa$

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b) The posterior distribution of $(\theta_{t-1} | V, \boldsymbol{Y}_{t-1})$ is given by

$$(\theta_{t-1} | V, \boldsymbol{Y}_{t-1}) \sim N\left(\hat{\theta}_{t-1}, B_{t-1}V\right)$$

with pdf

$$f(\theta_{t-1} | V, \boldsymbol{Y}_{t-1}) = \frac{1}{\sqrt{2\pi B_{t-1} v}} \exp\left(-\frac{\left(\theta_{t-1} - \hat{\theta}_{t-1}\right)^2}{2B_{t-1}v}\right)$$

where $B_{t-1} = \dfrac{A_{t-1}}{A_{t-1}+1}$ and $\hat{\theta}_{t-1} = B_{t-1} y_{t-1} + (1 - B_{t-1})\, \hat{\theta}_{t-2}$

c) The joint prior distribution of $(\theta_{t-1}, V | \boldsymbol{Y}_{t-2})$ is NIG$\left(\hat{\theta}_{t-2}, A_{t-1}V, a, \beta\right)$, with pdf

$$f(\theta_{t-1}, V | \boldsymbol{Y}_{t-2}) = \frac{1}{\sqrt{2\pi A_{t-1} v}} \exp\left(-\frac{\left(\theta_{t-1} - \hat{\theta}_{t-2}\right)^2}{2A_{t-1}v}\right) \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \frac{1}{v^{\alpha+1}} \exp\left(-\frac{1}{\beta v}\right)$$

d) The joint posterior distribution of $(\theta_{t-1}, V | \boldsymbol{Y}_{t-1})$ is NIG$\left(\hat{\theta}_{t-1}, B_{t-1}V, \dfrac{2a+1}{2}, N_{t-1}^{-1}\right)$,

with pdf given by

$$f(\theta_{t-1}, V | \boldsymbol{Y}_{t-1}) = \frac{1}{\sqrt{2\pi B_{t-1} v}} \exp\left(-\frac{\left(\theta_{t-1} - \hat{\theta}_{t-1}\right)^2}{2B_{t-1}v}\right) \times$$

$$\times \frac{1}{\Gamma\left(\dfrac{2a+1}{2}\right)\left[N_{t-1}^{-1}\right]^{\frac{2a+1}{2}}} \frac{1}{v^{\left(\frac{2a+1}{2}\right)+1}} \exp\left(-\frac{1}{N_{t-1}^{-1} v}\right)$$

where $N_{t-1} = \dfrac{\left(y_{t-1} - \hat{\theta}_{t-2}\right)^2}{2\left(A_{t-1}+1\right)} + \dfrac{1}{\beta}$

e) $(\theta_{t-1} | \boldsymbol{Y}_{t-1})$ has the following distribution

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$$f(\theta_{t-1}|\boldsymbol{Y}_{t-1})=\frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)}\frac{1}{\sqrt{\pi}}\left(2B_{t-1}N_{t-1}\right)^{\frac{2a+1}{2}}\frac{1}{\left(\left(\theta_{t-1}-\hat{\theta}_{t-1}\right)^2+2B_{t-1}N_{t-1}\right)^{\frac{(2a+1)+1}{2}}}$$

which is a transformation of the t-distribution with $2\alpha+1$ degrees of freedom.

$\hat{\theta}_{t-1}$ is the expected value of $(\theta_{t-1}|\boldsymbol{Y}_{t-1})$ described by

$$E[\theta_{t-1}|\boldsymbol{Y}_{t-1}]=\hat{\theta}_{t-1}=B_{t-1}y_{t-1}+\left(1-B_{t-1}\right)\hat{\theta}_{t-2}$$

C. We will show that the relationships a) – e) hold for t

a) The prior distribution of $(\theta_t|V,\boldsymbol{Y}_{t-1})$ and its pdf, will be resulted, using the following equation :

$$f(\theta_t|V,\boldsymbol{Y}_{t-1})=\int_{-\infty}^{+\infty}f\left(\theta_t\mid\theta_{t-1},V,Y_{t-1}\right)f\left(\theta_{t-1}\mid V,Y_{t-1}\right)d\theta_{t-1} \qquad (2.4.18)$$

According to the assumptions of the theorem

$$(\theta_t|\theta_{t-1},V,\boldsymbol{Y}_{t-1})\sim N(\theta_{t-1},\kappa V)$$

with pdf

$$f(\theta_t|\theta_{t-1},V,\boldsymbol{Y}_{t-1})=\frac{1}{\sqrt{2\pi\kappa v}}\exp\left(-\frac{(\theta_t-\theta_{t-1})^2}{2\kappa v}\right) \qquad (2.4.19)$$

In addition, according to the induction hypothesis, the posterior distribution of $(\theta_{t-1}|V,\boldsymbol{Y}_{t-1})$ is given by

$$(\theta_{t-1}|V,\boldsymbol{Y}_{t-1})\sim N\left(\hat{\theta}_{t-1},B_{t-1}V\right)$$

with pdf

$$f(\theta_{t-1}|V,\boldsymbol{Y}_{t-1})=\frac{1}{\sqrt{2\pi B_{t-1}v}}\exp\left(-\frac{\left(\theta_{t-1}-\hat{\theta}_{t-1}\right)^2}{2B_{t-1}v}\right) \qquad (2.4.20)$$

From the relationships (2.4.18), (2.4.19), (2.4.20), it follows that

$$f(\theta_t|V,\boldsymbol{Y}_{t-1})=\int_{-\infty}^{+\infty} f(\theta_t\mid\theta_{t-1},V,Y_{t-1})f(\theta_{t-1}\mid V,Y_{t-1})d\theta_{t-1}=$$

$$=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi\kappa v}}\exp\left(-\frac{(\theta_t-\theta_{t-1})^2}{2\kappa v}\right)\frac{1}{\sqrt{2\pi B_{t-1}v}}\exp\left(-\frac{\left(\theta_{t-1}-\hat{\theta}_{t-1}\right)^2}{2B_{t-1}v}\right)d\theta_{t-1}=$$

$$=\frac{1}{\sqrt{2\pi\kappa v}}\frac{1}{\sqrt{2\pi B_{t-1}v}}\int_{-\infty}^{+\infty}\exp\left(-\frac{B_{t-1}(\theta_{t-1}-\theta_t)^2+\kappa\left(\theta_{t-1}-\hat{\theta}_{t-1}\right)^2}{2\kappa B_{t-1}v}\right)d\theta_{t-1}=$$

$$=\frac{1}{2\pi v\sqrt{\kappa B_{t-1}}}\int_{-\infty}^{+\infty}\exp\left(-\frac{B_{t-1}\left(\theta_{t-1}^2+\theta_t^2-2\theta_t\theta_{t-1}\right)+\kappa\left(\theta_{t-1}^2+\hat{\theta}_{t-1}^2-2\theta_{t-1}\hat{\theta}_{t-1}\right)}{2\kappa B_{t-1}v}\right)d\theta_{t-1}=$$

$$=\frac{1}{2\pi v\sqrt{\kappa B_{t-1}}}\int_{-\infty}^{+\infty}\exp\left(-\frac{(B_{t-1}+\kappa)\theta_{t-1}^2-2\left(B_{t-1}\theta_t+\kappa\hat{\theta}_{t-1}\right)\theta_{t-1}+B_{t-1}\theta_t^2+\kappa\hat{\theta}_{t-1}^2}{2\kappa B_{t-1}v}\right)d\theta_{t-1}=$$

$$=\frac{1}{2\pi v\sqrt{\kappa B_{t-1}}}\int_{-\infty}^{+\infty}\exp\left(-\frac{\theta_{t-1}^2-2\dfrac{B_{t-1}\theta_t+\kappa\hat{\theta}_{t-1}}{B_{t-1}+\kappa}\theta_{t-1}+\dfrac{B_{t-1}\theta_t^2+\kappa\hat{\theta}_{t-1}^2}{B_{t-1}+\kappa}}{\dfrac{2\kappa B_{t-1}v}{B_{t-1}+\kappa}}\right)d\theta_{t-1}=$$

$$=\frac{1}{2\pi v\sqrt{\kappa B_{t-1}}}\times$$

$$\times\int_{-\infty}^{+\infty}\exp\left(-\frac{\left(\theta_{t-1}-\dfrac{B_{t-1}\theta_t+\kappa\hat{\theta}_{t-1}}{B_{t-1}+\kappa}\right)^2}{2\dfrac{\kappa B_{t-1}v}{B_{t-1}+\kappa}}\right)\exp\left(-\frac{\dfrac{B_{t-1}\theta_t^2+\kappa\hat{\theta}_{t-1}^2}{B_{t-1}+\kappa}-\left(\dfrac{B_{t-1}\theta_t+\kappa\hat{\theta}_{t-1}}{B_{t-1}+\kappa}\right)^2}{2\dfrac{\kappa B_{t-1}v}{B_{t-1}+\kappa}}\right)d\theta_{t-1}=$$

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$$=\frac{1}{2\pi v\sqrt{\kappa B_{t-1}}}\sqrt{\frac{2\pi\kappa B_{t-1}v}{B_{t-1}+\kappa}}\exp\left(-\frac{\dfrac{B_{t-1}\theta_t^2+\kappa\hat{\theta}_{t-1}^2}{B_{t-1}+\kappa}-\left(\dfrac{B_{t-1}\theta_t+\kappa\hat{\theta}_{t-1}}{B_{t-1}+\kappa}\right)^2}{2\dfrac{\kappa B_{t-1}v}{B_{t-1}+\kappa}}\right)=$$

$$=\frac{1}{\sqrt{2\pi\cdot(B_{t-1}+\kappa)\cdot v}}\exp\left(-\frac{(B_{t-1}+\kappa)\left(B_{t-1}\theta_t^2+\kappa\hat{\theta}_{t-1}^2\right)-\left(B_{t-1}\theta_t+\kappa\hat{\theta}_{t-1}\right)^2}{2\kappa B_{t-1}v(B_{t-1}+\kappa)}\right)=$$

$$=\frac{1}{\sqrt{2\pi\cdot(B_{t-1}+\kappa)\cdot v}}\times$$

$$\times\exp\left(-\frac{B_{t-1}(B_{t-1}+\kappa)\theta_t^2+\kappa(B_{t-1}+\kappa)\hat{\theta}_{t-1}^2-B_{t-1}^2\theta_t^2-\kappa^2\hat{\theta}_{t-1}^2-2\kappa B_{t-1}\theta_t\hat{\theta}_{t-1}}{2\kappa B_{t-1}v(B_{t-1}+\kappa)}\right)=$$

$$=\frac{1}{\sqrt{2\pi\cdot(B_{t-1}+\kappa)\cdot v}}\exp\left(-\frac{B_{t-1}\kappa\theta_t^2-2\kappa B_{t-1}\theta_t\hat{\theta}_{t-1}+\kappa B_{t-1}\hat{\theta}_{t-1}^2}{2\kappa B_{t-1}v(B_{t-1}+\kappa)}\right)=$$

$$=\frac{1}{\sqrt{2\pi\cdot(B_{t-1}+\kappa)\cdot v}}\exp\left(-\frac{\theta_t^2-2\dfrac{\kappa B_{t-1}\theta_t\hat{\theta}_{t-1}}{\kappa B_{t-1}}+\dfrac{\kappa B_{t-1}\hat{\theta}_{t-1}^2}{\kappa B_{t-1}}}{\dfrac{2\kappa B_{t-1}v(B_{t-1}+\kappa)}{\kappa B_{t-1}}}\right)=$$

$$=\frac{1}{\sqrt{2\pi\cdot(B_{t-1}+\kappa)\cdot v}}\exp\left(-\frac{\left(\theta_t-\hat{\theta}_{t-1}\right)^2}{2(B_{t-1}+\kappa)v}\right)$$

Consequently,

$$f(\theta_t|V,\boldsymbol{Y}_{t-1})=\frac{1}{\sqrt{2\pi A_t v}}\exp\left(-\frac{\left(\theta_t-\hat{\theta}_{t-1}\right)^2}{2A_t v}\right),\text{ where } A_t=B_{t-1}+\kappa$$

b) To prove that the posterior distribution of $(\theta_t|V,\boldsymbol{Y}_t)$ is given by

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$$(\theta_t | V, \boldsymbol{Y}_t) \sim N\left(\hat{\theta}_t, B_t V\right)$$

with pdf

$$f(\theta_t | V, \boldsymbol{Y}_t) = \frac{1}{\sqrt{2\pi B_t v}} \exp\left(-\frac{\left(\theta_t - \hat{\theta}_t\right)^2}{2B_t v}\right)$$

where $B_t = \frac{A_t}{A_t + 1}$ and $\hat{\theta}_t = B_t y_t + (1 - B_t)\,\hat{\theta}_{t-1}$, we note that

$$f(\theta_t | V, \boldsymbol{Y}_t) \propto f(Y_t | \theta_t, V, \boldsymbol{Y}_{t-1}) f(\theta_t | V, \boldsymbol{Y}_{t-1}) \tag{2.4.21}$$

The distributions of $(Y_t | \theta_t, V, \boldsymbol{Y}_{t-1})$ and $(\theta_t | V, \boldsymbol{Y}_{t-1})$ are also known, from the assumptions of the theorem and the last part of the theorem, respectively.
Specifically, the likelihood of $(Y_t | \theta_t, V, \boldsymbol{Y}_{t-1})$ is given by

$$f(Y_t | \theta_t, V, \boldsymbol{Y}_{t-1}) = \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{(y_t - \theta_t)^2}{2v}\right) \tag{2.4.22}$$

and the prior distribution of $(\theta_t | V, \boldsymbol{Y}_{t-1})$ is described by

$$f(\theta_t | V, \boldsymbol{Y}_{t-1}) = \frac{1}{\sqrt{2\pi A_t v}} \exp\left(-\frac{\left(\theta_t - \hat{\theta}_{t-1}\right)^2}{2A_t v}\right) \tag{2.4.23}$$

Combining the relationships (2.4.21), (2.4.22), (2.4.23), it follows that

$$f(\theta_t | V, \boldsymbol{Y}_t) \propto$$

$$\propto \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{(y_t - \theta_t)^2}{2v}\right) \frac{1}{\sqrt{2\pi A_t v}} \exp\left(-\frac{\left(\theta_t - \hat{\theta}_{t-1}\right)^2}{2A_t v}\right) \propto$$

$$\propto \exp\left(-\frac{A_t\left(\theta_t^2 + y_t^2 - 2y_t\theta_t\right) + \theta_t^2 + \hat{\theta}_{t-1}^2 - 2\hat{\theta}_{t-1}\theta_t}{2A_t v}\right) \propto$$

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$$\propto \exp\left(-\frac{(A_t+1)\theta_t^2 - 2\left(A_t y_t + \hat{\theta}_{t-1}\right)\theta_t + A_t y_t^2 + \hat{\theta}_{t-1}^2}{2A_t v}\right) \propto$$

$$\propto \exp\left(-\frac{\theta_t^2 - 2\frac{A_t y_t + \hat{\theta}_{t-1}}{A_t+1}\theta_t}{2\frac{A_t}{A_t+1}v}\right) \propto \exp\left(-\frac{\left(\theta_t - \frac{A_t y_t + \hat{\theta}_{t-1}}{A_t+1}\right)^2}{2\frac{A_t}{A_t+1}v}\right) \propto$$

$$\propto \frac{1}{\sqrt{2\pi\frac{A_t}{A_t+1}v}} \exp\left(-\frac{\left(\theta_t - \left(\frac{A_t}{A_t+1}y_t + \frac{\hat{\theta}_{t-1}}{A_t+1}\right)\right)^2}{2\frac{A_t}{A_t+1}v}\right)$$

Hence,

$$f(\theta_t | V, \boldsymbol{Y}_t) = \frac{1}{\sqrt{2\pi B_t v}} \exp\left(-\frac{\left(\theta_t - \hat{\theta}_t\right)^2}{2B_t v}\right)$$

where $B_t = \frac{A_t}{A_t+1}$ and $\hat{\theta}_t = B_t y_t + (1-B_t)\,\hat{\theta}_{t-1}$

c) In conformity with the case for $t=1$, the joint prior pdf of $(\theta_t, V|\boldsymbol{Y}_{t-1})$ is represented by

$$f(\theta_t, V|\boldsymbol{Y}_{t-1}) = f(\theta_t | V, \boldsymbol{Y}_{t-1})\, f(V|\boldsymbol{Y}_{t-1})$$

where the pdf of $(\theta_t | V, \boldsymbol{Y}_{t-1})$ has already been established in part a) and is given by

$$f(\theta_t | V, \boldsymbol{Y}_{t-1}) = \frac{1}{\sqrt{2\pi A_t v}} \exp\left(-\frac{\left(\theta_t - \hat{\theta}_{t-1}\right)^2}{2A_t v}\right)$$

Furthermore, $V \sim IG(\alpha,\beta)$ with pdf

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$$f(V|\boldsymbol{Y}_{t-1}) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \frac{1}{v^{\alpha+1}} \exp\left(-\frac{1}{\beta v}\right)$$

Hence, the joint prior distribution of $(\theta_t, V|\boldsymbol{Y}_{t-1})$ is $\text{NIG}\left(\hat{\theta}_{t-1}, A_t V, a, \beta\right)$, with pdf

$$f(\theta_t, V|\boldsymbol{Y}_{t-1}) = \frac{1}{\sqrt{2\pi A_t v}} \exp\left(-\frac{\left(\theta_t - \hat{\theta}_{t-1}\right)^2}{2A_t v}\right) \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \frac{1}{v^{\alpha+1}} \exp\left(-\frac{1}{\beta v}\right)$$

d) In this part, our aim is to prove that the joint posterior distribution of $(\theta_t, V|\boldsymbol{Y}_t)$ is $\text{NIG}\left(\hat{\theta}_t, B_t V, \frac{2a+1}{2}, N_t^{-1}\right)$, with pdf given by (2.4.9):

$$f(\theta_t, V|\boldsymbol{Y}_t) = \frac{1}{\sqrt{2\pi B_t v}} \exp\left(-\frac{\left(\theta_t - \hat{\theta}_t\right)^2}{2B_t v}\right) \times$$

$$\times \frac{1}{\Gamma\left(\frac{2a+1}{2}\right)\left[N_t^{-1}\right]^{\frac{2a+1}{2}}} \frac{1}{v^{\left(\frac{2a+1}{2}\right)+1}} \exp\left(-\frac{1}{N_t^{-1} v}\right)$$

where $N_t = \frac{\left(y_t - \hat{\theta}_{t-1}\right)^2}{2\left(A_t + 1\right)} + \frac{1}{\beta}$

To obtain the last equation, we first note that the distribution of $(\theta_t, V|\boldsymbol{Y}_t)$ arises through a direct application of Bayes theorem, i.e.

$$f(\theta_t, V|\boldsymbol{Y}_t) \propto f(Y_t|\theta_t, V, \boldsymbol{Y}_{t-1}) f(\theta_t, V|\boldsymbol{Y}_{t-1})$$

where the likelihood of $(Y_t|\theta_t, V, \boldsymbol{Y}_{t-1})$ is described by

$$f(Y_t|\theta_t, V, \boldsymbol{Y}_{t-1}) = \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{(y_t - \theta_t)^2}{2v}\right)$$

and the joint prior distribution of $(\theta_t, V|\boldsymbol{Y}_{t-1})$ is $\text{NIG}\left(\hat{\theta}_{t-1}, A_t V, a, \beta\right)$, with pdf

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$$f(\theta_t, V|\boldsymbol{Y}_{t-1}) = \frac{1}{\sqrt{2\pi A_t v}} \exp\left(-\frac{\left(\theta_t - \hat{\theta}_{t-1}\right)^2}{2A_t v}\right) \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{1}{v^{\alpha+1}} \exp\left(-\frac{1}{\beta v}\right)$$

Then,

$$f(\theta_t, V|\boldsymbol{Y}_t) \propto$$

$$\propto \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{(y_t - \theta_t)^2}{2v}\right) \frac{1}{\sqrt{2\pi A_t v}} \exp\left(-\frac{\left(\theta_t - \hat{\theta}_{t-1}\right)^2}{2A_t v}\right) \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{1}{v^{\alpha+1}} \exp\left(-\frac{1}{\beta v}\right) \propto$$

$$\propto \frac{1}{\sqrt{v}} \exp\left(-\frac{(y_t - \theta_t)^2}{2v}\right) \exp\left(-\frac{\left(\theta_t - \hat{\theta}_{t-1}\right)^2}{2A_t v}\right) \frac{1}{\sqrt{v}} \frac{1}{v^{\alpha+1}} \exp\left(-\frac{1}{\beta v}\right) \propto$$

$$\propto \frac{1}{\sqrt{v}} \exp\left(-\frac{A_t(\theta_t - y_t)^2 + \left(\theta_t - \hat{\theta}_{t-1}\right)^2}{2A_t v}\right) \frac{1}{v^{\left(a+\frac{1}{2}\right)+1}} \exp\left(-\frac{1}{\beta v}\right) \propto$$

$$\propto \frac{1}{\sqrt{v}} \exp\left(-\frac{A_t\left(\theta_t^2 + y_t^2 - 2\theta_t y_t\right) + \theta_t^2 + \hat{\theta}_{t-1}^2 - 2\theta_t \hat{\theta}_{t-1}}{2A_t v}\right) \frac{1}{v^{\left(a+\frac{1}{2}\right)+1}} \exp\left(-\frac{1}{\beta v}\right) \propto$$

$$\propto \frac{1}{\sqrt{v}} \exp\left(-\frac{(A_t + 1)\theta_t^2 - 2\left(A_t y_t + \hat{\theta}_{t-1}\right)\theta_t + A_t y_t^2 + \hat{\theta}_{t-1}^2}{2A_t v}\right) \frac{1}{v^{\left(a+\frac{1}{2}\right)+1}} \exp\left(-\frac{1}{\beta v}\right) \propto$$

$$\propto \frac{1}{\sqrt{v}} \exp\left(-\frac{\theta_t^2 - 2\dfrac{A_t y_t + \hat{\theta}_{t-1}}{A_t + 1}\theta_t + \dfrac{A_t y_t^2 + \hat{\theta}_{t-1}^2}{A_t + 1}}{2\dfrac{A_t}{A_t + 1}v}\right) \frac{1}{v^{\left(a+\frac{1}{2}\right)+1}} \exp\left(-\frac{1}{\beta v}\right) \propto$$

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$$\propto \frac{1}{\sqrt{v}} \exp\left(-\frac{\left(\theta_t - \frac{A_t y_t + \hat{\theta}_{t-1}}{A_t+1}\right)^2}{2\frac{A_t}{A_t+1}v}\right)\frac{1}{v^{\left(a+\frac{1}{2}\right)+1}}\exp\left(-\frac{1}{\beta v}\right)\times$$

$$\times \exp\left(-\frac{\frac{A_t y_t^2 + \hat{\theta}_{t-1}^2}{A_t+1} - \left(\frac{A_t y_t + \hat{\theta}_{t-1}}{A_t+1}\right)^2}{2\frac{A_t}{A_t+1}v}\right)\propto$$

$$\propto \frac{1}{\sqrt{2\pi\frac{A_t}{A_t+1}v}}\exp\left(-\frac{\left(\theta_t-\left(\frac{A_t}{A_t+1}y_t+\frac{1}{A_t+1}\hat{\theta}_{t-1}\right)\right)^2}{2\frac{A_t}{A_t+1}v}\right)\frac{1}{v^{\left(a+\frac{1}{2}\right)+1}}\exp\left(-\frac{1}{\beta v}\right)\times$$

$$\times\exp\left(-\frac{(A_t+1)\left(A_t y_t^2+\hat{\theta}_{t-1}^2\right)-\left(A_t y_t+\hat{\theta}_{t-1}\right)^2}{2A_t(A_t+1)v}\right)$$

Let $D=\frac{1}{\sqrt{2\pi\frac{A_t}{A_t+1}v}}\exp\left(-\frac{\left(\theta_t-\left(\frac{A_t}{A_t+1}y_t+\frac{1}{A_t+1}\hat{\theta}_{t-1}\right)\right)^2}{2\frac{A_t}{A_t+1}v}\right)\frac{1}{v^{\left(a+\frac{1}{2}\right)+1}}$

Then

$f(\theta_t, V|\boldsymbol{Y}_t)\propto$

$$\propto D\times\exp\left(-\frac{1}{\beta v}\right)\exp\left(-\frac{A_t^2y_t^2+A_t\hat{\theta}_{t-1}^2+A_t y_t^2+\hat{\theta}_{t-1}^2-A_t^2y_t^2-\hat{\theta}_{t-1}^2-2A_t y_t\hat{\theta}_{t-1}}{2A_t(A_t+1)v}\right)\propto$$

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$$\propto D \times \exp\left(-\frac{1}{\beta v}\right) \exp\left(-\frac{A_t \hat{\theta}_{t-1}^2 + A_t y_t^2 - 2A_t y_t \hat{\theta}_{t-1}}{2A_t(A_t+1)v}\right) \propto$$

$$\propto D \times \exp\left(-\frac{1}{\beta v}\right) \exp\left(-\frac{\hat{\theta}_{t-1}^2 + y_t^2 - 2y_t \hat{\theta}_{t-1}}{2(A_t+1)v}\right) \propto$$

$$\propto D \times \exp\left(-\frac{1}{\beta v}\right) \exp\left(-\frac{\left(y_t - \hat{\theta}_{t-1}\right)^2}{2(A_t+1)v}\right) \propto D \times \exp\left(-\left(\frac{1}{\beta} + \frac{\left(y_t - \hat{\theta}_{t-1}\right)^2}{2(A_t+1)}\right) \cdot \frac{1}{v}\right) \propto$$

$$\propto D \times \exp\left(-\frac{1}{\left(\frac{\left(y_t - \hat{\theta}_{t-1}\right)^2}{2(A_t+1)} + \frac{1}{\beta}\right)^{-1} \cdot v}\right)$$

Therefore,

$$f(\theta_t, V | \boldsymbol{Y}_t) = \frac{1}{\sqrt{2\pi B_t v}} \exp\left(-\frac{\left(\theta_t - \hat{\theta}_t\right)^2}{2B_t v}\right) \times$$

$$\times \frac{1}{\Gamma\left(\frac{2a+1}{2}\right)\left[\left(\frac{\left(y_t - \hat{\theta}_{t-1}\right)^2}{2(A_t+1)} + \frac{1}{\beta}\right)^{-1}\right]^{\frac{2a+1}{2}}} \frac{1}{v^{\left(\frac{2a+1}{2}\right)+1}} \exp\left(-\frac{1}{\left(\frac{\left(y_t - \hat{\theta}_{t-1}\right)^2}{2(A_t+1)} + \frac{1}{\beta}\right)^{-1} \cdot v}\right)$$

which is equal to (2.4.9)

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e) To calculate the marginal posterior pdf of $(\theta_t|\boldsymbol{Y}_t)$, we integrate, over v, the joint posterior pdf of $(\theta_t, V|\boldsymbol{Y}_t)$, which is given by (2.4.9) :

$f(\theta_t, V|\boldsymbol{Y}_t)$=

$$=\frac{1}{\sqrt{2\pi B_t v}}\exp\left(-\frac{\left(\theta_t-\hat{\theta}_t\right)^2}{2B_t v}\right)\frac{1}{\Gamma\left(\frac{2a+1}{2}\right)\cdot\left(N_t^{-1}\right)^{\frac{2a+1}{2}}}\frac{1}{v^{\frac{2a+1}{2}+1}}\exp\left(-\frac{1}{N_t^{-1}v}\right)$$

where $N_t=\dfrac{\left(y_t-\hat{\theta}_{t-1}\right)^2}{2\left(A_t+1\right)}+\dfrac{1}{\beta}$

Thus,

$f(\theta_t|\boldsymbol{Y}_t)=\int_0^{+\infty} f\left(\theta_t, V\mid Y_t\right)dv=$

$$=\int_0^{+\infty}\frac{1}{\sqrt{2\pi B_t v}}\exp\left(-\frac{\left(\theta_t-\hat{\theta}_t\right)^2}{2B_t v}\right)\frac{1}{\Gamma\left(\frac{2a+1}{2}\right)\left(N_t^{-1}\right)^{\frac{2a+1}{2}}}\frac{1}{v^{\frac{2a+1}{2}+1}}\exp\left(-\frac{1}{N_t^{-1}v}\right)dv=$$

$$=\frac{1}{\sqrt{2\pi B_t}}\frac{1}{\Gamma\left(\frac{2a+1}{2}\right)\left(N_t^{-1}\right)^{\frac{2a+1}{2}}}\int_0^{+\infty}\frac{1}{v^{(a+1)+1}}\exp\left(-\frac{\frac{\left(\theta_t-\hat{\theta}_t\right)^2}{2B_t}}{v}\right)\exp\left(-\frac{N_t}{v}\right)dv=$$

$$=\frac{1}{\sqrt{2\pi B_t}}\frac{1}{\Gamma\left(\frac{2a+1}{2}\right)\left(N_t^{-1}\right)^{\frac{2a+1}{2}}}\times\int_0^{+\infty}\frac{1}{v^{(a+1)+1}}\exp\left(-\frac{1}{\left(\frac{\left(\theta_t-\hat{\theta}_t\right)^2}{2B_t}+N_t\right)^{-1}\cdot v}\right)dv=$$

$$=\frac{1}{\sqrt{2\pi B_t}}\frac{1}{\Gamma\left(\frac{2a+1}{2}\right)\left(N_t^{-1}\right)^{\frac{2a+1}{2}}}\times\Gamma\left(\alpha+1\right)\left(\left(\frac{\left(\theta_t-\hat{\theta}_t\right)^2}{2B_t}+N_t\right)^{-1}\right)^{\alpha+1}=$$

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$$=\frac{\Gamma\left(\alpha+1\right)}{\Gamma\left(\frac{2a+1}{2}\right)}\frac{1}{\sqrt{2\pi B_t}}\frac{\left(\left(\frac{\left(\theta_t-\hat{\theta}_t\right)^2}{2B_t}+N_t\right)^{-1}\right)^{\alpha+1}}{\left(N_t^{-1}\right)^{\frac{2a+1}{2}}}=$$

$$=\frac{\Gamma\left(\frac{\left(2\alpha+1\right)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)}\frac{1}{\sqrt{2\pi B_t}}N_t^{\frac{2a+1}{2}}\frac{1}{\left(\frac{\left(\theta_t-\hat{\theta}_t\right)^2}{2B_t}+N_t\right)^{\alpha+1}}=$$

$$=\frac{\Gamma\left(\frac{\left(2\alpha+1\right)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)}\frac{1}{\sqrt{2\pi B_t}}N_t^{\frac{2a+1}{2}}\left(2B_t\right)^{a+1}\frac{1}{\left(\left(\theta_t-\hat{\theta}_t\right)^2+2B_tN_t\right)^{a+1}}=$$

$$=\frac{\Gamma\left(\frac{\left(2\alpha+1\right)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)}\frac{1}{\sqrt{\pi}}N_t^{\frac{2a+1}{2}}\left(2B_t\right)^{a+\frac{1}{2}}\frac{1}{\left(\left(\theta_t-\hat{\theta}_t\right)^2+2B_tN_t\right)^{\frac{(2a+1)+1}{2}}}=$$

$$=\frac{\Gamma\left(\frac{\left(2\alpha+1\right)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)}\frac{1}{\sqrt{\pi}}\left(2B_tN_t\right)^{\frac{2a+1}{2}}\frac{1}{\left(\left(\theta_t-\hat{\theta}_t\right)^2+2B_tN_t\right)^{\frac{(2a+1)+1}{2}}}=$$

Thus,

$$f(\theta_t\,|\,\boldsymbol{Y}_t)=\frac{\Gamma\left(\frac{\left(2\alpha+1\right)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)}\frac{1}{\sqrt{\pi}}\left(2B_tN_t\right)^{\frac{2a+1}{2}}\frac{1}{\left(\left(\theta_t-\hat{\theta}_t\right)^2+2B_tN_t\right)^{\frac{(2a+1)+1}{2}}}$$

where $N_t=\frac{\left(y_t-\hat{\theta}_{t-1}\right)^2}{2\left(A_t+1\right)}+\frac{1}{\beta}$

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We make the transformation

$$t_t = \frac{\theta_t - \hat{\theta}_t}{\frac{\sqrt{2B_t \mathrm{N}_t}}{\sqrt{2\alpha+1}}} \tag{2.4.24}$$

The Jacobian of the transformation is $\frac{\sqrt{2B_t N_t}}{\sqrt{2a+1}}$ and the marginal pdf of $(t_t | \boldsymbol{Y}_t)$ is given by

$$f(t_t | \boldsymbol{Y}_t) = \frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)} \frac{1}{\sqrt{\pi}} \left(2B_t N_t\right)^{\frac{2a+1}{2}} \frac{1}{\left(t_t^2 \cdot \frac{2B_t N_t}{2a+1} + 2B_t N_t\right)^{\frac{(2a+1)+1}{2}}} \frac{\sqrt{2B_t N_t}}{\sqrt{2a+1}} =$$

$$= \frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)} \frac{1}{\sqrt{\pi}} \left(2B_t N_t\right)^{\frac{2a+1}{2}} \frac{1}{\left(2B_t N_t\right)^{a+1}} \frac{1}{\left(\frac{t_t^2}{2a+1} + 1\right)^{\frac{(2a+1)+1}{2}}} \frac{\sqrt{2B_t N_t}}{\sqrt{2a+1}} =$$

$$= \frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{2a+1}} \frac{1}{\left(\frac{t_t^2}{2a+1} + 1\right)^{\frac{(2a+1)+1}{2}}} \Leftrightarrow$$

$$\Leftrightarrow f(t_t | \boldsymbol{Y}_t) = \frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)} \frac{1}{\sqrt{(2a+1)\pi}} \frac{1}{\left(\frac{t_t^2}{2a+1} + 1\right)^{\frac{(2a+1)+1}{2}}} \ ,$$

which means that $(t_t | \boldsymbol{Y}_t)$ has the *t*-distribution with *2α+1* degrees of freedom; hence, the expected value of the distribution is zero, i.e.

$$E[t_t | \boldsymbol{Y}_t] = 0 \tag{2.4.25}$$

From (2.4.24), (2.4.25), it follows that

$$E\left[\left. \frac{\theta_t - \hat{\theta}_t}{\frac{\sqrt{2B_t \mathrm{N}_t}}{\sqrt{2\alpha+1}}} \right| \mathrm{Y}_t \right] = 0 \Leftrightarrow E[\theta_t | \boldsymbol{Y}_t] = \hat{\theta}_t = B_t y_t + \left(1 - B_t\right) \hat{\theta}_{t-1}$$

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which is equal to (2.4.11).

2.4.4. Discussion

The theorem which has just been proved, enables us to calculate the estimate $\hat{\theta}_t$ of the systematic error between observed and forecasted temperature at time t, given the data Y_t (Y_t denote the difference between observed and forecasted temperature at time *t*). Thus, the estimate $\hat{\theta}_t$ arises recursively, through the equation (2.4.11) :

$$\hat{\theta}_t = B_t Y_t + (1 - B_t)\,\hat{\theta}_{t-1}$$

where $A_t = B_{t-1} + \kappa$, $B_t = \dfrac{A_t}{A_t + 1}$ and $B_0 = \kappa$.

Based on the above recursion, we could provide $\hat{\theta}_t$ as a function of $\hat{\theta}_0$ and the data $Y_1, Y_2, \ldots, Y_t$.

Finally, $\hat{\theta}_t$ is added to the prediction of the Numerical Weather Prediction model for the next time-step and the sum gives the final improved forecast for time *t+1*.

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Chapter 3

Specification of the parameters

3.1 Introduction

In the previous chapter we did focus on the extensive proof of the new proposed recursive formula for the systematic error $\hat{\theta}_t$ between observed and forecasted temperature.

According to this, at each time *t* we have:

$$\hat{\theta}_t = B_t Y_t + (1 - B_t)\hat{\theta}_{t-1}$$

where the data Y_t denote the difference between observed and forecasted temperature at time *t*, $B_t = \frac{A_t}{A_t + 1}$, $A_t = B_{t-1} + \kappa$, $B_0 = \kappa$, with both $\kappa > 0$ and $\hat{\theta}_0$ assumed to be known.

With the creation of this formula, we manage to provide an algorithm to improve the prediction of the NWP model for the temperature at each time *t+1*. Specifically, as it has already mentioned in section 2.4.4, the final improved forecast for each time-step *t+1* is equal to the sum of the prediction of the NWP model for the time-step *t+1* and the estimation $\hat{\theta}_t$.

However, before being able to apply the algorithm described above, we have to specify the initial value $\hat{\theta}_0$ and the parameter κ.

The initial value $\hat{\theta}_0$ of the systematic error can be assumed to be zero because we have no indications of its previous behavior (Homleid, 1995). After all, due to the sequential updating scheme of $\hat{\theta}_t$, even inaccurate choices of $\hat{\theta}_0$ will fairly quickly become accurate.

In this chapter and especially in the section 3.3, we will focus on the specification of the parameter κ. In this point, we epigrammatically refer that the parameter κ is

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updated. Every m time-steps, the differences between the observations Y_t and the estimations $\hat{Y}_t$, for $t = im, (i+1)m, \ldots, (i+m-1)m$, will be used to determine the value of κ which will be used for the correction of the temperature forecasts for the next m time-steps. For the calculation of the estimations $\hat{Y}_t$, we will use the predictive distribution.

3.2 The predictive distribution

Definition 3.1 :

We assume $\boldsymbol{Y}_t = (Y_1, Y_2, \ldots, Y_t)$ be the values of a random variable with pdf $f(Y_t|\theta)$. If the parameter θ has a prior distribution $f(\theta)$, then the predictive distribution (Geisser, 1993) for the future observation Y_{t+1} is given by

$$f(Y_{t+1}|\boldsymbol{Y}_t) = \int f(Y_{t+1}|\theta) f(\theta|Y) d\theta$$

where $f(\theta|\boldsymbol{Y}_t)$ denotes the posterior distribution.

The predictive distribution summarizes the information concerning the distribution of a future observable given the data we have already observed.

In our study, the data $\boldsymbol{Y}_t = (Y_1, Y_2, \ldots, Y_t)$ denote the differences between observed and forecasted temperatures.

The predictive distribution, defined above, enables us to provide a distribution of the future observation Y_t, given that we have observed the data $\boldsymbol{Y}_{t-1} = (Y_1, Y_2, \ldots, Y_{t-1})$.

In the following theorem we calculate the distribution of $(Y_t|\boldsymbol{Y}_{t-1})$ through the equation

$$f(Y_t|\boldsymbol{Y}_{t-1}) = \int_0^{+\infty} f\left(Y_t | V, Y_{t-1}\right) f\left(V | Y_{t-1}\right) dv \qquad (3.2.1)$$

In order to obtain a prediction of Y_t, we ought to calculate a point estimation of Y_t, which will rise from the distribution of $(Y_t|\boldsymbol{Y}_{t-1})$. The Minimum Mean Square Error

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(MMSE) is the expected value of the distribution $f(Y_t|\mathbf{Y}_{t-1})$, i.e. $\hat{Y}_t = E[Y_t|\mathbf{Y}_{t-1}]$ (Bernardo and Smith, 1995).

Lemma 3.1 :

The distribution of $(Y_t|V,\mathbf{Y}_{t-1})$ is given by

$$(Y_t|V,\mathbf{Y}_{t-1}) \sim N\left(\hat{\theta}_{t-1}, (A_t+1)V\right)$$

with pdf

$$f(Y_t|V,\mathbf{Y}_{t-1}) = \frac{1}{\sqrt{2\pi(A_t+1)v}} \exp\left(-\frac{\left(y_t-\hat{\theta}_{t-1}\right)^2}{2(A_t+1)v}\right) \tag{3.2.2}$$

Proof :

The distribution of $(Y_t|V,\mathbf{Y}_{t-1})$ and its pdf, will be obtained, using the following equation :

$$f(Y_t|V,\mathbf{Y}_{t-1}) = \int_{-\infty}^{+\infty} f(Y_t|\theta_t,V,Y_{t-1}) \cdot f(\theta_t|V,Y_{t-1})d\theta_t \tag{3.2.3}$$

According to the theorem 2.1, the probability density functions of the distributions $(Y_t|\theta_t,V,\mathbf{Y}_{t-1})$ and $(\theta_t|V,\mathbf{Y}_{t-1})$ are given by the equations (3.2.4) and (3.2.5), respectively :

$$f(Y_t|\theta_t,V,\mathbf{Y}_{t-1}) = \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{(y_t-\theta_t)^2}{2v}\right) \tag{3.2.4}$$

$$f(\theta_t|V,\mathbf{Y}_{t-1}) = \frac{1}{\sqrt{2\pi A_t v}} \exp\left(-\frac{\left(\theta_t-\hat{\theta}_{t-1}\right)^2}{2A_t v}\right) \tag{3.2.5}$$

Combining the relationships (3.2.3), (3.2.4), (3.2.5), it follows that

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$$f(Y_t|V,\boldsymbol{Y}_{t-1})=\int_{-\infty}^{+\infty} f(Y_t\,|\,\theta_t,V,Y_{t-1})\cdot f(\theta_t\,|\,V,Y_{t-1})d\theta_t=$$

$$=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi v}}\exp\left(-\frac{\left(y_t-\theta_t\right)^2}{2v}\right)\frac{1}{\sqrt{2\pi A_t v}}\exp\left(-\frac{\left(\theta_t-\hat{\theta}_{t-1}\right)^2}{2A_t v}\right)d\theta_t=$$

$$=\frac{1}{\sqrt{2\pi v}}\;\frac{1}{\sqrt{2\pi A_t v}}\;\int_{-\infty}^{+\infty}\exp\left(-\frac{A_t\left(\theta_t^2+y_t^2-2y_t\theta_t\right)+\theta_t^2+\hat{\theta}_{t-1}^2-2\hat{\theta}_{t-1}\,\theta_t}{2A_t v}\right)d\theta_t=$$

$$=\frac{1}{2\pi v}\;\frac{1}{\sqrt{A_t}}\;\int_{-\infty}^{+\infty}\exp\left(-\frac{\left(A_t+1\right)\theta_t^2-2\left(A_t y_t+\hat{\theta}_{t-1}\right)\theta_t+A_t y_t^2+\hat{\theta}_{t-1}^2}{2A_t v}\right)d\theta_t=$$

$$=\frac{1}{2\pi v}\;\frac{1}{\sqrt{A_t}}\;\int_{-\infty}^{+\infty}\exp\left(-\frac{\theta_t^2-2\dfrac{A_t y_t+\hat{\theta}_{t-1}}{A_t+1}\theta_t+\dfrac{A_t y_t^2+\hat{\theta}_{t-1}^2}{A_t+1}}{2\dfrac{A_t}{A_t+1}v}\right)d\theta_t=$$

$$=\frac{1}{2\pi v}\;\frac{1}{\sqrt{A_t}}\;\int_{-\infty}^{+\infty}\exp\left(-\frac{\left(\theta_t-\dfrac{A_t y_t+\hat{\theta}_{t-1}}{A_t+1}\right)^2}{2\dfrac{A_t}{A_t+1}v}\right)\exp\left(-\frac{\dfrac{A_t y_t^2+\hat{\theta}_{t-1}^2}{A_t+1}-\left(\dfrac{A_t y_t+\hat{\theta}_{t-1}}{A_t+1}\right)^2}{2\dfrac{A_t}{A_t+1}v}\right)d\theta_t=$$

$$=\frac{1}{2\pi v}\;\frac{1}{\sqrt{A_t}}\;\sqrt{2\pi\frac{A_t}{A_t+1}v}\,\exp\left(-\frac{\dfrac{A_t y_t^2+\hat{\theta}_{t-1}^2}{A_t+1}-\left(\dfrac{A_t y_t+\hat{\theta}_{t-1}}{A_t+1}\right)^2}{2\dfrac{A_t}{A_t+1}v}\right)=$$

$$=\frac{1}{\sqrt{2\pi\left(A_t+1\right)v}}\exp\left(-\frac{\left(A_t y_t^2+\hat{\theta}_{t-1}^2\right)\left(A_t+1\right)-\left(A_t^2 y_t^2+\hat{\theta}_{t-1}^2+2A_t y_t\,\hat{\theta}_{t-1}\right)}{2A_t\left(A_t+1\right)v}\right)=$$

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$$=\frac{1}{\sqrt{2\pi(A_t+1)v}}\exp\left(-\frac{A_t^2y_t^2+A_ty_t^2+A_t\hat{\theta}_{t-1}^2+\hat{\theta}_{t-1}^2-A_t^2y_t^2-\hat{\theta}_{t-1}^2-2A_ty_t\hat{\theta}_{t-1}}{2A_t(A_t+1)v}\right)=$$

$$=\frac{1}{\sqrt{2\pi(A_t+1)v}}\exp\left(-\frac{A_ty_t^2-2A_t\hat{\theta}_{t-1}y_t+A_t\hat{\theta}_{t-1}^2}{2A_t(A_t+1)v}\right)=$$

$$=\frac{1}{\sqrt{2\pi(A_t+1)v}}\exp\left(-\frac{\left(y_t-\hat{\theta}_{t-1}\right)^2}{2(A_t+1)v}\right),$$

which is equal to (3.2.2)

Theorem 3.1 :

$(Y_t|\boldsymbol{Y}_{t-1})$ has the transformed t-distribution with pdf given by

$$f(Y_t|Y_{t-1})=\frac{\Gamma\left(\frac{2a+1}{2}\right)}{\Gamma\left(\frac{2\alpha}{2}\right)}\frac{1}{\sqrt{\pi}}\left(2(A_t+1)\beta^{-1}\right)^{\frac{2a}{2}}\frac{1}{\left(\left(\theta_t-\hat{\theta}_{t-1}\right)^2+2(A_t+1)\beta^{-1}\right)^{\frac{2a+1}{2}}}$$

and the expected value of $Y_t|\boldsymbol{Y}_{t-1}$ is given by

$$E[Y_t|\boldsymbol{Y}_{t-1}]=\hat{Y}_t=\hat{\theta}_{t-1} \qquad (3.3.6)$$

Proof :

The distribution of $(Y_t|\boldsymbol{Y}_{t-1})$ will be calculated through the equation (3.2.1):

$$f(Y_t|\boldsymbol{Y}_{t-1})=\int_0^{+\infty}f\left(Y_t|V,Y_{t-1}\right)f\left(V|Y_{t-1}\right)dv$$

According to Lemma 3.1

$$(Y_t|V,\boldsymbol{Y}_{t-1})\sim N\left(\hat{\theta}_{t-1},(A_t+1)V\right)$$

with pdf given by

$$f(Y_t \mid V, \boldsymbol{Y}_{t-1}) = \frac{1}{\sqrt{2\pi(A_t+1)v}} \exp\left(-\frac{\left(y_t - \hat{\theta}_{t-1}\right)^2}{2(A_t+1)v} \right)$$

and the distribution of $(V|\boldsymbol{Y}_{t-1})$ is given by

$$(V|\boldsymbol{Y}_{t-1}) \sim IG(\alpha,\beta)$$

with pdf

$$f\left(V \mid Y_{t-1}\right) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \frac{1}{v^{\alpha+1}} \exp\left(-\frac{1}{\beta v}\right)$$

Therefore,

$$(Y_t, V \mid Y_{t-1}) \sim NIG\left(\hat{\theta}_{t-1}, \left(A_t+1\right)V, a, \beta\right)$$

with pdf

$$f\left(Y_t, V \mid Y_{t-1}\right) = \frac{1}{\sqrt{2\pi(A_t+1)v}} \exp\left(-\frac{\left(y_t - \hat{\theta}_{t-1}\right)^2}{2(A_t+1)v} \right) \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \frac{1}{v^{\alpha+1}} \exp\left(-\frac{1}{\beta v}\right)$$

Then, the equation (3.2.1) takes the form

$$f(Y_t \mid \boldsymbol{Y}_{t-1}) = \int_0^{+\infty} f\left(Y_t \mid V, Y_{t-1}\right) f\left(V \mid Y_{t-1}\right) dv = \int_0^{+\infty} f\left(Y_t, V \mid Y_{t-1}\right) dv =$$

$$= \int_0^{+\infty} \frac{1}{\sqrt{2\pi\left(A_t+1\right)v}} \exp\left(-\frac{\left(y_t - \hat{\theta}_{t-1}\right)^2}{2\left(A_t+1\right)v} \right) \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \frac{1}{v^{\alpha+1}} \exp\left(-\frac{1}{\beta v}\right) dv$$

The calculation of the above integral, i.e. the distribution of $(Y_t \mid \boldsymbol{Y}_{t-1})$ and the expected value $\hat{Y}_t = E[Y_t \mid \boldsymbol{Y}_{t-1}]$ will be resulted as a corollary of the part e) of Theorem 2.1.

In that part of Theorem, we proved that if

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$f(\theta_t|\boldsymbol{Y}_t)=\int_0^{+\infty} f(\theta_t,V\,|\,Y_t)dv$ and $(\theta_t,V|\boldsymbol{Y}_t)$~NIG$\left(\hat{\theta}_t, B_t V, \frac{2a+1}{2}, N_t^{-1}\right)$

then

$(\theta_t|\boldsymbol{Y}_t)$ has the transformed t-distribution with *2α+1* degrees of freedom with pdf given by

$$f(\theta_t|\boldsymbol{Y}_t)=\frac{\Gamma\left(\frac{(2\alpha+1)+1}{2}\right)}{\Gamma\left(\frac{2\alpha+1}{2}\right)}\frac{1}{\sqrt{\pi}}\left(2B_tN_t\right)^{\frac{2a+1}{2}}\frac{1}{\left(\left(\theta_t-\hat{\theta}_t\right)^2+2B_tN_t\right)^{\frac{(2a+1)+1}{2}}}$$

and the expected value of $(\theta_t|\boldsymbol{Y}_t)$ is

$$E[\theta_t|\boldsymbol{Y}_t]=\hat{\theta}_t$$

Similarly,

$(Y_t|\boldsymbol{Y}_{t-1})$ has the transformed t-distribution with *2α* degrees of freedom with pdf given by

$$f(Y_t|Y_{t-1})=\frac{\Gamma\left(\frac{2a+1}{2}\right)}{\Gamma\left(\frac{2\alpha}{2}\right)}\frac{1}{\sqrt{\pi}}\left(2(A_t+1)\beta^{-1}\right)^{\frac{2a}{2}}\frac{1}{\left(\left(\theta_t-\hat{\theta}_{t-1}\right)^2+2(A_t+1)\beta^{-1}\right)^{\frac{2a+1}{2}}}$$

and the expected value of $(\theta_t|\boldsymbol{Y}_t)$ is

$$\hat{Y}_t=E[Y_t\,|\,Y_{t-1}]=\hat{\theta}_{t-1}$$

3.3 Estimating κ via the predictive distribution

We assume that a NWP model is run and presents the temperature forecasts for times *1,2,…,m*. In addition, after the end of the time *m*, we have available data $Y_1, Y_2, \ldots, Y_m$ which are defined as the differences between the forecasted and observed temperatures.

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The improved forecast for the next time *m+1* will be obtained by the sum of the prediction of the NWP model for the time-step *m+1* and the estimate of the systematic error $\hat{\theta}_m = B_m Y_m + (1 - B_m)\,\hat{\theta}_{m-1}$,

where $B_m = \dfrac{A_m}{A_m + 1}$, $A_m = B_{m-1} + \kappa$ and $B_0 = \kappa$.

To estimate the parameter κ we make use of the m most recent errors $\left(Y_t - \hat{Y}_t\right)$ obtained with the help of the predictive distribution.

The algorithm for estimating the "optimal" κ is as follows:

- κ takes values in a grid
- for each value of κ, we calculate the estimate $\hat{Y}_t$, for $t=1,2,\ldots,m$, where, according to the theorem 3.1, the (predictive distribution based) estimate of $\hat{Y}_t$ is equal to $\hat{\theta}_{t-1}$, for $t=1,2,\ldots,m$
- for each value of κ, we calculate the quantity Sum Absolute Error:

  SAE= $\sum_{i=1}^{m} \left| Y_i - \hat{Y}_i \right|$
- we select κ (from the grid) for which SAE is minimized

This value of κ is used for the correction of the temperature forecast for the time-steps $t=m+1,m+2,\ldots,2m$.

For the correction of the forecasts for the next m time-steps, i.e. for time $t=2m+1,2m+2,\ldots,3m$, the procedure described above is repeated, using the window of the m previous data, $Y_{m+1}, Y_{m+2}, \ldots, Y_{2m}$.

In general, for the calculation of the improved temperature forecasts of $nm+1,nm+2,\ldots,(n+1)m$ time-steps (n:positive integral), we use the value of the parameter κ that minimizes the quantity $\sum_{i=(n-1)m+1}^{nm} \left| Y_i - \hat{Y}_i \right|$.

Sensitivity analysis regarding the choice of m will be done in the application section.

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Chapter 4

Application to real data

4.1 The NWP model SKIRON5

The process described in the previous chapters is applied to the correction of 2m-temperature forecasts, which are related to the station of Thessaloniki (station 16622) and the time period from 01 June 2007 to 30 April 2009. These forecasts are obtained by the NWP model SKIRON5, using the method of "interpolation from the 4 surrounding grid-points" (Pytharoulis, 2009; Mpaltzi, 2008). According to this method, the temperature forecast T_i which corresponds to the i station-in this case to Thessaloniki station (16622)-results from the weighted mean of forecasts of the 4 surrounding grid-points of the model and is estimated by the relationship

$$T_i = \frac{\sum_{k=1}^{4} w_k M_k}{\sum_{k=1}^{4} w_k}$$

where M_k are the corresponding temperature forecasts for the k surrounding grid-points and w_k are the respective weight. The w_k depend on the distance (r_k) of each of the grid-points from the exact position of the observation station and is calculated through the equation $w_k = \frac{1}{r_k^2}$.

The model variables are represented on the staggered Arakawa E-grid (Nickovic et al., 1998) which is not a regular latitude-longitude grid. E-grid resembles a rhomb with the thermodynamic (e.g. temperature) and dynamic (e.g. wind) variables being stored at its corners and center, respectively. It has been shown by (Winninghoff, 1968; Arakawa and Lamb, 1977) that various horizontal grids simulate differently large and synoptic scale atmospheric processes.

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The model SKIRON5 does the forecast procedure daily with data which correspond to 12:00 Universal Time Coordinated (UTC) and it takes 1-2 hours to be completed.
Every afternoon the model produces forecasts for the temperatures of the next three days. The forecasts are given by the model for every six real hours. Consequently, we have the forecasting periods of 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72 hours. For example, the model starts with the data of 12:00 UTC 01/06/2007 and provides the temperature forecasts as they appear in Table 1.

Date	01/06/07	02/06/07	02/06/07	02/06/07	02/06/07	03/06/07
Time	18:00	00:00	06:00	12:00	18:00	00:00
Forecasting period	6 hours	12 hours	18 hours	24 hours	30 hours	36 hours
Forecasts	21,809°C	18,456°C	21,584°C	26,793°C	22,305°C	19,142°C
Date	03/06/07	03/06/07	03/06/07	04/06/07	04/06/07	04/06/07
Time	06:00	12:00	18:00	00:00	06:00	12:00
Forecasting period	42 hours	48 hours	54 hours	60 hours	66 hours	72 hours
Forecasts	22,072°C	28,560°C	24,256°C	21,054°C	23,802°C	30,262°C

Table 1: The temperature forecasts obtained by the NWP model on 01/06/2007 at 12:00 UTC which correspond to the referred days and hours

As the prediction is done daily for the next three days, there is overlap in the data. For example, for the temperature of 12:00 UTC 04/06/2007 there is a 24-hours forecast which starts at 12:00 03/06/2007, a 48-hours forecast which starts at 12:00 02/06/2007 and a 72-hours forecast which starts at 12:00 01/06/2007, as it is shown in Table 2.

				Forecasting Periods											
Hour	Day	Month	Year	6	12	18	24	30	36	42	48	54	60	66	72
12	1	6	2007												30,262°C
12	2	6	2007								30,430°C				
12	3	6	2007				29,463°C								

Table 2: The 24-hours, 48-hours and 72-hours NWP model forecast for the temperature of 12:00 UTC 04/06/2007, as they appear in third, second and first line of the forecasting periods, respectively

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4.2 The presentation of the data

For the application of the procedure developed in this thesis, we have available the following data: first are the temperature forecasts of 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72-hours prediction that cover the time period from 01/06/2007 to 30/04/2009, and subsequently we have the corresponding observations. Consequently, our procedure is used for the correction of 2m-temperature forecasts of 700 days and for 12 different forecasting periods. Due to the fact that the range of temperature values is quite large (over the period of 700 days), for better visualization, we choose to present the graphs that describe the variation in time of the temperature values, during the period of three summer months, from 01/06/2008 to 31/08/2008.

At this point, we need to mention that in the available data, approximately 2% of the values are missing. Every forecasted or real temperature missing value is imputed by the average temperature of the previous and next days, during the same time, respectively. For example, in the following part of the data (Table 3), the forecasted by the NWP model temperature that corresponds to 15/06/2007 and to 6-hours forecasting period was missing.

HOUR	DAY	MONTH	YEAR	Ftime	6
12	12	6	2007	T2m.	24,276°C
12	13	6	2007	T2m.	24,287°C
12	14	6	2007	T2m.	24,784°C
12	15	6	2007	T2m.	M.V.
12	16	6	2007	T2m.	26,138°C
12	17	6	2007	T2m.	28,041°C

Table 3: The 6-hours temperature forecasts obtained by the NWP model at 12:00 UTC 12/06/2007-17/06/2007

This missing value will be imputed by $\frac{24,784+26,138}{2}=25,461$

In the case that the temperatures of two continuing days *t*, *t+1* are missing, they are imputed by the average temperature of the days *t-2* and *t+2*, (*t+1*)*-2* and (*t+1*)*+2*,

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during the same time, respectively. We did not encounter any cases with more than two successive missing values.
Given that the percentage of the missing values is very small, the imputation scheme does not affect the final results.

In Figures 1-6 below we present the variation in time of the real temperature values (observations) and the temperature forecasts obtained by the NWP model SKIRON5 of the summer season from 01/06/2008 to 31/08/2008 (i.e. from the 367th to the 458th day of the total data set of 700 days), for the different forecasting periods of 6, 12, 18, 24, 48, 72 hours, respectively.

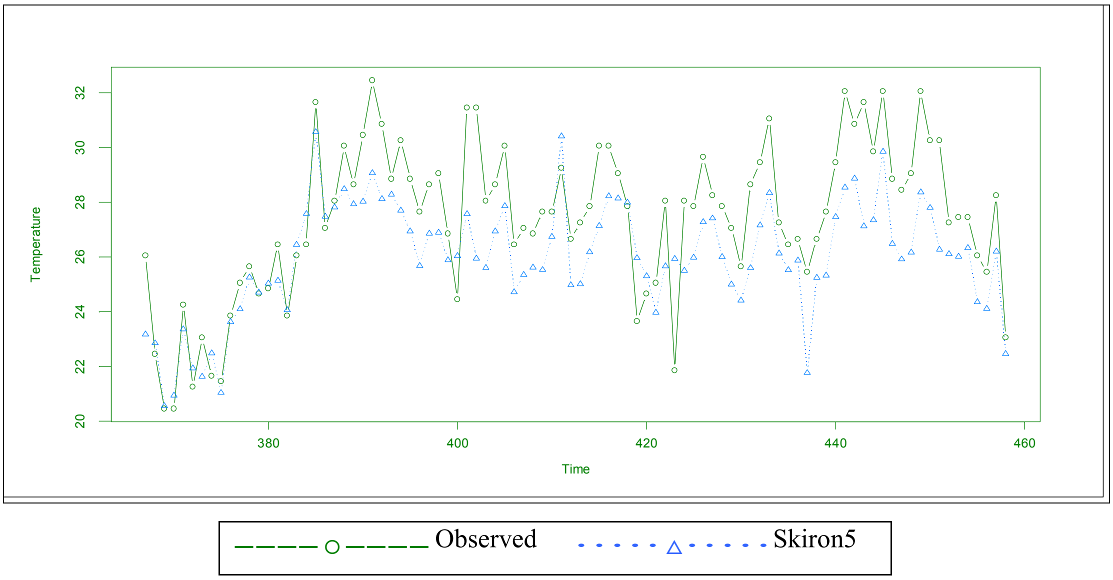

Figure 1: Observed temperatures (°C) and forecasts of the NWP model SKIRON5 in Thessaloniki (Station 16622) for the time period from 01/06/2008 to 31/08/2008 (forecasting period of 6 hours)

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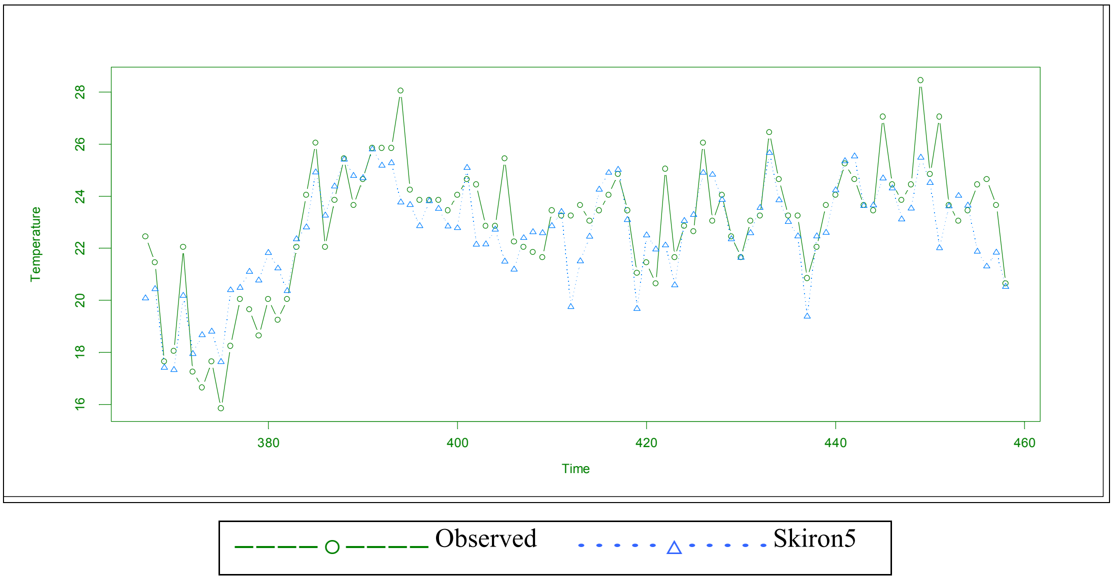

Figure 2: Observed temperatures (°C) and forecasts of the NWP model SKIRON5 in Thessaloniki (Station 16622) for the time period from 01/06/2008 to 31/08/2008 (forecasting period of 12 hours)

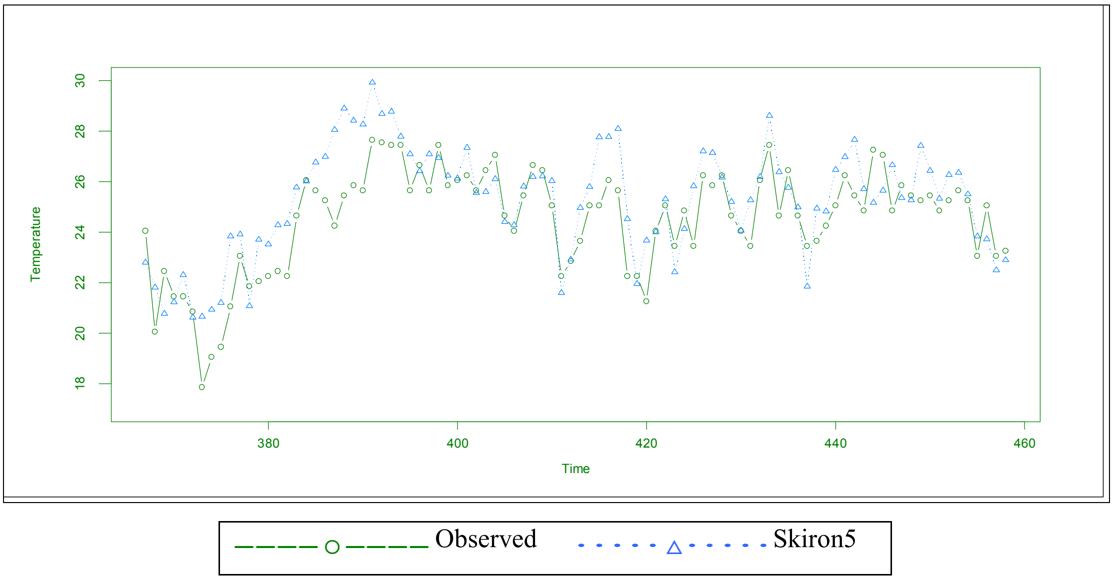

Figure 3: Observed temperatures (°C) and forecasts of the NWP model SKIRON5 in Thessaloniki (Station 16622) for the time period from 01/06/2008 to 31/08/2008 (forecasting period of 18 hours)

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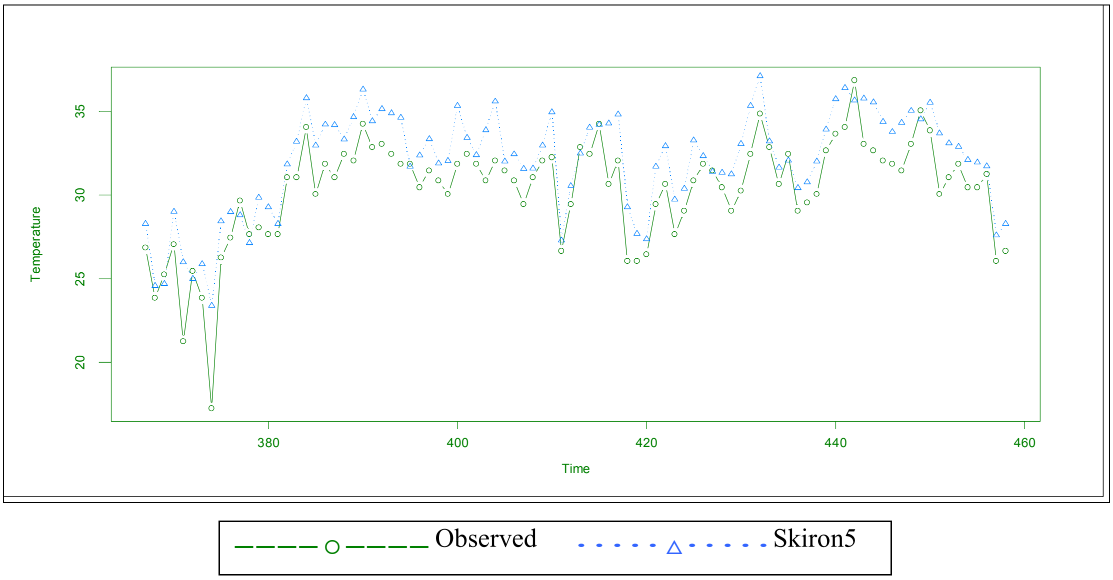

Figure 4: Observed temperatures (°C) and forecasts of the NWP model SKIRON5 in Thessaloniki (Station 16622) for the time period from 01/06/2008 to 31/08/2008 (forecasting period of 24 hours)

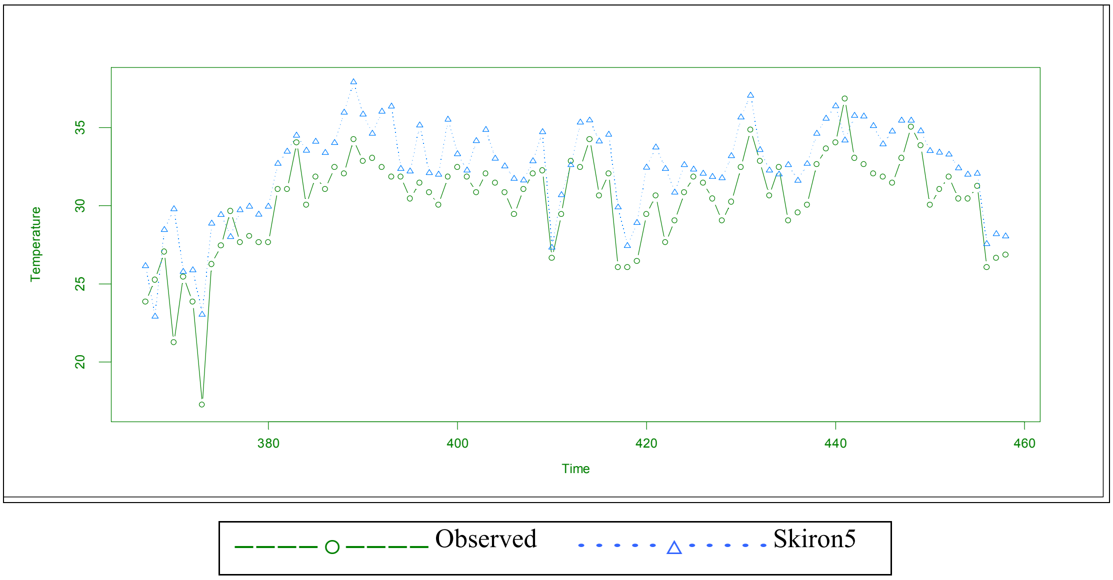

Figure 5: Observed temperatures (°C) and forecasts of the NWP model SKIRON5 in Thessaloniki (Station 16622) for the time period from 01/06/2008 to 31/08/2008 (forecasting period of 48 hours)

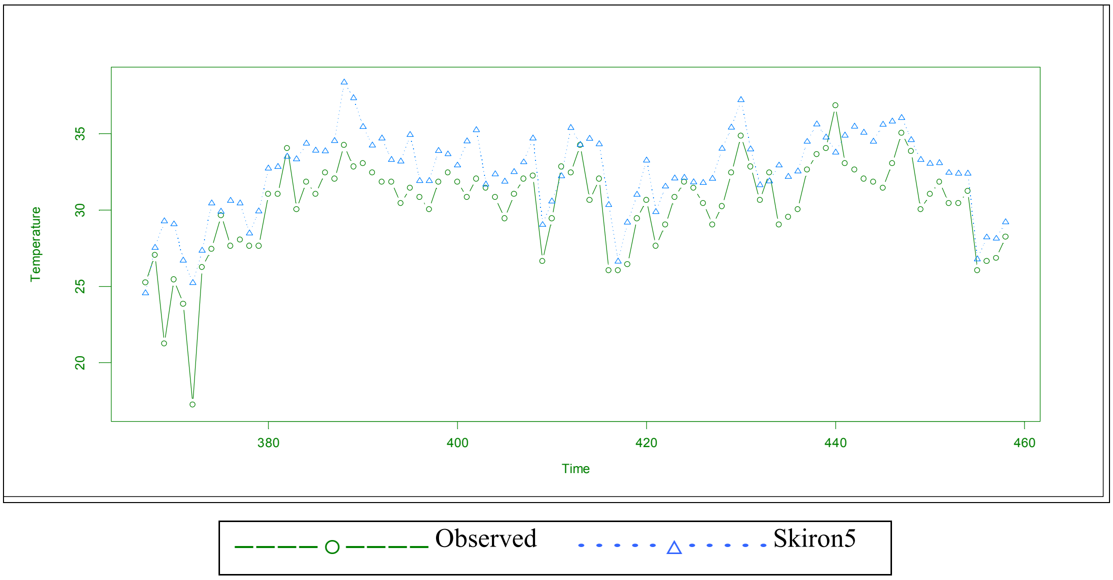

Figure 6: Observed temperatures (°C) and forecasts of the NWP model SKIRON5 in Thessaloniki (Station 16622) for the time period from 01/06/2008 to 31/08/2008 (forecasting period of 72 hours)

From the above Figures 1-6 we notice that SKIRON5 forecasts are relatively close to the observed temperatures of the time period from 01/06/08 to 31/08/08. However, we can also observe that in a few time steps there is a large deviation between forecasts and real temperatures. For example, in Figure 1 that represents the forecasting period of 6 hours, a lot of the forecasts are systematically about $1°C - 4°C$ below the observations. In contrary, for the forecasting period of 24, 48 and 72 hours the model over-estimates by about $0.5°C - 5°C$ during the summer season from 01/06/08 to 31/08/08.

4.3 Results

In this section we will discuss about the results of the proposed procedure which has been developed in this thesis for correcting 2m-temperature forecasts, and we will present the improved forecasts for the station of Thessaloniki and the time period from 01 June 2007 to 30 April 2009.

Based on our procedure, we obtain the improved temperature forecast for time *t+1*, using the prediction of the model SKIRON5 for time *t+1*, on which we add the

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Kalman Filter correction $\hat{\theta}_t$. We remind that the correction $\hat{\theta}_t$ is coming from the recursive formula

$$\hat{\theta}_t = B_t Y_t + (1 - B_t)\,\hat{\theta}_{t-1}$$

where $A_t = B_{t-1} + \kappa$, $B_t = \dfrac{A_t}{A_t + 1}$ and $B_0 = \kappa$

In chapter 3 we discussed about the specification of the parameter *κ*. The parameter *κ* is updated every *m* time-steps. As we referred in the previous chapter, we use the data $Y_1, Y_2, ..., Y_m$ to calculate the first value of *κ*, which will be used for the correction of the temperature forecast for the time-steps *t=m+1,m+2,…,2m*. Continuously, for the next *m* time-steps, i.e. for time *t=2m+1,2m+2,…,3m,* the specification of the parameter κ is obtained from the *m* previous data, $Y_{m+1}, Y_{m+2}, \ldots, Y_{2m}$, etc.

The open question at this point is what should be the choice of the window *m*.
For the estimation of *m*, we used the data that correspond to the forecasting time period of 6 hours. Concretely, the algorithm for estimating the "optimal" *m* is as follows:

- *m* takes values in a grid
- for every m we calculate the corrected temperatures
- for every *m* we calculate the quantity $MAE_m = \dfrac{\sum_{i=1}^{700} |Treal_i - Tfinal_i|}{700}$, where *Treal* is the vector which consists of the real temperatures, whereas *Tfinal* is the vector of the corrected temperatures (we remind here that at time *t* $Tfinal_t = Tmodel_t + \hat{\theta}_{t-1}$ and $Y_t = Treal_t - Tmodel_t$, where $Tmodel_t$ is the NWP model temperature forecast)
- select m that minimizes MAE_m

For illustration purposes, we provide values of *m* in a grid from 10 to 100 with step=10.
Empirically, turned out that the value of *m*=60 was providing the best results (see Figure 7 for an illustration).

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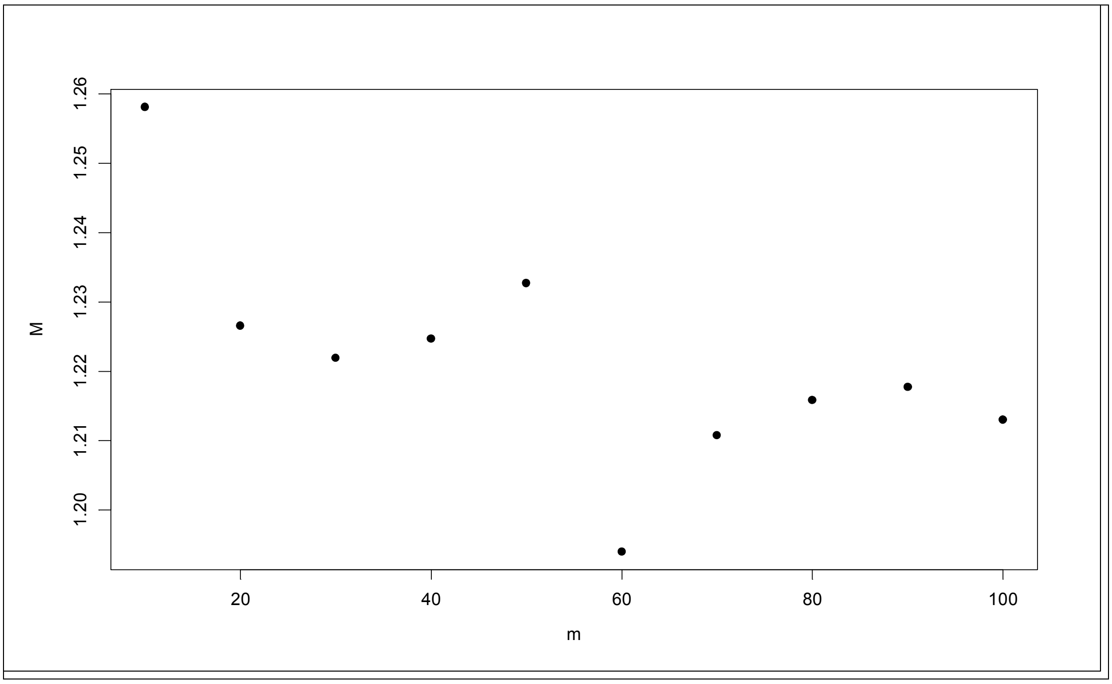

Figure 7: The values of the quantity MAE_m for m=10,20,…,100

According to what we have written above, we calculate, using the Kalman Filter recursive equations, the improved 2m-temperature forecasts, regarding the NWP model SKIRON5, at the station of Thessaloniki, for the time period from 01 June 2007 to 30 April 2009.

In Figures 8, 9, 10, 11, 12, and 13 below we present the time series of the temperature forecasts obtained by the NWP model SKIRON5 and the improved forecasts, along with the corresponding observations for the summer season from 01/06/2008 to 31/08/2008.

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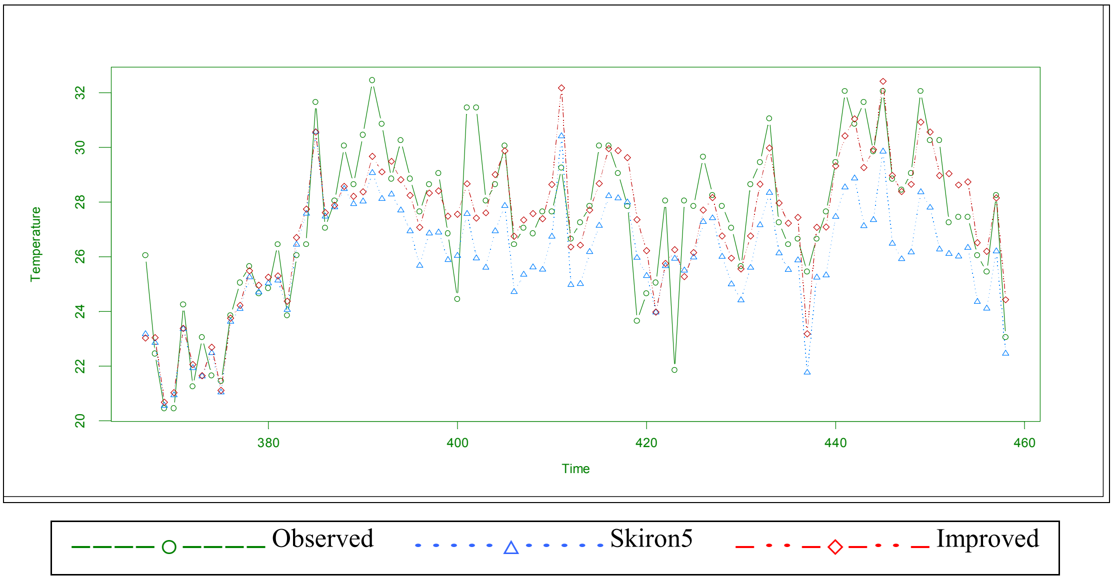

Figure 8: Observed temperatures (°C), forecasts of SKIRON5 and improved forecasts in Thessaloniki (Station 16622) for the time period from 01/06/2008 to 31/08/2008 (forecasting period of 6 hours)

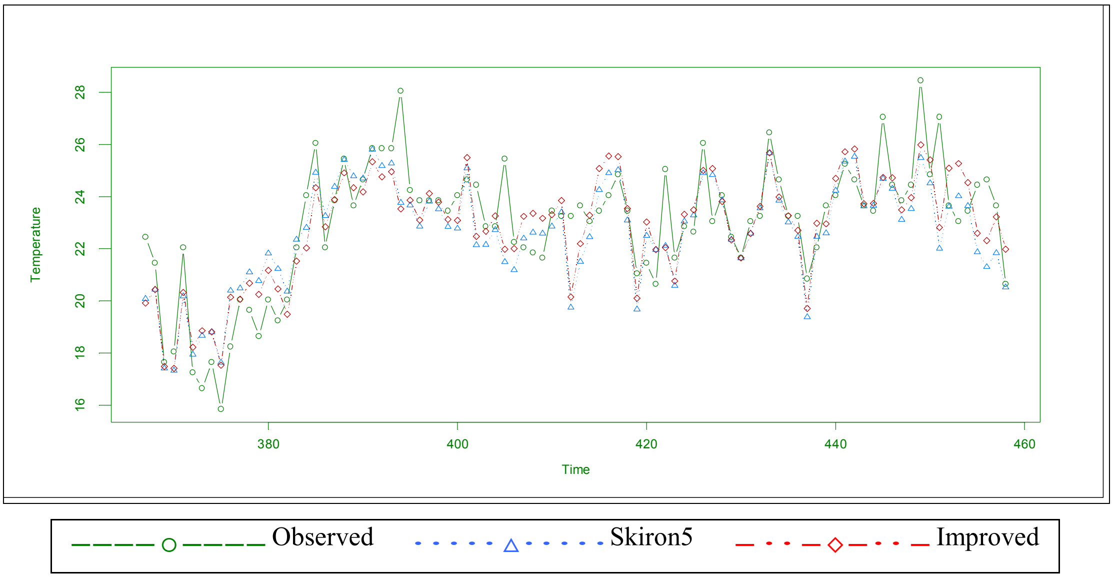

Figure 9: Observed temperatures (°C), forecasts of SKIRON5 and improved forecasts in Thessaloniki (Station 16622) for the time period from 01/06/2008 to 31/08/2008 (forecasting period of 12 hours)

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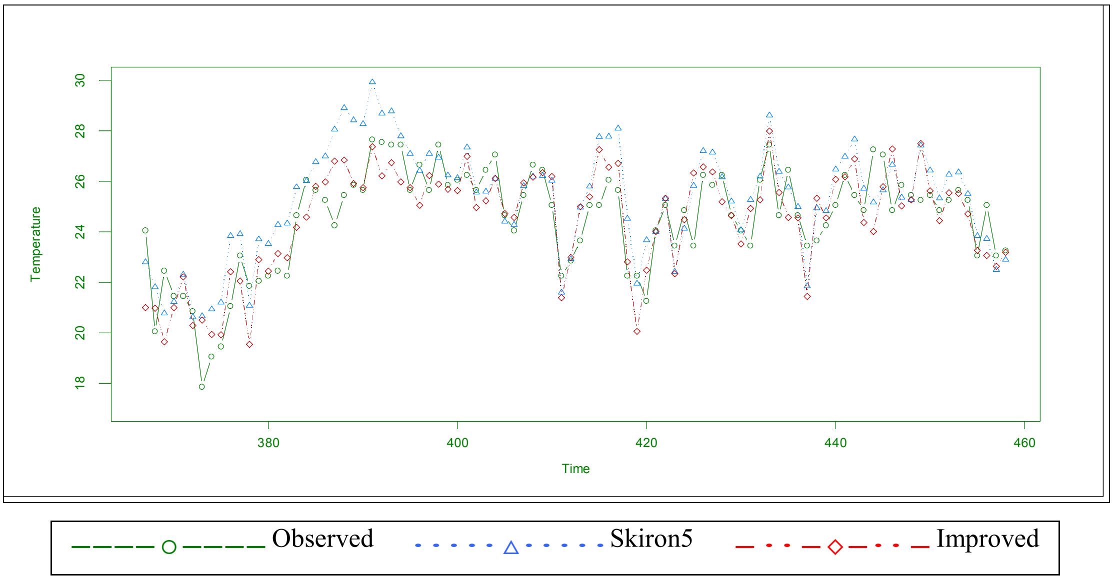

Figure 10: Observed temperatures (°C), forecasts of SKIRON5 and improved forecasts in Thessaloniki (Station 16622) for the time period from 01/06/2008 to 31/08/2008 (forecasting period of 18 hours)

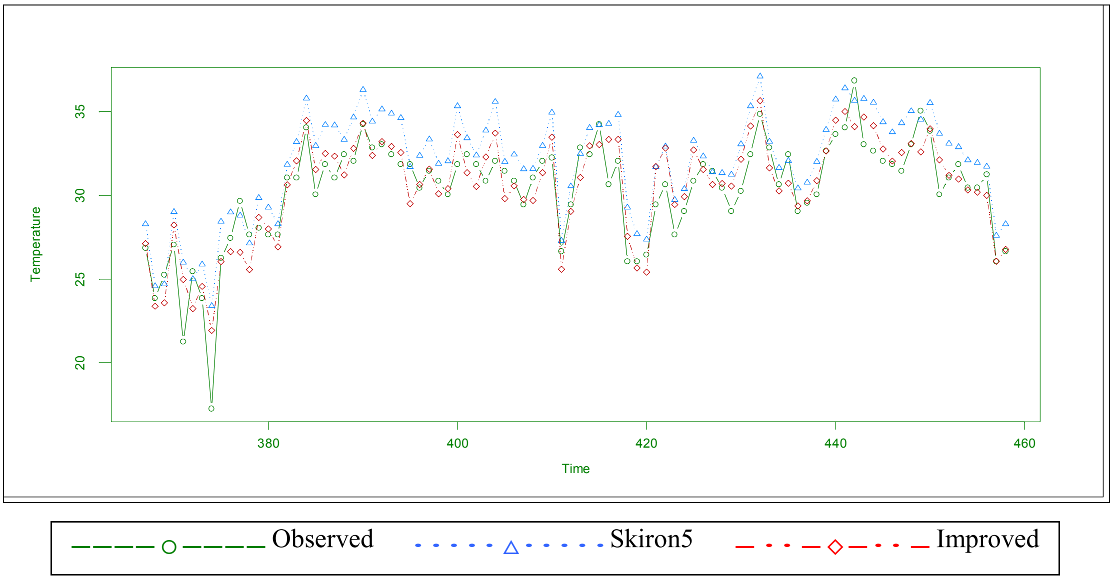

Figure 11: Observed temperatures (°C), forecasts of SKIRON5 and improved forecasts in Thessaloniki (Station 16622) for the time period from 01/06/2008 to 31/08/2008 (forecasting period of 24 hours)

https://doi.org/10.26219/heal.aueb.6651

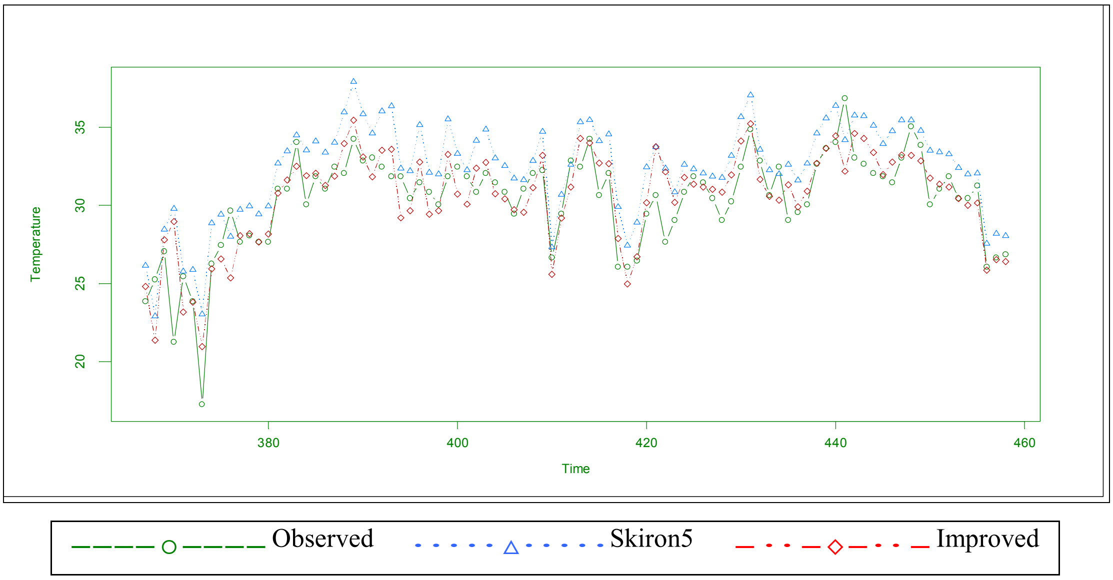

Figure 12: Observed temperatures (°C), forecasts of SKIRON5 and improved forecasts in Thessaloniki (Station 16622) for the time period from 01/06/2008 to 31/08/2008 (forecasting period of 48 hours)

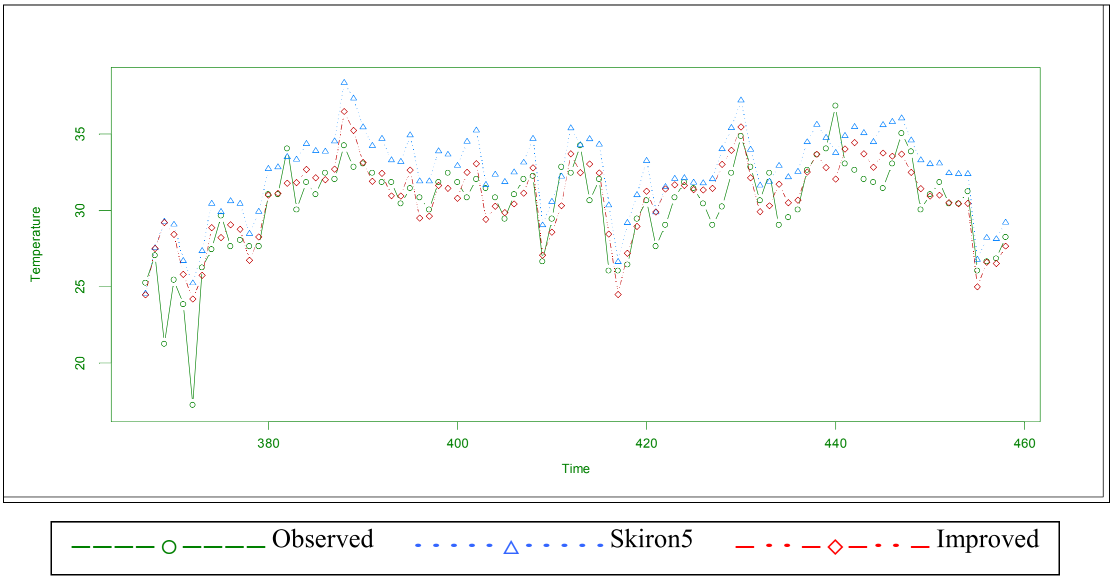

Figure 13: Observed temperatures (°C), forecasts of SKIRON5 and improved forecasts in Thessaloniki (Station 16622) for the time period from 01/06/2008 to 31/08/2008 (forecasting period of 72 hours)

As shown in Figures 8, 9, 10, 11, 12 and 13 the new proposed procedure gives corrected temperature forecasts that are closer to the observed temperatures than SKIRON5 forecasts. Finally, apart from graphical evidence (which for reasons of simplicity was over a short period of time), we will provide some numerical summary of the improvement over the entire period of time.

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a) The mean absolute error

$$MAE_{Skiron} = \frac{\sum_{i=1}^{700} \left| Treal_i - Tmodel_i \right|}{700} \quad , \quad MAE_{Kalman} = \frac{\sum_{i=1}^{700} \left| Treal_i - Tfinal_i \right|}{700}$$

where *Treal* is the vector which consists of the real temperatures, *Tmodel* is the vector of the forecasted temperatures and *Tfinal* is the vector of the corrected temperatures,

b) the root mean square error

$$RMSE_{Skiron} = \sqrt{\frac{\sum_{i=1}^{700} \left| Treal_i - Tmodel_i \right|^2}{700}} \quad , \quad RMSE_{Kalman} = \sqrt{\frac{\sum_{i=1}^{700} \left| Treal_i - Tfinal_i \right|^2}{700}}$$

c) the mean error

$$ME_{Skiron} = \frac{\sum_{i=1}^{700} \left(Treal_i - Tmodel_i \right)}{700} \quad , \quad ME_{Kalman} = \frac{\sum_{i=1}^{700} \left(Treal_i - Tfinal_i \right)}{700}$$

have been estimated for the forecasted and the corrected temperatures, for the time period from 01/06/2007 to 30/04/2009 (700 days) and for 12 different forecasting periods (6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72 hours), i.e. over all of the data, for the station of Thessaloniki.

These are presented in Table 4.

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	MAE		RMSE		ME	
Forecasting period	Forecasted by SKIRON5	Kalman Filter Correction	Forecasted by SKIRON5	Kalman Filter Correction	Forecasted by SKIRON5	Kalman Filter Correction
6 hours	1,300	1,182	1,630	1,506	-0,205	-0,104
12 hours	1,579	1,501	2,011	1,901	0,635	0,114
18 hours	1,751	1,515	2,163	1,921	1,112	0,191
24 hours	1,916	1,416	2,410	1,874	1,693	0,326
30 hours	1,176	1,141	1,497	1,462	0,143	-0,009
36 hours	1,604	1,500	1,997	1,907	0,793	0,135
42 hours	1,847	1,489	2,237	1,905	1,203	0,193
48 hours	2,081	1,588	2,553	2,057	1,752	0,316
54 hours	1,231	1,184	1,586	1,544	0,176	-0,006
60 hours	1,713	1,565	2,093	1,971	0,844	0,151
66 hours	1,941	1,563	2,316	1,983	1,218	0,202
72 hours	2,141	1,680	2,654	2,206	1,731	0,389

Table 4: MAE, RMSE and ME for forecasts of SKIRON5 and Kalman Filter corrections over all of the data

In addition the numerical summaries of MAE, RMSE and ME at the forecast times of 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72 hours are also presented graphically in Figures 14, 15 and 16, respectively.

The diurnal cycle observed in the patterns of ME, MAE and RMSE is due to measurement (human and instrumental) and model errors. (Gofa et al., 2008) showed that the errors of 2m temperature forecasts of SKIRON/Eta, COSMO-GR (both operated by the Hellenic National Meteorological Service) and ECMWF models exhibit a similar behaviour. The model errors are mainly due to a) the formulation of all the atmospheric numerical models that represent the area averaged value of a meteorological parameter as a grid-point value and b) the model physical parameterizations. More specifically, the 2m temperature forecasts are sensitive to the schemes that parameterize the soil, surface and boundary layer processes. Therefore, the increased errors observed in the 24, 48 and 72 hours temperature forecasts (that all correspond to 1200 UTC) and especially the positive values of ME indicate that the model systematically overestimates the maximum daily temperature at Thessaloniki, probably because of overestimated upward land surface fluxes of heat during noon.

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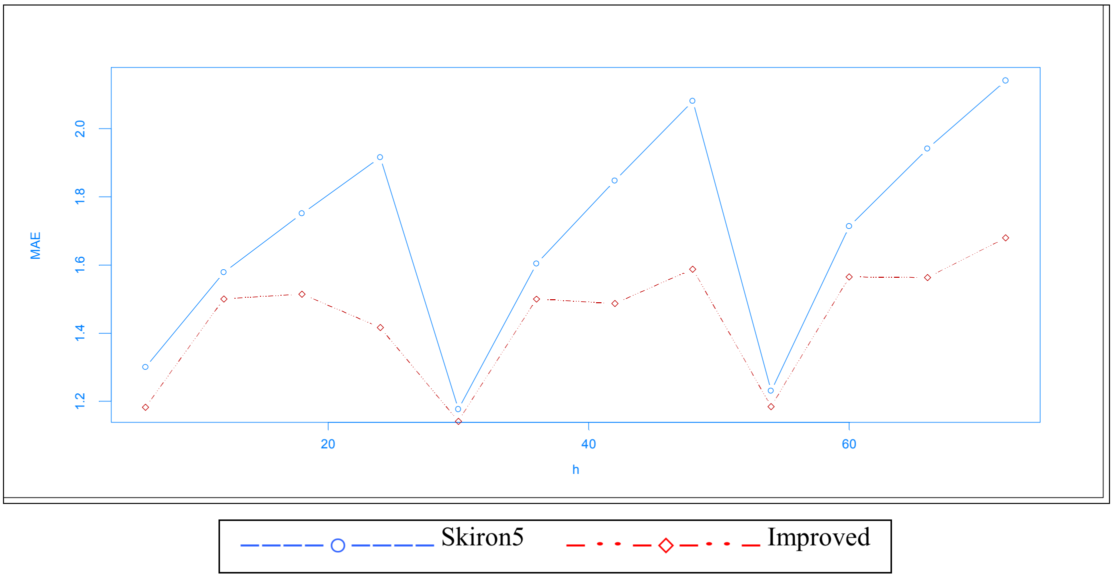

Figure 14: Variation of mean absolute error with forecasting periods for the time period of 700 days

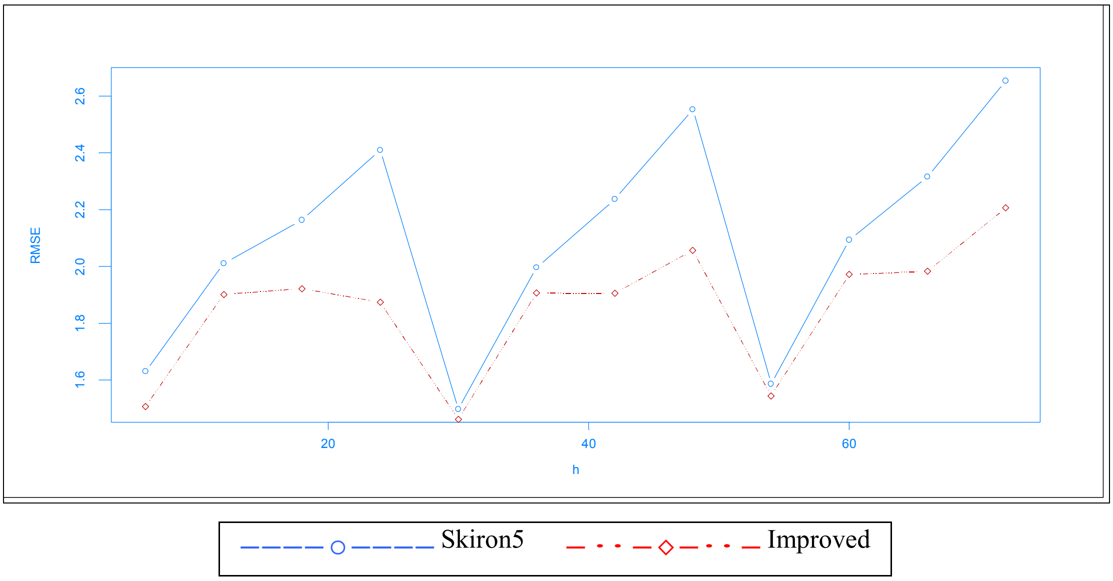

Figure 15: Variation of root mean square error with forecasting periods for the time period of 700 days

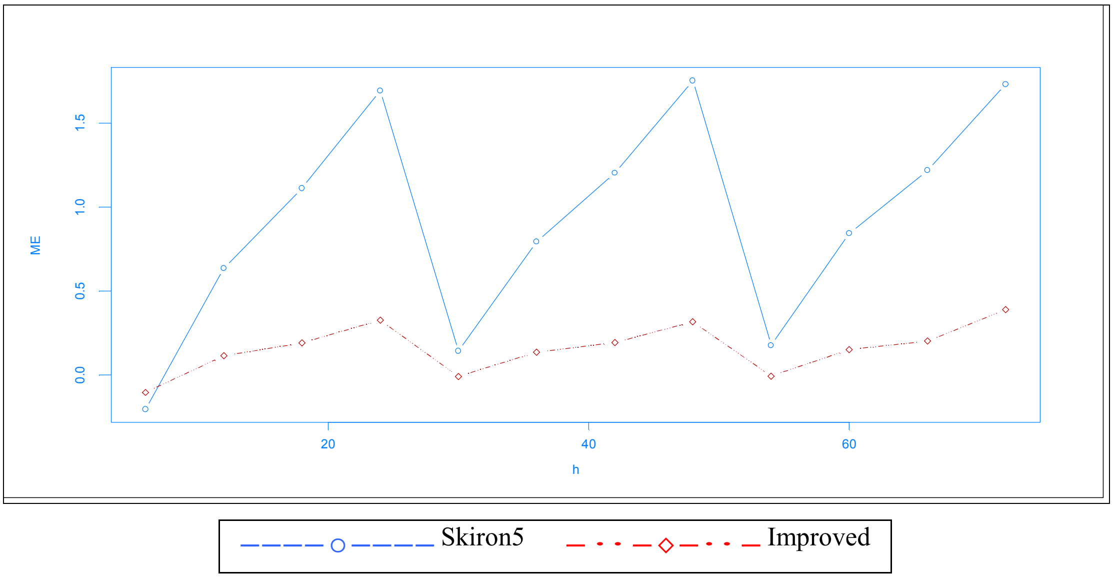

Figure 16: Variation of mean error with forecasting periods for the time period of 700 days

The improvement in 2m-temperature forecasts is clearly illustrated in the graphs presented above. In Figures 14 and 15 we can see that the mean absolute error (MAE) and the root mean square error (RMSE) of the temperature forecasts are decreased after the correction by our procedure for all 12 different forecasting periods. Whereas the MAE of the temperature forecasts came over the 2,14°C in the predicted period of 72 hours, after the correction the calculated MAE ranges between 1,141°C and 1,680°C. It is worth noting the fact that, after correction, the mean error of the temperature forecasts decreases to values close to zero, as it is shown in Figure 16.

Finally, the developed Kalman Filter correction was also applied to all the available operational meteorological stations of the Hellenic National Meteorological Service throughout Greece (Aleksandroupoli, Kerkura, Aktio-Preveza, Larissa, Limnos, Mutilini, Andravida, Skyros, Athina-Elliniko, Souda, Rodos, Hrakleio) and the improvement of the KF correction over the NWP model (Skiron5 and Skiron20) values was persistent to all stations.

4.4 Conclusion

In this study a new Bayesian Kalman Filter based method was developed to correct near surface temperature forecasts. The results which were presented in the

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previous chapter, indicate that our method is able to decrease the NWP model prediction error and lead to a significant improvement of the forecasts of the weather parameter in study. The effectiveness of this procedure, as well as the fact that it can be easily run on any PC, gives the opportunity for further applications. Although in this thesis we focus on near surface temperature forecasts, our method could be applied also for the improvement of the predictions of other weather forecasting parameters, such as wind speed, air pressure or humidity.

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Appendix

In this appendix we present the functions which have been used for the conducting of the results of Chapter 4.

1. The functions "Imput.Observ" and "Imput.Skiron"

We have noticed that in most of the sets of data which have been used for the correction of the next time-step temperature forecast, a percentage approximately of 2% of values is missing.

The functions "Imput.Observ" and "Imput.Skiron" impute these missing values.

###The function "Imput.Observ"###

```
Imput.Observ<-function(Treal,n){
#Treal is the vector that consists of n real temperatures. The vector Treal contains missing values which have been filled by the number -999#
TrealD<-c(rep(0,n))
for(i in 1:n){
if(Treal[i]==-999){
TrealD[i]<-mean(c(Treal[i-1],Treal[i+1]))
}
else{
TrealD[i]<-Treal[i]
}
}
TrealDD<-c(rep(0,n))
for(i in 1:n){
if(TrealD[i]<(-100)){
TrealDD[i]<-mean(c(TrealD[i-2],TrealD[i+2]))
}
```

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```
else{
TrealDD[i]<-TrealD[i]
}
}
TrealDD
}
```

###The function "Imput.Skiron"###

```
Imput.Skiron<-function(Tmodel,n){
#Tmodel is the vector that consists of n temperature forecasts. The vector Tmodel contains missing values which have been filled by the number -999#
TmodelD<-c(rep(0,n))
for(i in 1:n){
if(Tmodel[i]==-999){
TmodelD[i]<-mean(c(Tmodel[i-1],Tmodel[i+1]))
}
else{
TmodelD[i]<-Tmodel[i]
}
}
TmodelDD<-c(rep(0,n))
for(i in 1:n){
if(TmodelD[i]<(-100)){
TmodelDD[i]<-mean(c(TmodelD[i-2],TmodelD[i+2]))
}
else{
TmodelDD[i]<-TmodelD[i]
}
}
TmodelDD
}
```

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2. The function "Ck"

The function "Ck", using the observations and the temperature forecasts for the time-steps 1,2,…,m (Treal, Tmodel, respectively), calculates the value of κ which will be used for the correction of the temperature forecast for the time-steps m+1,m+2,…,2m. The function "Ck" calculates the quantity $D=\sum_{i=1}^{m}\left|Y_i-\hat{Y}_i\right|$ for 1000 different values of κ and selects the value of κ (kmin) for which the quantity D is minimized.

###The function "Ck"###

```
Ck<-function(m,Treal,Tmodel){
Yt<-Treal-Tmodel
thetahat0<-0
At<-c(rep(0,m))
Bt<-c(rep(0,m))
thetahat<-c(rep(0,m))
Ythat<-c(thetahat0,rep(0,m-1))
D<-c(rep(0,1000))
counter<-1
for(k in seq(0.01,10,0.01)){
B0<-k
At[1]<-B0+k
Bt[1]<-At[1]/(At[1]+1)
thetahat[1]<-Bt[1]*Yt[1]+thetahat0/(At[1]+1)
for(i in 2:m){
At[i]<-Bt[i-1]+k
Bt[i]<-At[i]/(At[i]+1)
thetahat[i]<-Bt[i]*Yt[i]+thetahat[i-1]/(At[i]+1)
Ythat[i]<-thetahat[i-1]
}
D[counter]<-sum(abs(Yt-Ythat))
```

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```
counter<-counter+1
}
k<-seq(0.01,10,0.01)
kmin<-k[order(D)[1]]
kmin
}
```

3. The function “S”

According to our algorithm, the improved temperature forecast for every future time t+1 is resulted by the sum of the prediction of the NWP model for the time-step t+1 and the estimate of the systematic error $\hat{\theta}_t$. The systematic error $\hat{\theta}_t$, for time t, is given by the equation

$$\hat{\theta}_t = B_t Y_t + \left(1 - B_t\right) \hat{\theta}_{t-1}$$

where

Y_t is the difference between observed and forecasted temperature at time t,

$B_t = \frac{A_t}{A_t + 1}$, $A_t = B_{t-1} + \kappa$, $B_0 = \kappa$ and $\hat{\theta}_0 = 0$

The function “S”, using the temperature forecasts for time t=1,2,…,m (Tmodel) and the observations (Treal), applies the algorithm and calculates the vector Tfinal which consists of the improved temperature forecasts for the time steps t=1,2,…,m.

###The function “S”###

```
S<-function(m,k,Treal,Tmodel){
n<-m-1
Yt<-Treal-Tmodel
Tfinal<-c(rep(0,m))
thetahat0<-0
At<-c(rep(0,n))
```

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```
Bt<-c(rep(0,n))
thetahat<-c(rep(0,n))
B0<-k
At[1]<-B0+k
Bt[1]<-At[1]/(At[1]+1)
thetahat[1]<-Bt[1]*Yt[1]+thetahat0/(At[1]+1)
Tfinal[1]<-Tmodel[1]+thetahat0
Tfinal[2]<-Tmodel[2]+thetahat[1]
for(i in 2:n){
At[i]<-Bt[i-1]+k
Bt[i]<-At[i]/(At[i]+1)
thetahat[i]<-Bt[i]*Yt[i]+thetahat[i-1]/(At[i]+1)
Tfinal[i+1]<-Tmodel[i+1]+thetahat[i]
}
Tfinal
}
```

4. The function "FF"'

The function "FF" includes the functions "Ck" and "S". It has as output a data frame that consists of the observed, forecasted and improved temperatures for the time-steps 1,2,…,t.

###The function "FF"###

```
FF<-function(m,t,Treal,Tmodel){
Tfinal<-c(rep(0,t))
Tfinal[1:m]<-Tmodel[1:m]
kmin1<-Ck(m,Treal[1:m],Tmodel[1:m])
Tfinal[(m+1):(m+m)]<-S(m,kmin1,Treal[(m+1):(m+m)],Tmodel[(m+1):(m+m)])
for(j in seq((m+m+1),t,m)){
```

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```
kmin<-Ck(m,Treal[(j-m):(j-1)],Tmodel[(j-m):(j-1)])
Tfinal[j:(j+m-1)]<-S(m,kmin,Treal[j:(j+m-1)],Tmodel[j:(j+m-1)])
}
data.frame(Date=1:t,Treal=Treal,Tmodel=Tmodel,Tfinal=Tfinal[1:t])
}
```

5. The function “W”’

The function “W” has as output the optimal value of the window m.

###The function “W”###

```
W<-function(t,Treal,Tmodel){
Tfinal700DD<-c(rep(0,t))
M<-0
counter<-1
for(m in seq(10,100,10)){
Tfinal700DD<-FF(m,t,Treal,Tmodel)[,4]
M[counter]<-(sum(abs(Treal[(m+1):t]-Tfinal700DD[(m+1):t])))/(t-m)
counter<-counter+1
print(Tfinal700DD)
}
m<-seq(10,100,10)
mmin<-m[order(M)[1]]
list(mmin,plot(m,M))

}
```

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References

Anadranistakis, M., Lagouvardos, K., Kotroni, V., Skouras, K. (2002). Combination of Kalman filter and an empirical method for the correction of near-surface temperature forecasts: Applications over Greece. *Geophysical Research Letters*, Vol. 29, No. 16

Arakawa, A. and Lamb, V.R. (1977). Computational desing of the basic dynamical processes of the UCLA general circulation model. *Methods in Computational Physics*, 17, *Academic Press*, 173-265

Bernardo, J.M. and Smith, A.F.M. (1995). *Bayesian Theory*. Wiley, Chichester

Carlin, B.P. and Louis, T.A. (1996). *Bayes and Empirical Bayes Methods for Data Analysis*. Chapman and Hall, London

Casella, G. and Berger, R.L. (2001). *Statistical Inference*. Duxbury, Belmot, CA

Galanis, G. and Anadranistakis, M. (2002). A one dimensional Kalman filter for the correction of near surface temperature forecasts. *Meteorol. Appl*., 9, 437-441

Galanis, G., Louka, P., Katsafados, P., Pytharoulis, I. and Kallos, G. (2006). Applications of Kalman filters based on non-linear functions to numerical weather predictions. *Ann. Geophys*., 24, 2451-2460

Geisser, S. (1993). *Predictive Inference: An Introduction*. Chapman and Hall, London

Gofa, F., Pytharoulis, I., Andreadis, T., Papageorgiou, I., Fragkouli, P., Louka, P., Avgoustoglou, E. and Tirli, V. (2008). Evaluation of the operational numerical weather forecasts of the Hellenic National Meteorological Service. *Proceedings of the 9th conference in Meteorology, Climatology and Atmospheric Physics, Thessaloniki, Greece*, 51-58.

 https://doi.org/10.26219/heal.aueb.6651

Hellenic National Meteorological Service
(http://www.hnms.gr/hnms/greek/index_html)

Homleid, M. (1995). Diurnal corrections of short-term surface temperature forecasts using the Kalman Filter. *Weather and Forecasting*, 10, 689-707

http://forecast.uoa.gr

http://meteo.geo.auth.gr/skiron

Janjic, Z.I. (1994). The Step-mountain Eta Coordinate Model: Further Developments of the Convection, Viscous Sublayer and Turbulence Closure Schemes. *Monthly Weather Review*, 122, 927-945

Janjic, Z.I., Gerrity, J.P., Nickovic, S. (2001). An alternative approach to nonhydrostatic modeling. *Monthly Weather Review*, 129, 1164-129

Kallos, G., Nickovic, S., Sakellaridis, G., Anadranistakis, E., Jovic, D., Papageorgiou, J. (1997). The regional weather forecasting system Skiron: An overview. *Proc. of the Symposium on Regional Weather Prediction on Parallel Computer Environments*, Athens, Greece

Kallos, G., Pytharoulis, I. (2005). Short-term predictions (weather forecasting purposes). In M.G. Anderson (Ed.). *Encyclopedia of Hydrological Sciences*, Wiley, 2791-2811

Kalman, R.E. (1960). A new approach to linear filtering and prediction problems. *Transactions of the American Society of Mechanical Engineers*, Ser. D 82, 35-45

Meinhold, R.J. and Singpurwalla, N.D. (1983). Understanding the Kalman filter. *The American Statistician*, Vol. 37, No. 2

Mesinger, F. (1984). A blocking technique for representation of mountains in atmospheric models. *Riv. Meteor. Aeronaut.*, 44, 195-202

https://doi.org/10.26219/heal.aueb.6651

Mpaltzi, F. (2008). Στατιστική αξιολόγηση αριθμητικού μοντέλου πρόγνωσης καιρού περιορισμένης περιοχής. *Διπλωματική διατριβή, Σχολή Εφαρμοσμένων Μαθηματικών και Φυσικών Επιστημών, Εθνικό Μετσόβιο Πολυτεχνείο*, 82

Nickovic, S., Mihailovic, D., Rajkovic, B. and Papadopoulos, A. (1998). The weather forecasting system SKIRON. Vol. II: Description of the model. *Atmospheric Modelling and Weather Forecasting Group, Department of Physics, University of Athens*, pp. 228

Papadopoulos, A. (2001). A regional numerical model with special capabilities in the use of the initial and boundary conditions. *Ph. D. Thesis, School of Physics, University of Athens*, Athens, 272

Papadopoulos, A., Katsafados, P., Kallos, G., Nickovic, S. (2002). The weather forecasting system for POSEIDON-An overview. *GAOS*, 8, 219-237

Pytharoulis, I. (2009). Υψηλής ανάλυσης επιχειρησιακές αριθμητικές προγνώσεις καιρού για αγρομετεωρολογικές εφαρμογές. *Πρακτικά 6ου Πανελλήνιου Συνεδρίου Γεωργικής Μηχανικής*, Θεσσαλονίκη, 75-82

Winninghof, F.J. (1968). On the adjustment toward a geostrophic balance in a simple primitive equation model with application to the problems of initialization and objective analysis. *PhD Thesis, Dept of Meteorology, Univ. of California*

https://doi.org/10.26219/heal.aueb.6651