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ΣΧΟΛΗ ΟΙΚΟΝΟΜΙΚΩΝ ΕΠΙΣΤΗΜΩΝ

ΤΜΗΜΑ ΟΙΚΟΝΟΜΙΚΗΣ ΕΠΙΣΤΗΜΗΣ

"ANALYSIS AND NUMERICAL SIMULATIONS OF A CLASS OF CLOSED ECONOMY NEW KEYNESIAN MODELS"

ΛΑΖΑΡΑΚΗΣ ΣΠΥΡΙΔΩΝ

Διατριβή υποβληθείσα προς μερική εκπλήρωση των απαραιτήτων προϋποθέσεων για την απόκτηση του Μεταπτυχιακού Διπλώματος Ειδίκευσης

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Αθήνα

Εγκρίνουμε τη διατριβή του: ΛΑΖΑΡΑΚΗ ΣΠΥΡΙΔΩΝ

ΓΕΩΡΓΙΟΣ ΑΛΟΓΟΣΚΟΥΦΗΣ [ΥΠΟΓΡΑΦΗ]

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This Master thesis deals with the New Keynesian Model as it is presented in Jordi Gali's book "Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework" putting down not only all the missing algebra in units 3,6 but also the study on the impulse responses on shocks in any case using MATLAB. As far as MATLAB is concerned, I compare the numerical results of three different methods of solving rational expectations problems (Undetermined coefficients, Blanchard & Kahn, Klein).

Introduction

The last thirty years there has been massive research on causes of economic fluctuations and on which are the optimal policies to stabilize the economies, but these topics are still in hot dispute between the economists, especially after 2007, when the depression reminded everyone its enormous costs to National Economies and consequently to social welfare. The two major schools of thought about business cycles are the Real Business Cycles and New Keynesian Perspective. The former was first to emerge after the famous Lucas Critique and the latter was, at least at the beginning, an extension of the RBC methodology. Based on the frictionless neoclassical growth model, the RBC approach aimed to explain economic fluctuations as an optimal response of rational agents to real disturbances, particularly to technology shocks. With the exception of the oil price shocks, however, it is difficult to identify negative technology shocks that are large enough to cause recessions. As King and Rebelo (1999) ask in their seminal paper, "If these shocks are large and important, why can't we read about them in the Wall Street Journal?". The New Keynesian models of Business Cycles provide an alternative explanation which suggests that fluctuations are often driven by monetary and fiscal policy shocks (many times along with technology shocks) which can cause real effects due to nominal rigidities. But of course the New Keynesian models are not a panacea: identifying many of these empirically assumptions have proven to be challenging e.g. Chari, Kehoe and McGrattan (2008), reminding that further research should be conducted in DSGE modeling.

One the latest progression to New Keynesian DSGE modeling was the Gali's perspective as it was presented in "Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework". The three main aspects of the Gali's New Keynesian models are:

Monopolistic Competition: The prices of goods and inputs are set by private economic agents in order to maximize their objectives, as opposed to being determined by an anonymous Walrasian auctioneer seeking to clear all (competitive) markets at once. The monopolistically competitive firms do not produce at the lowest possible cost per unit (they have excess capacity) and that they produce smaller quantities of the good from the point of view of society as a whole (inefficient allocation of resources). But the monopolistically competitive firms provide an endless variety of products, something the competitive firms cannot do. Since consumers' tastes and incomes are different, it is likely worth it to have the great variety of choices and to suffer the burdens of excess capacity and of the inefficient allocation of resources.

Nominal Rigidities: Firms are subject to some constraints on the frequency with which they can adjust the prices of the goods and services they sell and the same kind of friction applies to workers in the presence of sticky wages. Following the trend I make use of the popular Calvo model for sticky prices. It is a time dependent model since whether or not a firm adjusts prices is a function of time and does not depends on the state of the economy.

Short run non-neutrality of monetary policy: As a consequence of the presence of nominal rigidities, changes in short term nominal interest rates lead to variations in real interest rates, consumption and output. In the

long run however, all prices and wages adjust, and the economy reverts back to its natural equilibrium. However the nominal rigidities “restore” the role of the monetary policy.

In chapter 1 is the analysis of the benchmark New Keynesian model assuming only sticky prices i.e nominal rigidities in Goods Market and I exploit the IRF tool to show the response of the economy to a monetary shock. Under certain assumption the model provides that the policy shock leads to an increase in the nominal interest rate, an increase to the real interest rate and a decrease in inflation and output. In order to bring about the observed interest rate response, the central bank must engineer a reduction in the money supply and so the calibrated model consequently display a liquidity effect.

In chapter 2 I use analogous analysis to the previous model but adding also sticky wages i.e. there are nominal rigidities in the Labour Market. The presence of sticky wages and prices generates a much smaller inflation decline than with only sticky prices . Here since also wages are sticky, the inflationary contraction is divided between prices and wages and as a result real wage does not change at all or too little. This in turn reduces the impact of decline in activity on the real marginal cost and, hence, the limited size on inflation response.

In chapter 3 I cite a quick review of some extensions or omissions conserning the New Keynesian models. Also I cite some critical reviews about the assumptions and conclusions of the NKM.

1. The New Keynesian Model with price rigidities

1.1Households

Consider an economy consisting of many identical, infinitely lived households (with measure normalized to one). The representative household has an instantaneous, (and time separable) money in utility function of the form: $U\left(C_t, N_t, \frac{M_t}{P_t}\right)$ (1.1)

under the standard assumptions : $U_{Ct} > 0, U_{Nt} < 0, U_{Mt} > 0, U_{CCt} < 0, U_{NNt} < 0, U_{MMt} < 0$

along with the Inada Conditions: $\lim_{C\to 0} U_C = \infty, \lim_{C\to\infty} U_C = 0, \lim_{\frac{M}{P}\to 0} U_M = \infty, \lim_{\frac{M}{P}\to\infty} U_M = 0$

where:

C_t : consumption level

N_t : Labour

$\frac{M_t}{P_t}$: Real money

Also I assume separable preferences meaning that cross differentiations are zero i.e $U_{CNt} = U_{CMt} = U_{NMt} = 0$

A representative household maximizes utility, and discounts the future by a factor β :

$$E_o \sum_{t=0}^{\infty} \beta^t U\left(C_t, N_t, \frac{M_t}{P_t}\right) \quad (1.2)$$

The consumption index C_t is the sum of consumption of all goods i , and there exists a continuum of goods represented by the interval $[0,1]$: $C_t = \left(\int_0^1 C_{it}^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}}$ (1.3)

Given this assumption, goods become imperfect substitutes, a feature which equips firms with market poIr. Equation (1.3) also nests free competition as a special case. In particular, taking the limit as $\varepsilon \to \infty$ becomes: $\lim_{\varepsilon\to\infty} C_t = \int_0^1 C_{it} di$

Households maximization problem is subject to a one period budget constraint:

$$\int_0^1 P_{it} C_{it} di + M_t + Q_t B_t \le M_{t-1} + B_{t-1} + W_t N_t + T_t \quad (1.4)$$

In this setup, B_{t-1} is the number of bonds purchased last period, each yielding a payoff of one, and $Q \equiv \frac{1}{1+i_t}$ is the price per bond bought today. T_t denotes the dividends from ownership of firms.

Also, the following "No-Ponzi" condition is imposed: $\lim_{T\to\infty} Q^t A_t \ge 0$

Where $A_t \equiv B_t + \frac{M_t}{P_t}$ is the stock of financial wealth.

The representative household's decision problem can be dealt with in two stages. First, for any given level of consumption expenditures it will be optimal to purchase the consumption bundle that maximizes total consumption C_t . Second, given this optimal bundle (mix) of consumption goods, the household must choose the utility maximizing combination of consumption, labour and money.

First Stage

Assume an arbitrary level of consumption expenditures e_t such us $\int_0^1 P_{it}C_{it}di = e_t$ (1.5), then

$$\max_{C_{it}} \left(\int_0^1 C_{it}^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$
$$s.t \quad \int_0^1 P_{it}C_{it}di \le e_t$$

Assume that the constraint holds with equality and using Lagrange method

$$L = \left(\int_0^1 C_{it}^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} - \lambda_t \left(\int_0^1 P_{it}C_{it}di - q_t \right)$$

FONCs

$$wrt\ C_{it} : \frac{\varepsilon}{\varepsilon-1}\left(\int_0^1 C_{it}^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}-1} \frac{\varepsilon-1}{\varepsilon} C_{it}^{\frac{\varepsilon-1}{\varepsilon}-1} P_{it} - \lambda_t P_{it} = 0 \Leftrightarrow \left(\int_0^1 C_{it}^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{1}{\varepsilon-1}} C_{it}^{-\frac{1}{\varepsilon}} = \lambda_t P_{it} \Leftrightarrow \left(\int_0^1 C_{it}^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}\frac{1}{\varepsilon}} C_{it}^{-\frac{1}{\varepsilon}} = \lambda_t P_{it} \Leftrightarrow$$

$$\Leftrightarrow C_t^{\frac{1}{\varepsilon}} C_{it}^{-\frac{1}{\varepsilon}} = \lambda_t P_{it} \Leftrightarrow \left(\frac{C_t}{C_{it}} \right)^{\frac{1}{\varepsilon}} = \lambda_t P_{it}$$

The equality must hold for all goods, so the relationship between two different goods must be:

$$\frac{\left(\frac{C_t}{C_{it}} \right)^{\frac{1}{\varepsilon}}}{\left(\frac{C_t}{C_{jt}} \right)^{\frac{1}{\varepsilon}}} = \frac{\lambda_t P_{it}}{\lambda_t P_{jt}} \forall i, j \Rightarrow \left(\frac{C_{it}}{C_{jt}} \right)^{-\frac{1}{\varepsilon}} = \frac{P_{it}}{P_{jt}} \Rightarrow C_{it} = C_{jt} \left(\frac{P_{it}}{P_{jt}} \right)^{-\varepsilon} \quad (1.6)$$

I multiply both sides with P_{it} and then I integrate both sides over i. Thus:

$$\Rightarrow P_{it}C_{it} = P_{it}C_{jt} \left(\frac{P_{it}}{P_{jt}} \right)^{-\varepsilon} \Rightarrow \underbrace{\int_0^1 P_{it}C_{it}di}_{e_t} = \int_0^1 P_{it}C_{jt} \left(\frac{P_{it}}{P_{jt}} \right)^{-\varepsilon} di = C_{jt}P_{jt}^{\varepsilon} \int_0^1 P_{it}^{1-\varepsilon} di \Rightarrow C_{jt} = \frac{e_t P_{jt}^{-\varepsilon}}{\int_0^1 P_{it}^{1-\varepsilon} di} \quad (1.7)$$

This is the household's demand function for good j. It depends on the price P_{jt}, on the (arbitrary for now) expenditure budget for consumption, e_t, and on an expression that is a function of all prices in the economy. This expression can be shown that this is the cost for obtaining one unit of (total) consumption.
Let $C_t = 1$ and define P_t as the expenditure needed to purchase a unit level of C_t, that is $P_t \equiv e_t\big|_{C_t=1}$

Thus, $C_t = 1 = \left(\int_0^1 C_{it}^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} = \left[\int_0^1 \left(C_{jt} \left(\frac{P_{it}}{P_{jt}} \right)^{-\varepsilon} \right)^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} = \left[\int_0^1 C_{jt} P_{ji}^{\varepsilon-1} P_{it}^{1-\varepsilon} di \right]^{\frac{\varepsilon}{\varepsilon-1}} =$

$$= \left[C_{jt}^{\frac{\varepsilon-1}{\varepsilon}} P_{ji}^{\varepsilon-1} \int_0^1 P_{it}^{1-\varepsilon} di \right]^{\frac{\varepsilon}{\varepsilon-1}} = C_{jt} P_{ji}^{\varepsilon} \left[\int_0^1 P_{it}^{1-\varepsilon} di \right]^{\frac{\varepsilon}{\varepsilon-1}} \Leftrightarrow C_{jt} = C_t P_{ji}^{-\varepsilon} \left[\int_0^1 P_{it}^{1-\varepsilon} di \right]^{\frac{\varepsilon}{1-\varepsilon}}$$

I multiply both sides with P_{jt} this time and then I integrate both sides over j. Thus:

$$P_{ji} C_{jt} = C_t P_{ji}^{1-\varepsilon} \left[\int_0^1 P_{it}^{1-\varepsilon} di \right]^{\frac{\varepsilon}{1-\varepsilon}} \Leftrightarrow \int_0^1 P_{ji} C_{jt} dj = \int_0^1 C_t P_{ji}^{1-\varepsilon} \left[\int_0^1 P_{it}^{1-\varepsilon} di \right]^{\frac{\varepsilon}{1-\varepsilon}} dj \Leftrightarrow P_t = \int_0^1 P_{ji}^{1-\varepsilon} \left[\int_0^1 P_{it}^{1-\varepsilon} di \right]^{\frac{\varepsilon}{1-\varepsilon}} dj$$

But, the sum over all $j's$ and over all $i's$ involves summing over all prices, so:

$$P_t = \int_0^1 P_{ji}^{1-\varepsilon} dj \left[\int_0^1 P_{it}^{1-\varepsilon} di \right]^{\frac{\varepsilon}{1-\varepsilon}} \Leftrightarrow \boxed{P_t = \left(\int_0^1 P_{it}^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}} \quad (1.7)$$

And this is the **Aggregate price index**.

Now we can substitute the aggregate price index into the expression (1.7) to get:

$$C_{jt} = \frac{e_t P_{jt}^{-\varepsilon}}{\int_0^1 P_{it}^{1-\varepsilon} di} \Rightarrow C_{jt} = \frac{e_t P_{jt}^{-\varepsilon}}{P_t^{1-\varepsilon}} = P_{it}^{-\varepsilon} P_t^{\varepsilon-1} e_t \quad (1.8)$$

Then, by (3) and using (8) I have:

$$C_t = \left(\int_0^1 C_{it}^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} = \left(\int_0^1 \left[P_{it}^{-\varepsilon} P_t^{\varepsilon-1} e_t \right]^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} = e_t P_t^{\varepsilon-1} \left(\int_0^1 P_{it}^{1-\varepsilon} di \right)^{\frac{\varepsilon}{\varepsilon-1}} = e_t P_t^{\varepsilon-1} \left[\left(\int_0^1 P_{it}^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}} \right]^{-\varepsilon} =$$

$$= e_t P_t^{\varepsilon-1-\varepsilon} = e_t P_t^{-1} \quad \Leftrightarrow \quad e_t = C_t P_t \Leftrightarrow \quad \int_0^1 P_{it} C_{it} di = C_t P_t \quad (1.9)$$

Thus using the fact that $e_t = C_t P_t$ and (1.8) $\Rightarrow$ $\boxed{C_{it} = \left(\frac{P_{it}}{P_t} \right)^{-\varepsilon} C_t}$ (1.10)

(1.10) is the **Demand Function for Good** i

Equation (1.10) is the solution to the household's first step maximization problem. Once the household knows prices and has decided on C_{it}, it also knows how much to consume of each good. The next step is decide C_t, the aggregate consumption.

<u>Second Stage</u>

Now, the household can make its consumptions and leisure decisions optimally to maximize its Expected Utility function:

$$\max_{C_t, N_t, \frac{M_t}{P_t}, B_t} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t U \left(C_t, N_t, \frac{M_t}{P_t} \right) \right\} \quad (1.11)$$

$$st \quad P_t C_t + M_t + Q_t B_t \le M_{t-1} + B_{t-1} + W_t N_t + T_t$$

To use this dynamic problem I use the Langrangean method

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ U\left(C_{t,} N_t, \frac{M_t}{P_t} \right) + \lambda_t \left(-P_t C_t - M_t - Q_t B_t + M_{t-1} + B_{t-1} + W_t N_t + T_t \right) \right\} \quad (1.12)$$

FONCs

$$wrt \;\; C_t : \;\; \beta^t \left(U_{Ct} - \lambda_t P_t \right) = 0 \quad (1.13)$$

$$wrt \;\; N_t : \beta^t \left(U_{Nt} + \lambda_t W_t \right) = 0 \quad (1.14)$$

$$wrt \frac{M_t}{P_t} : \beta^t \left(U_{Mt} - \lambda_t P_t \right) + \beta^{t+1} E_t \lambda_{t+1} P_t = 0 \quad (1.15)$$

$$wrt \;\; B_t : \; -\beta^t \lambda_t Q_t + \beta^{t+1} E_t \lambda_{t+1} = 0 \quad (1.16)$$

$$(1.13) \Rightarrow U_{Ct} = \lambda_t P_t \Leftrightarrow \frac{U_{Ct}}{P_t} = \lambda_t \quad (1.13')$$

$$(1.16) \Rightarrow Q_t = \frac{\beta E_t \lambda_{t+1}}{\lambda_t} \quad (1.16')$$

Because I'm going to linearize the optimality conditions and additionally I am not going to make welfare analysis I assume that there is no covariance between λ_t and C_t or linearity that after linearization it will be true. In other words I can write (1.13') one period forward to get $E_t U_{Ct+1} = E_t \lambda_{t+1} E_t P_{t+1}$ (1.13'')

By (1.13'), (1.13'') and (1.16') $\Rightarrow Q = \beta E_t \left[\frac{U_{Ct+1}}{U_{Ct}} \frac{P_t}{P_{t+1}} \right]$

If I use the assumption that $Q_t = \frac{1}{1+i}$ then I get: $1 = \beta (1+i) E_t \left[\frac{U_{Ct+1}}{U_{Ct}} \frac{P_t}{P_{t+1}} \right]$ (1.18)

This is the Euler with respect to bonds.

By (1.13) and (1.14) I get: $\Rightarrow \frac{U_{Nt}}{U_{Ct}} = -\frac{W_t}{P_t}$ (1.19)

This is the Labour-leisure choice

By (1.15) and (1.13) I get: $\Rightarrow \frac{U_{Mt}}{U_{Ct}} = 1 - Q_t \Rightarrow \frac{U_{Mt}}{U_{Ct}} = 1 - \frac{1}{1+i_t} \Rightarrow \frac{U_{Mt}}{U_{Ct}} = \frac{i_t}{1+i_t}$ (1.20)

This is the money demand

Equations (1.18), (1.19),(1.20) determine intertemporal consumption allocation, the labour leisure choice and the money demand, respectively. Together, those equations determine the rational, forward looking household's allocation decisions.

Let the utility function be of a functional form as following:

$$U\left(C_t, N_t, \frac{M_t}{P_t} \right) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} + \frac{\left(\frac{M_t}{P_t} \right)^{1-\nu}}{1-\nu} \quad (1.21)$$

So, (1.18),(1.19),(1.20) become:

$$\Rightarrow 1 = \beta (1+i) E_t \left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{P_t}{P_{t+1}} \right] (1.18')$$

$$\Rightarrow C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t} \text{ (1.19')}$$

$$\text{(1.20)} \Rightarrow \frac{\left(\frac{M_t}{P_t}\right)^{-\nu}}{C_t^{-\sigma}} = \frac{i_t}{1+i_t} \Rightarrow \left(\frac{M_t}{P_t}\right)^{-\nu} = C_t^{-\sigma} \frac{i_t}{1+i_t} \Rightarrow \frac{M_t}{P_t} = C_t^{\frac{\sigma}{\nu}} \left(\frac{i_t}{1+i_t}\right)^{-\frac{1}{\nu}} \quad \text{(1.20')}$$

1.2 Firms

The second part of the model consists of firms' decisions. Firms try to minimize the costs of production and maximize profits.

Assume a continuum of firms indexed by $i \in [0,1]$, each producing a differentiated final good. All firms have access to an identical Cobb-Douglas technology: $Y_{it} = A_t N_{it}^{1-\alpha}$ (1.22)

Where:

Y_{it} : the output produced by firm i in period t

A_t : is the state of technology, assumed to be common across firms and to vary exogenously over time

N_{it} : is the labour force used by firm i

All firms face an identical isoelastic demand schedule given by (1.10), and take the aggregate price level P_t and aggregate consumption index C_t as given.

Cost Minimization

Firms i minimize costs by choosing choosing the lowest possible level of labour subject to producing the firm specific good Y_{it} , which results from the production function:

$$\min_{N_{it}} \left(\frac{W_t}{P_t}\right) N_{it} \quad \text{(1.23)}$$

$$st \;\; Y_{it} = A_t N_{it}^{1-\alpha}$$

Using the Langrangean: $L = \left(\frac{W_t}{P_t}\right) N_{it} - \xi_t \left(A_t N_{it}^{1-\alpha} - Y_{it}\right)$

FONC

$$\left(\frac{W_t}{P_t}\right) - \xi_t (1-\alpha) A_t N_{it}^{-\alpha} = 0 \Leftrightarrow \xi_t = \left(\frac{W_t}{P_t}\right) \frac{1}{(1-\alpha) A_t N_{it}^{-\alpha}} = \left(\frac{W_t}{P_t}\right) \frac{1}{MPN_t^n} \text{ (1.24)}$$

As we know from microeconomics and especially from cost theory that the optimum where the envelope theorem holds: $\frac{dC(Y_{it}, W_t, P_t)}{dY_{it}} = \xi_t$

i.e. at the optimum we can interpret ξ_t as the firm's real marginal cost. This is the "price" of one additional unit of output $\Rightarrow \xi_t = MC_{t+k|t} = \left(\frac{W_t}{P_t}\right) \frac{1}{MPN_t^n} = \frac{W_t}{P_t MPN_t^n}$ (1.25)

The nominal marginal gain is the income increase, that is the price times the marginal increase in production by adding a little more labour. Thus, the real marginal cost is the nominal cost relative to the nominal gain.

Calvo pricing Model

Gali follows the model proposed in Calvo (1983), each firm may reset its price only with probability $(1-\theta)$ in any given period, and independent of the time elapsed since the last adjustment. Calvo assumes that price changes, rather than arriving deterministically, arrive stochastically. Specifically, he assumes that opportunities to change prices follow a Poisson process: the probability that a firm is able to change its price is the same each period, regardless of when it was last able to change its price .Thus, each period a measure $(1-\theta)$ of producers reset their prices, while a fraction θ keep their prices unchanged. As a result, the expected duration of a price is given by $\frac{1}{1-\theta}$ (Appendix 1). In this context, θ becomes a natural index of price stickiness. Let $S(t)\subset[0,1]$ represent the set of firms not reoptimizing their posted price in period t. Using the definition of the aggregate price level and the fact that all firms resetting prices will choose an identical price P_t^*:

$$P_t=\left[\int_{S(t)}^{1}P_{it-1}^{1-\varepsilon}di+(1-\theta)P_t^{*1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}=\left[\theta P_{t-1}^{1-\varepsilon}+(1-\theta)P_t^{*1-\varepsilon}\right]^{1-\varepsilon}\Leftrightarrow P_t^{1-\varepsilon}=\left[\theta P_{t-1}^{1-\varepsilon}+(1-\theta)P_t^{*1-\varepsilon}\right]\Leftrightarrow$$

$$P_t^{1-\varepsilon}=P_{t-1}^{1-\varepsilon}\left[\theta+(1-\theta)\left(\frac{P_t^*}{P_{t-1}^{1-\varepsilon}}\right)^{1-\varepsilon}\right]\Leftrightarrow\frac{P_t^{1-\varepsilon}}{P_{t-1}^{1-\varepsilon}}=\theta+(1-\theta)\left(\frac{P_t^*}{P_{t-1}^{1-\varepsilon}}\right)^{1-\varepsilon}\Leftrightarrow\Pi_t^{1-\varepsilon}=\theta+(1-\theta)\left(\frac{P_t^*}{P_{t-1}^{1-\varepsilon}}\right)^{1-\varepsilon}\quad(1.26)$$

Π_t **is the Aggregate Gross Inflation**

How that works? The linearization of (1.26) around the steady state would give the answer. Steady state is defined by zero inflation, implying that $\Pi_t=\Pi=1$ and $P_t^*=P_t=P_{t-1}\ \ \forall t$, and using a first Taylor expansion yields:

$$\Pi+(1-\varepsilon)\Pi_t^{-\varepsilon}\pi_t=\left[\theta+(1-\theta)\left(\frac{P}{P}\right)^{1-\varepsilon}\right]+(1-\varepsilon)(1-\theta)\left(\frac{P}{P}\right)^{-\varepsilon}\left(\frac{1}{P}Pp_t^*-\frac{P}{P^2}Pp_{t-1}\right)$$

$$\Pi=1\Rightarrow(1-\varepsilon)\pi_t=(1-\varepsilon)(1-\theta)\left(p_t^*-p_{t-1}\right)\Rightarrow\boxed{\pi_t=(1-\theta)\left(p_t^*-p_{t-1}\right)}\ (1.27)$$

After linearization it is clear that in this model inflation results from the difference between the "reoptimising" price and the last period's aggregate price index.

Profit Maximization and Optimal price setting

A firm reoptimising in period will choose the price P_t^* that maximizes the current market value of the profits generated while the price remains effective. Formally, it solves the problem

$$\max_{P_t^*}\sum_{k=0}^{\infty}\theta^k E_t\left\{Q_{t|t+k}\left(P_t^*Y_{t+k|t}-\Psi_{t+k}\left(Y_{t+k|t}\right)\right)\right\}$$
$$st\quad Y_{t+k|t}=\left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon}C_{t+k}\quad for\ \ k=0,1,2,\ldots\qquad(1.28)$$

Where the stochastic discount factor for nominal payoffs in period $t+k$ is $Q_{t|t+k}$ which is given by

$$Q_{t|t+k} = \beta^k \left(\frac{C_{t+k}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+k}} \qquad (1.29)$$

$Y_{t+k|t}$: is the output in period $t+k$ for a firm that last set its price in period t

$\Psi_{t+k}(.)$: is the cost function as a function of its output

$\Psi'_{t+k}(.)$: denotes the nominal marginal cost in period $t+k$ for a firm which last reset its price in period t

The firm must discount the expected profits by both the probability to adjust their price and the stochastic discount factor. Higher values of the θ means that probably the firm must wait for some time to adjust its price and that's why must put a lot of weight on the expected profits while keeping the price that set at time t. Very low values of θ means that probably very soon the firm will have the opportunity to adjust its price so it does not "care" so much about the expected profit, especially far in the time horizon.

Plugging the constraint and the stochastic discount factor into the objective function the problem is transformed into:

$$\max_{P_t^*} \sum_{k=0}^{\infty} \theta^k E_t \left\{ \beta^k \left(\frac{C_{t+k}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+k}} \left(P_t^* \left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} C_{t+k} - \Psi_{t+k} \left(\left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} C_{t+k} \right) \right) \right\} \Leftrightarrow$$

$$\max_{P_t^*} \sum_{k=0}^{\infty} \theta^k E_t \left\{ \beta^k \left(\frac{C_{t+k}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+k}} P_t^* \left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} C_{t+k} - \beta^{\kappa} \left(\frac{C_{t+k}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+k}} \Psi_{t+k} \left(\left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} C_{t+k} \right) \right\} \Leftrightarrow$$

$$\Leftrightarrow \boxed{\max_{P_t^*} \sum_{k=0}^{\infty} \theta^k E_t \left\{ \beta^k C_{t+k}^{1-\sigma} C_t^{\sigma} P_t P_{t+k}^{\varepsilon-1} P_t^{*1-\varepsilon} - \beta^k C_{t+k}^{-\sigma} C_t^{\sigma} P_t P_{t+k}^{-1} \Psi_{t+k} \left(\left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} C_{t+k} \right) \right\}} \quad (1.29')$$

FONCs

$$\sum_{\kappa=0}^{\infty} \theta^k E_t \left\{ \beta^k C_{t+k}^{1-\sigma} C_t^{\sigma} P_t P_{t+k}^{\varepsilon-1} P_t^{*1-\varepsilon-1} (1-\varepsilon) - \beta^k C_{t+k}^{-\sigma} C_t^{\sigma} P_t P_{t+k}^{-1} \Psi'_{t+k|t}(.) \frac{(-\varepsilon)}{P_{t+k}} \left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon-1} C_{t+k} \right\} = 0 \Leftrightarrow$$

$$\sum_{\kappa=0}^{\infty} \theta^k E_t \left\{ \beta^k C_{t+k}^{1-\sigma} C_t^{\sigma} P_t P_{t+k}^{\varepsilon-1} P_t^{*-\varepsilon} (1-\varepsilon) + \varepsilon \beta^k C_{t+k}^{1-\sigma} C_t^{\sigma} P_t P_{t+k}^{\varepsilon-1} P_t^{*-\varepsilon-1} \Psi'_{t+k|t}(.) \right\} = 0 \Leftrightarrow$$

$$(1-\varepsilon) \sum_{\kappa=0}^{\infty} \theta^k E_t \left\{ \beta^k C_{t+k}^{1-\sigma} C_t^{\sigma} P_t P_{t+k}^{\varepsilon-1} P_t^{*-\varepsilon} \right\} + \varepsilon \sum_{k=0}^{\infty} \theta^k E_t \left\{ \beta^k C_{t+k}^{1-\sigma} C_t^{\sigma} P_t P_{t+k}^{\varepsilon-1} P_t^{*-\varepsilon-1} \Psi'_{t+k|t}(.) \right\} = 0 \Leftrightarrow$$

$$(\varepsilon-1) \sum_{\kappa=0}^{\infty} \theta^k E_t \left\{ \beta^k C_{t+k}^{1-\sigma} C_t^{\sigma} P_t P_{t+k}^{\varepsilon-1} P_t^{*-\varepsilon} \right\} = \varepsilon \sum_{k=0}^{\infty} \theta^k E_t \left\{ \beta^k C_{t+k}^{1-\sigma} C_t^{\sigma} P_t P_{t+k}^{\varepsilon-1} P_t^{*-\varepsilon-1} \Psi'_{t+k|t}(.) \right\} \Leftrightarrow$$

$$P_t C_t^{\sigma} P_t^{*-\varepsilon} (\varepsilon-1) \sum_{\kappa=0}^{\infty} \theta^k E_t \left\{ \beta^k C_{t+k}^{1-\sigma} P_{t+k}^{\varepsilon-1} \right\} = P_t C_t^{\sigma} P_t^{*-\varepsilon-1} \varepsilon \sum_{k=0}^{\infty} \theta^k E_t \left\{ \beta^k C_{t+k}^{1-\sigma} P_{t+k}^{\varepsilon-1} \Psi'_{t+k|t}(.) \right\} \Leftrightarrow$$

$$P_t^* = \frac{\varepsilon}{(\varepsilon-1)} \frac{\sum_{k=0}^{\infty} \theta^k E_t \left\{ \beta^k C_{t+k}^{1-\sigma} P_{t+k}^{\varepsilon-1} \Psi'_{t+k|t}(.) \right\}}{\sum_{k=0}^{\infty} \theta^k E_t \left\{ \beta^k C_{t+k}^{1-\sigma} P_{t+k}^{\varepsilon-1} \right\}} = \frac{\varepsilon}{(\varepsilon-1)} \frac{E_t \sum_{k=0}^{\infty} \theta^k \beta^k C_{t+k}^{1-\sigma} P_{t+k}^{\varepsilon-1} \Psi'_{t+k|t}(.)}{E_t \sum_{k=0}^{\infty} \theta^k \beta^k C_{t+k}^{1-\sigma} P_{t+k}^{\varepsilon-1}} \quad (1.30)$$

Gali defines $\frac{\Psi'_{t+k|t}(.)}{P_{t+k}} \equiv MC_{t+k|t}$ as the real marginal cost in period $t+k$ for a firm whose price was last set in period t.

Also, under the assumption that there are no rigidities in prices i.e. $\theta \to 0$, then the firm's maximizing collapses to a static problem where : $P_t^* = \frac{\varepsilon}{(\varepsilon-1)} \Psi'_{t|t}(.) \equiv \mathrm{M} \Psi'_{t|t}(.)$ (The exact derivation is on the Appendix 2). The monopolistically competitive firms charge a markup $\mathrm{M} \equiv \frac{\varepsilon}{\varepsilon-1}$ above the marginal cost Ψ' or in other words above the competitive equilibrium. So, I can rewrite (1.30) as:

$$\boxed{P_t^* = \mathrm{M} \frac{E_t \sum_{k=0}^{\infty} \theta^k \beta^k C_{t+k}^{1-\sigma} P_{t+k}^{\varepsilon} MC_{t+k|t}}{E_t \sum_{k=0}^{\infty} \theta^k \beta^k C_{t+k}^{1-\sigma} P_{t+k}^{\varepsilon-1}}} \quad (1.30')$$

The "reoptimising" price is a weighted average of current and future real marginal costs which means that the firms are forward looking. The firms are "weighting" the current and expected marginal costs by the factor $(\beta\theta)^k$ which reflecting the uncertainty and risk to set their prices at period t and keep them until the next opportunity to arise at the $t+k$ period.

1.3 Log-Linearization

(1.18') can be rewritten as: $1 = E_t \left\{ e^{\ln\left(\beta(1+i_t)\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}}\frac{P_t}{P_{t+1}}\right)} \right\} = E_t \left\{ e^{\ln\beta + \ln(1+i_t) - \sigma \ln\frac{C_{t+1}}{C_t} + \ln\frac{P_t}{P_{t+1}}} \right\}$

I can define:

$$\ln\beta \equiv -\rho, \quad \ln(1+i_t) \approx i_t, \quad \ln\frac{C_{t+1}}{C_t} = \ln C_{t+1} - \ln C_t = c_{t+1} - c_t,$$

$$\ln\frac{P_t}{P_{t+1}} = \ln P_t - \ln P_{t+1} = -(\ln P_{t+1} - \ln P_t) = -(p_{t+1} - p_t) = -\pi_{t+1}$$

So: $1 = E_t \left\{ e^{-\rho + i_t - \sigma(c_{t+1} - c_t) - \pi_{t+1}} \right\}$

In the steady state I assume that : $c_{t+1} - c_t = \gamma$, $i_t = i$, $\pi_t = \pi$ and $i \equiv r + \pi = \rho + \sigma\gamma$

By first order Taylor expansion:

$$1 = E_t\left\{e^{-\rho+i_t-\sigma(c_{t+1}-c_t)-\pi_{t+1}}\right\} \approx E_t\left\{1+(\rho-\rho)+(i_t-i)-\sigma(c_{t+1}-c_t-\gamma)-(\pi_{t+1}-\pi)\right\} \Rightarrow$$

$$1 = E_t\left\{1-i+\pi+\sigma\gamma+i_t-\sigma(c_{t+1}-c_t)-\pi_{t+1}\right\} \Rightarrow 0 = -i+\pi+\sigma\gamma+E_t i_t-\sigma E_t(c_{t+1}-c_t)-E_t\pi_{t+1} \Rightarrow$$

by $i - \sigma\gamma - \pi = -\rho$ and $E_t i_t = i_t$

$$0 = -\rho + E_t i_t - \sigma E_t(c_{t+1}-c_t) - E_t\pi_{t+1} \Rightarrow \sigma c_t = \rho - i_t + \sigma E_t c_{t+1} + E_t\pi_{t+1} \Rightarrow$$

$$\boxed{c_t = E_t c_{t+1} + \frac{1}{\sigma}(\rho - i_t + E_t\pi_{t+1})} \quad (1.31)$$

$$(1.19') \Rightarrow \ln\left(C_t^{\sigma} N_t^{\varphi}\right) = \ln\left(\frac{W_t}{P_t}\right) \Rightarrow \boxed{\sigma c_t + \varphi n_t = w_t - p_t} \quad (1.32)$$

$$(1.20') \Rightarrow \ln\left(\frac{M_t}{P_t}\right) = \ln\left(C_t^{\frac{\sigma}{\nu}}\left(\frac{i_t}{1+i_t}\right)^{-\frac{1}{\nu}}\right) \Rightarrow m_t - p_t = \frac{\sigma}{\nu}c_t - \frac{1}{\nu}\ln\left(\frac{i_t}{1+i_t}\right)$$

First order Taylor expansion

$$m_t - p_t - \frac{\sigma}{\nu}c_t = -\frac{1}{\nu}\ln\left(\frac{i_t}{1+i_t}\right) \approx -\frac{1}{\nu}\left\{\ln\frac{i}{1+i} + \frac{1+i}{i}\frac{1}{(1+i)^2}(i_t-i)\right\} \Rightarrow$$

$$m_t - p_t - \frac{\sigma}{\nu}c_t = -\frac{1}{\nu}\left\{\ln\frac{i}{1+i} + \frac{1}{(1+i)i}(i_t-i)\right\} = -\frac{1}{\nu}\ln\frac{i}{1+i} + \frac{1}{(1+i)} - \frac{1}{\nu}\frac{i_t}{(1+i)i} \Rightarrow$$

$$m_t - p_t = \frac{\sigma}{\nu}c_t - \left[\frac{1}{\nu}\ln\frac{i}{1+i} + \frac{1}{(1+i)}\right] - \frac{1}{\nu}\frac{i_t}{(1+i)i}$$

If I discard the constant term and assume an income elasticity of one, where this assumption implies σ=v, the money demand equation can be written as:

$$\boxed{m_t - p_t = c_t - \eta i_t} \quad (1.33) \text{ where } \eta = \frac{1}{\nu(1+i)i}$$

In this setup and because of separability of utility function the money demand only allow to determine modey supply to support the nominal interest rate implied by the Taylor Rule that Gali imposes to the model (see below).

Assuming zero inflation steady state:

$$\Pi_t = \frac{P_t^*}{P_{t-1}} = \frac{P_t^*}{P_t} = \frac{P_t^*}{P_{t+1}} = 1$$

$$Y_{t+k|t} = Y_{t|t}$$

$$Q_{t,t+k} = \beta^k$$

$$MC_{t+k|t} = MC_{t|t} = \frac{\Psi'_{t|t}}{P_t} = \frac{\varepsilon - 1}{\varepsilon} \equiv MC$$

$$P_t^* E_t \sum_{\kappa=0}^{\infty} \theta^k \left\{ \beta^k C_{t+k}^{1-\sigma} P_{t+k}^{\varepsilon-1} \right\} = ME_t \sum_{k=0}^{\infty} \theta^k \left\{ \beta^k C_{t+k}^{1-\sigma} P_{t+k}^{\varepsilon} MC_{t+k|t}(.) \right\}$$

The linearization of the left hand side:

$$P_t^* E_t \sum_{\kappa=0}^{\infty} \theta^k \beta^k C_{t+k}^{1-\sigma} P_{t+k}^{\varepsilon-1} \approx P \sum_{k=0}^{\infty} \theta^k \beta^k C^{1-\sigma} P^{\varepsilon-1} + E_t \sum_{k=0}^{\infty} \theta^k \beta^k C^{1-\sigma} P^{\varepsilon-1} (P_t^* - P) +$$

$$+ P(\varepsilon - 1) E_t \sum_{k=0}^{\infty} \theta^k \beta^k C^{1-\sigma} P^{\varepsilon-2} (P_{t+k} - P) + P(1-\sigma) E_t \sum_{k=0}^{\infty} \theta^k \beta^k C^{-\sigma} P^{\varepsilon-1} (C_{t+k} - C) =$$

$$= P \sum_{k=0}^{\infty} \theta^k \beta^k C^{1-\sigma} P^{\varepsilon-1} + E_t \sum_{k=0}^{\infty} \theta^k \beta^k C^{1-\sigma} P^{\varepsilon-1} (P_t^* - P) +$$

$$+ (\varepsilon - 1) E_t \sum_{k=0}^{\infty} \theta^k \beta^k C^{1-\sigma} P^{\varepsilon-1} (P_{t+k} - P) + P(1-\sigma) E_t \sum_{k=0}^{\infty} \theta^k \beta^k C^{-\sigma} P^{\varepsilon-1} (C_{t+k} - C) =$$

$$= C^{1-\sigma} P^{\varepsilon-1} \sum_{k=0}^{\infty} \theta^k \beta^k \left[P + (P_t^* - P) + (\varepsilon - 1)(P_{t+k} - P) + \frac{P}{C}(1-\sigma)(C_{t+k} - C) \right] =$$

$$= C^{1-\sigma} P^{\varepsilon} \sum_{k=0}^{\infty} \theta^k \beta^k \left[1 + (p_t^* - p) + (\varepsilon - 1)(p_{t+k} - p) + (1-\sigma)(c_{t+k} - c) \right]$$

the linearization of the right hand side yields:

$$ME_t \sum_{k=0}^{\infty} \theta^k \left\{ \beta^k C_{t+k}^{1-\sigma} P_{t+k}^{\varepsilon} MC_{t+k|t} \right\} \approx M \sum_{k=0}^{\infty} \theta^k \beta^k C^{1-\sigma} P^{\varepsilon} MC + (\varepsilon) ME_t \sum_{k=0}^{\infty} \theta^k \beta^k C^{1-\sigma} P^{\varepsilon-1} MC (P_{t+k} + P) +$$

$$+ (1-\sigma) ME_t \sum_{k=0}^{\infty} \theta^k \beta^k C^{-\sigma} P^{\varepsilon} MC (C_{t+k} - C) + ME_t \sum_{k=0}^{\infty} \theta^k \beta^k C^{1-\sigma} P^{\varepsilon} \left(MC_{t+k|t} - MC \right) =$$

$$= ME_t \sum_{k=0}^{\infty} \theta^k \beta^k C^{1-\sigma} P^{\varepsilon} MC + (\varepsilon) ME_t \sum_{k=0}^{\infty} \theta^k \beta^k C^{1-\sigma} P^{\varepsilon-1} MC \frac{1}{P}(P_{t+k} + P) +$$

$$+ (1-\sigma) ME_t \sum_{k=0}^{\infty} \theta^k \beta^k C^{1-\sigma} P^{\varepsilon-1} MC \frac{1}{C}(C_{t+k} - C) + ME_t \sum_{k=0}^{\infty} \theta^k \beta^k C^{1-\sigma} P^{\varepsilon-1} \left(MC_{t+k|t} - MC \right) =$$

$$= MP^{\varepsilon} C^{1-\sigma} MCE_t \sum_{k=0}^{\infty} \theta^k \beta^k \left[1 + (\varepsilon - 1)\frac{1}{P}(P_{t+k} + P) + (1-\sigma)\frac{1}{C}(C_{t+k} - C) + \frac{1}{MC}\left(MC_{t+k|t} - MC \right) \right]$$

$$= P^{\varepsilon} C^{1-\sigma} E_t \sum_{k=0}^{\infty} \theta^k \beta^k \left[1 + (\varepsilon - 1)(p_{t+k} - p) + (1-\sigma)(c_{t+k} - c) + \left(mc_{t+k|t} - mc \right) \right]$$

The sides must be equal:

$$C^{1-\sigma}P^{\varepsilon}E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[1+(p_t^*-p)+(\varepsilon-1)(p_{t+k}-p)+(1-\sigma)(c_{t+k}-c)\right]=$$

$$=P^{\varepsilon}C^{1-\sigma}E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[1+(\varepsilon)(p_{t+k}-p)+(1-\sigma)(c_{t+k}-c)+\left(mc_{t+k|t}-mc\right)\right]\Leftrightarrow$$

$$\Leftrightarrow E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[1+(p_t^*-p)+(\varepsilon-1)(p_{t+k}-p)+(1-\sigma)(c_{t+k}-c)\right]=$$

$$=E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[1+(\varepsilon)(p_{t+k}-p)+(1-\sigma)(c_{t+k}-c)+\left(mc_{t+k|t}-mc\right)\right]\Leftrightarrow$$

$$\Leftrightarrow E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[p_t^*-p_{t+k}\right]=E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[mc_{t+k|t}-mc\right]\Leftrightarrow E_t\sum_{k=0}^{\infty}\theta^k\beta^k p_t^*=E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[mc_{t+k|t}-mc+p_{t+k}\right]$$

If $\beta\theta<1$ and because p_t^* is constant for k periods $\sum_{k=0}^{\infty}\theta^k\beta^k p_t^*=p_t^*\sum_{k=0}^{\infty}\theta^k\beta^k=p_t^*\frac{1}{1-\theta\beta}$

$$\Rightarrow\frac{p_t^*}{1-\theta\beta}=E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[mc_{t+k|t}-mc+p_{t+k}\right]\Leftrightarrow p_t^*=(1-\theta\beta)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[mc_{t+k|t}-mc+p_{t+k}\right]\Leftrightarrow$$

$$\boxed{p_t^*=\mu+(1-\theta\beta)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[mc_{t+k|t}+p_{t+k}\right]} \quad (1.34)$$

The firms will set a price that corresponds to the desired markup (or frictionless markup), given by $\mu\equiv -mc$, over a weighted average of their current and expected nominal marginal costs, with the weights being proportional to the probability of the price remaining effective for k consecutive periods.

1.4 General Equilibrium

I solve for a decentralized equilibrium:

Given the paths of one policy instrument $\{i_t\}_{t=0}^{\infty}$ and the initial conditions M_0 and B_0, a decentralized equilibrium is defined to be an allocation $\left\{C_t, N_t, Y_t, T_t, P_t, B_t, \frac{M_t}{P_t}\right\}_{t=0}^{\infty}$ such that:

1. Households undertake their optimization problem outlined above
2. All budget constraints are satisfied
3. All markets clear:
   - I. Market clearing in the goods market implies: $Y_{it}=C_{it}$ (1.35)
   - II. Market Clearing Condition in the labour market: $N_t=\int_0^1 N_{it}di$ (1.36)
   - III. Market Clearing Condition in the Bonds market, total supply of bonds: $B_t=0$

IV. Market Clearing Condition in the Dividends market: $T_t = 0$ the free entrance of the firms causes the profit to become equal to zero.

V. Market Clearing Condition in the money market, total supply of money equals to demand od money : $M_t^s = M_t^d$ where $\frac{M_t^d}{P_t} = C_t^{\frac{\sigma}{v}} \left(\frac{i_t}{1+i_t} \right)^{-\frac{1}{v}}$

Let the aggregate output be defined as: $Y_t = \left(\int_0^1 Y_{it}^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$ (1.37)

By inserting (1.10) and (1.37) into (1.35) I get:

$$Y_t = \left(\int_0^1 C_{it}^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} = \left(\int_0^1 \left[\left(\frac{P_{it}}{P_t} \right)^{-\varepsilon} C_t \right]^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} = \left(\int_0^1 P_{it}^{1-\varepsilon} P_t^{\varepsilon-1} C_t^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} = \left(P_t^{\varepsilon-1} C_t^{\frac{\varepsilon-1}{\varepsilon}} \int_0^1 P_{it}^{1-\varepsilon} di \right)^{\frac{\varepsilon}{\varepsilon-1}} =$$

$$= \left(P_t^{\varepsilon-1} C_t^{\frac{\varepsilon-1}{\varepsilon}} \int_0^1 P_{it}^{1-\varepsilon} di \right)^{\frac{\varepsilon}{\varepsilon-1}} = P_t^{\varepsilon} C_t \left(\int_0^1 P_{it}^{1-\varepsilon} di \right)^{\frac{\varepsilon}{\varepsilon-1}} == P_t^{\varepsilon} C_t P_t^{-\varepsilon} = C_t \Rightarrow Y_t = C_t \Rightarrow \ln Y_t = \ln C_t \Rightarrow \boxed{y_t = c_t}$$ (1.38)

Equation (1.37) is the aggregate market clearing condition.

(1.31) by (1.38) $y_t = E_t y_{t+1} - \frac{1}{\sigma}(i_t - \rho - E_t \pi_{t+1})$ (1.39)

From (1.22) we have that $N_{it} = \left(\frac{Y_{it}}{A_t} \right)^{\frac{1}{1-\alpha}}$,

Insert this into (1.36) as well as the goods market clearing condition (1.38) and the consumption demand (1.10)

$$N_t = \int_0^1 \left(\frac{Y_{it}}{A_t} \right)^{\frac{1}{1-\alpha}} di = \int_0^1 \left(\frac{C_{it}}{A_t} \right)^{\frac{1}{1-\alpha}} di = \int_0^1 \left(\frac{\left(\frac{P_{it}}{P_t} \right)^{-\varepsilon} C_t}{A_t} \right)^{\frac{1}{1-\alpha}} di = \int_0^1 \left(\left(\frac{P_{it}}{A_t} \right)^{-\varepsilon} \frac{C_t}{A_t} \right)^{\frac{1}{1-\alpha}} di = \left(\frac{C_t}{A_t} \right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_{it}}{P_t} \right)^{\frac{-\varepsilon}{1-\alpha}} di =$$

$$= \left(\frac{Y_t}{A_t} \right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_{it}}{P_t} \right)^{\frac{-\varepsilon}{1-\alpha}} di \quad \Leftrightarrow \ln N_t = \ln \left[\left(\frac{Y_t}{A_t} \right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_{it}}{P_t} \right)^{\frac{-\varepsilon}{1-\alpha}} di \right] \Leftrightarrow$$

$$\Leftrightarrow n_t = \frac{1}{1-\alpha}\left(\ln Y_t - \ln A_t\right) + \ln\left(\int_0^1 \left(\frac{P_{it}}{P_t}\right)^{\frac{-\varepsilon}{1-\alpha}} di\right) \Leftrightarrow$$

$$y_t = a_t + (1-\alpha)n_t - (1-\alpha)\ln\left(\int_0^1 \left(\frac{P_{it}}{P_t}\right)^{\frac{-\varepsilon}{1-\alpha}} di\right) = a_t + (1-\alpha)n_t - d_t \quad (1.40)$$

As Gali proves that $d_t \approx 0$ because $\int_0^1 \left(\frac{P_{it}}{P_t}\right)^{\frac{-\varepsilon}{1-\alpha}} di \approx 1$ up to a first order approximation around $P_{it} = P_t$

Thus: $y_t = a_t + (1-\alpha)n_t$ (1.40')

1.5 Derivation of the New Keynesian Phillips Curve

In the new Keynesian era is very popular to express all the variables in gaps, as deviation from their natural values i.e $\tilde{y}_t = y_t - y_t^n$. Usually the natural value is the counterpart from the frictionless problem. The nominal marginal cost by using labour is the wage w_t. The real marginal cost is given by equation (1.25):

$$MC_t = \frac{W_t}{P_t MPN_t^n} \Leftrightarrow \ln MC_t = \ln \frac{W_t}{P_t MPN_t^n} \Leftrightarrow$$

$$\Leftrightarrow mc_t = w_t - p_t - mpn_t^n = w_t - p_t - a_t - \ln(1-\alpha) + \alpha n_t = w_t - p_t - a_t - \ln(1-\alpha) + \alpha \frac{y_t - a_t}{1-\alpha} \Rightarrow$$

$$\Rightarrow mc_t = w_t - p_t - \frac{a_t - \alpha y_t}{1-\alpha} - \ln(1-\alpha) \quad (1.41)$$

Which is an expression for individual firm's marginal cost as a function of the economy's average real marginal cost.

The real marginal cost in period $t+k$ is:

$$mc_{t+k|t} = w_{t+k} - p_{t+k} - mpn_{t+k|t}^n = w_{t+k} - p_{t+k} - \frac{a_{t+k} - \alpha y_{t+k|t}}{1-\alpha} - \ln(1-\alpha) \quad (1.42)$$

$$mc_{t+k|t} - mc_t = w_{t+k} - p_{t+k} - \frac{a_{t+k} - \alpha y_{t+k|t}}{1-\alpha} - \ln(1-\alpha) - w_{t+k} - p_{t+k} - \frac{a_{t+k} - \alpha y_{t+k}}{1-\alpha} - \ln(1-\alpha) =$$

$$= \frac{\alpha(y_{t+k|t} - y_{t+k})}{1-\alpha}$$

$$(1.28) \Rightarrow y_{t+k|t} = \ln\left[\left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} C_{t+k}\right] = -\varepsilon\left(p_t^* - p_{t+k}\right) + c_{t+k} = -\varepsilon\left(p_t^* - p_{t+k}\right) + y_{t+k}$$

$$mc_{t+k|t} - mc_t = \frac{-\alpha\varepsilon\left(p_t^* - p_{t+k}\right)}{1-\alpha} \Leftrightarrow mc_{t+k|t} = mc_t - \frac{\alpha\varepsilon\left(p_t^* - p_{t+k}\right)}{1-\alpha} \quad (1.43)$$

$$p_t^* = (1-\theta\beta)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[mc_{t+k|t} - mc + p_{t+k}\right] = (1-\theta\beta)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[mc_{t+k} - \frac{\alpha\varepsilon\left(p_t^* - p_{t+k}\right)}{1-\alpha} - mc + p_{t+k}\right] =$$

$$= (1-\theta\beta)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[mc_{t+k} - \frac{\alpha\varepsilon\left(p_t^* - p_{t+k}\right)}{1-\alpha} + p_{t+k}\right] = -\frac{\alpha\varepsilon p_t^*}{1-\alpha} + (1-\theta\beta)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[mc_{t+k}\frac{(1-\alpha+\alpha\varepsilon)p_{t+k}}{1-\alpha}\right] \Leftrightarrow$$

$$p_t^* + \frac{\alpha\varepsilon p_t^*}{1-\alpha} = (1-\theta\beta)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[mc_{t+k} + \frac{(1-\alpha+\alpha\varepsilon)p_{t+k}}{1-\alpha}\right] \Leftrightarrow$$

$$\frac{(1-\alpha+\alpha\varepsilon)}{1-\alpha}p_t^* = (1-\theta\beta)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[mc_{t+k} + \frac{(1-\alpha+\alpha\varepsilon)p_{t+k}}{1-\alpha}\right] \Leftrightarrow$$

$$p_t^* = (1-\theta\beta)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[\frac{1-\alpha}{1-\alpha+\alpha\varepsilon}mc_{t+k} + p_{t+k}\right]$$

Subtracting from both sides p_{t-1} I have: $p_t^* - p_{t-1} = -p_{t-1} + (1-\theta\beta)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[\Theta mc_{t+k} + p_{t+k}\right] \Leftrightarrow$

$$p_t^* - p_{t-1} = (1-\theta\beta)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[\Theta mc_{t+k} + p_{t+k} - p_{t-1}\right] = (1-\theta\beta)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\Theta mc_{t+k} + (1-\theta\beta)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[p_{t+k} - p_{t-1}\right]$$

Firstly, I will analyze the second term of the right hand side:

$$(1-\theta\beta)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left[p_{t+k} - p_{t-1}\right] = (1-\theta\beta)E_t\left\{\theta^0\beta^0(p_t - p_{t-1}) + \theta^1\beta^1(p_{t+1} - p_{t-1}) + \theta^2\beta^2(p_{t+2} - p_{t-1}) + \ldots\right\} =$$

$$= (1-\theta\beta)E_t\left\{\theta^0\beta^0(p_t - p_{t-1}) + \theta^1\beta^1(p_{t+1} - p_t + p_t - p_{t-1}) + \theta^2\beta^2(p_{t+2} - p_{t+1} + p_{t+1} - p_t + p_t - p_{t-1}) + \ldots\right\} =$$

$$= E_t\left\{\theta^0\beta^0(p_t - p_{t-1}) + \theta^1\beta^1(p_{t+1} - p_t + p_t - p_{t-1}) + \theta^2\beta^2(p_{t+2} - p_{t+1} + p_{t+1} - p_t + p_t - p_{t-1}) + \ldots\right\} -$$

$$- \theta\beta E_t\left\{\theta^0\beta^0(p_t - p_{t-1}) + \theta^1\beta^1(p_{t+1} - p_t + p_t - p_{t-1}) + \theta^2\beta^2(p_{t+2} - p_{t+1} + p_{t+1} - p_t + p_t - p_{t-1}) + \ldots\right\} =$$

$$= E_t\left\{\theta^0\beta^0\pi_t + \theta^1\beta^1(\pi_{t+1} + \pi_t) + \theta^2\beta^2(\pi_{t+2} + \pi_{t+1} + \pi_t) + \ldots\right\} -$$

$$- E_t\left\{\theta^1\beta^1\pi_t + \theta^2\beta^2(\pi_{t+1} + \pi_t) + \theta^3\beta^3(\pi_{t+2} + \pi_{t+1} + \pi_t) + \ldots\right\} =$$

$$= E_t\left\{\theta^0\beta^0\pi_t + \theta^1\beta^1(\pi_{t+1} + \pi_t) - \theta^1\beta^1\pi_t + \theta^2\beta^2(\pi_{t+2} + \pi_{t+1} + \pi_t) - \theta^2\beta^2(\pi_{t+1} + \pi_t) + \ldots\right\} =$$

$$= E_t\left\{\theta^0\beta^0\pi_t + \theta^1\beta^1(\pi_{t+1} + \pi_t) - \theta^1\beta^1\pi_t + \theta^2\beta^2(\pi_{t+2} + \pi_{t+1} + \pi_t) - \theta^2\beta^2(\pi_{t+1} + \pi_t) + \ldots\right\} =$$

$$= E_t\left\{\theta^0\beta^0\pi_t + \theta^1\beta^1\pi_{t+1} + \theta^2\beta^2\pi_{t+2} + \ldots\right\} = E_t\sum_{k=0}^{\infty}\theta^k\beta^k\pi_{t+k}$$

So,

$$p_t^* - p_{t-1} = (1-\theta\beta) E_t \sum_{k=0}^{\infty} \theta^k \beta^k \Theta mc_{t+k} + E_t \sum_{k=0}^{\infty} \theta^k \beta^k \pi_{t+k} \Leftrightarrow$$

$$p_t^* - p_{t-1} = (1-\theta\beta) E_t \sum_{k=1}^{\infty} \theta^k \beta^k \Theta mc_{t+k} + (1-\theta\beta)\Theta mc_t + E_t \sum_{k=1}^{\infty} \theta^k \beta^k \pi_{t+k} + \pi_t \Leftrightarrow$$

$$p_t^* - p_{t-1} = \theta\beta(1-\theta\beta) E_t \sum_{k=0}^{\infty} \theta^k \beta^k \Theta mc_{t+k+1} + \theta\beta E_t \sum_{k=0}^{\infty} \theta^k \beta^k \pi_{t+k+1} + (1-\theta\beta)\Theta mc_t + \pi_t \Leftrightarrow$$

$$p_t^* - p_{t-1} = \theta\beta\left[E_t\left(p_{t+1}^* - p_t \right)\right] + \pi_t + (1-\theta\beta)\Theta mc_t \Leftrightarrow$$

$$\pi_t (1-\theta)^{-1} = \theta\beta\left[E_t \pi_{t+1} (1-\theta)^{-1}\right] + \pi_t + (1-\theta\beta)\Theta mc_t \Leftrightarrow$$

$$\pi_t = \theta\beta\left[E_t \pi_{t+1}\right] + + (1-\theta)\pi_t (1-\theta)(1-\theta\beta)\Theta mc_t \Leftrightarrow$$

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(1-\theta)(1-\theta\beta)\Theta}{\theta} mc_t \Leftrightarrow \qquad \boxed{\pi_t = \beta E_t \pi_{t+1} + \lambda mc_t} \quad (1.44)$$

Next, a relation is derived from the economy's real marginal cost and a measure of economic activity. I have shown before that $mpn_t^n = a_t + \ln(1-\alpha) - \alpha n_t$. If I insert this into $mc_t = w_t - p_t - mpn_t$ and make use of the production function (in logs) and the Labour-leisure choice (1.32) I get:

$$mc_t = \sigma c_t + \varphi n_t - a_t - \ln(1-\alpha) + \alpha n_t = \sigma c_t + (\varphi+\alpha) n_t - a_t - \ln(1-\alpha) \Leftrightarrow$$

$$\Leftrightarrow mc_t = \sigma y_t + (\varphi+\alpha)\frac{y_t - a_t}{1-\alpha} - a_t - \ln(1-\alpha) = \frac{(1-\alpha)\sigma + \varphi + \alpha}{1-\alpha} y_t - \frac{1+\varphi}{1-\alpha} a_t - \ln(1-\alpha) \quad (1.45)$$

In the case of flexible pricing I know from before that $mc \equiv -\mu$. Define natural output y_t^n as the equilibrium level under full price flexibility. Accordingly the natural marginal cost is defined when the output is at its natural level, so:

$$mc \equiv -\mu = \frac{(1-\alpha)\sigma + \varphi + \alpha}{1-\alpha} y_t^n - \frac{1+\varphi}{1-\alpha} a_t - \ln(1-\alpha) \quad (1.46)$$

I can solve for y_t^n and:

$$\frac{(1-\alpha)\sigma + \varphi + \alpha}{1-\alpha} y_t^n = -\mu + \frac{1+\varphi}{1-\alpha} a_t + \ln(1-\alpha) \Leftrightarrow y_t^n = \frac{1+\varphi}{(1-\alpha)\sigma + \varphi + \alpha} a_t - \frac{1-\alpha}{(1-\alpha)\sigma + \varphi + \alpha}\mu + \frac{1-\alpha}{(1-\alpha)\sigma + \varphi + \alpha}\ln(1-\alpha)$$

$$\Leftrightarrow y_t^n = \psi_{ya}^n a_t + \theta_y^n \quad (1.47)$$

If I subtract (66) from (65) I get a measure of the real marginal cost gap

$$mc_t = mc_t - mc = \frac{(1-\alpha)\sigma + \varphi + \alpha}{1-\alpha} y_t - \frac{1+\varphi}{1-\alpha} a_t - \ln(1-\alpha) - \frac{(1-\alpha)\sigma + \varphi + \alpha}{1-\alpha} y_t^n + \frac{1+\varphi}{1-\alpha} a_t + \ln(1-\alpha) \Leftrightarrow$$

$$mc_t = \frac{(1-\alpha)\sigma + \varphi + \alpha}{1-\alpha}\left(y_t - y_t^n \right) = \frac{(1-\alpha)\sigma + \varphi + \alpha}{1-\alpha} \tilde{y}_t \quad (1.48)$$

Where $\tilde{y}_t$ denotes the **Output Gap.**

Hence (1.44) becomes:

$$\pi_t = \beta E_t\pi_{t+1} + \lambda mc_t = \beta E_t\pi_{t+1} + \lambda\frac{(1-\alpha)\sigma+\varphi+\alpha}{1-\alpha}\tilde{y}_t \Leftrightarrow \boxed{\pi_t = \beta E_t\pi_{t+1} + \kappa\tilde{y}_t} \quad (1.49)$$

Where $\kappa = \lambda\frac{(1-\alpha)\sigma+\varphi+\alpha}{1-\alpha}$

This is the **New Keynesian Phillips Curve** , shows that inflation depends only on expected future inflation and current output gap, with no role for past inflation or more complicated dynamics.

1.6 Derivation of the Dynamic IS equation

if I use the definition of the real interest rate $r_t \equiv i_t - E_t\pi_{t+1}$ equation (1.38) becomes

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}(r_t - \rho) \text{ and } y_t^n = E_t y_{t+1}^n - \frac{1}{\sigma}(r_t^n - \rho) \quad (1.50)$$

subtracting (1.50) from (1.38) gives the output gap from the natural output, i.e. the **dynamic IS equation**:

$$\tilde{y}_t \equiv y_t - y_t^n = E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t\pi_{t+1} - \rho) - E_t y_{t+1}^n + \frac{1}{\sigma}(r_t^n - \rho) = E_t\tilde{y}_{t+1} - \frac{1}{\sigma}(i_t - r_t^n - E_t\pi_{t+1}) \Leftrightarrow$$

$$\boxed{\tilde{y}_t = E_t\tilde{y}_{t+1} - \frac{1}{\sigma}(i_t - r_t^n - E_t\pi_{t+1})} \quad (1.51)$$

The DIS is the aggregate-demand equation of the model.

The only thing that remains to be defined is a process for the r_t^n.

If I take first difference of (1.47) then : $E_t(y_{t+1}^n - y_t^n) = \psi_{ya}^n E_t(a_{t+1} - a_t)$

And doing the same on (1.38): $E_t(y_{t+1} - y_t) = \frac{1}{\sigma}(i_t - E_t\pi_{t+1} - \rho)$ (1.52)

$$r_t^n = i_t - \sigma E_t(\tilde{y}_{t+1} - \tilde{y}_t) - E_t\pi_{t+1} = i_t - E_t\pi_{t+1} - \sigma E_t(y_{t+1} - y_{t+1}^n - y_t + y_t^n) =$$
$$= i_t - E_t\pi_{t+1} - \sigma E_t\left[(y_{t+1} - y_t) - (y_{t+1}^n - y_t^n)\right] = i_t - E_t\pi_{t+1} - (i_t - E_t\pi_{t+1} - \rho) + \psi_{ya}^n E_t(a_{t+1} - a_t) \Leftrightarrow$$
$$\boxed{r_t^n = \rho + \psi_{ya}^n E_t(a_{t+1} - a_t)} \quad (1.53)$$

Thus, the natural real rate is a function of households' discount rate and expected technological process (72).

I can be rewritten as: $\boxed{\widehat{r}_t^n = r_t^n - \rho = \sigma\psi_{ya}^n E_t(a_{t+1} - a_t)}$ (1.54)

If there are no technological shocks in the economy , the real rate becomes the discount rate.

In order to close the model I assume that the central bank follows a rule, similar to Taylor's rule:

$$\boxed{i_t = \rho + \varphi_\pi \pi_t + \varphi_y \tilde{y}_t + \upsilon_t} \quad (1.55)$$

Standard reasoning implies that φ_π and φ_y are non-negative.

Assume also that the exogenous processes are:

$$\rho_\upsilon \in [0,1): \upsilon_t = \rho_\upsilon \upsilon_{t-1} + \varepsilon_t^\upsilon \text{ where } \varepsilon_t^\upsilon \sim WN(s^2) \quad (1.56)$$

$$\rho_a \in [0,1): a_t = \rho_a a_{t-1} + \varepsilon_t^a \text{ where } \varepsilon_t^a \sim WN(s^2) \quad (1.57)$$

Hence, we end up solving for a decentralized equilibrium:

Given the paths of one policy instrument $\{i_t\}_{t=0}^\infty$, the initial conditions m_0 and b_0 and the exogenous processes $\{\upsilon_t, a_t\}_{t=0}^\infty$, a decentralized equilibrium is defined to be an allocation $\{\tilde{y}_t, \pi_t, m_t\}_{t=0}^\infty$ such that:

1. Households undertake their optimization problem outlined above
2. All budget constraints are satisfied
3. All markets clear.

The decentralized general equilibrium consists of 3 equations (1.49), (1.51), (1.33') given the policy instrument i_t which follows residually and given the exogenous processes. It can be observed that the money market clear condition (1.33') also follows residually so we can ignore it for a while and we can continue the analysis by using only (1.49), (1.51) and (1.55).

I first derive a forward looking version of the dynamic IS equation. Insert (1.55) and then (1.49) into (1.51):

$$\tilde{y}_t = E_t\tilde{y}_{t+1} - \frac{1}{\sigma}\left(i_t - r_t^n - E_t\pi_{t+1}\right) = E_t\tilde{y}_{t+1} - \frac{1}{\sigma}\left(\rho + \varphi_\pi\pi_t + \varphi_y\tilde{y}_t + \upsilon_t - r_t^n - E_t\pi_{t+1}\right) =$$

$$= E_t\tilde{y}_{t+1} - \frac{1}{\sigma}\left(\varphi_\pi\left(\beta E_t\pi_{t+1} + \kappa\tilde{y}_t\right) + \varphi_y\tilde{y}_t + \upsilon_t - \hat{r}_t^n - E_t\pi_{t+1}\right) =$$

$$= E_t\tilde{y}_{t+1} + \frac{1-\beta\varphi_\pi}{\sigma}E_t\pi_{t+1} - \frac{\varphi_\pi\kappa + \varphi_y}{\sigma}\tilde{y}_t + \frac{\hat{r}_t^n - \upsilon_t}{\sigma} \Leftrightarrow$$

$$\Leftrightarrow \tilde{y}_t + \frac{\varphi_\pi\kappa + \varphi_y}{\sigma}\tilde{y}_t = E_t\tilde{y}_{t+1} + \frac{1-\beta\varphi_\pi}{\sigma}E_t\pi_{t+1} + \frac{\hat{r}_t^n - \upsilon_t}{\sigma} \Leftrightarrow$$

$$\Leftrightarrow \frac{\sigma + \varphi_\pi\kappa + \varphi_y}{\sigma}\tilde{y}_t = E_t\tilde{y}_{t+1} + \frac{1-\beta\varphi_\pi}{\sigma}E_t\pi_{t+1} + \frac{\hat{r}_t^n - \upsilon_t}{\sigma} \Leftrightarrow$$

$$\Leftrightarrow \boxed{\tilde{y}_t = \frac{\sigma}{\sigma+\varphi_\pi\kappa+\varphi_y}E_t\tilde{y}_{t+1} + \frac{1-\beta\varphi_\pi}{\sigma+\varphi_\pi\kappa+\varphi_y}E_t\pi_{t+1} + \frac{\hat{r}_t^n-\upsilon_t}{\sigma+\varphi_\pi\kappa+\varphi_y}} \quad (1.58)$$

The equation (1.56) shows the current output gap as a function of expected output gap, expected inflation and shocks.

(1.49) by (1.58) becomes:

$$\pi_t = \beta E_t\pi_{t+1} + \kappa\tilde{y}_t = \beta E_t\pi_{t+1} + \kappa\left[\frac{\sigma}{\sigma+\varphi_\pi\kappa+\varphi_y}E_t\tilde{y}_{t+1} + \frac{1-\beta\varphi_\pi}{\sigma+\varphi_\pi\kappa+\varphi_y}E_t\pi_{t+1} + \frac{\hat{r}_t^n-\upsilon_t}{\sigma+\varphi_\pi\kappa+\varphi_y}\right] =$$

$$= \frac{\sigma\kappa}{\sigma+\varphi_\pi\kappa+\varphi_y}E_t\tilde{y}_{t+1} + \frac{\kappa\left(1-\beta\varphi_\pi\right)+\beta\left(\sigma+\varphi_\pi\kappa+\varphi_y\right)}{\sigma+\varphi_\pi\kappa+\varphi_y}E_t\pi_{t+1} + \frac{\kappa\left(\hat{r}_t^n-\upsilon_t\right)}{\sigma+\varphi_\pi\kappa+\varphi_y} \Leftrightarrow$$

$$\Leftrightarrow \boxed{\pi_t = \frac{\sigma\kappa}{\sigma+\varphi_\pi\kappa+\varphi_y}E_t\tilde{y}_{t+1} + \frac{\kappa+\beta\left(\sigma+\varphi_y\right)}{\sigma+\varphi_\pi\kappa+\varphi_y}E_t\pi_{t+1} + \frac{\kappa\left(\hat{r}_t^n-\upsilon_t\right)}{\sigma+\varphi_\pi\kappa+\varphi_y}} \quad (1.59)$$

The two equations (1.56) and (1.57) can be written as a system of forward looking difference equations:

$$\begin{bmatrix}\tilde{y}_t\\ \pi_t\end{bmatrix} = \frac{1}{\sigma+\varphi_\pi\kappa+\varphi_y}\begin{pmatrix}\sigma & 1-\beta\varphi_\pi\\ \sigma\kappa & \kappa+\beta\left(\sigma+\varphi_y\right)\end{pmatrix}\begin{pmatrix}E_t\tilde{y}_{t+1}\\ E_t\pi_{t+1}\end{pmatrix} + \frac{1}{\sigma+\varphi_\pi\kappa+\varphi_y}\begin{pmatrix}1\\ \kappa\end{pmatrix}\left(\hat{r}_t^n-\upsilon_t\right) \Leftrightarrow$$

Let $\frac{1}{\sigma+\varphi_\pi\kappa+\varphi_y} \equiv \Omega$

$$\Leftrightarrow \begin{bmatrix}\tilde{y}_t\\ \pi_t\end{bmatrix} = \underbrace{\Omega\begin{pmatrix}\sigma & 1-\beta\varphi_\pi\\ \sigma\kappa & \kappa+\beta\left(\sigma+\varphi_y\right)\end{pmatrix}}_{A_T}\begin{pmatrix}E_t\tilde{y}_{t+1}\\ E_t\pi_{t+1}\end{pmatrix} + \underbrace{\Omega\begin{pmatrix}1\\ \kappa\end{pmatrix}}_{B_T}\left(\hat{r}_t^n-\upsilon_t\right) \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix}\tilde{y}_t\\ \pi_t\end{bmatrix} = A_T\begin{pmatrix}E_t\tilde{y}_{t+1}\\ E_t\pi_{t+1}\end{pmatrix} + B_T\left(\hat{r}_t^n-\upsilon_t\right) \quad (1.60)$$

The system given in (1.58) is a reduced form representation of the dynamic IS curve and the New Keynesian Phillips Curve, which takes into account effects from the policy defined in equation (1.55).

1.7 Stability Conditions

Here I have two control variables , $\tilde{y}_t$ and π_t and no predetermined variable. Following Blanchard Kahn (1980) the system has a locally unique equilibrium if and only if both eigenvalues of the 2×2 matrix A_T $\left(A_T^{-1}\right)$ are inside(outside) the unit circle or in other words the system is a “sink” (“source”)

I am to examine what are the sufficient and necessary conditions for this property to hold. The two eigenvalues denoted as λ_1 and λ_2, are generally solutions to the following system written in matrix form, where $\underset{2\times2}{\mathrm{I}}$ is the identity matrix.

$$\left|\underset{2\times2}{\mathrm{A_T}}-\lambda\ \underset{2\times2}{\mathrm{I}}\right|=\underset{2\times2}{\mathrm{O}}\ \text{ or }\quad \left|\Omega\begin{pmatrix}\sigma & 1-\beta\varphi_\pi\\ \sigma\kappa & \kappa+\beta\left(\sigma+\varphi_y\right)\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right|=\underset{2\times2}{\mathrm{O}}$$

Characteristic equation of the system is:

$$\psi(\lambda)=\lambda^2-tr(\mathrm{A_T})\lambda+\det(\mathrm{A_T})=\lambda^2-\frac{\sigma+\kappa+\beta\left(\sigma+\varphi_y\right)}{\sigma+\varphi_\pi\kappa+\varphi_y}+\frac{\sigma\beta}{\sigma+\varphi_\pi\kappa+\varphi_y}=0$$

There are 3 conditions that have to hold simultaneously

$i)\Psi(1)>0$
$ii)\Psi(-1)>0$
$iii)\lambda_1\lambda_2<1$

Or in terms of determinant and trace:

$i)1-tr(\mathrm{A_T})+\det(\mathrm{A_T})>0$
$ii)1+tr(\mathrm{A_T})+\det(\mathrm{A_T})>0$
$iii)\det(\mathrm{A_T})<1$

$$i)\Rightarrow 1-\frac{\sigma+\kappa+\beta\left(\sigma+\varphi_y\right)}{\sigma+\varphi_\pi\kappa+\varphi_y}+\frac{\sigma\beta}{\sigma+\varphi_\pi\kappa+\varphi_y}>0\Leftrightarrow\frac{\sigma+\varphi_\pi\kappa+\varphi_y-\sigma-\kappa-\sigma\beta-\beta\varphi_y+\sigma\beta}{\sigma+\varphi_\pi\kappa+\varphi_y}>0\Leftrightarrow$$

$$\Leftrightarrow\frac{\varphi_\pi\kappa+\varphi_y-\kappa-\beta\varphi_y}{\sigma+\varphi_\pi\kappa+\varphi_y}>0$$ since $\sigma>0,\varphi_\pi>0,0<\kappa<1,\varphi_y>0$ i.e the denominator is positive ,so for this condition to hold must: $\left(\varphi_\pi-1\right)\kappa+\left(1-\beta\right)\varphi_y>0$

$$ii)\Rightarrow 1+\frac{\sigma+\kappa+\beta\left(\sigma+\varphi_y\right)}{\sigma+\varphi_\pi\kappa+\varphi_y}+\frac{\sigma\beta}{\sigma+\varphi_\pi\kappa+\varphi_y}>0\Leftrightarrow\frac{\sigma+\varphi_\pi\kappa+\varphi_y+\sigma+\kappa+\sigma\beta+\beta\varphi_y+\sigma\beta}{\sigma+\varphi_\pi\kappa+\varphi_y}>0$$

Since every single element of this fraction is positive then the inequality and consequently the condition holds.

$$iii)\Rightarrow\frac{\sigma\beta}{\sigma+\varphi_\pi\kappa+\varphi_y}<1\Leftrightarrow\sigma\beta<\sigma+\varphi_\pi\kappa+\varphi_y\Leftrightarrow\sigma(\beta-1)<\varphi_\pi\kappa+\varphi_y$$

This inequality holds since $\beta<1,\sigma>0\Rightarrow\sigma(\beta-1)<0$ and $0<\varphi_\pi\kappa+\varphi_y$

Concluding, the uniqueness holds as long as condition $i)$ holds or as long as the policy parameter have sufficiently high values i.e. as long as monetary authorities respond to deviations of inflation and output with

adequate strength. Considering our assumptions about the parameters $\sigma>0,\varphi_\pi>0,0<\kappa<1,\varphi_y>0,0<\beta<1$ imply that $\varphi_\pi>1$ is sufficient for $i)$ to hold.

1.8 Effects of a Monetary Policy Shock

Assume that the exogenous component of (1.55) follows an AR(1) process , where
$\rho_\upsilon\in[0,1):\upsilon_t=\rho_\upsilon\upsilon_{t-1}+\varepsilon_t^\upsilon$ (UC.1) where $\varepsilon_t^\upsilon\sim WN(s^2)$

A positive (negative) realization of ε_t^υ is interpreted as a contractionary (expansionary) monetary policy shock, leading to a rise (decline) in the nominal interest rate for given levels of inflation and output gap. Simultaneously for analytical reasons I assume that the technology shock is annihilated i.e $a_{t+1}-a_t=0$ and consequently $r_t^n=\rho$

Analytical solution using the method of undetermined coefficients:

Guess: $\tilde{y}_t=\psi_{y\upsilon}\upsilon_t$ (UC.2) and $\pi_t=\psi_{\pi\upsilon}\upsilon_t$ (UC.3)

I substitute these into NKPC (1.49):

$$\pi_t=\beta E_t\pi_{t+1}+\kappa\tilde{y}_t\Rightarrow$$

$$\Rightarrow\psi_{\pi\upsilon}\upsilon_t=\beta E_t\psi_{\pi\upsilon}\upsilon_{t+1}+\kappa\psi_{y\upsilon}\upsilon_t=\beta E_t\psi_{\pi\upsilon}\left(\rho_\upsilon\upsilon_t+\varepsilon_{t+1}^\upsilon\right)+\kappa\psi_{y\upsilon}\upsilon_t\Rightarrow$$

$$\Rightarrow\psi_{\pi\upsilon}\upsilon_t=\beta\psi_{\pi\upsilon}\rho_\upsilon\upsilon_t+\kappa\psi_{y\upsilon}\upsilon_t\Rightarrow(1-\beta\rho_\upsilon)\psi_{\pi\upsilon}=\kappa\psi_{y\upsilon}\Rightarrow\boxed{\psi_{y\upsilon}=\frac{1-\beta\rho_\upsilon}{\kappa}\psi_{\pi\upsilon}}\ \text{(UC.4)}$$

Then insert (1.55) into the Dynamic IS (1.51)

$$\tilde{y}_t=E_t\tilde{y}_{t+1}-\frac{1}{\sigma}\left(i_t-r_t^n-E_t\pi_{t+1}\right)=E_t\tilde{y}_{t+1}-\frac{1}{\sigma}\left(\rho+\varphi_\pi\pi_t+\varphi_y\tilde{y}_t+\upsilon_t-r_t^n-E_t\pi_{t+1}\right)=E_t\tilde{y}_{t+1}-\frac{1}{\sigma}\left(\varphi_\pi\pi_t+\varphi_y\tilde{y}_t+\upsilon_t-E_t\pi_{t+1}\right)$$

and again I substitute the Guess into the last equation:

$$\Rightarrow\psi_{y\upsilon}\upsilon_t=E_t\psi_{y\upsilon}\upsilon_{t+1}-\frac{1}{\sigma}\left(\varphi_\pi\psi_{\pi\upsilon}\upsilon_t+\varphi_y\psi_{y\upsilon}\upsilon_{t+1}+\upsilon_t-E_t\psi_{\pi\upsilon}\upsilon_{t+1}\right)\Rightarrow$$

$$\psi_{y\upsilon}\upsilon_t=E_t\psi_{y\upsilon}(\rho_\upsilon\upsilon_t+\varepsilon_{t+1}^\upsilon)-\frac{1}{\sigma}\left(\varphi_\pi\psi_{\pi\upsilon}\upsilon_t+\varphi_y\psi_{y\upsilon}(\rho_\upsilon\upsilon_t+\varepsilon_{t+1}^\upsilon)+\upsilon_t-E_t\psi_{\pi\upsilon}(\rho_\upsilon\upsilon_t+\varepsilon_{t+1}^\upsilon)\right)\Rightarrow$$

$$\sigma\psi_{y\upsilon}\upsilon_t=\sigma E_t\psi_{y\upsilon}\rho_\upsilon\upsilon_t-\varphi_\pi\psi_{\pi\upsilon}\upsilon_t+\varphi_y\psi_{y\upsilon}\rho_\upsilon\upsilon_t+\upsilon_t-E_t\psi_{\pi\upsilon}\rho_\upsilon\upsilon_t\Rightarrow$$

$$\sigma\psi_{y\upsilon}\upsilon_t=\sigma E_t\psi_{y\upsilon}\rho_\upsilon\upsilon_t-\varphi_\pi\psi_{\pi\upsilon}\upsilon_t+\varphi_y\psi_{y\upsilon}\rho_\upsilon\upsilon_t+\upsilon_t-E_t\psi_{\pi\upsilon}\rho_\upsilon\upsilon_t\Rightarrow$$

$$\left[\sigma(1-\rho_\upsilon)+\varphi_y\right]\psi_{y\upsilon}+(\varphi_\pi-\rho_\upsilon)\psi_{\pi\upsilon}=-1$$

And I insert (UC.4) $\Rightarrow\left[\sigma(1-\rho_\upsilon)+\varphi_y\right]\frac{1-\beta\rho_\upsilon}{\kappa}\psi_{\pi\upsilon}+(\varphi_\pi-\rho_\upsilon)\psi_{\pi\upsilon}=-1\Rightarrow$

$$\frac{\left[\sigma(1-\rho_\upsilon)+\varphi_y\right](1-\beta\rho_\upsilon)+\kappa(\varphi_\pi-\rho_\upsilon)}{\kappa}\psi_{\pi\upsilon}=-1\Rightarrow$$

$$\boxed{\psi_{\pi\upsilon} = -\frac{\kappa}{\left[\sigma\left(1-\rho_\upsilon\right)+\varphi_y\right]\left(1-\beta\rho_\upsilon\right)+\kappa\left(\varphi_\pi-\rho_\upsilon\right)} = -\kappa\Lambda_\upsilon} \text{ (UC.5)}$$

Where $\Lambda_\upsilon \equiv \frac{1}{\left[\sigma\left(1-\rho_\upsilon\right)+\varphi_y\right]\left(1-\beta\rho_\upsilon\right)+\kappa\left(\varphi_\pi-\rho_\upsilon\right)}$

And after inserting (UC.5) into (UC.4) I get: $\psi_{y\upsilon} = \frac{1-\beta\rho_\upsilon}{\kappa}\left(-\kappa\Lambda_\upsilon\right) = -\left(1-\beta\rho_\upsilon\right)\Lambda_\upsilon$ (UC.6)

Thus, the solutions to (UC.2) and (UC.3) are of linear form with respect to monetary shock and specifically:

$$\boxed{\tilde{y}_t = -\left(1-\beta\rho_\upsilon\right)\Lambda_\upsilon\upsilon_t} \text{ (UC.7) and } \boxed{\pi_t = -\kappa\Lambda_\upsilon\upsilon_t} \text{ (UC.8)}$$

Or by substituting the AR(1) process we get:

$$\tilde{y}_t = -\left(1-\beta\rho_\upsilon\right)\Lambda_\upsilon\left(\rho_\upsilon\upsilon_{t-1}+\varepsilon_t^\upsilon\right) = \underbrace{-\left(1-\beta\rho_\upsilon\right)\Lambda_\upsilon\rho_\upsilon\upsilon_{t-1}}_{\rho_\upsilon\tilde{y}_{t-1}} - \left(1-\beta\rho_\upsilon\right)\Lambda_\upsilon\varepsilon_t^\upsilon \Leftrightarrow \boxed{\tilde{y}_t = \rho_\upsilon\tilde{y}_{t-1} - \left(1-\beta\rho_\upsilon\right)\Lambda_\upsilon\varepsilon_t^\upsilon} \text{ (UC.7')}$$

$$\pi_t = -\kappa\Lambda_\upsilon\left(\rho_\upsilon\upsilon_{t-1}+\varepsilon_t^\upsilon\right) = \underbrace{-\kappa\Lambda_\upsilon\rho_\upsilon\upsilon_{t-1}}_{\rho_\upsilon\pi_{t-1}} - \kappa\Lambda_\upsilon\varepsilon_t^\upsilon \Leftrightarrow \boxed{\pi_t = \rho_\upsilon\pi_{t-1} - \kappa\Lambda_\upsilon\varepsilon_t^\upsilon} \text{ (UC.8')}$$

But what is the sign of the factors $-\left(1-\beta\rho_\upsilon\right)\Lambda_\upsilon$ and $-\kappa\Lambda_\upsilon$?

$\left(1-\beta\rho_\upsilon\right) > 0$ because: $0 < \beta < 1$ and by assumption $0 < \rho_\upsilon < 1$.

$\sigma\left(1-\rho_\upsilon\right)+\varphi_y > 0$ because: $\sigma > 0$ by economic theory, $\varphi_y > 0$ by stability condition and $0 < \rho_\upsilon < 1$

$\kappa\left(\varphi_\pi-\rho_\upsilon\right) > 0$ because: $\varphi_\pi > 1$ by stability condition and $0 < \rho_\upsilon < 1$

Also $0 < \kappa < 1$.

So finally: $-\left(1-\beta\rho_\upsilon\right)\Lambda_\upsilon < 0$ and $-\kappa\Lambda_\upsilon < 0$

Hence, an exogenous increase in the interest rate leads to a persistent decline in both output gap and inflation. Because the natural level of output is unaffected by the monetary policy, the response of output matches that of the output gap.

Also:

$$\hat{r}_t \equiv r_t - r_t^n = i_t - E_t\pi_{t+1} - \rho$$

And making use of (71)

$$\Rightarrow \hat{r}_t = \sigma E_t(\tilde{y}_{t+1} - \tilde{y}_t) = -\sigma\left(1-\beta\rho_\upsilon\right)\Lambda_\upsilon\rho_\upsilon\upsilon_t + \sigma\left(1-\beta\rho_\upsilon\right)\Lambda_\upsilon\upsilon_t = \sigma\left(1-\rho_\upsilon\right)\left(1-\beta\rho_\upsilon\right)\Lambda_\upsilon\upsilon_t \text{ (UC.9)}$$

Using the same reasoning as above the factor $\sigma(1-\rho_\upsilon)(1-\beta\rho_\upsilon)\Lambda_\upsilon$ is positive and a positive monetary policy shock increases real rate.

$$\hat{i}_t = \hat{r}_t + E_t\pi_{t+1} = \hat{r}_t - \kappa\Lambda_\upsilon\rho_\upsilon\upsilon_t = -\kappa\Lambda_\upsilon\rho_\upsilon\upsilon_t + \sigma(1-\rho_\upsilon)(1-\beta\rho_\upsilon)\Lambda_\upsilon\upsilon_t = \left[\sigma(1-\rho_\upsilon)(1-\beta\rho_\upsilon) - \kappa\rho_\upsilon\right]\Lambda_\upsilon\upsilon_t \quad \text{(UC.10)}$$

The sign of the factor $\left[\sigma(1-\rho_\upsilon)(1-\beta\rho_\upsilon) - \kappa\rho_\upsilon\right]\Lambda_\upsilon$ is ambiguous. As shown above, if the stability conditions hold then Λ_υ is positive. But if the persistence of the monetary policy shock is sufficiently high then the nominal interest rate will decline rather than increase in a response to a rise in υ_t. In that case, and despite the lower nominal rate, the policy shock still has a contractionary effect on output, because the latter is inversely related to the real rate, which goes up unambiguously.

From the money demand function (1.33) and market clearing condition results to: $m_t = p_t + y_t - \eta i_t$ which is the equilibrium in the money market.

Differentiating (1.33) with respect to the monetary shock yields:

$$\frac{dm_t}{d\varepsilon_t^\upsilon} = \frac{dp_t}{d\varepsilon_t^\upsilon} + \frac{dy_t}{d\varepsilon_t^\upsilon} - \eta\frac{di_t}{d\varepsilon_t^\upsilon} =$$

$$= -\kappa\Lambda_\upsilon - (1-\beta\rho_\upsilon)\Lambda_\upsilon - \eta\left[\sigma(1-\rho_\upsilon)(1-\beta\rho_\upsilon) - \kappa\rho_\upsilon\right]\Lambda_\upsilon = \left[-\kappa - (1-\beta\rho_\upsilon) - \eta\left[\sigma(1-\rho_\upsilon)(1-\beta\rho_\upsilon) - \kappa\rho_\upsilon\right]\right]\Lambda_\upsilon \quad \text{(UC.11)}$$

Hence, the sign of the change in the money supply that supports the exogenous policy intervention is, in principle, ambiguous. Even though the money supply needs to be tightened to raise the nominal rate given output and prices, the decline in the latter induces by the policy shocks combined with the possibility of an induced nominal rate decline make it impossible to rule out a countercyclical movement in money in response to an interest rate shock. However, for $\frac{di_t}{d\varepsilon_t^u} > 0$,which seems to be the most relevant case, the money supply falls in response to a monetary policy shock (liquidity effect).The (UC.11) does not describe the equilibrium dynamics of the money supply, but the impact reaction in the first period! For the presentation of the money growth in MATLAB I use the first differences of prices, output gap and nominal interest rate.

In conclusion, in order to understand the model and to correctly interpret the reaction to one of the shocks, it is important to observe the allocation distortions in the new Keynesian baseline model: Different to the RBC model, the allocation distortions in the New Keynesian model imply that resource allocation is inefficient. In equilibrium, the market power of the companies leads to suboptimal high prices and suboptimal low production. After exogenous shocks, the existence of staggered price adjustment prevents that the steady state will be reached immediately. This implies that also the allocation distortion is not constant but fluctuating.

Numerical solution using Klein's method:

The economy is described by 3 equations:

$$\left.\begin{aligned}\tilde{y}_t &= E_t\tilde{y}_{t+1} - \frac{1}{\sigma}\left(i_t - r_t^n - E_t\pi_{t+1}\right)\\ \pi_t &= \beta E_t\pi_{t+1} + \kappa\tilde{y}_t\\ i_t &= \rho + \varphi_\pi\pi_t + \varphi_y\tilde{y}_t + \upsilon_t\end{aligned}\right\}\Leftrightarrow$$

$$\Leftrightarrow\left.\begin{aligned}\tilde{y}_t &= E_t\tilde{y}_{t+1} - \frac{1}{\sigma}\left(\rho + \varphi_\pi\pi_t + \varphi_y\tilde{y}_t + \upsilon_t - r_t^n - E_t\pi_{t+1}\right)\\ \pi_t &= \beta E_t\pi_{t+1} + \kappa\tilde{y}_t\end{aligned}\right\}\Leftrightarrow$$

$$\Leftrightarrow\left.\begin{aligned}\sigma\tilde{y}_t &= \sigma E_t\tilde{y}_{t+1} - \rho - \varphi_\pi\pi_t - \varphi_y\tilde{y}_t - \upsilon_t + \rho + \psi_{ya}^n E_t(a_{t+1} - a_t) + E_t\pi_{t+1}\\ \pi_t &= \beta E_t\pi_{t+1} + \kappa\tilde{y}_t\end{aligned}\right\}\Leftrightarrow$$

$$\Leftrightarrow\left.\begin{aligned}\sigma E_t\tilde{y}_{t+1} + E_t\pi_{t+1} &= (\sigma + \varphi_y)\tilde{y}_t + \varphi_\pi\pi_t + \upsilon_t - \psi_{ya}^n E_t(a_{t+1} - a_t)\\ \beta E_t\pi_{t+1} &= -\kappa\tilde{y}_t + \pi_t\end{aligned}\right\}$$

Assume: $\rho_\upsilon \in [0,1): \upsilon_t = \rho_\upsilon\upsilon_{t-1} + \varepsilon_t^\upsilon$ (K.1) where $\varepsilon_t^\upsilon \sim WN(s^2)$ and $a_{t+1} - a_t = 0$

I will make use of Klein's Method, so I will write the system as follows:

$$\left.\begin{aligned}\upsilon_t &= \rho_\upsilon\upsilon_{t-1} + \varepsilon_t^\upsilon\\ \sigma E_t\tilde{y}_{t+1} + E_t\pi_{t+1} &= (\sigma + \varphi_y)\tilde{y}_t + \varphi_\pi\pi_t + \upsilon_t - \psi_{ya}^n E_t(a_{t+1} - a_t)\\ \beta E_t\pi_{t+1} &= -\kappa\tilde{y}_t + \pi_t\end{aligned}\right\}$$

$$\Updownarrow$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & \sigma & 1\\ 0 & 0 & \beta\end{bmatrix}E_t\begin{bmatrix}\upsilon_t\\ \tilde{y}_{t+1}\\ \pi_{t+1}\end{bmatrix} = \begin{bmatrix}\rho_\upsilon & 0 & 0\\ 1 & (\sigma + \varphi_y) & \varphi_\pi\\ 0 & -\kappa & 1\end{bmatrix}\begin{bmatrix}\upsilon_{t-1}\\ \tilde{y}_t\\ \pi_t\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\varepsilon_t^\upsilon$$

$$\Updownarrow$$

$$AE_tS_{t+1}^o = BS_t^o + C\varepsilon_t^\upsilon \text{ (K.2)}$$

Where: $S_t^o = \begin{bmatrix}S_{1t}^o\\ S_{2t}^o\end{bmatrix}$ and so $E_tS_{t+1}^o = \begin{bmatrix}S_{1t+1}^o\\ E_tS_{2t+1}^o\end{bmatrix}$

Let $S_t^o = \begin{bmatrix}\upsilon_{t-1} & \tilde{y}_t & \pi_t\end{bmatrix}'$ and $A \equiv \begin{bmatrix}1 & 0 & 0\\ 0 & \sigma & 1\\ 0 & 0 & \beta\end{bmatrix}, B \equiv \begin{bmatrix}\rho_\upsilon & 0 & 0\\ 1 & (\sigma + \varphi_y) & \varphi_\pi\\ 0 & -\kappa & 1\end{bmatrix}, C \equiv \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$

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Schur Decomposition: $QAZ = S$ and $QBZ = T$ (K.3)

Q and Z are unitary and S and T are upper triangular matrices. The generalized eigenvalues of A and B can be recovered as the ratios of the diagonal elements of T and S :

$$\lambda(B, A) = \left\{ {}^{T_{ii}}\!/_{S_{ii}} \,\middle|\, i = 1, 2, 3 \right\}$$

The matrices Q, Z, S and T can always be arranged so that the generalized eigenvalues are ordered in increasing value in moving from left to right. In the vector S_t^o one variable is predetermined and two are control variables.

- Proposition 1:if the number of eigenvalues outside the unit circle is equal to the number of control variables, then there is a unique solution to the system
- Proposition 2:if the number of eigenvalues outside the unit circle is greater than the number of control variables, then there is no solution to the system
- Proposition 3:if the number of eigenvalues outside the unit circle is less than the number of control variables, than there is an infinite number of solutions.

The matrices Q, Z, S and T are partitioned, so that:

$$Q = \begin{bmatrix} Q_1 \\ 1\times 3 \\ Q_2 \\ 2\times 3 \end{bmatrix}, \quad Z = \begin{bmatrix} Z_{11} & Z_{12} \\ 1\times 1 & 1\times 2 \\ Z_{21} & Z_{22} \\ 2\times 1 & 2\times 2 \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} & S_{12} \\ 1\times 1 & 1\times 2 \\ \mathrm{O} & S_{22} \\ 2\times 1 & 2\times 2 \end{bmatrix}, \quad T = \begin{bmatrix} T_{11} & T_{12} \\ 1\times 1 & 1\times 2 \\ \mathrm{O} & T_{22} \\ 2\times 1 & 2\times 2 \end{bmatrix}$$

In order to "triangularize" the system I first define the vector of auxiliary variables as: $\boxed{S_t^1 = Z^H S_t^o}$ (K.4)

Letting Z^H denote the Hermitian conjugate transpose of Z , so that $S_t^1 = \begin{bmatrix} S_{1t}^1 \\ S_{2t}^1 \end{bmatrix}$ where

$$\underset{1\times 1}{S_{1t}^1} = \underset{1\times 1}{Z_{11}^H}\,\underset{1\times 1}{\upsilon_{t-1}} + \underset{2\times 1}{Z_{21}^H}\underset{2\times 1}{\begin{bmatrix} \tilde{y}_t \\ \pi_t \end{bmatrix}} \qquad \underset{2\times 1}{S_{2t}^1} = \underset{2\times 1}{Z_{12}^H}\,\underset{1\times 1}{\upsilon_{t-1}} + \underset{2\times 1}{Z_{22}^H}\underset{2\times 1}{\begin{bmatrix} \tilde{y}_t \\ \pi_t \end{bmatrix}} \qquad \text{(K.5)}$$

Since Z is unitary, $Z^H Z = I$ or $Z^H = Z^{-1}$ and hence $S_t^1 = Z^{-1} S_t^o \Rightarrow Z S_t^1 = \underset{I}{Z Z^{-1}} S_t^o \Rightarrow \boxed{S_t^o = Z S_t^1}$

I use this property to rewrite the system as:

$$AZE_t S_{t+1}^o = BZS_t^o + C\varepsilon_t^{\upsilon} \text{ (K.6)}$$

Premultiplying (K.6) by Q and then using (K.3) yields:

$$QAZE_t S_{t+1}^o = QBZS_t^o + QC\varepsilon_t^{\upsilon} \Rightarrow SE_t S_{t+1}^o = TS_t^o + QC\varepsilon_t^{\upsilon} \Rightarrow$$

$$\begin{bmatrix} S_{11} & S_{12} \\ 1\times1 & 1\times2 \\ O & S_{22} \\ 2\times1 & 2\times2 \end{bmatrix} E_t \begin{bmatrix} S^1_{1t+1} \\ 1\times1 \\ S^1_{2t+1} \\ 2\times1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 1\times1 & 1\times2 \\ O & T_{22} \\ 2\times1 & 2\times2 \end{bmatrix} \begin{bmatrix} S^1_{1t} \\ 1\times1 \\ S^1_{2t} \\ 2\times1 \end{bmatrix} + \begin{bmatrix} Q_1 \\ 1\times3 \\ Q_2 \\ 2\times3 \end{bmatrix} \underset{3\times1}{C} \underset{1\times1}{\varepsilon^{\upsilon}_t}$$

$$\Updownarrow$$

$$\left\{ \begin{array}{c} \underset{1\times1}{S_{11}} \underset{1\times1}{E_t S^1_{1t+1}} + \underset{1\times2}{S_{12}} \underset{2\times1}{E_t S^1_{1t+1}} = \underset{1\times1}{T_{11}} \underset{1\times1}{S^1_{1t}} + \underset{1\times2}{T_{12}} \underset{2\times1}{S^1_{2t}} + \underset{1\times3}{Q_1} \underset{3\times1}{C} \underset{1\times1}{\varepsilon^{\upsilon}_t} \\ \underset{2\times2}{S_{22}} \underset{2\times1}{E_t S^1_{2t+1}} = \underset{2\times2}{T_{22}} \underset{2\times1}{S^1_{2t}} + \underset{2\times3}{Q_2} \underset{3\times1}{C} \underset{1\times1}{\varepsilon^{\upsilon}_t} \end{array} \right\}$$

Due to block diagonal (recursive) structure, the process $\underset{2\times1}{S^1_{2t}}$ is unrelated with $\underset{1\times1}{S^1_{1t}}$ and I can solve it independently forward :

$$\underset{2\times2}{S_{22}} \underset{2\times1}{E_t S^1_{2t+1}} = \underset{2\times2}{T_{22}} \underset{2\times1}{S^1_{2t}} + \underset{2\times3}{Q_2} \underset{3\times1}{C} \underset{1\times1}{\varepsilon^{\upsilon}_t} \Rightarrow \underset{2\times2}{S_{22}} \underset{2\times1}{E_t S^1_{2t+1}} - \underset{2\times2}{T_{22}} \underset{2\times1}{S^1_{2t}} = \underset{2\times3}{Q_2} \underset{3\times1}{C} \underset{1\times1}{\varepsilon^{\upsilon}_t} \Rightarrow \left(\underset{2\times2}{S_{22}} F - \underset{2\times2}{T_{22}} \right) \underset{2\times1}{S^1_{2t}} = \underset{2\times3}{Q_2} \underset{3\times1}{C} \underset{1\times1}{\varepsilon^{\upsilon}_t} \Rightarrow$$

$$\left(\underset{2\times2}{T_{22}} - \underset{2\times2}{S_{22}} F \right) \underset{2\times1}{S^1_{2t}} = - \underset{2\times3}{Q_2} \underset{3\times1}{C} \underset{1\times1}{\varepsilon^{\upsilon}_t} \Rightarrow \left(\underset{2\times2}{I} - \underset{2\times2}{S_{22}} \underset{2\times2}{T^{-1}_{22}} F \right) \underset{2\times2}{T_{22}} \underset{2\times1}{S^1_{2t}} = - \underset{2\times3}{Q_2} \underset{3\times1}{C} \underset{1\times1}{\varepsilon^{\upsilon}_t} \Rightarrow$$

$$\underset{2\times2}{T_{22}} \underset{2\times1}{S^1_{2t}} = - \left(\underset{2\times2}{I} - \underset{2\times2}{S_{22}} \underset{2\times2}{T^{-1}_{22}} F \right)^{-1} \underset{2\times3}{Q_2} \underset{3\times1}{C} \underset{1\times1}{\varepsilon^{\upsilon}_t} \Rightarrow \boxed{\underset{2\times1}{S^1_{2t}} = - \underset{2\times2}{T^{-1}_{22}} \left(\underset{2\times2}{I} - \underset{2\times2}{S_{22}} F \right)^{-1} \underset{2\times3}{Q_2} \underset{3\times1}{C} \underset{1\times1}{\varepsilon^{\upsilon}_t}}$$

where $\left(\underset{2\times2}{I} - \underset{2\times2}{S_{22}} F \right)^{-1} \underset{2\times3}{Q_2} \underset{3\times1}{C} = \left[\sum_{j=0}^{\infty} \left(S_{22} T_{22}^{-1} \right)^j Q_2 C \right] = \underset{2\times1}{R}$

$$\underset{2\times1}{vec(R)} = vec \left[\sum_{j=0}^{\infty} \left(S_{22} T_{22}^{-1} \right)^j Q_2 C \right] = \sum_{j=0}^{\infty} vec \left[\left(S_{22} T_{22}^{-1} \right)^j Q_2 C \right] =$$

And I use the property: $vec(AXB) = (B^T \otimes A) vec(X)$,where X here is $\underset{1\times1}{I^j}$

$$= \sum_{j=0}^{\infty} \left[\underset{1\times1}{I^j} \otimes \left(S_{22} T_{22}^{-1} \right)^j \right] vec \left(Q_2 C \right) = \sum_{j=0}^{\infty} \left[\underset{1\times1}{I} \otimes \left(S_{22} T_{22}^{-1} \right) \right]^j vec \left(Q_2 C \right) =$$

$$= \underbrace{\left[\underset{2\times2}{I} - \underset{1\times1}{I} \otimes \underset{2\times2}{\left(S_{22} T_{22}^{-1} \right)} \right]^{-1}}_{2\times2} \underbrace{vec \left(\underset{2\times3}{Q_2} \underset{3\times1}{C} \right)}_{2\times1}$$

Finally:

$$\boxed{\underset{2\times1}{S^1_{2t}} = - \underset{2\times2}{T^{-1}_{22}} \underset{2\times1}{R} \underset{1\times1}{\varepsilon^{\upsilon}_t}} \text{ (K.7)}$$

Using the above result allows to solve for $\tilde{y}_t$ and π_t from equation (K.5):

$$\underset{2\times 1}{\begin{bmatrix}\tilde{y}_t\\ \pi_t\end{bmatrix}} = -\left(\underset{2\times 2}{Z_{22}^H}\right)^{-1} \underset{2\times 1}{Z_{12}^H}\ \underset{1\times 1}{\upsilon_{t-1}} - \left(\underset{2\times 1}{Z_{22}^H}\right)^{-1} \underset{2\times 2}{T_{22}^{-1}}\ \underset{2\times 1}{R}\ \underset{1\times 1}{\varepsilon_t^{\upsilon}} \quad \text{(K.8)}$$

In order to find some expressions for $-\left(\underset{2\times 2}{Z_{22}^H}\right)^{-1} \underset{2\times 1}{Z_{12}^H}$ and $\left(\underset{2\times 2}{Z_{22}^H}\right)^{-1}$ we proceed as follows:

From assumption we have that Z is unitary, i.e.

$$\underbrace{\begin{bmatrix}\underset{1\times 1}{Z_{11}^H} & \underset{1\times 2}{Z_{21}^H}\\ \underset{2\times 1}{Z_{12}^H} & \underset{2\times 2}{Z_{22}^H}\end{bmatrix}}_{Z^H}\underbrace{\begin{bmatrix}\underset{1\times 1}{Z_{11}} & \underset{1\times 2}{Z_{12}}\\ \underset{2\times 1}{Z_{21}} & \underset{2\times 2}{Z_{22}}\end{bmatrix}}_{Z} = \begin{bmatrix}\underset{1\times 1}{I} & \underset{1\times 2}{O}\\ \underset{2\times 1}{O} & \underset{2\times 2}{I}\end{bmatrix} \Leftrightarrow$$

$$\underset{1\times 1}{Z_{11}^H}\ \underset{1\times 1}{Z_{11}} + \underset{1\times 2}{Z_{21}^H}\ \underset{2\times 1}{Z_{21}} = \underset{1\times 1}{I} \Rightarrow \underset{1\times 1}{Z_{11}^H} + \underset{1\times 2}{Z_{21}^H}\ \underset{2\times 1}{Z_{21}}\ \underset{1\times 1}{Z_{11}}^{-1} = \underset{1\times 1}{Z_{11}}^{-1} \quad \text{(K.9)}$$

$$\underset{1\times 1}{Z_{11}^H}\ \underset{1\times 2}{Z_{12}} + \underset{1\times 2}{Z_{21}^H}\ \underset{2\times 2}{Z_{22}} = \underset{1\times 2}{\mathrm{O}} \Rightarrow \underset{1\times 2}{Z_{21}^H}\ \underset{2\times 2}{Z_{22}} = -\underset{1\times 1}{Z_{11}^H}\ \underset{1\times 2}{Z_{12}} \quad \text{(K.10)}$$

$$\underset{2\times 1}{Z_{12}^H}\ \underset{1\times 1}{Z_{11}} + \underset{2\times 2}{Z_{22}^H}\ \underset{2\times 1}{Z_{21}} = 0 \Rightarrow \underset{2\times 2}{Z_{22}^{H\,-1}}\ \underset{2\times 1}{Z_{12}^H}\ \underset{1\times 1}{Z_{11}} + \underset{2\times 2}{Z_{22}^{H\,-1}}\ \underset{2\times 2}{Z_{22}^H}\ \underset{2\times 1}{Z_{21}} = 0 \Rightarrow -\underset{2\times 2}{Z_{22}^{H\,-1}}\ \underset{2\times 1}{Z_{12}^H} = \underset{2\times 1}{Z_{21}}\ \underset{1\times 1}{Z_{11}}^{-1} \quad \text{(K.11)}$$

$$\underset{2\times 1}{Z_{12}^H}\ \underset{1\times 2}{Z_{12}} + \underset{2\times 2}{Z_{22}^H}\ \underset{2\times 2}{Z_{22}} = \underset{2\times 2}{I} \Rightarrow \underset{2\times 2}{Z_{22}^{H\,-1}}\ \underset{2\times 1}{Z_{12}^H}\ \underset{1\times 2}{Z_{12}} + \underset{2\times 2}{Z_{22}} = \underset{2\times 2}{Z_{22}^{H\,-1}} \text{ and by (K.9) } \underset{2\times 2}{Z_{22}^{H\,-1}} = \underset{2\times 2}{Z_{22}} - \underset{2\times 1}{Z_{21}}\ \underset{1\times 1}{Z_{11}}^{-1}\ \underset{1\times 2}{Z_{12}} \quad \text{(K.12)}$$

So now by using (K.11) and (K.12), (K.8) can be rewritten as:

$$\underset{2\times 1}{\begin{bmatrix}\tilde{y}_t\\ \pi_t\end{bmatrix}} = \underset{2\times 1}{Z_{21}}\ \underset{1\times 1}{Z_{11}}^{-1}\ \underset{1\times 1}{\upsilon_{t-1}} - \left[\underset{2\times 2}{Z_{22}} - \underset{2\times 1}{Z_{21}}\ \underset{1\times 1}{Z_{11}}^{-1}\ \underset{1\times 2}{Z_{12}}\right] \underset{2\times 2}{T_{22}^{-1}}\ \underset{2\times 1}{R}\ \underset{1\times 1}{\varepsilon_t^{\upsilon}} \Rightarrow \quad \underset{2\times 1}{\begin{bmatrix}\tilde{y}_t\\ \pi_t\end{bmatrix}} = \underset{2\times 1}{\mathrm{M}_1}\ \underset{1\times 1}{\upsilon_{t-1}} + \underset{2\times 1}{\mathrm{M}_2}\ \underset{1\times 1}{\varepsilon_t^{\upsilon}} \quad \text{(K.13)}$$

Where: $\underset{2\times 1}{\mathrm{M}_1} \equiv \underset{2\times 1}{Z_{21}}\ \underset{1\times 1}{Z_{11}}^{-1}$ and $\underset{2\times 1}{\mathrm{M}_2} \equiv -\left[\underset{2\times 2}{Z_{22}} - \underset{2\times 1}{Z_{21}}\ \underset{1\times 1}{Z_{11}}^{-1}\ \underset{1\times 2}{Z_{12}}\right] \underset{2\times 2}{T_{22}^{-1}}\ \underset{2\times 1}{R}$

Now I can solve for $\underset{1\times 1}{S_{1t}^1}$:

$$\underset{1\times 1}{S_{1t}^1} = \underset{1\times 1}{Z_{11}^H}\ \underset{1\times 1}{\upsilon_{t-1}} + \underset{1\times 2}{Z_{21}^H}\underset{2\times 1}{\begin{bmatrix}\tilde{y}_t\\ \pi_t\end{bmatrix}} = \underset{1\times 1}{Z_{11}^H}\ \underset{1\times 1}{\upsilon_{t-1}} + \underset{1\times 2}{Z_{21}^H}\left[\underset{2\times 1}{\mathrm{M}_1}\ \underset{1\times 1}{\upsilon_{t-1}} + \underset{2\times 1}{\mathrm{M}_2}\ \underset{1\times 1}{\varepsilon_t^{\upsilon}}\right] = \left[\underset{1\times 1}{Z_{11}^H} + \underset{1\times 2}{Z_{21}^H}\ \underset{2\times 1}{\mathrm{M}_1}\right]\underset{1\times 1}{\upsilon_{t-1}} + \underset{1\times 2}{Z_{21}^H}\ \underset{2\times 1}{\mathrm{M}_2}\ \underset{1\times 1}{\varepsilon_t^{\upsilon}} \Rightarrow$$

$$\underset{1\times 1}{S_{1t}^1} = \left[\underset{1\times 1}{Z_{11}^H} + \underset{1\times 2}{Z_{21}^H}\ \underset{2\times 1}{Z_{21}}\ \underset{1\times 1}{Z_{11}}^{-1}\right]\underset{1\times 1}{\upsilon_{t-1}} - \underset{1\times 2}{Z_{21}^H}\left[\underset{2\times 2}{Z_{22}} - \underset{2\times 1}{Z_{21}}\ \underset{1\times 1}{Z_{11}}^{-1}\ \underset{1\times 2}{Z_{12}}\right] \underset{2\times 2}{T_{22}^{-1}}\ \underset{2\times 1}{R}\ \underset{1\times 1}{\varepsilon_t^{\upsilon}} \text{ by equation (K.9)} \Rightarrow$$

$$S^1_{1t} = Z_{11}^{-1}\upsilon_{t-1} - \left[Z_{21}^H Z_{22} - Z_{21}^H Z_{21} Z_{11}^{-1} Z_{12}\right] T_{22}^{-1} R\,\varepsilon_t^{\upsilon} = Z_{11}^{-1}\upsilon_{t-1} - \left[-Z_{11}^H Z_{12} - Z_{21}^H Z_{21} Z_{11}^{-1} Z_{12}\right] T_{22}^{-1} R\,\varepsilon_t^{\upsilon} \Rightarrow$$

$$S^1_{1t} = Z_{11}^{-1}\upsilon_{t-1} - \left[-Z_{11}^H - Z_{21}^H Z_{21} Z_{11}^{-1}\right] Z_{12} T_{22}^{-1} R\,\varepsilon_t^{\upsilon} \Rightarrow S^1_{1t} = Z_{11}^{-1}\upsilon_{t-1} + \underbrace{\left[Z_{11}^H + Z_{21}^H Z_{21} Z_{11}^{-1}\right]}_{Z_{11}^{-1}} Z_{12} T_{22}^{-1} R\,\varepsilon_t^{\upsilon} \Rightarrow$$

$$\boxed{S^1_{1t} = Z_{11}^{-1}\upsilon_{t-1} + Z_{11}^{-1} Z_{12} T_{22}^{-1} R\,\varepsilon_t^{\upsilon}} \quad \text{(K.14)}$$

And I plug (K.14) into the upper part of the system (K.6) and get:

$$\boxed{\upsilon_t = \mathrm{M}_3\,\upsilon_{t-1} + \mathrm{M}_4\,\varepsilon_t^{\upsilon}} \quad \text{(K.15)}$$

Where $\mathrm{M}_3 = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1}$ and

$$\mathrm{M}_4 = Z_{11} S_{11}^{-1}\left[T_{11} Z_{11}^{-1} Z_{12} T_{22}^{-1} R + Q_1 C + S_{12} T_{22}^{-1} R - T_{12} T_{22}^{-1} R\right] - Z_{12} T_{22}^{-1} R$$

Hence, the model's solution can be written compactly in State Space form by combining the above as:

$$\left.\begin{cases} k_{t+1} = \Gamma_0 k_t + \Gamma_1\,\varepsilon_{t+1}^{\upsilon} \\ f_t = \Gamma_2 k_t \end{cases}\right\} \begin{matrix} \text{predetermined variables} \\ \text{control variables} \end{matrix}$$

Where here $k_t = \begin{bmatrix} \upsilon_{t-1} \\ \varepsilon_t^{\upsilon} \end{bmatrix}$, $f_t = \begin{bmatrix} \tilde{y}_t \\ \pi_t \end{bmatrix}$, $\Gamma_0 = \begin{bmatrix} \mathrm{M}_3 & \mathrm{M}_4 \\ 0 & 1 \end{bmatrix}$, $\Gamma_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\Gamma_2 = \begin{bmatrix} \mathrm{M}_1 & \mathrm{M}_2 \end{bmatrix}$

Consistent with the analytical results, it is seen that the policy shock leads to an increase in the real interest rate, and a decrease in inflation and output. The latter two effects correspond to that of the output gap because the natural level of output is not affected by the monetary policy shock. Using the calibration that Gali provides, the nominal interest rate goes up, though by less than its exogenous component, as a result of the downward adjustment induced by the decline in inflation and output gap. In order to bring about the observed interest rate response , the central bank must engineer a reduction in the money supply. The calibrated model thus display a liquidity effect. Also the response of the real rate is higher than that of the nominal rate as a result of the decrease in expected inflation. Overall, figure 1 shows dynamic responses which are qualitatively similar to those estimated using structural VAR methods.

Three different solution methods were implement in MATLAB (undetermined Coefficients, Klein, Blanchard Kahn) and yielded identical results up to 15 decimals (the codes are provided in Appendix 3)

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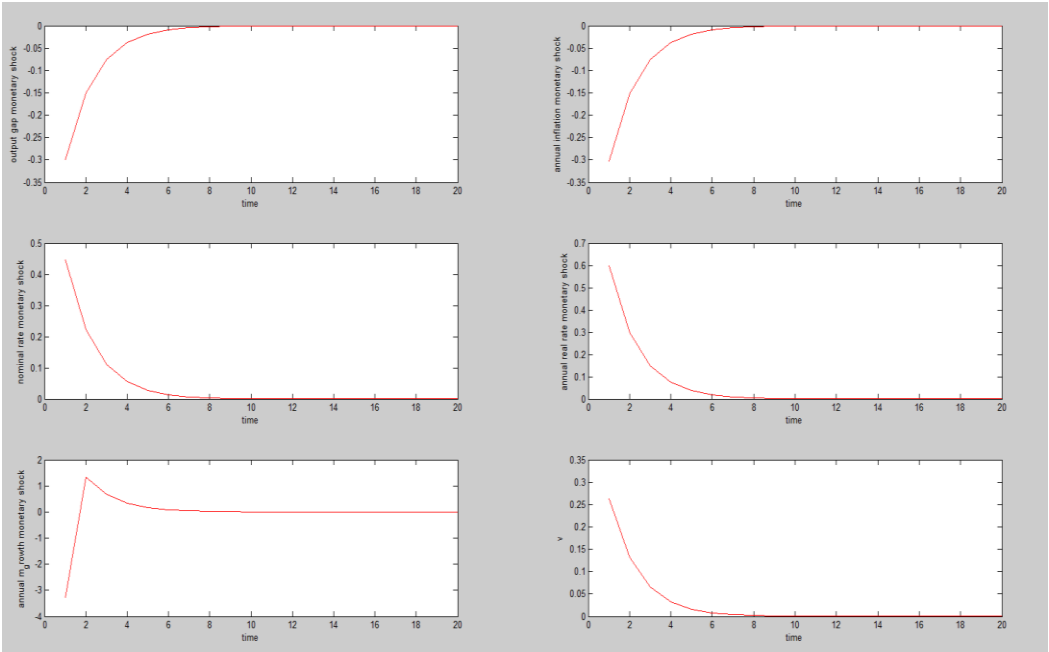

(Figure 1.)

2. The New Keynesian Model with price and wage rigidities

This section expands the baseline model with monopoly supply power and sticky and sticky wages in the labor market. A continuum of differentiated labor services is assumed, all of which are used by each firm. Each household specialize in one type of labour, which it supplies monopolistically. Wages are sticky in an analogous way to goods prices and in each period only a constant fraction of households can adjust their posted nominal wage. As a result, the aggregate nominal wage responds sluggishly to shocks, generating inefficient variations in the wage markup.

2.1 Households

Assume a continuum of households indexed by $j \in [0,1]$. As before, given a sequence of budget constraints household j seeks to maximize lifetime utility:

$$E_o \sum_{t=0}^{\infty} \beta^t U\left(C_{jt}, N_{jt}\right) \quad (2.1)$$

The consumption index is now given by:

$$C_{jt} \equiv \left(\int_0^1 C_{ijt}^{\frac{\varepsilon_p - 1}{\varepsilon_p}} di \right)^{\frac{\varepsilon_p}{\varepsilon_p - 1}} \quad (2.2)$$

In this model there is the additional assumption that households specialize in different types of labour, and therefore face some monopoly power in the labour market given by $\varepsilon_w > 1$. They post the nominal wage at which they are willing to supply labour services to firms who demand them, and because of monopoly power this wage contains a markup. However, households also face a constraint with respect to the frequency in which they can change wages. A constant fraction θ_w of the household is stuck with the wage they had last period, while the remaining $1-\theta_w$ households can reoptimize the price of their labour services. Thus, a household who reset its wage in period t will choose W_t^* in order to maximize:

$$E_t \sum_{k=0}^{\infty} \left(\theta_w \beta\right)^k U\left(C_{t+k|t}, N_{t+k|t}\right) \quad (2.3)$$

$$st \quad N_{t+k|t} = \left(\frac{W_t^*}{W_{t+k}} \right)^{-\varepsilon_w} N_{t+k} \quad (2.4)$$

$$P_{t+k|t} C_{t+k|t} + E_{t+k}\left\{ Q_{t+k,t+k+1} D_{t+k+1|t} \right\} \leq D_{t+k|t} + W_t^* N_{t+k|t} - T_t \quad for \;\; k = 0,1,2,\ldots \quad (2.5)$$

$C_{t+k|t}$ and $N_{t+k|t}$ denote consumption and labour at time $t+k$ for a household that last set the wage at time t. Because of the wage rigidity, (2.3) can be interpreted as "the expected discounted sum of utilities generated over the uncertain period during which the wage remains unchanged at level W_t^* set the current peri-

od. Note that the utility generated under any other wage set in the future is irrelevant from the point of view of the optimal setting of the current wage, and thus can be ignored in (2.3). maximization of (2.3) is subject to the sequence of labour demand schedules and flow budget constraints that are effective while W_t^* remains in place where $N_{t+k} \equiv \int_0^1 N_{i,t+k} di$ denotes aggregate employment in period $t+k$, $D_{t+k|t}$ is the market value in period $t+k$ of the portfolio o f securities held at the beginning of that period by households that last reoptimized their wage in period t, while $E_{t+k}\left\{Q_{t+k,t+k+1} D_{t+k+1|t}\right\}$ is the corresponding market value as of period $t+k$ of the portfolio purchased in that period, which yields a random payoff $D_{t+k+1|t}$.

Given the concavity of the utility function, (2.5) holds with equality. Furthermore, it can be solved for $C_{t+k|t}$

$$C_{t+k|t} = \frac{1}{P_{t+k|t}}\left[D_{t+k|t} + W_t^* N_{t+k|t} - T_t - E_{t+k} Q_{t+k,t+k+1} D_{t+k+1|t} \right] \quad (2.6)$$

Insert (2.6) and then (2.4) into (2.3) to get an unconstrained problem:

$$\max_{W_t^*} E_t \sum_{k=0}^{\infty} (\theta_w \beta)^k U\left(C_{t+k|t}, N_{t+k|t}\right) =$$

$$= \max_{W_t^*} E_t \sum_{k=0}^{\infty} (\theta_w \beta)^k U\left(\frac{1}{P_{t+k|t}}\left[D_{t+k|t} + W_t^* N_{t+k|t} - T_t - E_{t+k} Q_{t+k,t+k+1} D_{t+k+1|t} \right], \left(\frac{W_t^*}{W_{t+k}}\right)^{-\varepsilon_w} N_{t+k} \right) =$$

$$= \max_{W_t^*} E_t \sum_{k=0}^{\infty} (\theta_w \beta)^k U\left(\frac{1}{P_{t+k|t}}\left[D_{t+k|t} + W_t^* \left(\frac{W_t^*}{W_{t+k}}\right)^{-\varepsilon_w} N_{t+k} - T_t - E_{t+k} Q_{t+k,t+k+1} D_{t+k+1|t} \right], \left(\frac{W_t^*}{W_{t+k}}\right)^{-\varepsilon_w} N_{t+k} \right) =$$

$$= \max_{W_t^*} E_t \sum_{k=0}^{\infty} (\theta_w \beta)^k U\left(\frac{1}{P_{t+k|t}}\left[D_{t+k|t} + \left(W_t^*\right)^{1-\varepsilon_w} \left(W_{t+k}\right)^{\varepsilon_w} N_{t+k} - T_t - E_{t+k} Q_{t+k,t+k+1} D_{t+k+1|t} \right], \left(W_t^*\right)^{-\varepsilon_w} \left(W_{t+k}\right)^{\varepsilon_w} N_{t+k} \right) \quad (2.7)$$

<u>FONC</u>

$$E_t \sum_{k=0}^{\infty} (\theta_w \beta)^k \left[U_{Ct}\left(C_{t+k|t}, N_{t+k|t}\right) \frac{1}{P_{t+k|t}} (1-\varepsilon_w) \left(W_t^*\right)^{-\varepsilon_w} \left(W_{t+k}\right)^{\varepsilon_w} N_{t+k} + (-\varepsilon_w) U_{Nt}\left(C_{t+k|t}, N_{t+k|t}\right) \left(W_t^*\right)^{-\varepsilon_w - 1} \left(W_{t+k}\right)^{\varepsilon_w} N_{t+k} \right] = 0 \Leftrightarrow$$

$$E_t \sum_{k=0}^{\infty} (\theta_w \beta)^k \left[U_{Ct}\left(C_{t+k|t}, N_{t+k|t}\right) \frac{1}{P_{t+k|t}} (1-\varepsilon_w) N_{t+k|t} - \varepsilon_w U_{Nt}\left(C_{t+k|t}, N_{t+k|t}\right) \frac{1}{W_t^*} N_{t+k|t} \right] = 0 \Leftrightarrow$$

$$E_t \sum_{k=0}^{\infty} (\theta_w \beta)^k N_{t+k|t} \left[U_{Ct}\left(C_{t+k|t}, N_{t+k|t}\right) \frac{W_t^*}{P_{t+k|t}} - \frac{\varepsilon_w}{(1-\varepsilon_w)} U_{Nt}\left(C_{t+k|t}, N_{t+k|t}\right) \right] = 0 \Leftrightarrow$$

Letting $MRS_{t+k|t} \equiv -\frac{U_{Nt}\left(C_{t+k|t}, N_{t+k|t}\right)}{U_{Ct}\left(C_{t+k|t}, N_{t+k|t}\right)}$ as the marginal rate of substitution between consumption and hours in period $t+k$ for a household resetting the wage in period t and $M_w \equiv \frac{\varepsilon_w}{\left(1-\varepsilon_w\right)}$, the FONC can be rewritten as:

$$E_t \sum_{k=0}^{\infty} \left(\theta_w \beta\right)^k N_{t+k|t} U_{Ct}\left(C_{t+k|t}, N_{t+k|t}\right)\left[\frac{W_t^*}{P_{t+k|t}} - M_w MRS_{t+k|t}\right] = 0 \quad (2.8)$$

And I can solve for W_t^* as it is independent of k and can be taken out of the sum and the expectational operator:

$$W_t^* E_t \sum_{k=0}^{\infty} \left(\theta_w \beta\right)^k U_{Ct}\left(C_{t+k|t}, N_{t+k|t}\right)\frac{N_{t+k|t}}{P_{t+k|t}} = M_w E_t \sum_{k=0}^{\infty} \left(\theta_w \beta\right)^k N_{t+k|t} U_{Ct}\left(C_{t+k|t}, N_{t+k|t}\right) MRS_{t+k|t} \Leftrightarrow (2.9)$$

$$\boxed{W_t^* = M_w \frac{E_t \sum_{k=0}^{\infty} \left(\theta_w \beta\right)^k N_{t+k|t} U_{Ct}\left(C_{t+k|t}, N_{t+k|t}\right) MRS_{t+k|t}}{E_t \sum_{k=0}^{\infty} \left(\theta_w \beta\right)^k U_{Ct}\left(C_{t+k|t}, N_{t+k|t}\right)\frac{N_{t+k|t}}{P_{t+k|t}}}} \quad (2.10)$$

In the limiting case of full wage flexibility $\left(\theta_w = 0\right)$, $\frac{W_t^*}{P_t} = \frac{W_t}{P_t} = M_w MRS_{t|t} \quad \forall t$

Thus M_w is the wedge between the real wage and the marginal rate of substitution that prevails in the absence of wage rigidities, i.e. the desired gross wage markup. Also in a perfect foresight zero inflation steady state $\frac{W^*}{P} = \frac{W}{P} = M_w MRS$

In a similar manner as I log-linearized the optimal price equation (1.30') around steady state, I now loglinearize the optimal wage equation (2.10). First, it is convenient set as starting point (2.9)

A first-order Taylor expansion of the left hand side of (2.9) yields:

$$W_t^* E_t \sum_{k=0}^{\infty} \left(\theta_w \beta\right)^k U_{Ct}\left(C_{t+k|t}, N_{t+k|t}\right)\frac{N_{t+k|t}}{P_{t+k|t}} \approx$$

$$\approx W \sum_{k=0}^{\infty} \left(\theta_w \beta\right)^k U_C\left(C,N\right)\frac{N}{P} + E_t \sum_{k=0}^{\infty} \left(\theta_w \beta\right)^k U_C\left(C,N\right)\frac{N}{P}\left(W_t^* - W\right) + WE_t \sum_{k=0}^{\infty} \left(\theta_w \beta\right)^k U_{CC}\left(C,N\right)\frac{N}{P}\left(C_{t+k|t} - C\right) +$$

$$+ WE_t \sum_{k=0}^{\infty} \left(\theta_w \beta\right)^k \left[U_{CN}\left(C,N\right)\frac{N}{P} + U_C\left(C,N\right)\frac{1}{P}\right]\left(N_{t+k|t} - N\right) + WE_t \sum_{k=0}^{\infty} \left(\theta_w \beta\right)^k U_C\left(C,N\right)\left(-\frac{N}{P^2}\right)\left(P_{t+k|t} - P\right) =$$

$$= W\sum_{k=0}^{\infty}(\theta_w\beta)^k U_C(C,N)\frac{N}{P} + WE_t\sum_{k=0}^{\infty}(\theta_w\beta)^k U_C(C,N)\frac{N}{P}\left(w_t^* - w\right) + WE_t\sum_{k=0}^{\infty}(\theta_w\beta)^k U_{CC}(C,N)\frac{N}{P}C\left(c_{t+k|t} - c\right) +$$

$$+ WE_t\sum_{k=0}^{\infty}(\theta_w\beta)^k\left[U_{CN}(C,N)\frac{N^2}{P} + U_C(C,N)\frac{N}{P}\right]\left(n_{t+k|t} - n\right) + WE_t\sum_{k=0}^{\infty}(\theta_w\beta)^k U_C(C,N)\left(-\frac{N}{P}\right)\left(p_{t+k|t} - p\right) =$$

$$= WU_C(C,N)\frac{N}{P}E_t\sum_{k=0}^{\infty}(\theta_w\beta)^k\left[1 + \left(w_t^* - w\right) + C\frac{U_{CC}(C,N)}{U_C(C,N)}\left(c_{t+k|t} - c\right) +\right.$$

$$\left. + \frac{U_{CN}(C,N)}{U_C(C,N)}N\left(n_{t+k|t} - n\right) + \left(n_{t+k|t} - n\right) - \left(p_{t+k|t} - p\right)\right] =$$

A first-order Taylor expansion of the right hand side of (2.9) yields:

$$M_w E_t\sum_{k=0}^{\infty}(\theta_w\beta)^k N_{t+k|t}U_{Ct}\left(C_{t+k|t}, N_{t+k|t}\right)MRS_{t+k|t} \approx$$

$$\approx M_w E_t\sum_{k=0}^{\infty}(\theta_w\beta)^k U_C(C,N)NMRS + M_w E_t\sum_{k=0}^{\infty}(\theta_w\beta)^k U_{CC}(C,N)NMRS\left(C_{t+k|t} - C\right) +$$

$$+ M_w E_t\sum_{k=0}^{\infty}(\theta_w\beta)^k\left[U_{CN}(C,N)N + U_C(C,N)\right]MRS\left(N_{t+k|t} - N\right) + M_w E_t\sum_{k=0}^{\infty}(\theta_w\beta)^k U_C(C,N)N\left(MRS_{t+k|t} - MRS\right) =$$

$$= M_w E_t\sum_{k=0}^{\infty}(\theta_w\beta)^k U_C(C,N)NMRS + M_w E_t\sum_{k=0}^{\infty}(\theta_w\beta)^k CU_{CC}(C,N)NMRS\left(c_{t+k|t} - c\right) +$$

$$+ M_w E_t\sum_{k=0}^{\infty}(\theta_w\beta)^k\left[U_{CN}(C,N)N^2 + U_C(C,N)N\right]MRS\left(n_{t+k|t} - n\right) + M_w E_t\sum_{k=0}^{\infty}(\theta_w\beta)^k U_C(C,N)NMRS\left(mrs_{t+k|t} - mrs\right) =$$

$$= M_w(N)(MRS)U_C(C,N)E_t\sum_{k=0}^{\infty}(\theta_w\beta)^k\left[C\frac{U_{CC}(C,N)}{U_C(C,N)}\left(c_{t+k|t} - c\right) + N\frac{U_{CN}(C,N)}{U_C(C,N)}\left(n_{t+k|t} - n\right) + \left(n_{t+k|t} - n\right) + \left(mrs_{t+k|t} - mrs\right)\right] =$$

The two sides must be equal:

$$WU_C(C,N)\frac{N}{P}E_t\sum_{k=0}^{\infty}(\theta_w\beta)^k\underbrace{\left[1 + \left(w_t^* - w\right) + C\frac{U_{CC}(C,N)}{U_C(C,N)}\left(c_{t+k|t} - c\right) + N\frac{U_{CN}(C,N)}{U_C(C,N)}\left(n_{t+k|t} - n\right) + \left(n_{t+k|t} - n\right) - \left(p_{t+k|t} - p\right)\right]}_{H}$$

$$= M_w(N)(MRS)U_C(C,N)E_t\sum_{k=0}^{\infty}(\theta_w\beta)^k\underbrace{\left[1 + C\frac{U_{CC}(C,N)}{U_C(C,N)}\left(c_{t+k|t} - c\right) + N\frac{U_{CN}(C,N)}{U_C(C,N)}\left(n_{t+k|t} - n\right) + \left(n_{t+k|t} - n\right) + \left(mrs_{t+k|t} - mrs\right)\right]}_{F} \Leftrightarrow$$

$$\frac{W}{P}E_t\sum_{k=0}^{\infty}(\theta_w\beta)^k H = M_w(MRS)E_t\sum_{k=0}^{\infty}(\theta_w\beta)^k F \Leftrightarrow$$

But as is shown earlier: $\frac{W}{P} = M_w MRS$, thus:

$$\Rightarrow E_t\sum_{k=0}^{\infty}\left(\theta_w\beta\right)^k H = E_t\sum_{k=0}^{\infty}\left(\theta_w\beta\right)^k F \Leftrightarrow$$

$$E_t\sum_{k=0}^{\infty}\left(\theta_w\beta\right)^k\left[\left(w_t^*-w\right)-\left(p_{t+k|t}-p\right)\right]=E_t\sum_{k=0}^{\infty}\left(\theta_w\beta\right)^k\left(mrs_{t+k|t}-mrs\right)\Leftrightarrow$$

$$E_t\sum_{k=0}^{\infty}\left(\theta_w\beta\right)^k\left[w_t^*-(w-p)\right]=E_t\sum_{k=0}^{\infty}\left(\theta_w\beta\right)^k\left(mrs_{t+k|t}-mrs+p_{t+k|t}\right)\Leftrightarrow$$

If $\beta\theta<1$ and because $w_t^*-(w-p)$ is constant for k periods

$$\sum_{k=0}^{\infty}\theta^k\beta^k\left[w_t^*-(w-p)\right]=\left[w_t^*-(w-p)\right]\sum_{k=0}^{\infty}\theta^k\beta^k=\frac{\left[w_t^*-(w-p)\right]}{1-\theta\beta}$$

$$\Rightarrow\frac{\left[w_t^*-(w-p)\right]}{1-\theta\beta}=E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left(mrs_{t+k|t}-mrs+p_{t+k|t}\right)\Leftrightarrow$$

$$\left[w_t^*-(w-p)\right]=\left(1-\theta\beta\right)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left(mrs_{t+k|t}-mrs+p_{t+k|t}\right)\Leftrightarrow$$

$$w_t^*=(w-p)-mrs+\left(1-\theta\beta\right)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left(mrs_{t+k|t}+p_{t+k|t}\right)\Leftrightarrow$$

$$\boxed{w_t^*=\mu^w+\left(1-\theta\beta\right)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left(mrs_{t+k|t}+p_{t+k|t}\right)} \quad (2.11)$$

Where $\mu^w\equiv(w-p)-mrs=\ln\left(\frac{W}{P*MRS}\right)=\ln\left(M_w\right)$

the intuition behind the wage setting rule is straight forward. First, w_t^* is increasing in expected future prices because households care about the purchasing power of their nominal wage. Second, w_t^* is increasing in the expected average marginal disutilities of labour in terms of goods over the life of the wage, because households want to adjust their expected average real wage accordingly, given expected future prices.

Let the utility function be of a functional form as following:

$$U\left(C_t,N_t\right)=\frac{C_t^{1-\sigma}}{1-\sigma}-\frac{N_t^{1+\varphi}}{1+\varphi} \quad (2.12)$$

The assumed separability between consumption and hours, combined with the assumption of complete asset markets, implies that consumption is independent of the wage history of a household, i.e. $C_{t+k|t}=C_{t+k}$ for any $k=0,1,2,...$

Thus from (2.12): $MRS_{t+k|t}\equiv-\frac{U_{Nt}\left(C_{t+k|t},N_{t+k|t}\right)}{U_{Ct}\left(C_{t+k|t},N_{t+k|t}\right)}=\frac{N_{t+k|t}^{\varphi}}{C_{t+k|t}^{-\sigma}}=N_{t+k|t}^{\varphi}C_{t+k|t}^{\sigma}\Rightarrow$

$$\Rightarrow mrs_{t+k|t} = \sigma c_{t+k|t} + \varphi n_{t+k|t} \quad (2.13)$$

Also, from (2.4) we get: $N_{t+k|t} = \left(\frac{W_t^*}{W_{t+k}}\right)^{-\varepsilon_w} N_{t+k} \Rightarrow n_{t+k|t} = n_{t+k} - \varepsilon_w\left(w_t^* - w_{t+k}\right)$ (2.14)

If I let the $mrs_{t+k|t} = \sigma c_{t+k|t} + \varphi n_{t+k|t}$ define the economy's average marginal rate of substitution, so that:

$$mrs_{t+k|t} = mrs_{t+k} + \varphi\left(n_{t+k|t} - n_{t+k}\right) = mrs_{t+k} - \varphi\varepsilon_w\left(w_t^* - w_{t+k}\right) \quad (2.15)$$

So, by (2.15) equation (2.11) becomes:

$$w_t^* = \mu^w + (1-\theta\beta)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left(mrs_{t+k} - \varphi\varepsilon_w\left(w_t^* - w_{t+k}\right) + p_{t+k|t}\right) \Leftrightarrow$$

$$w_t^* = \mu^w - \varphi\varepsilon_w w_t^* + (1-\theta\beta)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left(mrs_{t+k} + \varphi\varepsilon_w w_{t+k} + p_{t+k|t}\right) \Leftrightarrow$$

$$(1+\varphi\varepsilon_w)w_t^* = \mu^w + (1-\theta\beta)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left(mrs_{t+k} + \varphi\varepsilon_w w_{t+k} + p_{t+k|t}\right) \Leftrightarrow$$

$$(1+\varphi\varepsilon_w)w_t^* = (1-\theta\beta)E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left(\mu^w + mrs_{t+k} + \varphi\varepsilon_w w_{t+k} + p_{t+k|t}\right) \Leftrightarrow$$

$$w_t^* = \frac{(1-\theta\beta)}{(1+\varphi\varepsilon_w)}E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left(\mu^w + mrs_{t+k} + \varphi\varepsilon_w w_{t+k} + p_{t+k|t}\right)$$

But also: $\mu_{t+k}^w \equiv (w_{t+k} - p_{t+k}) - mrs_{t+k} \Leftrightarrow mrs_{t+k} + p_{t+k} = -\mu_{t+k}^w + w_{t+k}$, so:

$$w_t^* = \frac{(1-\theta\beta)}{(1+\varphi\varepsilon_w)}E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left(\mu^w - \mu_{t+k}^w + w_{t+k} + \varphi\varepsilon_w w_{t+k}\right) \Leftrightarrow$$

$$w_t^* = \frac{(1-\theta\beta)}{(1+\varphi\varepsilon_w)}E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left(\mu^w - \mu_{t+k}^w + (1+\varphi\varepsilon_w)w_{t+k}\right) \Leftrightarrow$$

$$w_t^* = \frac{(1-\theta\beta)}{(1+\varphi\varepsilon_w)}E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left(-\hat{\mu}_{t+k}^w + (1+\varphi\varepsilon_w)w_{t+k}\right) \Leftrightarrow$$

$$w_t^* = \frac{(1-\theta\beta)}{(1+\varphi\varepsilon_w)}E_t\sum_{k=1}^{\infty}\theta^k\beta^k\left(-\hat{\mu}_{t+k}^w + (1+\varphi\varepsilon_w)w_{t+k}\right) + \frac{(1-\theta\beta)}{(1+\varphi\varepsilon_w)}\left(-\hat{\mu}_t^w + (1+\varphi\varepsilon_w)w_t\right) \Leftrightarrow$$

$$w_t^* = w_t(1-\theta\beta) - \frac{(1-\theta\beta)}{(1+\varphi\varepsilon_w)}\hat{\mu}_t^w + \theta\beta\underbrace{\frac{(1-\theta\beta)}{(1+\varphi\varepsilon_w)}E_t\sum_{k=0}^{\infty}\theta^k\beta^k\left(-\hat{\mu}_{t+k+1}^w + (1+\varphi\varepsilon_w)w_{t+k+1}\right)}_{E_t w_{t+1}^*} \Leftrightarrow$$

$$w_t^* = w_t(1-\theta\beta) - \frac{(1-\theta\beta)}{(1+\varphi\varepsilon_w)}\hat{\mu}_t^w + \theta\beta E_t w_{t+1}^* \quad (2.16)$$

The next step is to derive the wage inflation equation. Given the wage setting structure, the evolution if the aggregate wage index is given by:

$$W_t = \left[\int_{S(t)}^{1} W_{it-1}^{1-\varepsilon_w} di + (1-\theta_w) W_t^{*1-\varepsilon_w} \right]^{\frac{1}{1-\varepsilon_w}} = \left[\theta_w W_{t-1}^{1-\varepsilon_w} + (1-\theta_w) W_t^{*1-\varepsilon_w} \right]^{\frac{1}{1-\varepsilon_w}} \quad (2.17)$$

A first order Taylor approximation around zero wage inflation steady state yields:

$$W_t \approx W + \frac{1}{1-\varepsilon_w} \left[\theta_w W + (1-\theta_w) W \right]^{\frac{1}{1-\varepsilon_w}-1} (1-\varepsilon_w) \theta_w W^{-\varepsilon_w} \left(W_{t-1} - W \right) + \frac{1}{1-\varepsilon_w} \left[\theta_w W + (1-\theta_w) W \right]^{\frac{1}{1-\varepsilon_w}-1} (1-\varepsilon_w)(1-\theta_w) W^{-\varepsilon_w} \left(W_t^* - W \right) =$$

$$= W + \left[\theta_w W^{1-\varepsilon_w} + (1-\theta_w) W^{1-\varepsilon_w} \right]^{\frac{\varepsilon_w}{1-\varepsilon_w}} \theta_w W^{-\varepsilon_w} \left(W_{t-1} - W \right) + \left[\theta_w W^{1-\varepsilon_w} + (1-\theta_w) W^{1-\varepsilon_w} \right]^{\frac{\varepsilon_w}{1-\varepsilon_w}} (1-\theta_w) W^{-\varepsilon_w} \left(W_t^* - W \right) =$$

$$= W + \theta_w W \left(W_{t-1} - W \right) + (1-\theta_w) W \left(W_t^* - W \right) \Leftrightarrow$$

$$W_t \approx W + \theta_w \left(W_{t-1} - W \right) + (1-\theta_w) \left(W_t^* - W \right) \Leftrightarrow$$

$$\frac{W_t - W}{W} \approx \theta_w \left(\frac{W_{t-1} - W}{W} \right) + (1-\theta_w) \left(\frac{W_t^* - W}{W} \right) \Leftrightarrow$$

$$w_t - w \approx \theta_w \left(w_{t-1} - w \right) + (1-\theta_w) \left(w_t^* - w \right) \Leftrightarrow \boxed{w_t \approx \theta_w w_{t-1} + (1-\theta_w) w_t^*} \quad (2.18)$$

From (2.18) we see that inflation $\pi_t^w = w_t - w_{t-1}$ is given by:

$$\pi_t^w = w_t - w_{t-1} \approx \theta_w w_{t-1} + (1-\theta_w) w_t^* - w_{t-1} = (1-\theta_w) w_t^* - (1-\theta_w) w_{t-1} = (1-\theta_w) \left(w_t^* - w_{t-1} \right) \quad (2.19)$$

Finally, insert (2.16) into 2.18 and by using (2.19):

$$w_t \approx \theta_w w_{t-1} + (1-\theta_w) w_t^* \Rightarrow w_t \approx \theta_w w_{t-1} + (1-\theta_w) \left[w_t \left(1-\theta_w \beta \right) - \frac{\left(1-\theta_w \beta \right)}{\left(1+\varphi \varepsilon_w \right)} \hat{\mu}_t^w + \theta_w \beta E_t w_{t+1}^* \right] \Leftrightarrow$$

$$w_t - \theta_w w_{t-1} = (1-\theta_w) \left[w_t - \theta_w \beta w_t - \frac{\left(1-\theta \beta \right)}{\left(1+\varphi \varepsilon_w \right)} \hat{\mu}_t^w + \theta_w \beta E_t w_{t+1}^* \right] \Leftrightarrow$$

$$w_t - \theta_w w_{t-1} = (1-\theta_w) w_t + \left[- \frac{\left(1-\theta \beta \right)}{\left(1+\varphi \varepsilon_w \right)} \hat{\mu}_t^w + \theta \beta E_t \left(w_{t+1}^* - w_t \right) \right] \Leftrightarrow$$

$$w_t - \theta_w w_{t-1} = w_t - \theta_w w_t - (1-\theta_w) \frac{\left(1-\theta_w \beta \right)}{\left(1+\varphi \varepsilon_w \right)} \hat{\mu}_t^w + (1-\theta_w) \theta_w \beta E_t \left(w_{t+1}^* - w_t \right) \Leftrightarrow$$

$$\theta_w w_t - \theta_w w_{t-1} = -(1-\theta_w) \frac{\left(1-\theta_w \beta \right)}{\left(1+\varphi \varepsilon_w \right)} \hat{\mu}_t^w + (1-\theta_w) \theta_w \beta E_t \left(w_{t+1}^* - w_t \right) \Leftrightarrow$$

$$\pi_t^w = (1-\theta_w) \beta E_t \frac{\pi_{t+1}^w}{(1-\theta_w)} - \frac{(1-\theta_w)\left(1-\theta_w \beta \right)}{\theta_w \left(1+\varphi \varepsilon_w \right)} \hat{\mu}_t^w \Leftrightarrow$$

$$\boxed{\pi_t^w = \beta E_t \pi_{t+1}^w - \lambda_w \hat{\mu}_t^w} \quad (2.20)$$

$$\text{Where } \lambda_w \equiv \frac{(1-\theta_w)(1-\theta_w\beta)}{\theta_w(1+\varphi\varepsilon_w)}$$

The wage inflation equation (2.20) has a form analogous to the goods price inflation equation (1.49). The intuition is also the same. When the average wage in the economy is below the level consistent with maintaining (on average) the desired markup, households readjusting their nominal wage will tend to increase the latter, thus generating positive wage inflation. Here, (2.20) replaces condition that real wage is equal to marginal rate of substitution, one of the optimality conditions associated with the household's problem used earlier. The imperfect adjustment of nominal wages will generally drive a wedge between the real wage and the marginal rate of substitution for each household and as a result, between the average real wage and the average marginal rate of substitution. This leads to variation in the average wage markup, and given (2.20), also to variation in wage inflation.

The Euler equation is almost the same as in the sticky prices model because $C_{t+k|t} = C_{t+k}$, but at the same the hours worked now depends on when the wage last was set, (1.18) is written as:

$$1 = \beta(1+i)E_t\left[\frac{U_{Ct+1}\left(C_{t+1}, N_{t+1|t-k}\right)}{U_{Ct}\left(C_t, N_{t|t-k}\right)}\frac{P_t}{P_{t+1}}\right]$$

However, due to separability the log linearization yield the same again the equation (1.18') or:

$$\boxed{c_t = E_t c_{t+1} + \frac{1}{\sigma}(\rho - i_t + E_t\pi_{t+1})} \quad (2.21)$$

i.e. the Euler equation does not depend on wage rigidities under this setup.

2.2 Firms

Now that households have monopoly power in the labour market, firms face a two dimensional problem as well. First, for any given level of labour costs, they have to find the output maximizing combination of labour. Second, and given this optimal labour vector, firms have to set prices such that profit is maximized. Firm's output is still given by:

$$Y_{it} = A_t N_{it}^{1-\alpha} \quad (1.22)$$

However, total labour N_{it} used by firm i, where the N_{ijt} is the quantity of type j labor employed, is now defined by:

$$N_t \equiv \left(\int_0^1 N_{iJt}^{\frac{\varepsilon_W - 1}{\varepsilon_W}} dj\right)^{\frac{\varepsilon_W}{\varepsilon_W - 1}} \quad (2.22)$$

The elasticity of substitution among labor types is denoted ε_w.specification (2.1) implies that labour is an imperfect substitute as long as $\varepsilon_w << \infty$. Note that the assumption of individual worker types implies an

indexation $j \in [0,1]$. Let W_{jt} denote the nominal wage for type j worker in period t. In an analogous way to the optimal consumption vector derived in (1.10), firm $i's$ demand for labour j is given by:

$$N_{ijt} = \left(\frac{W_{jt}}{W_t} \right)^{-\varepsilon_w} N_{it} \quad (2.23)$$

This holds for all $i, j \in [0,1]$ where :

$$W_t = \left(\int_0^1 w_{jt}^{1-\varepsilon_w} dj \right)^{\frac{1}{1-\varepsilon_w}} \quad (2.24)$$

In exactly some manner as I derived the aggregation result for consumption expenditures, one can derive the following aggregation result for firms' labour expenditures:

$$\int_0^1 W_{jt} N_{ijt} dj = W_t N_{it} \quad (2.25)$$

Denote θ_p as the price stickiness parameter in the goods market and ε_p the elasticity of substitution betweeen consumption goods. Then, the profit maximization problem given by (1.33') is:

$$\boxed{\max_{P_t^*} \sum_{k=0}^{\infty} \theta_p^k E_t \left\{ \beta^k C_{t+k}^{1-\sigma} C_t^{\sigma} P_t P_{t+k}^{\varepsilon_p - 1} P_t^{*1-\varepsilon_p} - \beta^k C_{t+k}^{-\sigma} C_t^{\sigma} P_t P_{t+k}^{-1} \Psi_{t+k} \left(\left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon_p} C_{t+k} \right) \right\}} \quad (2.26)$$

As shown in (1.) the aggregation of the resulting price setting rules yields, to a first order approximation and in a neighborhood of the zero inflation steady state, the following equation for price inflation π_t^p :

$$\pi_t^p = \beta E_t \pi_{t+1}^p + \frac{(1-\theta_p)(1-\theta_p \beta)\Theta_p}{\theta_p} mc_t \Leftrightarrow \pi_t^p = \beta E_t \pi_{t+1}^p + \lambda_p mc_t \Leftrightarrow \boxed{\pi_t^p = \beta E_t \pi_{t+1}^p - \lambda_p \hat{\mu}_t^p} \quad (2.27)$$

Where : $\hat{\mu}_t^p \equiv \mu_t^p - \mu^p = -mc_t$, $\lambda_p \equiv \frac{(1-\theta_p)(1-\theta_p \beta)\Theta_p}{\theta_p}$ and $\Theta_p = \frac{1-\alpha}{1-\alpha+\alpha\varepsilon_p}$

From (2.6) I see that when the average price markup is below its steady state value, firms that are adjusting prices set them higher, thus generating positive inflation.

2.3 Price and Wage Inflation Equations and the Dynamic IS equation

I will now derive the price inflation equation, the wage inflation equation and the output gap equation. Start with wage inflation.

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Let $\tilde{y}_t = y_t - y_t^n$ be the output gap from the level of natural output, with the latter now being defined as the equilibrium level of output in absence of both price and wage rigidities. Define also the real wage as $\omega_t \equiv w_t - p_t$ which implies that the real wage gap is $\tilde{\omega}_t = \omega_t - \omega_t^n$.

The natural real wage, denoted ω_t^n, is the real wage that would prevail in the absence of nominal rigidities. To get an expression for ω_t^n, note first that the production function implies that output, employment and marginal productivity in their linearized versions are given again by $y_t = a_t + (1-\alpha)n_t$ (2.28)and

$mpn_t = a_t + \ln(1-\alpha) - \alpha n_t$ (2.29)

$$-\mu^p = \omega_t^n - mpn_t^n \Rightarrow$$

$$-\mu^p = \omega_t^n - a_t - \ln(1-\alpha) + \alpha n_t^n = \omega_t^n - a_t - \ln(1-\alpha) + \alpha n_t^n + n_t^n - n_t^n \Leftrightarrow$$

$$-\mu^p = \omega_t^n - \ln(1-\alpha) + n_t^n \underbrace{-a_t - (1-\alpha)n_t^n}_{-y_t^n} \Leftrightarrow -\mu^p = \omega_t^n - \ln(1-\alpha) + \frac{y_t^n - a_t}{(1-\alpha)} - y_t^n \Leftrightarrow$$

$$-\mu^p = \omega_t^n - \ln(1-\alpha) + \frac{y_t^n - a_t - y_t^n + \alpha y_t^n}{(1-\alpha)} \Leftrightarrow -\mu^p = \omega_t^n - \ln(1-\alpha) + \frac{\alpha y_t^n - a_t}{(1-\alpha)} \Leftrightarrow$$

And by using 1.47 we get:

$$-\mu^p = \omega_t^n - \ln(1-\alpha) + \frac{\alpha(\psi_{ya}^n a_t + \theta_y^n) - a_t}{(1-\alpha)} \Leftrightarrow -\mu^p = \omega_t^n - \ln(1-\alpha) + \frac{\left(\alpha\psi_{ya}^n - 1\right)a_t}{(1-\alpha)} + \frac{\alpha\theta_y^n}{(1-\alpha)} \Leftrightarrow$$

$$\omega_t^n = -\mu^p + \ln(1-\alpha) - \frac{\left(\alpha\psi_{ya}^n - 1\right)a_t}{(1-\alpha)} - \frac{\alpha\theta_y^n}{(1-\alpha)} \Leftrightarrow$$

$$\omega_t^n = -\mu^p + \underbrace{\frac{\left(1-\alpha\psi_{ya}^n\right)}{(1-\alpha)}}_{\psi_{wa}^n} a_t - \underbrace{\frac{\alpha\theta_y^n}{(1-\alpha)}}_{-\theta_w^n} + \ln(1-\alpha) \Leftrightarrow \boxed{\omega_t^n = -\mu^p + \psi_{wa}^n a_t + \theta_w^n + \ln(1-\alpha)} \text{ (2.28)}$$

Where: $\psi_{wa}^n \equiv \frac{\left(1-\alpha\psi_{ya}^n\right)}{(1-\alpha)}$ and $\theta_w^n \equiv -\frac{\alpha\theta_y^n}{(1-\alpha)}$

The change in the natural real wage is given by: $\Delta\omega_t^n \equiv \omega_t^n - \omega_{t-1}^n = \psi_{wa}^n \Delta a_t$ (2.29)

$$\hat{\mu}_t^p = \mu_t^p - \mu^p = \left(-\omega_t + mpn_t\right) - \left(-\omega_t^n + mpn_t^n\right) = \left(-\omega_t + \omega_t^n\right) + \left(mpn_t - mpn_t^n\right) =$$

$$= -\tilde{\omega}_t + \left[a_t + \ln(1-\alpha) - \alpha n_t - a_t - \ln(1-\alpha) + \alpha n_t^n\right] =$$

$$= -\tilde{\omega}_t - \left[\alpha n_t - \alpha n_t^n\right] = -\tilde{\omega}_t - \alpha\left[\frac{y_t - a_t}{(1-\alpha)} - \frac{y_t^n - a_t}{(1-\alpha)}\right] =$$

$$= -\tilde{\omega}_t - \alpha \frac{y_t - y_t^n}{(1-\alpha)} \Leftrightarrow \boxed{\hat{\mu}_t^p = -\tilde{\omega}_t - \alpha \frac{\tilde{y}_t}{(1-\alpha)}} \text{ (2.30)}$$

Finally, by inserting (2.30) onto equation (2.27) we get the **New Keynesian Phillips curve**:

$$\pi_t^p = \beta E_t \pi_{t+1}^p - \lambda_p \hat{\mu}_t^p \Leftrightarrow \pi_t^p = \beta E_t \pi_{t+1}^p - \lambda_p \left(-\tilde{\omega}_t - \alpha \frac{\tilde{y}_t}{(1-\alpha)} \right) \Leftrightarrow$$

$$\pi_t^p = \beta E_t \pi_{t+1}^p + \lambda_p \tilde{\omega}_t + \lambda_p \alpha \frac{\tilde{y}_t}{(1-\alpha)} \Leftrightarrow \boxed{\pi_t^p = \beta E_t \pi_{t+1}^p + \lambda_p \tilde{\omega}_t + \kappa_p \tilde{y}_t} \text{ (2.31) Where } \kappa_p \equiv \lambda_p \frac{\alpha}{(1-\alpha)}$$

Increased output and increased real wage lead to higher price inflation

Given the utility function (2.12) marginal rate of substitution in its linearized form is given by:

$$mrs_t = \sigma c_t + \varphi n_t = \sigma y_t + \varphi n_t \text{ (2.32)}$$

Thus, given market clearing conditions in the goods market and in the labor market, the log deviation in the economy's average wage markup from its steady state counterpart, where the wage markup written as $\mu_t^w = \omega_t - mrs_t$ is given by:

$$\hat{\mu}_t^w = \mu_t^w - \mu^w = \omega_t - mrs_t - \omega_t^n + mrs_t^n = \tilde{\omega}_t - \left[mrs_t - mrs_t^n \right] = \tilde{\omega}_t - \left[\sigma y_t + \varphi n_t - \sigma y_t^n - \varphi n_t^n \right] =$$

$$= \tilde{\omega}_t - \left[\sigma y_t + \varphi \frac{y_t - a_t}{(1-\alpha)} - \sigma y_t^n - \varphi \frac{y_t^n - a_t}{(1-\alpha)} \right] = \tilde{\omega}_t - \left[\sigma y_t + \varphi \frac{y_t - a_t}{(1-\alpha)} - \sigma y_t^n - \varphi \frac{y_t^n - a_t}{(1-\alpha)} \right] =$$

$$= \tilde{\omega}_t - \left[\sigma y_t + \frac{\varphi}{(1-\alpha)} y_t - \sigma y_t^n - \frac{\varphi}{(1-\alpha)} y_t^n \right] = \tilde{\omega}_t - \left(\sigma + \frac{\varphi}{(1-\alpha)} \right) \left(y_t - y_t^n \right) \Leftrightarrow$$

$$\hat{\mu}_t^w = \tilde{\omega}_t - \left(\sigma + \frac{\varphi}{(1-\alpha)} \right) \tilde{y}_t \text{ (2.33)}$$

Finally, by inserting (2.33) into (2.20) we get the **New Keynesian Wage Phillips Curve:**

$$\pi_t^w = \beta E_t \pi_{t+1}^w - \lambda_w \hat{\mu}_t^w \Leftrightarrow \pi_t^w = \beta E_t \pi_{t+1}^w - \lambda_w \left[\tilde{\omega}_t - \left(\sigma + \frac{\varphi}{(1-\alpha)} \right) \tilde{y}_t \right] \Leftrightarrow \pi_t^w = \beta E_t \pi_{t+1}^w - \lambda_w \tilde{\omega}_t + \lambda_w \left(\sigma + \frac{\varphi}{(1-\alpha)} \right) \tilde{y}_t \Leftrightarrow$$

$$\Leftrightarrow \boxed{\pi_t^w = \beta E_t \pi_{t+1}^w - \lambda_w \tilde{\omega}_t + \kappa_w \tilde{y}_t} \text{ (2.34)}$$

$$\text{Where } \kappa_w \equiv \lambda_w \left(\sigma + \frac{\varphi}{(1-\alpha)} \right)$$

With wage rigidity in addition to price rigidity, there is an identity relating changes in the real wage gap to wage inflation, price inflation and changes in the real wage:

$$\Delta\tilde{\omega}_t \equiv \tilde{\omega}_t - \tilde{\omega}_{t-1} \equiv \left(\omega_t - \omega_t^n\right) - \left(\omega_{t-1} - \omega_{t-1}^n\right) \equiv \Delta\omega_t - \Delta\omega_t^n \equiv$$
$$\equiv \left[w_t - p_t - w_{t-1} + p_{t-1}\right] - \Delta\omega_t^n \equiv \pi_t^w - \pi_t^p - \Delta\omega_t^n \Rightarrow$$

$$\Rightarrow \boxed{\tilde{\omega}_t \equiv \tilde{\omega}_{t-1} + \pi_t^w - \pi_t^p - \Delta\omega_t^n} \quad (2.35)$$

Where $\Delta\omega_t^n$ is given by (2.29) i.e. $\Delta\omega_t^n \equiv \omega_t^n - \omega_{t-1}^n = \psi_{wa}^n \Delta a_t$

In order to complete the model the non-policy block of the model, equilibrium conditions (2.31), (2.34), (2.35) must be supplemented with a dynamic IS equation. Given (2.27) and market clearing condition in goods market, aggregate output is given by:

$$y_t = E_t y_{t+1} + \frac{1}{\sigma}(\rho - i_t + E_t \pi_{t+1}^p) \quad (2.36)$$

Following the same procedure as section 1 we get the **Dynamic IS Equation***:*

$$\boxed{\tilde{y}_t = E_t \tilde{y}_{t+1} - \frac{1}{\sigma}\left(i_t - r_t^n - E_t \pi_{t+1}\right)} \quad (2.37)$$

Where $\boxed{r_t^n = \rho + \psi_{ya}^n E_t(a_{t+1} - a_t)}$ (2.38) and $\boxed{\hat{r}_t^n = r_t^n - \rho = \sigma\psi_{ya}^n E_t(a_{t+1} - a_t)}$ (2.39)

In order to close the model I assume that the central bank follows a rule, similar to Taylor's rule:

$$\boxed{i_t = \rho + \varphi_\pi \pi_t + \varphi_y \tilde{y}_t + \upsilon_t} \quad (1.55)$$

Standard reasoning implies that φ_π, φ_y and φ_w are non-negative.

2.4 Stability Conditions

Hence, we end up solving for a decentralized equilibrium:

We now have 5 equations to solve for the paths of five endogenous variables $\left\{\tilde{y}_t, \pi_t^p, \pi_t^w, \tilde{\omega}_t\right\}_{t=0}^{\infty}$ as functions of the exogenous shocks a_t and υ_t, given $\tilde{\omega}_{t-1}$ and also given the path of one policy instrument $\left\{i_t\right\}_{t=0}^{\infty}$. To simplify slightly, the nominal interest rate is inserted into the Dynamic IS equation, and the system of 4 equilibrium conditions are written in matrix form as:

$A_{w,0} x_t = A_{w,1} E_t\left\{x_{t+1}\right\} + B_w z_t$ where $x_t = \left[\tilde{y}_t \quad \pi_t^p \quad \pi_t^w \quad \tilde{\omega}_{t-1}\right]'$, $z_t \equiv \left[r_t^n - \upsilon_t \quad \Delta\omega_t^n\right]'$

In order to examine the uniqueness of equilibrium, the relevant matrix is $A_w \equiv A_{w,0}^{-1} A_{w,1}$, and for the system to fulfill the Blanchard Kahn conditions must hold that:

- The number of the stable roots of difference equations must be equal to the number of the three endogenous "control" variables $\tilde{y}_t, \pi_t^p, \pi_t^w$, i.e. three roots must be within the unit circle .

- The number of the unstable roots of difference equations must be equal to the number of the one predetermined variable $\tilde{\omega}_{t-1}$
- This corresponds to A_w having three characteristic roots within the unit circle, and one root outside the unit circle.

In this case an analytical solution is not available, but following Gali, numerical analysis indicates that $\varphi_p + \varphi_w > 1$ is a sufficient condition for equilibrium determinacy, being alike to a generalized Taylor principle.

In this setup, closing all gaps and having zero inflation rates is not generally feasible. Even if $r_t^n = \upsilon_t, \tilde{y}_t = \pi_t^p = \pi_t^w = 0$ will not be possible as productivity shocks will require changes in real wages, so only if $\Delta\omega_t^n = 0$, is a ''gapless'' equilibrium possible.

2.5 Effects of a Monetary Policy Shock

Klein's method Solution:

The economy is described by 5 equations:

$$\left\{\begin{array}{l} \tilde{\omega}_t \equiv \tilde{\omega}_{t-1} + \pi_t^w - \pi_t^p - \Delta\omega_t^n \\ \tilde{y}_t = E_t\tilde{y}_{t+1} - \dfrac{1}{\sigma}\left(i_t - r_t^n - E_t\pi_{t+1}\right) \\ \pi_t^p = \beta E_t\pi_{t+1}^p + \lambda_p\tilde{\omega}_t + \kappa_p\tilde{y}_t \\ \pi_t^w = \beta E_t\pi_{t+1}^w - \lambda_w\tilde{\omega}_t + \kappa_w\tilde{y}_t \\ i_t = \rho + \varphi_\pi\pi_t + \varphi_y\tilde{y}_t + \varphi_w\tilde{\omega}_t + \upsilon_t \end{array}\right\} \Leftrightarrow$$

$$\Leftrightarrow \left\{\begin{array}{l} \tilde{\omega}_t \equiv \tilde{\omega}_{t-1} + \pi_t^w - \pi_t^p - \Delta\omega_t^n \\ \tilde{y}_t = E_t\tilde{y}_{t+1} - \dfrac{1}{\sigma}\left(\rho + \varphi_\pi\pi_t + \varphi_y\tilde{y}_t + \varphi_w\tilde{\omega}_t + \upsilon_t - r_t^n - E_t\pi_{t+1}\right) \\ \pi_t^p = \beta E_t\pi_{t+1}^p + \lambda_p\tilde{\omega}_t + \kappa_p\tilde{y}_t \\ \pi_t^w = \beta E_t\pi_{t+1}^w - \lambda_w\tilde{\omega}_t + \kappa_w\tilde{y}_t \end{array}\right\} \Leftrightarrow$$

$$\Leftrightarrow \left\{\begin{array}{l} \tilde{\omega}_t \equiv \tilde{\omega}_{t-1} + \pi_t^w - \pi_t^p - \Delta\omega_t^n \\ \sigma\tilde{y}_t = \sigma E_t\tilde{y}_{t+1} - \rho - \varphi_\pi\pi_t - \varphi_y\tilde{y}_t - \varphi_w\tilde{\omega}_t - \upsilon_t + \rho + \psi_{ya}^n E_t(a_{t+1} - a_t) + E_t\pi_{t+1} \\ \pi_t^p = \beta E_t\pi_{t+1}^p + \lambda_p\tilde{\omega}_t + \kappa_p\tilde{y}_t \\ \pi_t^w = \beta E_t\pi_{t+1}^w - \lambda_w\tilde{\omega}_t + \kappa_w\tilde{y}_t \end{array}\right\} \Leftrightarrow$$

$$\Leftrightarrow \left\{ \begin{array}{l} \tilde{\omega}_t \equiv \tilde{\omega}_{t-1} + \pi_t^w - \pi_t^p - \psi_{wa}^n E_t(a_t - a_{t-1}) \\ \sigma E_t \tilde{y}_{t+1} + E_t \pi_{t+1} = (\sigma + \varphi_y)\tilde{y}_t + \varphi_\pi \pi_t + \varphi_w \tilde{\omega}_t + \upsilon_t - \psi_{ya}^n E_t(a_{t+1} - a_t) \\ \beta E_t \pi_{t+1}^p = \pi_t^p - \lambda_p \tilde{\omega}_t - \kappa_p \tilde{y}_t \\ \beta E_t \pi_{t+1}^w = \pi_t^w + \lambda_w \tilde{\omega}_t - \kappa_w \tilde{y}_t \end{array} \right\} \Leftrightarrow$$

To examine the effects of a monetary shock assume: $\rho_\upsilon \in [0,1) : \upsilon_t = \rho_\upsilon \upsilon_{t-1} + \varepsilon_t^\upsilon$ (K.1) where $\varepsilon_t^\upsilon \sim WN(s^2)$ and $a_t = a_{t+1} = a_{t-1} = 1 \Rightarrow a_{t+1} - a_t = 0$ and $a_t - a_{t-1} = 0$

I will make use of Klein's Method, so I will write the system as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sigma & 1 & 0 \\ -\lambda_p & 0 & \beta & 0 \\ -\lambda_w & 0 & 0 & \beta \end{bmatrix} E_t \begin{bmatrix} \tilde{\omega}_t \\ \tilde{y}_{t+1} \\ \pi_{t+1}^P \\ \pi_{t+1}^w \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & (\sigma + \varphi_y) & \varphi_\pi & \varphi_w \\ 0 & -\kappa_P & 1 & 0 \\ 0 & -\kappa_w & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\omega}_{t-1} \\ \tilde{y}_t \\ \pi_t^P \\ \pi_t^w \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \upsilon_t$$

$$\Updownarrow$$

$$AE_t S_{t+1}^o = BS_t^o + C\upsilon_t \text{ (K.2)}$$

Where: $S_t^o = \begin{bmatrix} S_{1t}^o \\ S_{2t}^o \end{bmatrix}$ and so $E_t S_{t+1}^o = \begin{bmatrix} S_{1t+1}^o \\ E_t S_{2t+1}^o \end{bmatrix}$

Let $S_t^o = \begin{bmatrix} \tilde{\omega}_{t-1} & \tilde{y}_t & \pi_t^p & \pi_t^w \end{bmatrix}'$ and $A \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sigma & 1 & 0 \\ -\lambda_p & 0 & \beta & 0 \\ -\lambda_w & 0 & 0 & \beta \end{bmatrix}, B \equiv \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & (\sigma + \varphi_y) & \varphi_\pi & \varphi_w \\ 0 & -\kappa_P & 1 & 0 \\ 0 & -\kappa_w & 0 & 1 \end{bmatrix}, C \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Schur Decomposition: $QAZ = S$ and $QBZ = T$ (K.3)

Q and Z are unitary and S and T are upper triangular matrices. The generalized eigenvalues of A and B can be recovered as the ratios of the diagonal elements of T and S:

$$\lambda(B, A) = \left\{ {T_{ii}}/{S_{ii}} \,\middle|\, i = 1,2,3 \right\}$$

The matrices Q, Z, S and T can always be arranged so that the generalized eigenvalues are ordered in increasing value in moving from left to right. In the vector S_t^o one variable is predetermined and two are control variables.

- Proposition 1:if the number of eigenvalues outside the unit circle is equal to the number of control variables, then there is a unique solution to the system

- Proposition 2:if the number of eigenvalues outside the unit circle is greater than the number of control variables, then there is no solution to the system
- Proposition 3:if the number of eigenvalues outside the unit circle is less than the number of control variables, than there is an infinite number of solutions.

The matrices Q, Z, S and T are partitioned, so that:

$$Q = \begin{bmatrix} Q_1 \\ 1\times 4 \\ Q_2 \\ 3\times 4 \end{bmatrix}, \quad Z = \begin{bmatrix} Z_{11} & Z_{12} \\ 1\times 1 & 1\times 3 \\ Z_{21} & Z_{22} \\ 3\times 1 & 3\times 3 \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} & S_{12} \\ 1\times 1 & 1\times 3 \\ O & S_{22} \\ 3\times 1 & 3\times 3 \end{bmatrix}, \quad T = \begin{bmatrix} T_{11} & T_{12} \\ 1\times 1 & 1\times 3 \\ O & T_{22} \\ 3\times 1 & 3\times 3 \end{bmatrix}$$

In order to "triangularize" the system I first define the vector of auxiliary variables as: $\boxed{S_t^1 = Z^H S_t^o}$ (K.4)

Letting Z^H denote the Hermitian conjugate transpose of Z, so that $S_t^1 = \begin{bmatrix} S_{1t}^1 \\ S_{2t}^1 \end{bmatrix}$ where

$$\underset{1\times 1}{S_{1t}^1} = \underset{1\times 1}{Z_{11}^H}\ \underset{1\times 1}{\tilde{\omega}_{t-1}} + \underset{1\times 3}{Z_{21}^H} \begin{bmatrix} \tilde{y}_t \\ \pi_t^p \\ \pi_t^w \end{bmatrix}_{3\times 1} \qquad \underset{3\times 1}{S_{2t}^1} = \underset{3\times 1}{Z_{12}^H}\ \underset{1\times 1}{\tilde{\omega}_{t-1}} + \underset{3\times 3}{Z_{22}^H} \begin{bmatrix} \tilde{y}_t \\ \pi_t^p \\ \pi_t^w \end{bmatrix}_{3\times 1} \qquad \text{(K.5)}$$

Since Z is unitary, $Z^H Z = I$ or $Z^H = Z^{-1}$ and hence $S_t^1 = Z^{-1} S_t^o \Rightarrow Z S_t^1 = \underset{I}{Z Z^{-1}} S_t^o \Rightarrow \boxed{S_t^o = Z S_t^1}$

I use this property to rewrite the system as:

$$AZE_t S_{t+1}^o = BZS_t^o + C\upsilon_t \quad \text{(K.6)}$$

Premultiplying (K.6) by Q and then using (K.3) yields:

$$QAZE_t S_{t+1}^o = QBZS_t^o + QC\varepsilon_t^\upsilon \Rightarrow SE_t S_{t+1}^o = TS_t^o + QC\varepsilon_t^\upsilon \Rightarrow$$

$$\begin{bmatrix} S_{11} & S_{12} \\ 1\times 1 & 1\times 3 \\ O & S_{22} \\ 3\times 1 & 3\times 3 \end{bmatrix} E_t \begin{bmatrix} S_{1t+1}^1 \\ 1\times 1 \\ S_{2t+1}^1 \\ 3\times 1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 1\times 1 & 1\times 3 \\ O & T_{22} \\ 3\times 1 & 3\times 3 \end{bmatrix} \begin{bmatrix} S_{1t}^1 \\ 1\times 1 \\ S_{2t}^1 \\ 3\times 1 \end{bmatrix} + \begin{bmatrix} Q_1 \\ 1\times 4 \\ Q_2 \\ 3\times 4 \end{bmatrix} \underset{4\times 1}{C}\ \underset{1\times 1}{\upsilon_t}$$

$$\Updownarrow$$

$$\left\{ \begin{array}{c} \underset{1\times 1}{S_{11}}\ E_t\ \underset{1\times 1}{S_{1t+1}^1} + \underset{1\times 3}{S_{12}}\ E_t\ \underset{3\times 1}{S_{2t+1}^1} = \underset{1\times 1}{T_{11}}\ \underset{1\times 1}{S_{1t}^1} + \underset{1\times 3}{T_{12}}\ \underset{3\times 1}{S_{2t}^1} + \underset{1\times 4}{Q_1}\ \underset{4\times 1}{C}\ \underset{1\times 1}{\upsilon_t} \\ \underset{3\times 3}{S_{22}}\ E_t\ \underset{3\times 1}{S_{2t+1}^1} = \underset{3\times 3}{T_{22}}\ \underset{3\times 1}{S_{2t}^1} + \underset{3\times 4}{Q_2}\ \underset{4\times 1}{C}\ \underset{1\times 1}{\upsilon_t} \end{array} \right\}$$

Due to block diagonal (recursive) structure, the process $\underset{3\times 1}{S_{2t}^1}$ is unrelated with $\underset{1\times 1}{S_{1t}^1}$ and I can solve it independently forward :

$$\underset{3\times3}{S_{22}}\,\underset{3\times1}{E_t\,S^1_{2t+1}}=\underset{3\times3}{T_{22}}\,\underset{3\times1}{S^1_{2t}}+\underset{3\times4}{Q_2}\,\underset{4\times1}{C}\,\underset{1\times1}{\upsilon_t}\Rightarrow \underset{3\times3}{S_{22}}\,\underset{3\times1}{E_t\,S^1_{2t+1}}-\underset{3\times3}{T_{22}}\,\underset{3\times1}{S^1_{2t}}=\underset{3\times4}{Q_2}\,\underset{4\times1}{C}\,\underset{1\times1}{\upsilon_t}\Rightarrow\left(\underset{3\times3}{S_{22}}F-\underset{3\times3}{T_{22}}\right)\underset{3\times1}{S^1_{2t}}=\underset{3\times4}{Q_2}\,\underset{4\times1}{C}\,\underset{1\times1}{\upsilon_t}\Rightarrow$$

$$\left(\underset{3\times3}{T_{22}}-\underset{3\times3}{S_{22}}F\right)\underset{3\times1}{S^1_{2t}}=-\underset{3\times4}{Q_2}\,\underset{4\times1}{C}\,\underset{1\times1}{\upsilon_t}\Rightarrow\left(\underset{3\times3}{I}-\underset{3\times3}{S_{22}}\,\underset{3\times3}{T_{22}^{-1}}F\right)\underset{3\times3}{T_{22}}\,\underset{3\times1}{S^1_{2t}}=-\underset{3\times4}{Q_2}\,\underset{4\times1}{C}\,\underset{1\times1}{\upsilon_t}\Rightarrow$$

$$\underset{3\times3}{T_{22}}\,\underset{3\times1}{S^1_{2t}}=-\left(\underset{3\times3}{I}-\underset{3\times3}{S_{22}}\,\underset{3\times3}{T_{22}^{-1}}F\right)^{-1}\underset{3\times4}{Q_2}\,\underset{4\times1}{C}\,\underset{1\times1}{\upsilon_t}\Rightarrow\underset{3\times1}{S^1_{2t}}=-\underset{3\times3}{T_{22}^{-1}}\left(\underset{3\times3}{I}-\underset{3\times3}{S_{22}}F\right)^{-1}\underset{3\times4}{Q_2}\,\underset{4\times1}{C}\,\underset{1\times1}{\varepsilon_t^{\upsilon}}\Rightarrow$$

$$\underset{3\times1}{S^1_{2t}}=-\underset{3\times3}{T_{22}^{-1}}\left[E_t\sum_{j=0}^{\infty}\left(S_{22}T_{22}^{-1}\right)^j Q_2C\,\underset{1\times1}{\upsilon_{t+j}}\right]$$

But also if we iterate the $\upsilon_t=\rho_\upsilon\upsilon_{t-1}+\varepsilon_t^{\upsilon}$ forward we get that: $E_t\upsilon_{t+j}=\rho_\upsilon^j\upsilon_t$

$$\Rightarrow\underset{2\times1}{S^1_{2t}}=-\underset{2\times2}{T_{22}^{-1}}\left[E_t\sum_{j=0}^{\infty}\left(S_{22}T_{22}^{-1}\right)^j Q_2C\rho_\upsilon^j\upsilon_t\right]\Rightarrow-\underset{2\times2}{T_{22}^{-1}}\left[\sum_{j=0}^{\infty}\left(S_{22}T_{22}^{-1}\rho_\upsilon\right)^j Q_2C\upsilon_t\right]\Rightarrow$$

where $\underset{3\times1}{\left[\sum_{j=0}^{\infty}\left(S_{22}T_{22}^{-1}\rho_\upsilon\right)^j Q_2C\right]=R}$

$$\underset{2\times1}{vec(R)}=vec\left[\sum_{j=0}^{\infty}\left(S_{22}T_{22}^{-1}\rho_\upsilon\right)^j Q_2C\right]$$

And I use the property: $vec(AXB)=(B^T\otimes A)vec(X)$,where X here is $\underset{1\times1}{I^j}$

$$=\sum_{j=0}^{\infty}\left[\underset{1\times1}{\rho_\upsilon{}^j}\otimes\left(S_{22}T_{22}^{-1}\right)^j\right]vec\left(Q_2C\right)=\sum_{j=0}^{\infty}\left[\underset{1\times1}{\rho_\upsilon}\otimes\left(S_{22}T_{22}^{-1}\right)\right]^j vec\left(Q_2C\right)=$$

$$=\underbrace{\left[\underset{3\times3}{I}-\underset{1\times1}{\rho_\upsilon}\otimes\underset{3\times3}{\left(S_{22}T_{22}^{-1}\right)}\right]^{-1}}_{3\times3}\underbrace{vec\left(\underset{3\times4}{Q_2}\,\underset{4\times1}{C}\right)}_{3\times1}$$

Finally: $\boxed{\underset{3\times1}{S^1_{2t}}=-\underset{3\times3}{T_{22}^{-1}}\,\underset{3\times1}{R}\,\underset{1\times1}{\upsilon_t}}$ (K.7)

Using the above result allows to solve for $\tilde{y}_t$ and π_t from equation (K.5):

$$\underset{3\times1}{\begin{bmatrix}\tilde{y}_t\\ \pi_t^p\\ \pi_t^w\end{bmatrix}}=-\underset{3\times3}{\left(Z_{22}^H\right)^{-1}}\,\underset{3\times1}{Z_{12}^H}\,\underset{1\times1}{\tilde{\omega}_{t-1}}-\underset{3\times1}{\left(Z_{22}^H\right)^{-1}}\,\underset{3\times3}{T_{22}^{-1}}\,\underset{3\times1}{R}\,\underset{1\times1}{\upsilon_t}\quad\text{(K.8)}$$

I have to find some expressions for $-\underset{3\times1}{\left(Z_{22}^{H}\right)^{-1}}\underset{3\times1}{Z_{12}^{H}}$ and $\underset{3\times3}{\left(Z_{22}^{H}\right)^{-1}}$

under the assumption that Z us unitary, i.e.

$$\underbrace{\begin{bmatrix} \underset{1\times1}{Z_{11}^{H}} & \underset{1\times3}{Z_{21}^{H}} \\ \underset{3\times1}{Z_{12}^{H}} & \underset{3\times3}{Z_{22}^{H}} \end{bmatrix}}_{Z^{H}}\underbrace{\begin{bmatrix} \underset{1\times1}{Z_{11}} & \underset{1\times3}{Z_{12}} \\ \underset{3\times1}{Z_{21}} & \underset{3\times3}{Z_{22}} \end{bmatrix}}_{Z}=\begin{bmatrix} \underset{1\times1}{I} & \underset{1\times3}{O} \\ \underset{3\times1}{O} & \underset{3\times3}{I} \end{bmatrix}$$

I find that:

$$\underset{1\times1}{Z_{11}^{H}}\underset{1\times1}{Z_{11}}+\underset{1\times3}{Z_{21}^{H}}\underset{3\times1}{Z_{21}}=\underset{1\times1}{I}\Rightarrow\underset{1\times1}{Z_{11}^{H}}+\underset{1\times3}{Z_{21}^{H}}\underset{3\times1}{Z_{21}}\underset{1\times1}{Z_{11}^{-1}}=\underset{1\times1}{Z_{11}^{-1}}\ \text{(K.9)}$$

$$\underset{1\times1}{Z_{11}^{H}}\underset{1\times3}{Z_{12}}+\underset{1\times3}{Z_{21}^{H}}\underset{3\times3}{Z_{22}}=\underset{1\times3}{O}\Rightarrow\underset{1\times3}{Z_{21}^{H}}\underset{3\times3}{Z_{22}}=-\underset{1\times1}{Z_{11}^{H}}\underset{1\times3}{Z_{12}}\quad\text{(K.10)}$$

$$\underset{3\times1}{Z_{12}^{H}}\underset{1\times1}{Z_{11}}+\underset{3\times3}{Z_{22}^{H}}\underset{3\times1}{Z_{21}}=\underset{3\times1}{O}\Rightarrow\underset{3\times3}{Z_{22}^{H^{-1}}}\underset{3\times1}{Z_{12}^{H}}\underset{1\times1}{Z_{11}}+\underset{3\times3}{Z_{22}^{H^{-1}}}\underset{3\times3}{Z_{22}^{H}}\underset{3\times1}{Z_{21}}=\underset{3\times1}{O}\Rightarrow-\underset{3\times3}{Z_{22}^{H^{-1}}}\underset{3\times1}{Z_{12}^{H}}=\underset{3\times1}{Z_{21}}\underset{1\times1}{Z_{11}^{-1}}\ \text{(K.11)}$$

$$\underset{3\times1}{Z_{12}^{H}}\underset{1\times3}{Z_{12}}+\underset{3\times3}{Z_{22}^{H}}\underset{3\times3}{Z_{22}}=\underset{3\times3}{I}\Rightarrow\underset{3\times3}{Z_{22}^{H^{-1}}}\underset{3\times1}{Z_{12}^{H}}\underset{1\times3}{Z_{12}}+\underset{3\times3}{Z_{22}}=\underset{3\times3}{Z_{22}^{H^{-1}}}\ \text{and by (K.9)}\ \underset{3\times3}{Z_{22}^{H^{-1}}}=\underset{3\times3}{Z_{22}}-\underset{3\times1}{Z_{21}}\underset{1\times1}{Z_{11}^{-1}}\underset{1\times3}{Z_{12}}\ \text{(K.12)}$$

So now using (K.11) and (K.12), (K.8) can be rewritten as:

$$\underset{3\times1}{\begin{bmatrix}\tilde{y}_t\\ \pi_t^{p}\\ \pi_t^{w}\end{bmatrix}}=\underset{3\times1}{Z_{21}}\underset{1\times1}{Z_{11}^{-1}}\underset{1\times1}{\tilde{\omega}_{t-1}}-\left[\underset{3\times3}{Z_{22}}-\underset{3\times1}{Z_{21}}\underset{1\times1}{Z_{11}^{-1}}\underset{1\times3}{Z_{12}}\right]\underset{3\times3}{T_{22}^{-1}}\underset{3\times1}{R}\underset{1\times1}{\upsilon_t}\Rightarrow\quad\underset{3\times1}{\begin{bmatrix}\tilde{y}_t\\ \pi_t^{p}\\ \pi_t^{w}\end{bmatrix}}=\underset{3\times1}{\mathrm{M}_1}\underset{1\times1}{\tilde{\omega}_{t-1}}+\underset{3\times1}{\mathrm{M}_2}\underset{1\times1}{\upsilon_t}\quad\text{(K.13)Where:}$$

$$\underset{3\times1}{\mathrm{M}_1}\equiv\underset{3\times1}{Z_{21}}\underset{1\times1}{Z_{11}^{-1}}\ \text{and}\ \underset{3\times1}{\mathrm{M}_2}\equiv-\left[\underset{3\times3}{Z_{22}}-\underset{3\times1}{Z_{21}}\underset{1\times1}{Z_{11}^{-1}}\underset{1\times3}{Z_{12}}\right]\underset{3\times3}{T_{22}^{-1}}\underset{3\times1}{R}$$

Now I can solve for $\underset{1\times1}{S_{1t}^{1}}$:

$$\underset{1\times1}{S_{1t}^{1}}=\underset{1\times1}{Z_{11}^{H}}\underset{1\times1}{\tilde{\omega}_{t-1}}+\underset{1\times3}{Z_{21}^{H}}\underset{3\times1}{\begin{bmatrix}\tilde{y}_t\\ \pi_t^{p}\\ \pi_t^{w}\end{bmatrix}}=\underset{1\times1}{Z_{11}^{H}}\underset{1\times1}{\tilde{\omega}_{t-1}}+\underset{1\times3}{Z_{21}^{H}}\left[\underset{3\times1}{\mathrm{M}_1}\underset{1\times1}{\tilde{\omega}_{t-1}}+\underset{3\times1}{\mathrm{M}_2}\underset{1\times1}{\upsilon_t}\right]=\left[\underset{1\times1}{Z_{11}^{H}}+\underset{1\times3}{Z_{21}^{H}}\underset{3\times1}{\mathrm{M}_1}\right]\underset{1\times1}{\tilde{\omega}_{t-1}}+\underset{1\times3}{Z_{21}^{H}}\underset{3\times1}{\mathrm{M}_2}\underset{1\times1}{\upsilon_t}\Rightarrow$$

$$\underset{1\times1}{S_{1t}^{1}}=\left[\underset{1\times1}{Z_{11}^{H}}+\underset{1\times3}{Z_{21}^{H}}\underset{3\times1}{Z_{21}}\underset{1\times1}{Z_{11}^{-1}}\right]\underset{1\times1}{\tilde{\omega}_{t-1}}-\underset{1\times3}{Z_{21}^{H}}\left[\underset{3\times3}{Z_{22}}-\underset{3\times1}{Z_{21}}\underset{1\times1}{Z_{11}^{-1}}\underset{1\times3}{Z_{12}}\right]\underset{3\times3}{T_{22}^{-1}}\underset{3\times1}{R}\underset{1\times1}{\upsilon_t}\ \text{by equation (K.9)}\Rightarrow$$

$$\underset{1\times1}{S^1_{1t}} = \underset{1\times1}{Z_{11}}^{-1}\underset{1\times1}{\tilde{\omega}_{t-1}} - \left[\underset{1\times3}{Z^H_{21}}\underset{3\times3}{Z_{22}} - \underset{1\times3}{Z^H_{21}}\underset{3\times1}{Z_{21}}\underset{1\times1}{Z_{11}}^{-1}\underset{1\times3}{Z_{12}}\right]\underset{3\times3}{T_{22}^{-1}}\underset{3\times1}{R}\underset{1\times1}{\upsilon_t} = \underset{1\times1}{Z_{11}}^{-1}\underset{1\times1}{\tilde{\omega}_{t-1}} - \left[-\underset{1\times1}{Z^H_{11}}\underset{1\times3}{Z_{12}} - \underset{1\times3}{Z^H_{21}}\underset{3\times1}{Z_{21}}\underset{1\times1}{Z_{11}}^{-1}\underset{1\times3}{Z_{12}}\right]\underset{3\times3}{T_{22}^{-1}}\underset{3\times1}{R}\underset{1\times1}{\upsilon_t} \Rightarrow$$

$$\underset{1\times1}{S^1_{1t}} = \underset{1\times1}{Z_{11}}^{-1}\underset{1\times1}{\tilde{\omega}_{t-1}} - \left[-\underset{1\times1}{Z^H_{11}} - \underset{1\times3}{Z^H_{21}}\underset{3\times1}{Z_{21}}\underset{1\times1}{Z_{11}}^{-1}\right]\underset{1\times3}{Z_{12}}\underset{3\times3}{T_{22}^{-1}}\underset{3\times1}{R}\underset{1\times1}{\upsilon_t} \Rightarrow \underset{1\times1}{S^1_{1t}} = \underset{1\times1}{Z_{11}}^{-1}\underset{1\times1}{\tilde{\omega}_{t-1}} + \underbrace{\left[\underset{1\times1}{Z^H_{11}} + \underset{1\times3}{Z^H_{21}}\underset{3\times1}{Z_{21}}\underset{1\times1}{Z_{11}}^{-1}\right]}_{Z_{11}^{-1}}\underset{1\times3}{Z_{12}}\underset{3\times3}{T_{22}^{-1}}\underset{3\times1}{R}\underset{1\times1}{\upsilon_t} \Rightarrow$$

$$\boxed{\underset{1\times1}{S^1_{1t}} = \underset{1\times1}{Z_{11}}^{-1}\underset{1\times1}{\tilde{\omega}_{t-1}} + \underset{1\times1}{Z_{11}}^{-1}\underset{1\times3}{Z_{12}}\underset{3\times3}{T_{22}^{-1}}\underset{3\times1}{R}\underset{1\times1}{\upsilon_t}} \quad \text{(K.14)}$$

And I plug (K.14) into the upper part of the system (K.6) and get:

$$\boxed{\underset{1\times1}{\tilde{\omega}_t} = \mathrm{M}_3\underset{1\times1}{\tilde{\omega}_{t-1}} + \mathrm{M}_4\underset{1\times1}{\upsilon_t}} \quad \text{(K.15)}$$

Where $\underset{1\times1}{\mathrm{M}_3} = \underset{1\times1}{Z_{11}}\underset{1\times1}{S_{11}}^{-1}\underset{1\times1}{T_{11}}\underset{1\times1}{Z_{11}}^{-1}$ and

$$\underset{1\times1}{\mathrm{M}_4} = \underset{1\times1}{Z_{11}}\underset{1\times1}{S_{11}}^{-1}\left[\underset{1\times1}{T_{11}}\underset{1\times1}{Z_{11}}^{-1}\underset{1\times3}{Z_{12}}\underset{3\times3}{T_{22}^{-1}}\underset{3\times1}{R} + \underset{1\times4}{Q_1}\underset{4\times1}{C} + \underset{1\times3}{S_{12}}\underset{3\times3}{T_{22}^{-1}}\underset{3\times1}{R} - \underset{1\times3}{T_{12}}\underset{3\times3}{T_{22}^{-1}}\underset{3\times1}{R}\right] - \underset{1\times3}{Z_{12}}\underset{3\times3}{T_{22}^{-1}}\underset{3\times1}{R}$$

Hence, the model's solution can be written compactly in State Space form by combining the above as:

$$\left.\begin{cases} s_{t+1} = \Gamma_0 s_t + \Gamma_1\underset{1\times1}{\varepsilon^{\upsilon}_{t+1}} \\ f_t = \Gamma_2 s_t \end{cases}\right\} \begin{matrix} \text{predetermined variables} \\ \text{control variables} \end{matrix}$$

Where: $k_t = \begin{bmatrix} \tilde{\omega}_{t-1} \\ \upsilon_{t-1} \end{bmatrix}$, $f_t = \begin{bmatrix} \tilde{y}_t \\ \pi^p_t \\ \pi^w_t \end{bmatrix}$, $\Gamma_0 = \begin{bmatrix} \mathrm{M}_3 & \mathrm{M}_4 \\ 0 & \rho_\upsilon \end{bmatrix}$, $\Gamma_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\Gamma_2 = \begin{bmatrix} \mathrm{M}_1 & \mathrm{M}_2 \end{bmatrix}$

In the Appendix 3 I have the MATLAB coded putting on practice all the aforementioned.

Numerical Results

Simulated impulse responses are shown in figure 2. The presence of sticky wages and prices generates a much smaller inflation decline than with only sticky prices (figure 1). The reason is that when wages are flexible, a monetary policy shock leads to a large decline in wage inflation. Here, instead, since also wages are sticky, the inflationary contraction is divided between prices and wages. As a result, real wage does not change at all. This in turn reduces the impact of decline in activity on the real marginal cost and, hence, the limited size on inflation response. Overall, the limited response of price and wage inflation following a monetary policy shock seem more in line with data than the large responses we found in figure 1. Also, the effect on the real wage in figure 2 seems much more plausible.

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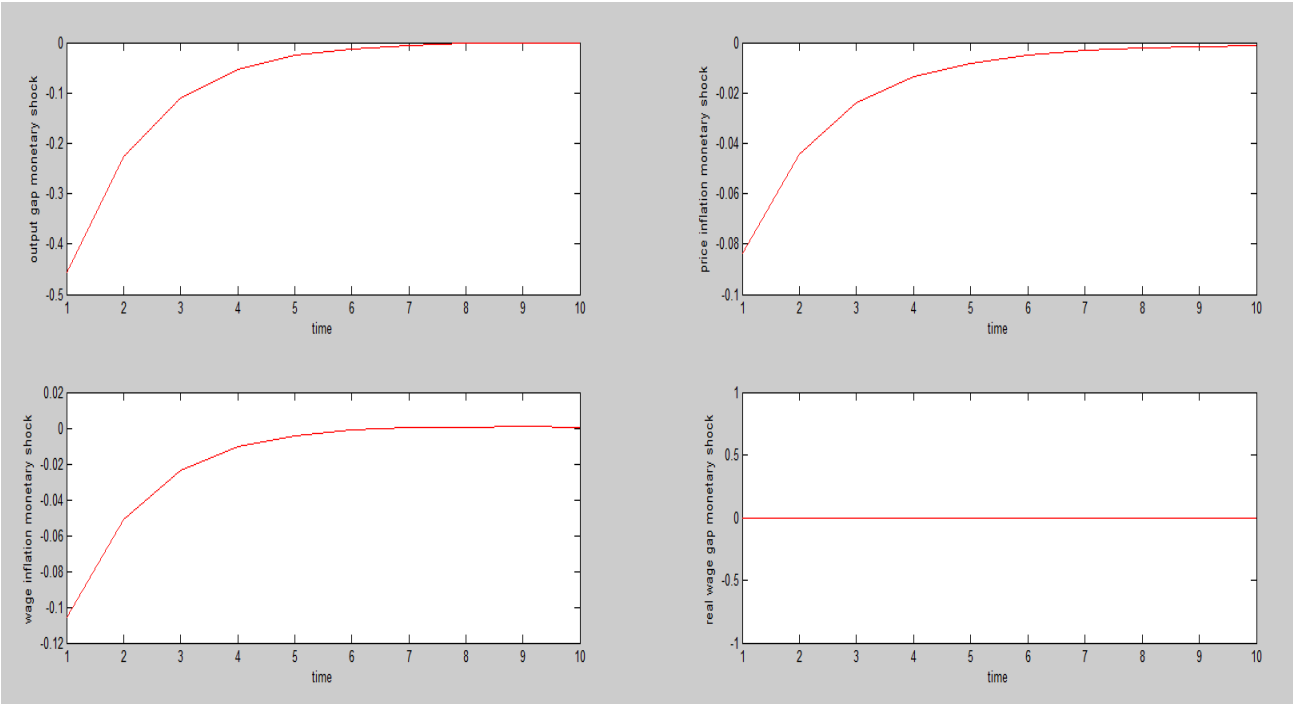

(Figure 2.)

Two different solution methods were implement in MATLAB (Klein, Blanchard Kahn) and yielded identical results up to 15 decimals.

Empirical facts

Empirical estimates of dynamic responses to monetary policy shocks, suggest that:

- The strong response of wage inflation and the very persistent and strong response of real wages in the sticky prices model seems to be unrealistic
- The response of the model including sticky prices and sticky wages seems to be in line with empirical findings concerning the paper of Christiano, Eichenbaum and Evans (2005) and they concluded “A key finding of the analysis is that stickiness in nominal wages is crucial for the model’s performance. Stickiness in prices plays a relatively small role”.

3. Critique-Extensions

There are two major limitations of the new Keynesian IS curve. First, and most obviously, it leaves out investment, government purchases, and net exports. Virtually every model intended for practical use includes investment modeled as arising from the decisions of profit maximizing firms and its absence seems odd since investments have huge business cycles. Government purchases are almost always included as well; they are generally modeled exogenous. Second the basic new Keynesian IS curve, even when it is extended to include other components of output, tends to imply large and rapid responses to shocks. To better match the data, the models therefore generally include ingredients that slow adjustment. With regard to consumption, the most common approach is to assume habit formation and under appropriate assumptions, this slows the response of consumption to shocks. If the model includes investments the most common way of slowing responses is to assume directly that there are costs of adjusting investment.

But also there are objections about the building blocks of this era of New Keynesian models:

- Are households and firms really as forward-looking and rational? Empirically: autocorrelated forecast errors. In Rudd-Whelan paper "Does the Labor Share of Income Drive Inflation?" it is presented that reveals a shockingly bad performance for the NKPC model of pure forward looking relationship. The present value of output gaps turns out to be negatively correlated with inflation. So, not only does inflation not appear to equal this present value, it does not even have the correctly-signed correlation.
- Calvo price setting cannot match all micro facts: Alvarez and Burriel (2010)
- Inflation rate is jump variable. Empirical research e.g. Rudd and Whelan "Can Rational Expectations Sticky-Price Models Explain Inflation Dynamics?" claims that inflation is very persistent. Many economists add lags in ad-hoc manner to the new Keynesian Phillips curve in order to achieve that desired characteristic (hybrid types of NKPC).
- Labour market is in equilibrium, as it was shown in both that models. The empirical evidence of course shows high unemployment rates. But, latest models, even Gali's (2011) or Blanchard and Gali (2008), possess that feature.
- Capital accumulation is excluded, as it is mainly a depiction of short run effects. Empirical evidence recommends that investments have huge business cycles.

Conclusions

The model with both sticky wages and prices compared to the benchmark case of flexible wages, but sticky prices, implies at first a very modest response of wage inflation and the real wage. Furthermore implies a very small decline in inflation, since the modest response of real wages reduces the impact of variations in real activity on real marginal costs and hence on inflation. Also implies a strong fall of the output gap, as the very small decline in inflation implies that the central bank keeps the interest rate higher than in the benchmark case. Finally as Christiano, Eichenbaum and Evans mention in their paper "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy" (2005) the responses of the model including sticky prices and wages seems to be in line with empirical findings.

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Appendix 1

The probability of getting to change prices every period is $1-\theta$ and consequently the probability of not changing prices is θ. So, if the firm have changed its prices now the probability to change prices is the next k period is:

k	Pr
1	$1-\theta$
2	$(1-\theta)\theta$
3	$(1-\theta)\theta^2$
4	$(1-\theta)\theta^3$
⋮	⋮
k	$(1-\theta)\theta^k$

The expected duration between price changes is then the sum of probabilities times duration i.e

$$E_t D = \sum_{k=1}^{\infty}(1-\theta)\theta^{k-1}k = (1-\theta)\sum_{k=1}^{\infty}\theta^{k-1}k$$

Where the sum $\sum_{k=1}^{\infty}\theta^{k-1}k$ is an arithmetico-geometric sequence and we can handle it like below:

$$S_n = \sum_{k=1}^{\infty}\theta^{k-1}k = 1+2\theta+3\theta^2+4\theta^3\ldots$$

Multiply S_n by θ:

$$\theta S_n = \theta+2\theta^2+3\theta^3+4\theta^4\ldots$$

Subtract θS_n from S_n to get:

$$S_n-\theta S_n = 1+(2-1)\theta+(3-2)\theta^2+(4-3)\theta^3+(5-4)\theta^4\ldots$$

$$(1-\theta)S_n = 1+\theta+\theta^2+\theta^3+\theta^4\ldots$$

And since $0<\theta<1$ it holds that: $(1-\theta)S_n = \frac{1}{(1-\theta)} \Leftrightarrow S_n = \frac{1}{(1-\theta)^2}$

Plugging this back in to the original expression yields:

Expected Duration between Price changes $= (1-\theta)\frac{1}{(1-\theta)^2} = \frac{1}{(1-\theta)}$

Appendix 2

The profit maximizing problem of the firms if they can adjust prices every period is:

$$\left.\begin{array}{l}\max\limits_{Y_{it}} \pi_t = P_{it}Y_{it} - \Psi' Y_{it} \\ st \quad Y_{it} = \left(\dfrac{P_{it}}{P_t}\right)^{-\varepsilon} Y_t\end{array}\right\} \Leftrightarrow \max_{P_{it}} \pi_t = P_{it}\left(\frac{P_{it}}{P_t}\right)^{-\varepsilon} Y_t - \Psi'\left(\frac{P_{it}}{P_t}\right)^{-\varepsilon} Y_t$$

FONC

$$(1-\varepsilon)\left(\frac{P_{it}}{P_t}\right)^{-\varepsilon} Y_t + \varepsilon \frac{1}{P_{it}} \Psi'\left(\frac{P_{it}}{P_t}\right)^{-\varepsilon} Y_t = 0 \Leftrightarrow P_{it} = \frac{\varepsilon}{\varepsilon - 1}\Psi' = \mathrm{M}\Psi'$$

But under the assumptions of symmetry and free entry the prices will be equal across the firms, thus:

$$P_t^* = \mathrm{M}\Psi'$$

Appendix 3

MATLAB CODE, IRF for Gali, Chapter 3 (2008) Undetermined Coefficients

```
% Calibrated parameter values
beta=0.99;
hetta=4;
sigma=1;
phipi=1.5;
phix=0.5/4;
theta=2/3;
phi=1;
epsilon=6;
alpha=1/3;
bigtheta=(1-alpha)/(1-alpha+alpha*epsilon);
lamda=bigtheta*(1-theta)*(1-beta*theta)/theta;
kappa=lamda*(sigma+(phi+alpha)/(1-alpha));
psiyanatural=(1+phi)/(sigma*(1-alpha)+phi+alpha);
bigomega=1/(sigma+phix+kappa*phipi);
rhou=0.5;
stdu=0.0525;
stdalpha=0.0525;
T=(1-beta*rhou)*(sigma*(1-rhou)+phix)+kappa*(phipi-rhou);
Glamda=1/T;

%monetary policy shock
xu=zeros(22,1);
piu=zeros(22,1);
v=zeros(22,1);
i=zeros(22,1);
r_real=zeros(22,1);
m_growth=zeros(22,1);
e =10*stdu;
rhoa=0;
```

```
rhou=0.5;
v(2,:) = e;

 for t = 3:22
   v(t,:)=rhou*v(t-1,:);
   xu(t,:) = -((1-beta*rhou)*Glamda*v(t,:)')';
   piu(t,:) = -(kappa*Glamda*v(t,:)')';
   i(t,:)=((sigma*(1-rhou)*(1-beta*rhou)-rhou*kappa)*Glamda*v(t,:)')';
   r_real(t,:)=(sigma*(1-rhou)*(1-beta*rhou)*Glamda*v(t,:)')';
   m_growth(t,:)=xu(t,:)-xu(t-1,:)-(hetta*(i(t,:)-i(t-1,:)))'+piu(t,:);

 end

outputgap = xu(3:22,1);
inflation = piu(3:22,1);
shock=v(3:22,1);
m_growthu=m_growth(3:22,1);
r_realu=r_real(3:22,1);
iu=i(3:22,1);


anninflation=4*inflation;
annnomRu=4*iu;
annr_real=4*r_realu;
annm_growth=4*m_growthu;
```

MATLAB CODE, IRF for Gali, Chapter 3 (2008) Klein's Method

```
%parameters
beta=0.99;
hetta=4;
sigma=1;
phipi=1.5;
phix=0.5/4;
theta=2/3;
phi=1;
epsilon=6;
alpha=1/3;
bigtheta=(1-alpha)/(1-alpha+alpha*epsilon);
lamda=bigtheta*(1-theta)*(1-beta*theta)/theta;
kappa=lamda*(sigma+(phi+alpha)/(1-alpha));
psiyanatural=(1+phi)/(sigma*(1-alpha)+phi+alpha);
bigomega=1/(sigma+phix+kappa*phipi);
% rhoa=0.9;
rhou=0.5;
stdu=0.0525;
stdalpha=0.0525;


A=[1 0 0;0 sigma 1;0 0 beta];
B=[rhou 0 0;1 (phix+sigma) phipi;0 -kappa 1];
```

```
C=[1 0 0]';
P=0.5;

%Klein's Method

[S,T,Q,Z] = qz(A,B);

[S,T,Q,Z] = reorder(S,T,Q,Z);

Q1 = Q(1:1,:);
Q2 = Q(2:3,:);

Z11 = Z(1:1,1:1);
Z12 = Z(1:1,2:3);
Z21 = Z(2:3,1:1);
Z22 = Z(2:3,2:3);

S11 = S(1:1,1:1);
S12 = S(1:1,2:3);
S21 = S(2:3,1:1);
S22 = S(2:3,2:3);

T11 = T(1:1,1:1);
T12 = T(1:1,2:3);
T21 = T(2:3,1:1);
T22 = T(2:3,2:3);

L1=T11(1,1)/S11(1,1);
L2=T22(1,1)/S22(1,1);
L3=T22(2,2)/S22(2,2);

 if abs(T11(1,1)/S11(1,1)) > 1
  'error - no solution'
 end

 if abs(T22(1,1)/S22(1,1)) < 1
  'error - multiple solutions'
 end

%towards a solution
 G = S22/T22;

 H = Q2*C;

 vecR = inv(eye(2)-kron(P,G))*H(:);

 R = reshape(vecR,2,1);

% M matrices
 M1 = Z21/Z11;

 M2 = -(Z22-(Z21/Z11)*Z12)*inv(T22)*R;
```

```
M3 = (Z11/S11)*(T11/Z11);

D = (T11/Z11)*(Z12/T22)*R+Q1*C + (S12/T22)*R*P-(T12/T22)*R;

M4 = (Z11/S11)*D-(Z12/T22)*R*P;

%the solution is St+1=GAMMA0*St+GAMMA1*et+1 & Ft=GAMMA2*St

GAMMA0I = [M3 M4;0 1];

GAMMA1 = [0 ; 1];

GAMMA2I=[M1 M2];

GAMMA0 = real(GAMMA0I);

GAMMA2 = real(GAMMA2I);

GAMMA211 = GAMMA2(1,1);
GAMMA212 = GAMMA2(1,2);
GAMMA221 = GAMMA2(2,1);
GAMMA222 = GAMMA2(2,2);

%monetary policy shock
Su=zeros(12,2);
Fu=zeros(12,2);
i=zeros(12,1);
r_real=zeros(12,1);
m_growth=zeros(12,1);
E =10*stdu;
rhoa=0;
rhou=0.5;
Su(2,:) = [E 0];

for t = 3:12

  Su(t,:) = (GAMMA0*Su(t-1,:)')';
  Fu(t,:) = (GAMMA2*Su(t,:)')';
  r_real(t,:)=(-sigma*(1-rhou)*GAMMA211*Su(t,1)')';
  i(t,:)=((-sigma*(1-rhou)*GAMMA211+rhou*GAMMA221)*Su(t,1)')';
  m_growth(t,:)=Fu(t,1)-Fu(t-1,1)-(hetta*(i(t,:)-i(t-1,:))')'+Fu(t,2);
end


xu = Fu(3:12,1);
inflation = Fu(3:12,2);
v=Su(3:12,1);
m_growthu=m_growth(3:12,1);
r_realu=r_real(3:12,1);
iu=i(3:12,1);
```

```
anninflation=4*inflation;
annnomRu=4*iu;
annr_real=4*r_realu;
annm_growth=4*m_growthu;
```

MATLAB CODE, IRF for Gali, Chapter 3 (2008) Blanchard -Kahn Method

```
% Calibrated parameter values
beta=0.99;
hetta=4;
sigma=1;
phipi=1.5;
phix=0.5/4;
theta=2/3;
phi=1;
epsilon=6;
alpha=1/3;
bigtheta=(1-alpha)/(1-alpha+alpha*epsilon);
lamda=bigtheta*(1-theta)*(1-beta*theta)/theta;
kappa=lamda*(sigma+(phi+alpha)/(1-alpha));
psiyanatural=(1+phi)/(sigma*(1-alpha)+phi+alpha);
bigomega=1/(sigma+phix+kappa*phipi);
rhou=0.5;
stdu=0.0525;
stdalpha=0.0525;


% Calculate kappa

% Define state space matrices
A0=[1 0 0;0 sigma 1;0 0 beta];
A1=[rhou 0 0;1 (phix+sigma) phipi;0 -kappa 1];
B0=[1;0;0];
% Calculate alternative state space matrices
A=inv(A0)*A1;
B=inv(A0)*B0;

% Jordan decomposition of A
[H,lambda] = jordan(A);
[lambda,order] = sort(diag(lambda),'ascend');  %# sort eigenvalues in ascending order
LAMBDA=diag(lambda);
H = H(:,order);
Hstar=inv(H);
%
% % Partition matrices
LAMBDA1=LAMBDA(1,1);
LAMBDA2=LAMBDA(2:3,2:3);
H11=Hstar(1,1);
H12=Hstar(1,2:3);
H21=Hstar(2:3,1);
H22=Hstar(2:3,2:3);
```

```
R=H\B;

% Print out matrices of recursive solution of model
D=real(inv(H11-(H12/H22)*H21)*LAMBDA1*(H11-(H12/H22)*H21)); real(inv(H11-(H12/H22)*H21)*R(1));
E=-real(H22\H21);


%monetary policy shock
Su=zeros(22,1);
Fu=zeros(22,2);
i=zeros(22,1);
r_real=zeros(22,1);
m_growth=zeros(22,1);
e =10*stdu;
rhoa=0;
rhou=0.5;
Su(2,:) = e;

 for t = 3:22

   Su(t,:) = (D*Su(t-1,:)')';
   Fu(t,:) = (E*Su(t,:)')';
   r_real(t,:)=-(sigma*(1-rhou)*E(1)*Su(t,:)')';
   i(t,:)=((rhou*E(2)-sigma*(1-rhou)*E(1))*Su(t,:)')';
   m_growth(t,:)=Fu(t,1)-Fu(t-1,1)-(hetta*(i(t,:)-i(t-1,:))')'+Fu(t,2);
 end

xu = Fu(3:22,1);
piu = Fu(3:22,2);
v=Su(3:22,1);
m_growthu=m_growth(3:22,1);
r_realu=r_real(3:22,1);
iu=i(3:22,1);

anninflation=4*piu;
annnomRu=4*iu;
annr_real=4*r_realu;
annm_growth=4*m_growthu;
```

MATLAB CODE, IRF for Gali, Chapter 6 (2008) Klein's Method

```
%parameters
beta=0.99;
sigma=1;
phipi=1.5;
phix=0;
phiw=0;
thetap=2/3;
thetaw=3/4;
phi=1;
epsilon=6;
```

```
epsilonw=6;
alpha=1/3;
mip=log(epsilon)-log(epsilon-1);
bigtheta=(1-alpha)/(1-alpha+alpha*epsilon);
lamdap=bigtheta*(1-thetap)*(1-beta*thetap)/thetap;
% lamdap=0;
lamdaw=(1-thetaw)*(1-beta*thetaw)/(thetaw*(1+epsilonw*phi));
% lamdaw=0;
kappap=alpha*lamdap/(1-alpha);
% kappap=0;
kappaw=lamdaw*(sigma+phi/(1-alpha));
% kappaw=0;
psiyanatural=(1+phi)/(sigma*(1-alpha)+phi+alpha);
psiwanatural=(1-alpha*psiyanatural)/(1-alpha);
thetaynatural=((1-alpha)*(mip-log(1-alpha)))/(alpha*(1-alpha)+phi+alpha);
rhown=0.5;
rhorn=0.5;
rhou=0.5;
rhoa=0;
stdu=0.25;
stdwn=0.025;
stdrn=0.025;


A=[1 0 0 0;0 sigma 1 0; -lamdap 0 beta 0 ;lamdaw 0 0 beta ];
B=[1 0 -1 1 ; 0 sigma+phix phipi phiw;0 -kappap 1 0;0 -kappaw 0 1];
C=[0 0 0;0 0 1;0 0 0;0 0 0];
P = [rhown 0 0;0 rhorn 0;0 0 rhou];

%Klein's Method

[S,T,Q,Z] = qz(A,B);

[S,T,Q,Z] = ordqz(S,T,Q,Z,'udo');

Q1 = Q(1:1,:);
Q2 = Q(2:4,:);

Z11 = Z(1:1,1:1);
Z12 = Z(1:1,2:4);
Z21 = Z(2:4,1:1);
Z22 = Z(2:4,2:4);

S11 = S(1:1,1:1);
S12 = S(1:1,2:4);
S21 = S(2:4,1:1);
S22 = S(2:4,2:4);

T11 = T(1:1,1:1);
T12 = T(1:1,2:4);
T21 = T(2:4,1:1);
T22 = T(2:4,2:4);
```

```
L1=T11(1,1)/S11(1,1);
L2=T22(1,1)/S22(1,1);
L3=T22(2,2)/S22(2,2);
L4=T22(3,3)/S22(3,3);

 if abs(T11(1,1)/S11(1,1)) > 1
  'error - no solution'
 end

 if abs(T22(1,1)/S22(1,1)) < 1
  'error - multiple solutions'
 end

%towards a solution
 G = S22/T22;

 H = Q2*C;

 vecR = inv(eye(9)-kron(P,G))*H(:);

 R = reshape(vecR,3,3);

% M matrices
 M1 = Z21/Z11;

 M2 = -(Z22-(Z21/Z11)*Z12)*inv(T22)*R;

 M3 = (Z11/S11)*(T11/Z11);

 D = (T11/Z11)*(Z12/T22)*R+Q1*C + (S12/T22)*R*P-(T12/T22)*R;

 M4 = (Z11/S11)*D-(Z12/T22)*R*P;

%can be written compactly in State Space form as:  St+1=GAMMA0*St+GAMMA1*et+1 & Ft=GAMMA2*St

 GAMMA0 = [ M3 M4 ; zeros(3,1) P ];

 GAMMA1 = [ zeros(1,3) ; eye(3) ];

 GAMMA2=[M1 M2];

 GAMMA0 = real(GAMMA0);

 GAMMA2 = real(GAMMA2);

%monetary policy shock
Su=zeros(4,12);
Fu=zeros(3,12);
Eu=zeros(3,12);
i=zeros(1,12);
n=zeros(1,12);
```

```
y=zeros(1,12);
yn=zeros(1,12);
w=zeros(1,12);
wn=zeros(1,12);
rn=zeros(1,12);
rhown=0;
rhorn=0;
rhou=0.5;
e =stdu;
Eu(3,3) = e;
 for t = 2:12
   Su(:,t) = GAMMA0*Su(:,t-1)+GAMMA1*Eu(:,t);
   Fu(:,t) =GAMMA2*Su(:,t);
   i(1,t) = -log(beta)+phipi*Fu(2,t)+phix*Fu(1,t)+phiw*Fu(3,t)+Su(4,t); % Monetary policy rule
   rn(1,t)=-log(beta);
   yn(1,t) = thetaynatural; % natural level of output
   wn(1,t)=log(1-alpha)-(alpha/(1-alpha))*thetaynatural-mip;% natural real wage
   w(1,t)=wn(1,t)+Su(2,t); %real wage
   y(1,t) = Fu(1,t)+yn(1,t); % output
   n(1,t)=y(1,t)/(1-alpha); %employment
 end

xu = Fu(1,3:12)';
pipu = Fu(2,3:12)';
piwu = Fu(3,3:12)';
wgapu= Su(2,3:12)';
v= Su(4,3:12)';
iu=i(1,3:12)';
nu=(1-alpha)*n(1,3:12)';
yu=y(1,3:12)';
wu=w(1,3:12)';

priceanninflation=4*pipu;
wageanninflation=4*piwu;
nominalrateannual=4*iu;
```

MATLAB CODE, IRF for Gali, Chapter 6 (2008) Blanchard Kahn Method

```
%parameters
beta=0.99;
sigma=1;
phipi=1.5;
phix=0;
phiw=0;
thetap=2/3;
thetaw=3/4;
phi=1;
epsilon=6;
epsilonw=6;
alpha=1/3;
bigtheta=(1-alpha)/(1-alpha+alpha*epsilon);
```

```
lamdap=bigtheta*(1-thetap)*(1-beta*thetap)/thetap;
lamdaw=(1-thetaw)*(1-beta*thetaw)/(thetaw*(1+epsilonw*phi));
kappap=alpha*lamdap/(1-alpha);
kappaw=lamdaw*(sigma+phi/(1-alpha));
psiyanatural=(1+phi)/(sigma*(1-alpha)+phi+alpha);
psiwanatural=(1-alpha*psiyanatural)/(1-alpha);
rhown=0.5;
rhorn=0.5;
rhou=0.5;
rhoa=0;
stdu=0.25;
stdwn=0.025;
stdrn=0.025;

%the model in state space: A0*EtSt+1=A1*St+BUt & Ut=P*Ut-1+et
A0=[1 0 0 0;0 sigma 1 0; lamdap 0 beta 0 ;-lamdaw 0 0 beta ];
A1=[1 0 -1 1 ; 0 sigma+phix phipi phiw;0 -kappap 1 0;0 -kappaw 0 1];
B0=[0 0 0;0 0 1;0 0 0;0 0 0];
P = [rhown 0 0;0 rhorn 0;0 0 rhou];


% Calculate alternative state space matrices
A=inv(A0)*A1;
B=inv(A0)*B0;

% Jordan decomposition of A
[ H, lambda ]= eig (A); % get eigenvectors and eigenvalues
lambda = diag ( lambda ); % vectorize lambda
[ unused, order ]= sort ( abs ( lambda ) ,'ascend'); % sort eigenvalues in ascending order
LAMBDA = diag ( lambda ( order )); % reorder lambda and make diagonal again
H = H (: , order ); % reorder eigenvectors
Hstar=inv(H);

% Partition matrices
H11=Hstar(1,1);
H12=Hstar(1,2:4);
H21=Hstar(2:4,1);
H22=Hstar(2:4,2:4);
LAMBDA1=LAMBDA(1,1);
LAMBDA2=LAMBDA(2:4,2:4);
Q=H\B;
Q1=Q(1,:);
Q2=Q(2:4,:);
vecQ2=reshape(Q2,9,1);
vecR = inv(eye(9)-kron(P,inv(LAMBDA2)))*vecQ2;
R = reshape(vecR,3,3);
M1=-H22\H21;
M2=-H22\(LAMBDA2\R);
M3=(H11+H12*M1)\(LAMBDA1*(H11+H12*M1));
M4=(H11+H12*M1)\(Q1+LAMBDA1*H12*M2-H12*M2*P);

%can be written compactly in State Space form as:  St+1=GAMMA0*St+GAMMA1*et+1 & Ft=GAMMA2*St
```

```
GAMMA0 = [ M3 M4 ; zeros(3,1) P ];

GAMMA1 = [ zeros(1,3) ; eye(3) ];

GAMMA2=[M1 M2];

GAMMA0 = real(GAMMA0);

GAMMA2 = real(GAMMA2);

%monetary policy shock
Su=zeros(4,12);
Fu=zeros(3,12);
Eu=zeros(3,12);
rhown=0;
rhorn=0;
rhou=0.5;
e =stdu;
Eu(3,3) = e;
 for t = 2:12
   Su(:,t) = GAMMA0*Su(:,t-1)+GAMMA1*Eu(:,t);
   Fu(:,t) =GAMMA2*Su(:,t);
 end

xu = Fu(1,3:12)';
pipu = Fu(2,3:12)';
piwu = Fu(3,3:12)';
omega= Su(2,3:12)';
v= Su(4,3:12)';

priceanninflation=4*pipu;
wageanninflation=4*piwu;
```