https://doi.org/10.26219/heal.aueb.6034

Department of Statistics

Master Thesis

On some recent developments of Time-Between-Events Control Charts

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A THESIS:
Submitted to the Department of Statistics of
the Athens University of Economics
and Business in partial fulfillment
of the requirements
for the degree of Master
of Science in Statistics

https://doi.org/10.26219/heal.aueb.6034

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Abstract

Time Between Events (TBE) control charts are used to monitor high-performance processes to quickly and accurately identify any changes in the process. For this reason, they are considered a useful tool in Statistical Process Control (SPC). The main objective of this method is to improve process efficiency by detecting deviations over time, allowing for prompt intervention. TBE control charts are considered valuable across various professional sectors, such as healthcare, customer service, and even simpler sectors like manufacturing. In this study, several different control charts were examined, including EWMA, CUSUM, and Shewhart charts, as well as combinations of these methods. Moreover, various techniques utilizing Time Between Events (TBE) control charts were applied, and simulations were conducted for some of these methods to compare their performance and results.

Chapter 1

Introduction

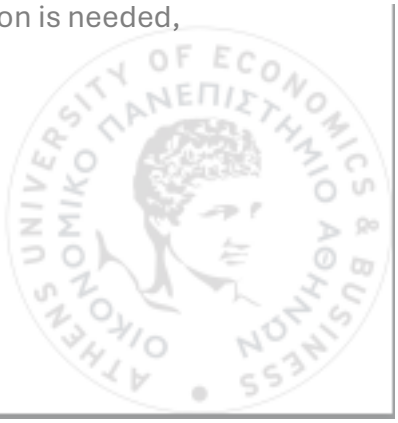
1.1 Statistical process control

Often, companies will need to use statistical process control (SPC). This is because statistical process control uses statistical methods to monitor and control production processes. When used by a company, its main objective is to keep the efficiency of a particular process, but apart from that, it tries to keep the product within the limits and standards set by the company. In this way they can satisfy the needs of their customers and have a high level of quality in their products. Finally, all of this has an impact on the market, since, based on statistical process control (SPC), there are also increased levels of competitiveness for higher-quality products. Also, according to Montgomery (2009), quality improves when variability is reduced, and although it is not possible to create the 'perfect' process, it can lead to better results when variability is reduced as much as possible.

On the other hand, to achieve the objective of product quality, other processes will be needed. Such as the use of statistical quality control (SQC), which focuses on the early detection of defective products. In this way, companies have the possibility to improve their production by reducing the number of defective products that can be created and, as a result, reduce costs. To achieve the above, they will use a valuable tool, which is the control chart. The control chart helps to monitor the process and identify defective products. If a deviation occurs in the process, the chart signals that it is not working properly. This process, known as Statistical Process Monitoring (SPM), helps to reduce variation over time and improve product quality. But how exactly does a control chart work?

The control chart is a valuable statistical tool used in quality control to monitor and analyze process changes over time. In particular, the control chart shows the average values of measurements of a quality attribute in relation to the number of samples. Furthermore, the control chart consists of the lower control limit, the upper control limit, and the central line. The central value is the mean value, which is the expected value in a stable process where there is no variation. The upper control limit (UCL) and the lower control limit (LCL) indicate the acceptable limits to which the process can move, and in the case where all points are within the limits, then the process is controlled. Conversely, if even one point is above the upper control limit or below the lower control limit, then the process is considered out of control, and some action is needed to correct it. However, this does not mean that even when all points are within control, no corrective action is needed, as the data pattern may be showing instability.

An example of a control chart is the following:



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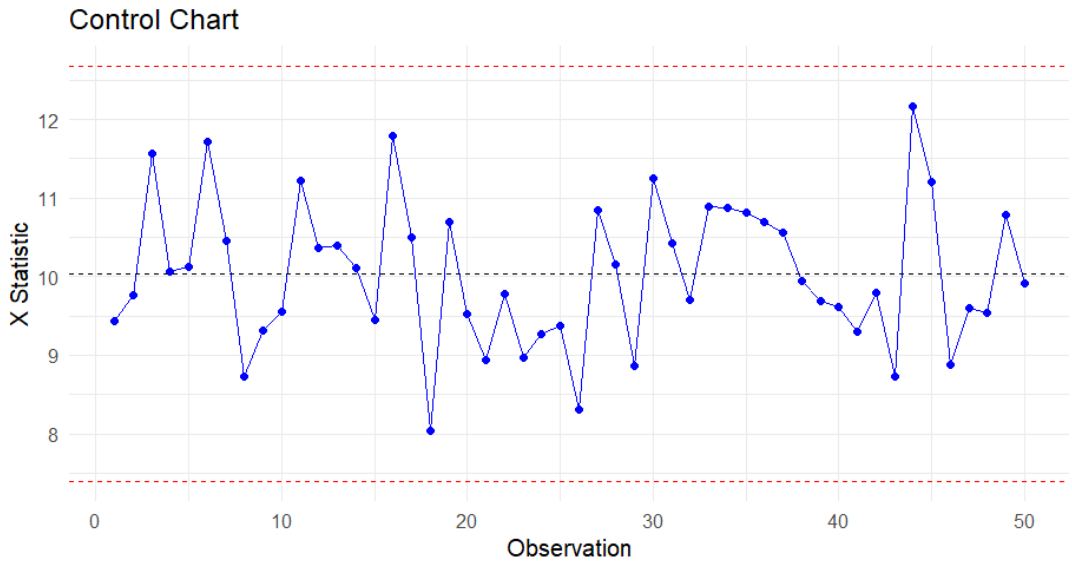

Figure 1.1: Control Chart

Figure 1.1 shows the upper control limit and the lower control limit with red dotted lines. Furthermore, the X-axis of the diagram is the successive observations, while the Y-axis shows the measured values (X Statistic). In addition, note the black line which is in the center of the diagram and represents the average value of the process. Finally, there are the blue points and the line track individual observations, helping to identify trends, shifts or outliers in the process.

All control charts are divided into two categories, which are memoryless and memory-based charts. More specifically, memoryless control charts are effective in detecting large changes and use more recent data. A specific example is the Shewhart control chart. Whereas memory-based charts are effective in cases where small changes need to be detected, since they use past and more recent data. Examples of this category are the Cusum and EWMA charts. The choice of the type of diagram varies according to different factors and conditions in which the process is found. The different types of charts will be discussed in more detail in Chapter 2.

1.2 Time between Events (TBE)

The time between events (TBE) is the time interval between two consecutive occurrences of an event in a process. It is quite useful in many processes, such as quality control, because it can improve the process by detecting possible variations. Some examples of understanding are that it can show how long an ATM machine has been running before it breaks down, so that it can plan maintenance. The time between events (TBE) is studied by using different distributions depending on the events or for the purpose of its use. For example, it is commonly used with the exponential distribution when events occur randomly, such as the arrival of a customer.

There are several applications where time between events can be used, such as healthcare and epidemiology. About healthcare, it offers significant help as it can calculate the time between the admission of patients to the hospital or predicting the time between different medical events. In this way, hospitals can allocate their resources properly, without shortages or differential problems. Another application is service management, as in medical care, the time between incidents can predict the arrival time between clients, which helps to recruit the right number of employees, so that the company does not have more costs, and the client is not dissatisfied with the service.

Different statistical models are used to analyze the time between events. In fact, the most common statistical model used is the exponential distribution, this is because it is relatively easy to use and is used in cases where events occur randomly. In other cases where failure rates vary over time, it is common to use the Weibull distribution. Whereas, in the example of health care where the time interval between patient arrivals was studied, the Poisson distribution is used. This is because by using the Poisson distribution it is possible to predict the number of events in each period.

In conclusion, using time between events (TBE) helps businesses or different services to get several benefits. Because in this way, regardless of the distribution they use, they will be able to offer to a business the prediction of a possible machine failure or to a service of a possible future malfunction. In this way, they will be able to take appropriate measures and avoid causing damage or additional costs.

1.3 Thesis Description

In this thesis we will try to explain methods found by many different researchers regarding Time Between Events diagrams. Apart from the theory they used and the mathematical equations, we will use different examples and simulations, for better understanding and to compare results. This thesis is structured as follows: Chapter 2 provides a description of the control charts that will be used to explain the following chapters. Chapter 3 explores the traditional Time Between Events (TBE) models, while Chapter 4 tries to describe as best as possible the advanced and non-parametric approaches. Chapter 5 introduces modern TBE techniques, followed by Chapter 6, which uses applications for some of the methods mentioned in the previous chapters. Finally, Chapter 7 creates a brief review of the chapters and reports the conclusions of the thesis.

Chapter 2

Control charts

Control charts vary according to their use, and apart from the two different types, which are divided into memory-based control charts and memoryless control charts, there are also several subcategories. In these subcategories they are divided according to their use, and they can be used with different statistical models. But what are the most common and useful control charts?

2.1 SHEWHART Control charts

One of the most popular diagrams used is the Shewhart diagrams created by Shewhart (1931), which are commonly used in static process control (SPC). Something that sets them apart from other charts is that they use predefined control limits and compare individual measurements in a particular process. As we have mentioned so far, most charts try to focus on small changes in a process, in this case the Shewhart chart is suitable for large sudden changes in the process. In this case, it is not so suitable for small changes in the process, which is why it is case-specific and industry-specific.

For the Shewhart chart we will need to create the lower and upper control limit and the centerline. Basically, the center line of the diagram, as seen in the previous diagrams, represents the mean value of the process. Whereas for the control limits, they represent either the maximum acceptable value or the minimum acceptable value, so that the process can continue to be monitored. The types for the upper or lower control limit are as follows:

$$UCL = \mu + 3\sigma$$

$$LCL = \mu - 3\sigma$$

As can be seen from the above formula, the procedure follows a normal distribution, as the specific control limits are set at ±3 standard deviations (σ) from the mean value. In cases where there is even one point that is either above the upper control limit or below the lower control limit, this means that the process has taken a shift and there is an error that needs further investigation. But this does not mean that even when we have no points that have not exceeded the limits, it means that the process is stable. More specifically, even between the control limits, the points can change direction abruptly, which means that the process is not stable.

The Shewhart chart is divided into several different categories, varying according to the type of data it observes. The Shewhart chart is divided into several different

categories, varying according to the type of data it observes. The categories that distinguish the Shewhart charts are the following:

- There is the X chart or the mean chart, whose purpose is to track the variance of the mean. The control limits for this one are as follows:

$$UCL = \bar{X} + A_2\bar{R}$$

$$LCL = \bar{X} - A_2\bar{R}$$

  where A_2 is a constant based on sample size, and $\bar{R}$ is the average range.
- The P chart, or range chart, which is used in cases where you need to monitor the variability within a sample. The control limits of this diagram are calculated from the types:

$$UCL = D_4\bar{R}$$

$$LCL = D_3\bar{R}$$

  where D_3 and D_4 are fixed values, which depend on the sample size.
- There is also the S chart, or standard deviation diagram, which is used in cases where we need to measure the dispersion of the process as time changes. The specific control limits are used to generate the graph, which are as follows:

$$UCL = B_4\bar{S}$$

$$LCL = B_3\bar{S}$$

  where B_3 and B_4 are fixed values, which depend on the sample size. The $\bar{S}$ is the average standard deviation.

These are some of the most important diagrams used by the Shewhart subcategories. There are also other charts that are created using the same method, but the purpose for which they are used changes.

For a better understanding of the Shewhart diagram, we created a simulation to show the specific diagram.

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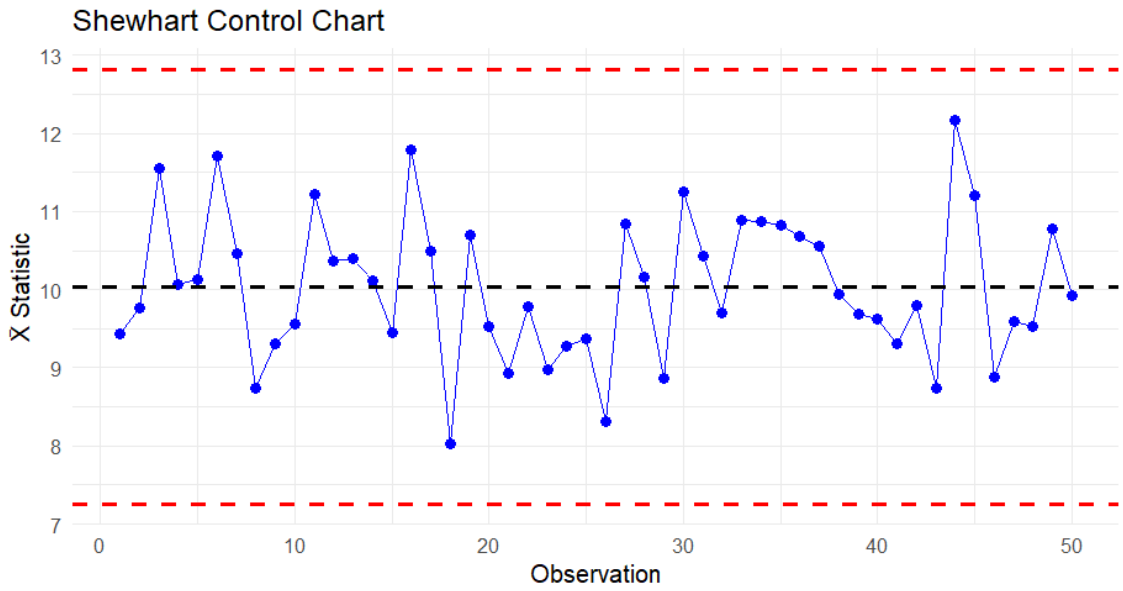

Figure 2.1: Shewhart Control Chart

For the creation of this particular diagram, we needed to take a sample of 50 random observations, which are modeled as a normal distribution with a mean equal to 10 and a standard deviation equal to 1. In Figure 2.1, there is the black dashed line which represents the meaning of the procedure, and also with red dashed lines are the upper and lower control limits. In addition, notice the blue dots between the control limits, which represent the individual sample averages, and the blue line joining them shows their trend over time. Regarding the trend of the process, we can see that we do not have points that are out of bounds, but the pattern has several fluctuations. Despite all this, the characterization we would give to the process is stable.

In conclusion, Shewhart charts are a useful tool that is used in different industries and sectors. More specifically, this chart will be used in cases where it is necessary to detect large process changes. It also falls into the category of diagrams which are suitable for production or quality control processes. On the other hand, the Shewhart chart is not preferred in cases where it is necessary to detect small changes in the process, such as for example in customer service. In these situations, we prefer the diagrams we have already mentioned such as the EWMA diagram.

2.2 Exponentially Weighted Moving Average Chart (EWMA)

The exponentially weighted moving average (EWMA), created by Roberts (1959), is a statistical method used to track processes by averaging data. In addition, this chart gives more emphasis on recent observations but also considers older values by gradually decreasing their importance in the process. In this way, the EWMA control chart is suitable for detecting small changes in a process over time.

In the computational part, the type of EWMA diagram is as follows:

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$$Z_t = \lambda X_t + (1 - \lambda) Z_{t-1}$$

in which type Z_t is the most recent observation at time t. The variable λ is the weighting factor, which also determines the meaning of the graph, because the variable is responsible for how much weight is given to recent data versus past values. The values that this variable can take are from 0 to 1. More specifically, if the value it takes is closer to 1, it means that it gives more emphasis to recent observations, while the opposite means that it gives emphasis to past data. It is also necessary to calculate the EWMA variation, which is:

$$\sigma_{Z_t}^2 = \frac{\lambda}{(2-\lambda)} \sigma^2.$$

The last part of the diagram is the control limits, which are very important to be able to calculate whether the process is in or out of control, which are calculated as follows:

$$UCL = \mu + L\sigma_{Z_t}$$

$$LCL = \mu - L\sigma_{Z_t}$$

where L is a constant, usually set to 3 for a 99.73% confidence level.

Based on the method and the types above, we created a simulation to show the EWMA diagram which is:

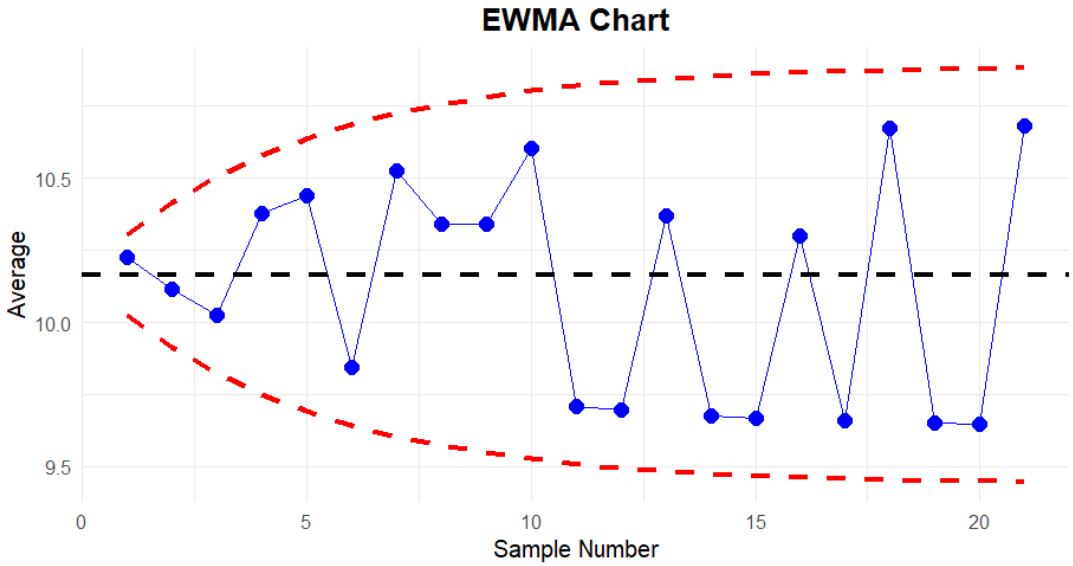

Figure 2.2: EWMA Control Chart

To design the graph, we generated 50 random observations from a normal distribution with mean (μ = 10) and standard deviation (σ = 2). For the variable λ we chose a value of 0.3 to emphasize past data without reducing recent observations, so that there is a balance. Finally, both the upper and lower control

limits are calculated based on the types and data we have. If we had points outside the limits, we would mark them with a different color and consider the procedure out of control. In this example, all elements are within control, and we consider the process fixed.

In conclusion, the EWMA diagram is a quite useful tool in several cases and in different industries. The reasons that make it effective and useful are that it can detect small changes in the process, but it makes use of old and recent data. This makes it more reliable in monitoring trends. In this way a company can improve its efficiency and reduce any defects that may arise.

2.3 Cumulative Sum Chart (CUSUM)

Another chart that many people use is the cumulative sum control chart (CUSUM) created by Page (1954), which is a statistical tool used to monitor the process. While many charts prefer to operate based on only recent data, the CUSUM accumulates small deviations over time, in this way trying to be effective in detecting small changes. This manages to make the CUSUM chart quite useful in cases where small variations need to be detected, such as in health care, where the variations may be very small because there may be a very frequent patient.

Regarding the calculation of the chart, we will start with the calculation of the CUSUM statistics, which is of the following formula:

$$C_t = C_{t-1} + (X_t - \mu)$$

where C_t is the CUSUM value at time t and X_t is a point in the data set at a particular time, while μ stands for the target value. Then to continue creating the chart we need to choose which CUSUM chart. The most common chart is the tabular CUSUM chart, which is more accurate in detecting shifts. It also uses two reference variables, which are the reference value k and the control limit h. These are more specifically used to determine the amount of displacement to be detected based on the value k, but also using the control limit h, which is set at 4 to 5 times the standard deviation of the process. Based on the previous values, we can calculate the control limits through the types:

$$C_t^+ = max(0, C_{t-1}^+ + (X_t - (\mu + k)))$$

$$C_t^- = max(0, C_{t-1}^- - (X_t - (\mu - k)))$$

In these cases, when $C_t^+ > h$ or C_t^- > h, where h is the threshold for detecting process changes, means that there are out-of-bound points, and the process is out of control.

So, we created a simulation based on the above to present an example of the CUSUM diagram which is as follows:

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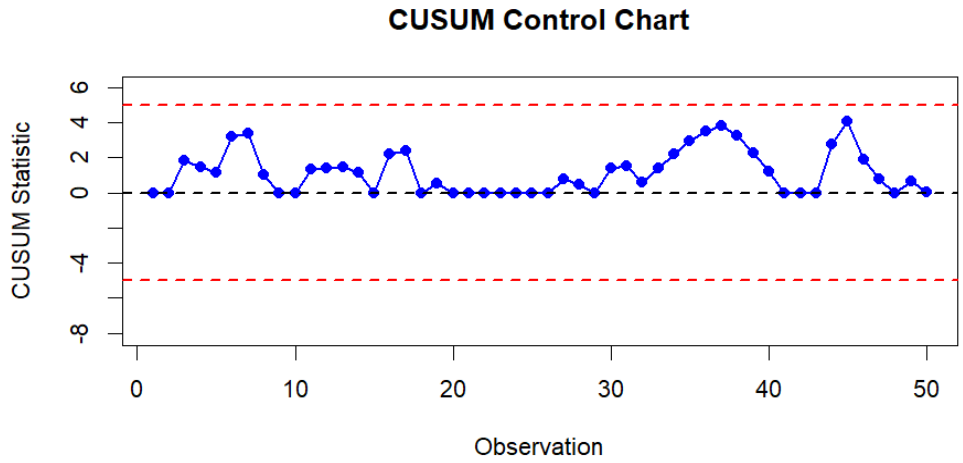


Figure 2.3: CUSUM Control Chart

Figure 2.3 shows a CUSUM diagram, in which we initially created a dataset of 50 observations, but also set a normal distribution with a mean of 10 and a standard deviation of 1.5. As mentioned above, we also use the reference variables, such as the value k set at 0.5 to detect small shifts and the value h set at 5. In this case we observe that the graph is stable because there are no out-of-bound values. However, in cases where even one point on the diagram would exceed either the upper control limit or the lower control limit, which are denoted by red dotted lines, we would indicate that the process is out of control.

In conclusion, the CUSUM diagram is a quite useful tool and can be used for several different areas. Basically, it is a chart that accumulates data over time, making it useful for trend detection, which makes it useful to avoid false associations in the process. It also helps with quality control and reliability monitoring, which are quite important in the health care sector.

Chapter 3

Parametric TBE Models

The parametric time between events (TBE) models are one of the sections in which we will divide the specific methods. More specifically, this category was created in the monitoring and analysis of the time between events in various processes. The specific methods were used for the pattern they provide, which makes it easier to monitor changes and identify problems. Furthermore, an important advantage of these methods is that they cover a wide variety of domains, which is why they have such different methods.

3.1 Exponential Time Between Events Control Charts

The following methods use the TBE control diagrams to explain several fields. For this explanation use different tools, such as different diagrams, CUSUM, EWMA, Shewhart and their combination. The exponential distribution is a natural fit for these situations, making it useful for detecting changes in event rates.

3.1.1 Monitoring exponentially distributed time between events data: self-starting perspective

As mentioned in Eralp Dogu and Muhammad Noor-ul-Amin ‘s (2021) article, traditional methods are no longer as effective as TBE charts. This is because the p-chart, n-chart and c-chart cannot cope with high probabilities of false alarm. More specifically, they use time-weighted self-starting diagrams such as EWMA and CUSUM to verify that they outperform traditional diagrams, especially in the detection of small displacements, based on the authors' claims.

The authors tried to create a method which, in addition to being effective, would be easy enough to calculate. To begin with, the first step for the method they follow is to determine the boundaries they use for the charts. For the control chart boundaries for exponential data (Shewhart’s starting chart) the upper and lower control limits are:

$$LCL = -ln(1 - \frac{a}{2})$$

$$CL = -ln(\frac{1}{2})$$

$$UCL = -ln(\frac{a}{2})$$

Where the variable a is the false alarm probability. For Self-Starting EWMA for Exponential Data they use:

$$Z_t = (1-k)Z_{t-1} + ku_t$$

where k is the smoothing constant, and the limits for the EWMA control chart are:

$$LCL_{EWMA} < Z_t < UCL_{EWMA}$$

The above limits were designed to detect specific shifts in the process.
For the Self-Starting CUSUM chart for exponential TBE data tracks the cumulative sum of the standardized observations. The statistics are:

$$S_t^+ = max(0, S_{t-1}^+ + (u_t - k_u))$$

$$S_t^- = max(0, S_{t-1}^- - (u_t - k_l))$$

The process is considered out-of-control if $S_t^+ \geq h_u$ or $S_t^- \leq h_l$, where k_u, k_l, h_u, and h_l are design parameters.

In fact, the authors reported that these charts have more advantages than the traditional Shewhart charts and the corresponding charts without automatic starting. They report that the EWMA and CUSUM charts with self-starting are ideal for faster detection of small process changes in TBE data. Furthermore, based on this advantage, the authors wanted to separate the reason why these charts are ideal. More specifically, the advantage of each is that SS-EWMA diagrams are well suitable for detecting gradual changes due to their time-weighted averaging mechanism, while SS-CUSUM is preferred in cases where they want to detect abrupt changes through the summation of deviations from the target.

The authors have continued their studies by carrying out some simulations to test the proposed diagrams in comparison with the SS-Shewhart diagrams, where they observe that the proposed diagrams perform better. The same is true in this case, as SS-EWMA is suitable for small displacements, while SS-CUSUM can also work effectively in cases with different displacement sizes. The difference in this case is that all self-starting diagrams depend on the quality and size of the original data. In processes where the initial data is poorer, the charts become less effective. For this reason, the authors emphasize the necessity of initial data examination. More simply, it is important enough to know the quality and quantity of the data to be able to select the process chart appropriately.

In conclusion, the authors mention that the choice between SS-EWMA and SS-CUSUM depends on what the monitoring process requires most. What they conclude is that both methods are useful tools for monitoring processes with exponentially distributed TBE data. What they mention at the end of their article is that their study may be useful for future studies where they try to use the

charts for poor quality data.

3.1.2 A one-sided exponentially weighted moving average control chart for time between events

The traditional EWMA diagram is quite efficient and has several advantages in a process. But this does not mean that there is no marginal room for improvement. So, based on FuPeng Xie et al. (2022), there is an improved version of the EWMA diagram for detecting the decrease or increase of the process average using the truncation method for inter-event time data. In addition, the truncation method counts the contribution of parameters to performance.

The method starts by taking the traditional EWMA chart and converting it to the One-Sided EWMA Statistic with Truncation Method. The EWMA statistic Q_t for TBE is calculated as follows:

$$Q_t = r \cdot \max(0, TBE_t - H) + (1 - r) \cdot Q_{t-1},$$

$$Q_1 = 0$$

Where:

- r is the smoothing parameter.
- H is the threshold truncation level.
- TBE_t is the observed time-between-events at time t.

Also, for this method they need to calculate only the UCL and set the LCL equal to zero, because article focuses on detecting a decrease in TBE. The type of UCL is:

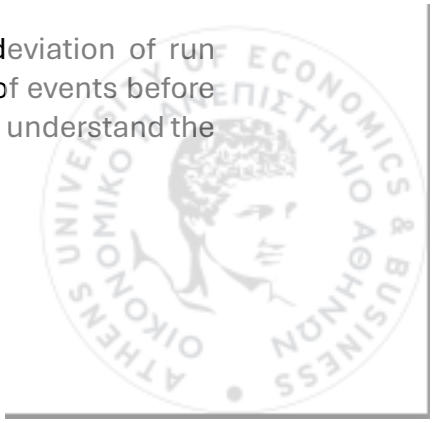
$$UCL = \mu_0 + L \cdot \sigma \sqrt{\frac{2r}{1-(1-r)^{2t}}}$$

Where:

- μ_0 is the in-control process mean.
- σ is the standard deviation of the TBE distribution.
- L is the control limit multiplier (typically set between 2.5 and 3 for sensitivity control).

To evaluate the performance of this diagram they calculate the two main parameters with the help of the Markov chain model:

- ARL (average run length): this is the average number of events before the diagram detects that the process is out of control. To create a good control chart, the authors mention that it should have a high ARL when the process is stable (under control) and a low ARL when there is a signal of a problem (out of control).
- SDRL (standard deviation of run length): the standard deviation of run length shows how much variation there is in the number of events before the chart signals a shift. The purpose of this parameter is to understand the consistency of the performance of the control chart.



The improved EWMA diagram has several advantages and is suitable for processes where the detection of TBE reductions is critical, such as when high quality processes exist. Furthermore, for this method the authors use the Markov chain model. This is because they wanted to use it for the evaluation of allowances, and to be able to apply the method to a wide range of working conditions.

The authors conclude that the one-sided EWMA control chart with the cut-off method is a rather useful tool in process monitoring with TBE data. This technique helps to increase the sensitivity to shifts while maintaining robustness against random variations by applying a cutoff and focusing on significant deviations. Also, the evaluation measures, ARL and SDRL, help the authors to ensure a detailed understanding of the reliability and responsiveness of the chart. The combination of all these mentioned shows that the improved EWMA chart is more effective than traditional charts and more specific in monitoring reliability.

3.1.3 Exponential renewal process control charts for TBE monitoring

Traditional TBE diagrams, in high quality processes, have many positives but can have problems in quickly identifying errors. So, for this reason, Ali et al. (2017) propose a new methodology, which exploits the update process with an exponential distribution approach. More specifically, the author consider alternative forms of exponential distribution, such as generalized exponential (GE), generalized Rayleigh (GR) and exponentiated Pareto (EP) distributions, but with more emphasis on the generalized exponential distribution. This is because this distribution is an improved version of the exponential with the inclusion of a non-constant risk rate.

To test the aforementioned methodology, the limit equations for the procedure were formulated. In this case, the calculation of the upper limit, the lower limit and the central limit differ. The differences in the equations are listed below:

- $UCL = -\lambda_1 \ln\left(1 - (2\alpha)^{\beta_1}\right)$
- $CL = -\lambda_1 ln(1 - (2)^{\beta_1})$
- $UCL = -\lambda_1 ln(1 - (1 - 2\alpha)^{\beta_1})$

where λ is the rate parameter, β is the shape parameter of the chosen distribution, and α is the probability of false alarm. This approach allows for customization depending on process requirements and the expected failure rate characteristics .

Also to measure the efficiency of the method Ali et al. used performance metrics, which are:

- Expected squared loss (EQL): the EQL helps calculate the loss or cost of a process they are considering. More specifically, this way they can calculate if the process has enough false synergies, this means that the process is detecting wrong shifts. When the EQL is high, it means that they have false synergies, which leads to a reduction in the quality of the process and to several economic losses.
- Continuous Rank Probability Score (CRPS): it is particularly useful for understanding how consistently the chart performs at different shift sizes. CRPS compares the predicted response of the control chart to the actual shifts by rating how well the chart detects shifts at different sizes. This makes CRPS valuable in settings where displacements of all sizes, both large and small, must be reliably detected.

Based on Ali et al. (2017), who applied the above techniques and methods to simulated and real data, they drew several conclusions about the effectiveness of their method. All the examples were carried out with the application of the GE diagram, showing that they can identify more quickly and efficiently the large or small shifts in the process compared to traditional methods.

More flexibility in different processes is made possible by the possibility of modifying the control limits according to both the rate parameter λ and the shape parameter β. While the performance metrics -ARL, EQL and CRPS- allow the tuning and evaluation of the effectiveness of the control chart under real conditions, this flexibility improves the sensitivity to shifts of different ratios. As a result, it increases the accuracy and reliability of this procedure.

This approach is particularly useful in environments with high quality processes, in which errors are few and accurate detection of drift is vital to avoid unnecessary costs that would cause damage to the business or service. Thus, the exponentiated renewal process method makes TBE monitoring applicable to increasingly dynamic and complex processes, establishing it as a useful tool for process control in modern industrial environments.

3.1.4 Boxplot-Based Phase I Control Chart for Time Between Events (TBE)

Another method that is added for time between event charts, which are more effective in high quality production processes, compared to traditional frequency-based charts. In this case, Dovoedo and Chakraborti (2011) propose a modified boxplot approach that enhances robustness by using the median instead of the mean and apply it to phase I. Where the goal of phase I, besides studying the behavior of the data, is to establish a baseline process within the control, but also to monitor the false alarms generated to help enhance the subsequent phases.

Considered above in a previous method analysis the one-sided control chart, in this case they use a two-sided control chart to monitor both increases and decreases in TBE. In this chart they construct both the upper and lower control limits so that deviations in either direction can be detected. This means that

increases and decreases can exist in the process. Here are the formulas for constructing the limits:

Lower Control Limit (LCL):

$$LCL = X_{(m1)} - k_l(X_{(m1)} - X_{(l)})$$

Upper Control Limit (UCL):

$$UCL = X_{(m1)} + k_u(X_{(u)} - X_{(m1)})$$

Where:

- $X_{(m1)}$ is the median of the sample,
- $X_{(u)}$ and $X_{(l)}$ are the upper and lower quartiles, respectively,
- k_l and k_u are constantly determined to control the false alarm probability.

They note that for their applications and examples, Dovoedo and Chakraborti (2011) define precomputed constants depending on the survey sample. This is done to ensure a smooth process and to maintain the reliability of the graph even when the data deviates slightly from the assumed exponential distribution, a phenomenon observed in real events.

According to the authors, the boxplot approach to the Phase I diagram offers several advantages. Basically, as they mention with this method there is more robustness. This is because the impact of extreme values is reduced with this diagram, resulting in more stable allowances within the control. In addition to this, with the boxplot whoever operates it can detect shifts in both directions (increases and decreases). Moreover, its ease of use is another advantage for the method, as it makes it easier to use by professionals.

In conclusion, the authors report that the boxplot-based Phase I control chart is a quite useful tool for monitoring TBE data. In particular, the use of quadrant-based limits helps to accurately detect process drift. Also, its application to Phase I analysis is useful in establishing a strong basis for subsequent process control efforts.

3.1.5 Generalized Group Runs (GGR) Control Chart for Time Between Events (TBE)

EWMA-T, T and Tr diagrams are the methods most used in processes where events occur infrequently, such as reliability monitoring in manufacturing or healthcare incident monitoring. This is an effective method, but Fang et al. proposed a method that combines a TBE chart with an advanced run length analysis to enhance the sensitivity to shifts in the TBE which is called generalized group runs (GGR), effectively outperforming traditional methods.

The GGR diagram includes two more sub-diagrams. Each sub-diagram has a different role in how help to detect shifts in the process. In more detail:

1. TBE Subchart (Tr/S Subchart): It is used to detect a potential shift in the process, which occurs when the total time for r consecutive events (denoted by Yr) falls below a calculated lower control limit (LCL), marked as a "non-compliant" TBE sample.
2. Generalized Group Conforming Run Length Subplot (GGCRL): The number of consecutive 'conforming' TBEs, which are those that meet the control conditions in the TBE subplot mentioned before the next non-conforming TBE is calculated in this subplot. More specifically, in the case that this count falls below a threshold L in consecutive iterations, it will operate by indicating a change in the average TBE and generating an out-of-control signal.

Related to the first sub-diagram mentioned above; to construct the control limits he use the cumulative distribution function (CDF) of the Erlang distribution. For an exponential distribution with mean θ_0, the cumulative time Y_r for r events follows an Erlang distribution. Therefore, to find the threshold Yr is obtained by solving the following equation:

$$F_{Y_r}\left(LCL_{T_r}\right) = \alpha$$

where:

- $F_{Y_r}(LCL_{T_r})$ is the cumulative probability up to LCL_{T_r} for an in-control mean TBE of θ_0,
- α is the probability of a false alarm (Type-I error).

For the second subroutine observe that when it detects a non-compliant sample Y_r, it starts to monitor the number of samples that are above the LCL. The number of these samples is called the conformal path length (CRL) and it is used to compare it with the threshold L. An out-of-control signal will be noted by the GGCRL when:

- For g=1: Any CRL less than or equal to L.
- For g=2: Two consecutive CRLs are each less than or equal to L.
- For g=3: Three consecutive CRLs are each less than or equal to L, and so on

The choice of the values g, L and r depends on the desired magnitude of the displacement. For this reason, Fang et al. suggest:

- The smaller the values of r (i.e. 1 or 2) the more effective they will be for detecting larger displacements.
- The larger the values of r (i.e. 3 or 4) then they will be more effective for detecting smaller displacements.
- Similarly large values of g (i.e. 3 or 4) allow finer control of sensitivity, but to do so requires a larger sample size.

The combination of aggregate event tracking with adaptive analysis of team execution time manages to improve traditional TBE tracking. The GGR chart may be customized for quick and sensitive shift detection by varying the parameters g, r, and L. This makes it especially useful in quality-critical applications where early alerts are crucial.

The GGR chart is a useful tool in several different industries based on Yen Yen Fang and Michael B. C. Khoo (2015). They mention in more detail that it is quite useful in the healthcare sector, this is because it is used to monitor the intervals between adverse events, such as medication errors, so that it can intervene more quickly. It can also be used to monitor activities that affect the environment, so that it can intervene or offer solutions to changes that happen in cases that pollute the environment. In this way it offers assistance that several sectors and services can benefit from.

In conclusion, the GGR control chart offers a robust and flexible solution for monitoring TBE, and as a result it is better in many cases than traditional charts. It is also a valuable tool that can be used in different industries, as the authors mentioned in their examples. This is because it can and does adjust detection parameters to enhance its usefulness in a variety of operational settings. Finally, as the authors mention and present in their examples, it is a method that can be used for many different reasons.

3.1.6 Median-Based Estimator for TBE Exponential Control Charts

In general, Shewhart charts have been mentioned as being the traditional charts, which cannot quickly identify the shifts that take place in the high-quality production process. In this case, however, the Shewhart-type exponential control chart is effective in high-performance production processes where events, otherwise referred to as defects or failures, occur infrequently. This is because this diagram depends on the accurate estimation of the exponential rate parameter λ. This parameter was derived from the data set in Phase I, which was most often contaminated by extreme values. Due to this contamination the effectiveness of the maximum likelihood estimator (MLE) or the minimum variance estimator (MVUE) was significantly reduced, because both estimators are sensitive to extreme values and are based on the mean term. For this reason, Nirpeksh Kumar and Sonam Jaiswal proposed a new estimator, which is Median-Based Estimator (MBE). On the other hand, Kumar and Chakraborti proposed the methodology based on media for construction of control limits and for Phase I to improve the reliability of the control charts. In Phase I, as mentioned above, there are extreme values, which can distort the control limits and create spurious correlations. Thus, the median-based diagram attempts to address this problem by setting more robust control limits. This improvement sets a more stable baseline for continuation in Phase II, allowing the MBE-based control chart to maintain reliable IC performance with fewer observations.

The median-based estimator (MBE) mentioned above, unlike MLE and MVUE, uses the sample median, which is much less affected by outliers. This feature improves the robustness of the control charts in Phase I and continues making them more reliable for Phase II control mapping. The estimator λ is defined as follows:

$$\widehat{\lambda_{MBE}} = \frac{K}{\widetilde{Y_m}}$$

where $\widetilde{Y_m}$ is the sample median of the Phase I observations and K is a constant that scales the estimator to match the desired control performance. The choice of K value was made to achieve a certain Conditional Average Run Length (CARL), which is needed to consider the variability in the performance of the chart.

For Phase 2 the control limits are constructed by a simple substitution of λ^MBE into the standard exponential control limit formulas:

Lower Control Limit (LCL):

$$\hat{L} = \frac{A_1}{\widehat{\lambda_{MBE}}} = \frac{A_1 \widetilde{Y_m}}{K}$$

Upper control limit (UCL):

$$\hat{U} = \frac{A_2}{\widehat{\lambda_{MBE}}} = \frac{A_2 \widetilde{Y_m}}{K}$$

where A_1 and A_2 are constant determined by the desired false alarm rate α. These limits are chosen such that $P(X < \hat{L}) = P(X > \hat{U}) = \frac{a}{2}$.

Performance within control (IC): Conditional Average Run Length (CARL), which accounts for the variability induced by the estimate, is used to assess the MBE-based chart's IC performance. Practitioners tend to perform more consistently in-control while using MBE since it usually produces a lower standard deviation of CARL values (σCARL) than MVUE. Since the MBE chart needs around 90 data points to attain dependable IC performance, it maintains a steady IC ARL with fewer Phase I observations than MVUE charts, which need larger samples (sometimes more than 500 observations) to reach the same level of confidence.

Out of control (OOC) Performance: The MVUE-based chart is especially good in identifying rises in the rate parameter λ, which typically indicate a breakdown in the process. The MBE chart works better than the MVUE as the event rate rises because it sends out OOC signals more quickly. The MVUE-based chart performs marginally better for shifts when the event rate drops, suggesting process improvement. When event rate increases are the main issue, customers can select the MBE for strong OOC detection thanks to this trade-off.

In conclusion, exponential control charts for TBE data are more robust, accurate and easy to use when the median-based estimator is used. It is highly recommended that in situations where there is a small or large probability of contamination of the data, the sample sizes of phase I samples may be limited, or direct detection of event rate increases is necessary. As a result, the MBE-based Phase II exponential chart is a reliable, effective and efficient means of monitoring TBE that enhances the capability of quality control procedures in high throughput and high reliability processes.

3.1.7 Statistical design of ATS-unbiased charts with runs rules for monitoring exponential time between events

This section presents an analysis of ATS-unbiased control charts, which are used for processes where events occur infrequently or unpredictably. These charts help monitor the time between events (TBE) by modeling it with an exponential distribution. The key metric used in this approach is an average time to signal (ATS), which is particularly useful when the intervals between events are not consistent. For accuracy, bias-free ATS control charts attempt to ensure that the ATS metric reaches the highest within-control (IC) value in the target rate parameter, and in this way allows for the most reliable detection of shifts in the event rate.

The control limits are set in this procedure to satisfy the desired percentage of false alarms, with the parameter λ being fixed. More specifically, for the tr diagram for phase II, the lower control limit (LCL) and the upper control limit (UCL) are given by:

- $LCL = \frac{X^2_{2r,a/2}}{2\lambda}$
- $UCL = \frac{X^2_{2r,1-a/2}}{2\lambda}$

where $X^2_{2r,\rho}$represents the p-th quadrant of the chi-square distribution with $2r$ degrees of freedom, allowing the diagram to respond to shifts while maintaining robustness.

Run rules, which were first used for Shewhart charts, have been modified for this purpose to enhance the identification of minor changes in TBE charts. The k/m rule states that if k out of m consecutive points fall beyond the control boundaries, an out-of-control (OOC) signal is generated. Modified side-sensitive (MSS) rules add a requirement that additional points stay within a certain range, whereas side-sensitive rules (SS) use successive points on the same side of the centerline to indicate an OOC situation.

In the following, the authors will use a Markov chain method to calculate the average time to signal reception (ATS) for the tr-charts. This is the case when the TBE data are exponential and the statistical chart Tr, which represents the time to the r-th event, has a gamma distribution. Thus, this approach allows accurate calculation of the ATS by recording the movement of data points between zones, which are in and out of control in a transition matrix.

For the selection of execution rules, Nirpeksh Kumara and Tribhuvan Singha (2024) suggested that it is done based on displacements. More specifically:

- **Small displacements (0.6 ≤ d ≤ 2)**: Symmetric rules, such as **M:3/5, M:3/5**, maintain sensitivity on both sides of the graph, making them effective for fixed displacements.
- **Large shifts (d < 0.6 or d > 2)**: Asymmetric rules, such as **1/1,4/4**, are recommended for rapid detection of sudden, significant changes, especially when monitoring the increase in event rates.

For high-performance tasks, where early detection of displacements is essential, ATS-unbiased tr-charts with running rules offer a convincing substitute for conventional exponential TBE charts. These charts can provide fast and accurate detection of process changes, which control different event intervals using run rules and even the ATS metric. These charts are flexible tools for quality control in areas such as manufacturing or healthcare monitoring, and this is due to the customized run rule settings.

3.1.8 Phase II control chart for monitoring Gumbel's bivariate exponential distribution

For processes with interdependent events, it is necessary to monitor the changes that will occur either between events or in the rhythm of events. Such changes are not helped by traditional charts, which focus on single-event data, which can miss important correlations. While the bivariate exponential distribution provides a way to model the interdependent TBE data, with the ability to capture the correlation between them. For this reason, Peile Chena and Zhi Songb make an introduction to the Phase II control chart for GBE distribution using log-likelihood ratio (LLR) tests. For example, in healthcare they use this method because it helps with quality control in systems where two processes are linked, such as in production and monitoring.

Log-likelihood ratio (LLR) tests are used in the proposed Phase II control chart to detect event rate and dependency departures from the in-control state. One-sample and two-sample LLR tests are essential components that focus on distinct facets of the TBE procedure. In the resuming, based on what is said in the article, they see the tests performed for the marginal distributions, for the dependence parameter and for the enhanced two-sample monitoring.

1. **One-sample log likelihood ratio test for marginal distributions**
   - The **marginal LLR test** is applied to each TBE variable, X and Y, to detect shifts in their scale parameters θ_1 and $\theta_2$representing event rates.
   - The LLR statistics are computed as:

$$\Lambda_X = -2\ln\left(\frac{L(\theta_1^0 \mid X_1, \dots, X_n)}{L(\widehat{\theta_1} \mid X_1, \dots, X_n)}\right)$$

$$\Lambda_Y = -2\ln\left(\frac{L(\theta_2^0 \mid Y_1, \dots, Y_n)}{L(\widehat{\theta_2} \mid Y_1, \dots, Y_n)}\right)$$

- Here, θ_1^0 and $\theta_2^0$are the in-control scale parameters, and $\widehat{\theta_1}$, $\widehat{\theta_2}$ are maximum likelihood estimates (MLEs). An out-of-control (OOC) signal occurs if Λ_X or Λ_Y exceeds a critical threshold, indicating a potential rate shift.

2. **One-Sample Log-Likelihood Ratio for Dependence Parameter**
   - The dependence parameter δ models the correlation between X and Y. An LLR test is used to detect shifts in δ, based on copula models.
   - The LLR statistic for δ is given by:

$$\Lambda_\delta = -2\ln\left(\frac{L\left(\delta^0 \mid (X_1, Y_1), \ldots, (X_n, Y_n)\right)}{L\left(\hat{\delta} \mid (X_1, Y_1), \ldots, (X_n, Y_n)\right)}\right)$$

   δ^0 is the in-control dependence parameter, and $\hat{\delta}$ is the MLE from the sample data. If Λ_δ exceeds the threshold, it indicates a shift in correlation, suggesting a change in the interdependence of events.

3. **Two-Sample Log-Likelihood Ratio Test for Enhanced Monitoring**
   - To address estimation variability, a **two-sample LLR test** compares Phase II data directly against Phase I baseline data, rather than relying on parameter estimates.
   - The two sample LLR statistics are:

$$\Lambda_\delta = -2\ln\left(\frac{L\left(\theta_0^1, \theta_0^2, \delta^0 \mid Phase\ I\right) \times L\left(\widehat{\theta_1}, \widehat{\theta_2}, \hat{\delta} \mid Phase\ II\right)}{L\left(\widehat{\theta_1}, \widehat{\theta_2}, \hat{\delta} \mid Combined\ Data\right)}\right)$$

   - This approach allows for robust monitoring by reducing parameter estimation errors, enhancing detection of small to large shifts.

To improve displacement detection for bivariate data they use, based on the article by Kimona Anojahatlo and Kamyar Sabri-Laghaie (2022), the Multivariate Rate Array (MRA), which provides a powerful alternative as it is more robust to extreme values. The MRA chart is used simultaneously with the traditional charts, for example MEWMA and MCUSUM, especially for moderate shifts. The MRA statistics, Ra_{1_i} and Ra_{2_i}, are calculated because of the measurements of the observations relative to the median:

$$Ra_{1_i} = \max\left(\frac{N_{M^+1i}}{i}, \frac{N_{M^-1i}}{i}\right),$$

$$Ra_{2_i} = \max\left(\frac{N_{M^+2i}}{i}, \frac{N_{M^-2i}}{i}\right),$$

where N_{M^+1i} and N_{M^-1i} are the counts of values above and below the median for variable X_1, with similar definitions for X_2. The combination of MRA with the traditional charts as mentioned above, based on simulation studies performed by

the authors, significantly improves the Average Signal Time (ATS) for small to moderate displacements.

Different Monte Carlo simulations were used for the effectiveness of the charts in the article. This evaluation was done for different displacement sizes in TBE processes. Below are the conclusions regarding these displacements :

- Small Shift Detection: The GBE chart offers early indications for slow process changes since it is sensitive to even the smallest changes in correlations and event rates.
- Large Shift Detection: The GBE chart reacts well to significant, abrupt changes, particularly in coupled systems, when sample sizes are raised.
- Comparison with Exponential CUSUM and MEWMA Charts: In situations with strong dependency, the GBE chart outperformed conventional CUSUM and MEWMA charts in identifying changes in interdependent TBE processes.

In relation to the above evaluations they suggested that it is more reliable for the monitoring of phase II to have up to 100 observations in phase I to reduce the variability in the parameter estimates.

A thorough technique for tracking bivariate TBE processes is the GBE control chart with log-likelihood ratio testing. This method is very useful for identifying process changes in applications where events are interdependent since it evaluates changes in both the marginal and dependency parameters. The GBE chart is a useful addition to quality control in domains such as manufacturing, reliability engineering, and healthcare monitoring because it combines resilience against estimating mistakes with sensitivity to changes through the use of one-sample and two-sample LLR tests. The GBE chart offers a more comprehensive picture of TBE processes than univariate approaches, enabling practitioners to identify significant changes in interdependence and individual event rates—two factors that are essential for upholding high standards in complex systems.

3.1.9 Economic Design of TBE Control Chart Systems for Multistage Manufacturing

The method in multi-stage production, based on H.Y. Zhang et al. (2011), involves designing a system that maximizes profitability while maintaining effective process monitoring. This method is used in the financial sector and what it does is to consider the process parameters at each stage but also aims to reduce spurious correlations and the cost of unidentified shifts. The primary objective for the method, however, is to find marginal control settings that will help to strike a balance between the sensitivity of detection in relation to cost-effectiveness.

For the design of this methodology, three basic assumptions will be needed, more specifically:

1) **Single stage out of control:** which means that during the process he assume that only one stage is out of control

2) **Incidence rates in control (IC) and out of control (OOC):** which indicates that each stage separately has predefined IC and OOC rates, allowing accurate predictions of the frequency of attributable causes. Which causes have the potential to affect the repair time of the system.
3) **False alarm and Type I error rates**: In traditional diagrams are used to adjust a uniform error rate at all stages, which results in inefficiency. For this reason, in this case the false alarm rate (a) and the relative probability of type I error are optimized separately for each diagram.

These are the basic conditions for the design of this methodology, so that there are no difficulties or differential errors.

The goal function that maximizes predicted profit per time unit I is the basis of the economic design. Costs associated with false alarms, in-control and out-of-control profits, assignable cause management, and TBE monitoring are all integrated into this function. The following is the formulation of economic optimization:

$$I = \max\{Profit\ (incontrol) - Costs\ (monitoring\ +\ false\ alarms\ +\ OOC\ losses)\}$$

To retain the system's sensitivity to process changes, the function maximizes I while considering limitations on the in-control average time to signal (ATS). The formulation of this constraint is:

$$Constraint: ATS_0\ \geq s$$

where s is the lowest permissible ATS for the in-control procedure, guarantees prompt shift detection without generating an excessive number of false alarms. The ATS limitations and individual optimization for the cost-benefit structure of each stage determine the control limits, LCL_i(lower control limit) and UCL_i (upper control limit). Limits that distribute power across charts based on the assignable cause rate and process shift magnitude of each stage are the aim of the design.

This economic model considers several different cost causes that affect the overall profit. Some of them are:

- Monitoring costs: this is the most basic cost in any research and in any process. More specifically, he refer to the costs associated with data collection, or the costs associated with analysis.
- False alarm costs: something that has been mentioned in all the articles so far is false alarm. As mentioned here, false alarm is when an out-of-control (OOC) signal is triggered incorrectly. As an example, there may be a false alarm about how often certain patients take medication. That is why in this case and method he is trying to optimise at each stage.
- Effect of in- and out-of-control situations on profit: in this case the authors of the article mention that it is important to have an improvement of the ATS (average time to signal). This is because when he is within control limits the gain is maximized, while in situations where they are out of control the gain decreases per unit time. For this reason, optimizing the ATS helps to reduce the gain loss during OOC periods.

The formula that used to calculate the profit based on the constellations mentioned above, considering the probability and cost of each item, is:

$$Expected\ Profit = (Profit\ (IC) - Monitoring\ Costs) - (False\ Alarm\ Costs + Assignable\ Cause\ Costs + OOC\ Losses)$$

Compared to conventional TBE systems, profit may be greatly increased by optimizing the Type I error rate and control limits at every stage:

- Customized False Alarm Rates: By adjusting false alarm rates for every stage, unwanted stoppages are decreased and sensitivity is matched to the particular needs of each stage.

- Stage-Specific Control Limits: These control limits are tailored to the particular risks and rewards of each stage, offering greater sensitivity where changes have the most effect on profit.

These modifications result in a responsive and reasonably priced control chart system that is ideal for complex multi-stage industrial processes, where each stage operates differently and each stage is important to the final result.

According to the author, the economic planning that proposes is more effective on several different issues than the traditional TBE chart system. This is because financial planning has adjusted the control limits and false positive rates to have positive results at each stage. One of the positive results is the high profitability achieved due to the system using adjustment for unique characteristics at each stage. To prove this, the author conducted studies on this topic, where it was able to show that profit increased by 8.3% compared to traditional charts, given a real company. Also point out the improved sensitivity achieved by this system, which implies more accurate detection of shifts at critical stages.

In conclusion, this economic planning is a quite useful tool, offering significant progress. According to the authors, companies or industries that aim to increase profits and improve quality use this system. It is also much more effective than traditional charts in various areas, but it is also more complicated to implement.

3.1.10 Designing a New TBE t-Chart Using Repetitive Sampling

Time-between-events (TBE) monitoring is essential for early shift detection in processes with exponential distributions, particularly in high-quality manufacturing and healthcare. Conventional t-charts frequently use single-sample limits, which may cause false alarms or overlook small changes. A new repeating sampling t-chart with inner and outside control boundaries has been created to overcome these restrictions. This method offers a workable, affordable alternative for settings where prompt, accurate shift detection is crucial by using a repeating sampling mechanism to increase sensitivity while maintaining manageable sample sizes.

Based on Muhammad Aslama and Nasrullah Khan (2014), for the repeated sampling t-plot to work, the TBE data must be transformed into a near-normal

form with the statistic $T^* = T^{1/3.6}$. With this transformation they can have symmetric control limits but also can detect deviations in the process more quickly and easily. The internal and external control limits are presented in more detail below based on the suitable equations:

1. **External control limits (LCL1, UCL1)**:
   - The external control limits serve as the primal threshold for detecting out-of-control (OOC) signals, signaling as directly as possible. They are defined in terms of the transformed mean value μ_{T^*} and the standard deviation σ_{T^*}:

$$LCL1 = \mu_{T^*} - k_1\sigma_{T^*},$$

$$UCL1 = \mu_{T^*} + k_1\sigma_{T^*}$$

   - Here, k_1 is chosen to achieve a target for the average within-control path length (ARL), balancing sensitivity with control stability.
2. **Inner control limits (LCL2, UCL2)**:
   - These internal boundaries create a secondary decision boundary. If an observation falls within the range between the inner and outer limits, in this situation the diagram starts an iterative sampling phase, more specifically this means that additional samples are collected until it is confirmed that the process is in or out of control.

$$LCL2 = \mu_{T^*} - k_2\sigma_{T^*},$$

$$UCL2 = \mu_{T^*} + k_2\sigma_{T^*}$$

   - The repeatable sampling capability minimizes redundant signals and sample size, as data points trigger further sampling only when they fall between the inner and outer limits, creating a flexible monitoring system that adapts to subtle process changes.

And in this case, they use performance counters, as has been used by other authors. In this methodology they use the average run length (ARL) and average sample size (ASS). The Average Run Length (ARL) of the chart measures the speed of detection. The out-of-control ARL (ARL_1) gauges how fast the chart picks up real changes, whereas the in-control ARL (ARL_0) shows the anticipated number of samples before a false alert. Greater sensitivity to shifts is indicated by lower ARL_1 levels.

The average number of samples required to achieve a control decision is indicated by the average sample size (ASS). Because the selective repeated sampling mentioned above is only activated in this chart design when the points fall between the inner and outer boundary, it has the effect of keeping the ASS low.

The t chart with repeated sampling compared to traditional charts has several advantages in a sample, as the authors report from different examples they have carried out. Such advantages are:

- **Improved sensitivity to small displacements**: This chart provides faster detection for small displacements by reducing ARL_1 values compared to

traditional charts, which they have often reported need larger displacements for fast detection. In one of the examples cited in the authors' article, they demonstrate the effectiveness of the repeated sampling chart, which detects displacements as small as 80% of the within-control average.

- **Efficient sampling through optimized ASS:** Because repeated sampling is selective and only takes further samples when observations fall within the inner and outer bounds, the ASS metric stays low. It is perfect for continuous monitoring in cost-sensitive situations because it maintains low sample costs without compromising detection power, which is not the aim of traditional charts.

The t chart for repeated sampling works particularly well in settings that need high sensitivity and immediate response to small changes in the process. It is an excellent choice for fields such as healthcare and high-throughput manufacturing because of its adaptable design, which attempts to keep a balance between sensitivity and sampling efficiency with both internal and external control limits. The repeatable sampling t-chart provides a useful solution for monitoring TBE in exponential processes, increasing the speed of detection for small variations, but also helping companies to keep process quality and reducing monitoring costs.

3.1.11 Sensitivity of the Shewhart-type Tr-chart to Non-Exponential Distributions

In high-yield processes, where failures or nonconforming products are rare, the Shewhart-type Tr-chart is frequently used to track time-between-events (TBE). Due to the simplicity and usefulness of Tr-charts in dependability contexts, it is common to assume that TBEs follow an exponential distribution. But real-world procedures frequently deviate somewhat from this presumption, making it difficult to maintain precise detection. Using the gamma and Weibull distributions as substitutes, this study investigates the Tr-chart's resilience in situations when the real TBE distribution differs from the exponential. The resilience of the chart to non-exponential behavior may be evaluated since these distributions, for certain parameters, create skewness differences while somewhat resembling the form of an exponential distribution. For this reason, Ranjeet Kumar Singh and Nirpeksh Kumar (2019) also generated several examples to confirm their theories.

The obliquities mentioned above also affect the control limits of the diagram, creating different equations from those that would result. The control limits are formulated below as follows:

Control limits for the diagram Tr: A cumulative TBE statistic is used to monitor the TBE of the Tr diagram, which follows a gamma distribution under exponential assumptions. The control limits are calculated as follows:

$$L_r = \frac{x^2_{a/2,2r}}{2\lambda_0}$$

$$U_r = \frac{x^2_{1-a/2,2r}}{2\lambda_0}$$

where λ_0 is the parameter of the percentage within control of the assumed exponential distribution and the quantiles χ^2 are calculated for a chosen false alarm rate (a).

After finding the control limits, the authors also examined the skewness deviations. For accuracy they examined the sensitivity of the Tr plot, which is analyzed by introducing changes in the skewness coefficient, defined as the shape difference from the exponential distribution, which has a constant skewness of 2. As ΔSk we will hereafter denote the skewness deviations, which are introduced with small increments between them (e.g. ±2%, ±5%, ±10%), simulating conditions where the TBE distribution is slightly more or less skewed.

For this research the Gamma and Weibull distributions will be used as alternative models for the real TBE data. In both cases they resemble the exponential distribution but offer controlled changes in skewness.

- **Gamma distribution**: When the shape parameter $k \approx 1$, the gamma distribution closely resembles the exponential distribution.
- **Weibull distribution**: Setting the Weibull shape parameter provides similar control, with the shape approximating the exponential when $k \approx 1$.

Then they also use well-known metrics such as average execution time (ARL), which is the main performance indicator, measuring the average time until a signal is out of control. The study considers both the ARL within control (ARL_0) and the ARL out of control (ARL_1):

$$ARL = \frac{1}{P(T_r < L_r \; or \; T_r > U_r)}$$

In more detail the results of the surveys are for in-control performance (ARL_0) and for out-of-control performance (ARL_1).

In-Control Performance ($\boldsymbol{ARL_0}$): The ARL within control of the Tr plot rises when the TBE data are somewhat less skewed than the exponentials (e.g., ΔSk=-0.1), suggesting more resilience and fewer false alarms. For instance, ARL_0 increased by up to 5% in gamma distributions with less skewness, which could be advantageous for stability.

Higher skewness deviations cause the Tr-chart assumptions to fail, resulting in a drop in within-control ARL values and more false alarms for highly skewed data (e.g., ΔSk=+0.1).

Out-of-Control Performance ($\boldsymbol{ARL_1}$): The Tr-chart performs poorly in identifying changes for higher skewness shifts, particularly when the shifts are tiny (e.g., $\delta = 0.4$, shift multiplier for %). For instance, the out-of-control ARL rose by around 10% when gamma distributions with a skewness deviation of $+5\%$ were considered, which delayed the identification of real changes. On the other hand, the Tr plot demonstrated better resilience when larger sample sizes (such as $r = 4$) were employed, with ARL deviations staying within allowable bounds even

when skewness rose by up to $\pm 5\%$. This implies that some sensitivity problems with non-exponential data can be mitigated by adjusting the sample size.

For processes within control for small reductions in skewness, the Tr diagram is quite robust to small deviations from the exponential distribution. On the other hand, in specific cases where there are increases in skewness and the distributions are Gamma or Weibull, it turns out that the sensitivity of the Tr diagram decreases with effective delays in detecting the shifts. The study above tries to suggest the use of CUSUM or EWMA diagrams for processes where non-exponential TBE behaviour is expected, and the reason is that they are more robust to the assumptions of the distributions and maintain constant sensitivity even in the presence of skewness. Finally, the proposal is said to ensure accurate and timely monitoring by providing a means of quality control in cases where the exact TBE distributions could deviate from the exponential distributions.

3.1.12 Exponential Cumulative Sums (CUSUM) Chart for Detecting Shifts in Time-Between-Events (TBE)

A practical approach to monitoring process stability involves using the Time-Between-Events (TBE) method, which is particularly effective for high-performance processes where failures are rare but significant. In such cases, cumulative sums of squares (CUSUM) plots are employed under the assumption that the TBE data follow an exponential distribution. This technique is chosen because CUSUM plots are well-suited for detecting small, gradual shifts in the process mean. In this article, however, Liang Qu et al. (2017) propose an improved version of the diagram, for accuracy the weighted CUSUM (WCUSUM) diagram, which is specifically designed for TBE data and has an adjustable power parameter fitted. This enhancement allows the detection of shifts, increasing or decreasing, in the exponential rate parameter λ.

The exponential distribution, characterized by a single parameter λ (representing the event occurrence rate), is widely used for TBE data due to its memoryless property, making it suitable for random and infrequent events. For example, according to the parameter λ, according to the authors, increasing the parameter λ indicates smaller intervals between failures, which means a decrease in the quality of the process, while the opposite is true when the parameter λ is reduced, which indicates longer intervals between failures, which means an increase in the quality of the process.

By adding a power-weighted parameter w, the WCUSUM chart improves upon the traditional CUSUM method and makes it more capable of identifying changes of different magnitudes. The WCUSUM diagram provides a solution for two sectors, one in industries such as healthcare and the other in high reliability manufacturing, where tracking both types of changes is crucial to maintaining optimal performance. This is achieved by balancing between two single-sided CUSUM diagrams (one for detecting increases in λ and one for decreases).

To design the WCUSUM diagram he need to define the control limits. This chart consists of two cumulative statistics, C^+ and C^-, which track the cumulative

deviations from a TBE baseline $T_0 = 1/\lambda_0$, the average time interval under constant conditions. These conditions are formulated as follows:

Upper CUSUM (for increases in λ)

$$C_i^+ = max(0, C_{i-1}^+ + (T_0 - T_i)^{w+} - k^+)$$

Lower CUSUM (for decreases in λ)

$$C_i^- = max(0, C_{i-1}^- + (T_i - T_0)^{w-} - k^-)$$

In these formulas:

- T_i is the observed TBE for the $i - th$ event.
- T_0 represents the expected TBE under normal (in-control) conditions.
- w^+ and w^- are power parameters for the upper and lower CUSUM charts, respectively.
- k^+, and k^- are reference values that control sensitivity to deviations for each CUSUM chart.

To adjust the sensitivity of WCUSUM to large deviations, the power parameters w^+ and w^- help, but also make them more effective in small deviations as well as for sudden, substantial changes in TBE. Setting the w parameter allows professionals to balance between fast detection of displacements and minimizing false alarms.

The parameters used so far and the control limits that have been set have been important enough for the effectiveness of the WCUSUM diagram. However, he also aim to optimize the Average Time Signal (ATS). For accuracy, the WCUSUM diagram is designed to minimize the ATS over a range of expected shifts, ensuring both early detection of deterioration in process quality and responsiveness to possible improvements. Thus, some additional optimizations of the parameters proposed by the authors for the smooth process are also derived. These are:

1. **Top-level optimization**: Using a z-factor, the WCUSUM chart first divides the detection power between the top and lower CUSUM charts. Depending on the requirements of the process, this allocation factor achieves symmetric or asymmetric sensitivity by balancing the detection power between increases and drops in λ.
2. **Setting parameters at a lower level:** The ideal parameters of k^+, k^-, w^+, and w^- for a particular z distribution is identified to provide the lowest ATS for anticipated changes in λ. These settings are changed to keep the ATS under control, guaranteeing reliable operation and reducing false alarms.
3. **Control limit setting:** The chart's false alarm rate is further optimized for process circumstances by adjusting the $H^+$and H^- control limits once the parameters k and w have been established.

Several studies conducted, as mentioned in the article, show that the WCUSUM chart is superior to the traditional two-sided CUSUM and Shewhart charts in detecting both upward and downward shifts. Because of the power parameter w's flexibility, it continuously exhibits reduced ATS values throughout a variety of shifts. Additionally, the WCUSUM chart offers a more balanced approach to both

tiny and large changes than traditional techniques by reducing the average loss (AL), a statistic that integrates ATS with possible costs of missed signals.

As it makes it possible to detect changes in the exponential rate parameters with greater sensitivity, the WCUSUM diagram marks an optimal solution to the mapping of TBE control. It can effectively react to both increases and decreases in λ thanks to its two-sided approach and variable power parameter, which makes it highly flexible for operations that need precise monitoring. The WCUSUM diagram is a powerful tool for monitoring high-performance processes, where early detection of both degradation and performance improvement is crucial due to its adaptability. To increase the flexibility of the WCUSUM approach in different industrial situations, further research could explore other parameter optimization strategies or extend its application to other more complex distributions. In this case, however, it would make it quite difficult to use.

3.1.13 Improved Exponential EWMA Chart with Power Transformation for TBE Monitoring

Exponentially weighted moving average (EWMA) control charts are quite useful for monitoring the time between events (TBE) in high-reliability processes. More specifically, these diagrams can detect changes in the process. For this reason, different EWMA diagrams have been proposed with several improvements or modifications. In the present thesis, one improvement of the EWMA diagram has already been considered, which is the one-sided EWMA diagram with a cut-off method. This method focuses on truncation to detect one-sided displacements, which can focus only on deviations exceeding a certain threshold. On the other hand, Yang Zhang and Yanfen Shang (2022) propose another improved version of the diagram, which is the exponential EWMA diagram with power transformation. This method achieves tailored sensitivity to both small and moderate displacements, providing a flexible solution for processes that require detailed monitoring of small variations. Each method has unique advantages for specific monitoring scenarios.

Related to the design of this EWMA diagram is that it applies a power transformation to the TBE data before calculating the diagram. In more detail, this transformation has the property of enhancing the sensitivity of the diagram, to small downward shifts of the TBE mean, which indicate shorter intervals between events and possible process degradation. So, it can and does provide a powerful tool for high quality processes with low defect rates. Let Ti denote the observed TBE for the $i-th$ event. In the power-transformed EWMA chart, each observation Ti is transformed as follows:

$$Y_i = T_i^{\beta}$$

where β is a power parameter selected to optimize the chart's sensitivity. The higher the value of the β variable, the more it amplifies the smaller shifts, making the graph more sensitive to the gradual decreases in the average TBE. After

transformation, the EWMA statistic Z_i is computed by updating them iteratively with the following formula:

$$Z_i = \lambda Y_i + (1 - \lambda) Z_{i-1}$$

where:

- λ is the smoothing constant, typically chosen between 0 and 1, that controls the weighting of recent data versus historical data.
- Z_0 is initialized to the mean of the transformed in-control data.

The better the choice of the parameter values, the better the method works, and they refer to the force parameter β and the smoothing constant λ. The parameters are adjusted in such a way as to minimize the ARL when the process is out of control, while maintaining a high ARL when the process is under control. For this reason, the authors propose different ways of optimizing the parameters.

1. **Determination of the power parameter β**: Several tests with different values of β shall be carried out to find the optimum value of the parameter. For example, a higher β typically increases the sensitivity, especially for shifts downwards from the mean value of the TBE, while a lower β decreases the sensitivity, especially for shifts upwards from the mean value of the TBE.
2. **Set smoothing constant λ**: Once the value of parameter b has been selected, the value of parameter λ needs to be selected. Higher values of λ give more weight to past observations, smoothing the response of the graph, while higher values increase sensitivity to recent shifts.
3. **Calculation of control limits**: Finally, once the values of the parameters have been selected, it is necessary to formulate the control limits. The Upper Control Limit (UCL) is calculated based on the desired within-control ARL. When Z_i exceeds the UCL, an out-of-control signal is triggered, indicating that the average TBE has probably decreased.

From the benefits of the EWMA diagram with transformation, and from the examples in the article, it seems that they have several advantages. The power-transformed EWMA chart's increased sensitivity to little changes gives it a few advantages over conventional EWMA charts. This figure is especially useful for identifying process degradation early on since it focuses on downward movements in the TBE mean. Compared to conventional EWMA and Shewhart-type charts, this chart routinely obtains a lower ARL in identifying downward movements, according to comparative studies. This makes it perfect for operations like health monitoring systems, electronics manufacturing, and other high-reliability applications where it's crucial to detect performance decline early. Additionally, by adjusting the parameter β, the power transformation enables the chart to adjust to varied shift magnitudes, offering a customized method for systems with varying sensitivities. For example, a lower β value could be enough for identifying large changes, whereas a higher β value might be utilized when minor shifts are of special interest.

3.1.14 TBE Exponential Method Using Generalized Multiple Dependent State (GMDS) Sampling

In this part of our study, the Generalized Multiple Dependent State (GMDS) sampling approach, which is the one that enhances the traditional TBE diagram by incorporating information from both current and past observations. This method improves the decision-making process by allowing faster and more reliable detection of changes in event frequency. This is because it uses a conditional sampling mechanism, combining historical data with real-time observations for a more dynamic analysis. Then, based on Muhammad Aslam and Muhammad Ali Raza (2018), they define the basic statistical parameters, such as the mean and standard deviation of the transformed values, to calculate the internal and external control limits and generate the graph. These limits allow the chart to use a scalar response to possible shifts, with decision rules ranging from direct signals to out-of-control signals.

When event times are modelled with an exponential distribution, they use the TBE diagram, which is the easiest choice in reliability analysis and quality engineering. The reason why it does this is its ability to model the times of rare or independent events. For the probability density function (PDF) of exponential distribution divided by:

$$f(t) = \frac{1}{\theta} e^{-\frac{t}{\theta}}, \qquad t > 0$$

where θ is identified with the scale parameter, which indicates the average time between events when the process is under control. Moreover, this parameter is common in several methods, as it effectively handles skewed data distributions, which are a common phenomenon in manufacturing industries.

The study continues to focus on the methodology associated with the diagram, but also on the GMDS sampling system. Using both current sample data and past sample data, the Generalized Multiple Dependent State (GMDS) sampling technique improves upon the conventional TBE control chart. Unlike static sampling, the GMDS system uses conditional sampling, which takes both recent and past sample results into account when making control options.

For all these to work it is important to calculate the basic statistics, which is a transformed version of the observed time between events T, denoted as $T^*$and calculated as follows:

$$T^* = T^{1/3.6}$$

With respect to the transformation, it has the property of approximating a symmetric distribution, facilitating the construction of more consistent and efficient control limits. Also, the types related to the mean and standard deviation are obtained as follows:

https://doi.org/10.26219/heal.aueb.6034

$$\mu_{T^*} = \theta_0^{1/3.6}\, \Gamma(1 + \frac{1}{3.6}),$$

$$\sigma_{T^*} = \theta_0^{1/3.6} \sqrt{\Gamma\left(1 + \frac{2}{3.6}\right) - (\Gamma\left(1 + \frac{1}{3.6}\right))^2}\,,$$

where θ_0 represents the in-control process mean, and Γ is the gamma function. These parameters could allow the application of graded decision rules by setting internal and external control limits in the GMDS-based TBE diagram. More specifically, two control factors, $k_1$and k_2, which are chosen according to the intended average path length within control (ARL), define these boundaries. $LCL1$ and $UCL1$ are more extreme external control limits, which, if exceeded, immediately activate an out-of-control signal. When a sample falls between the internal control limits $LCL2$ and $UCL2$, it means that more control is required. The control limits are calculated as follows:

$$LCL1 = \mu_{T^*} - k_1\sigma_{T^*}, \quad UCL1 = \mu_{T^*} + k_1\sigma_{T^*},$$

$$LCL2 = \mu_{T^*} - k_2\sigma_{T^*}, \quad UCL2 = \mu_{T^*} + k_2\sigma_{T^*}$$

Furthermore, based on the control limits, some decisions are taken regarding the operational procedure. There are three rules on which decisions depend:

1. **Design and initial assessment**: Where the process is under control when $LCL2 \leq T^* \leq UCL2$, indicating that no substantial shifts are detected.
2. **Immediate decision out of control**: In this case the T^* exceeds the external control limits $LCL1\ or\ UCL1$, a state of out-of-control is declared, indicating a shift in the process mean and warranting corrective action.
3. **Conditional sampling for intermediate results:** Finally, if T^* falls between the internal and external control limits, the diagram applies a majority rule by considering the last m samples. But if at least k of the last m samples fall within the internal control limits, the process is still under control. If this is not the case, the process is identified as potentially out of control and further investigation is required.

This is done by helping to match historical data with real-time observations. In addition, the authors state that the terms used by the mechanism ensure better and more sensitive monitoring of processes with TBE data but also try to ensure faster detection of process change. Finally, they state that, based on the examples and studies carried out, they conclude that the mechanism is suitable in the industry sector.

3.2 Poisson Time Between Events Control Charts

The following methods will use the Poisson concept to monitor processes where events occur randomly but at a constant average rate. In this case too, several tools will be used for the best results. Also, these methods are chosen because Poisson distribution models physical data based on measurements over time, making it more effective than traditional control charts for such processes.

3.2.1 TBE Poisson Method for Maintenance Policy in Service Facilities

Many times, various sectors, such as service industries, or machines, such as ticket machines, ATMs and scanners, for example, show several failures due to bad use, internal damage or other outside factors. For this reason, maintenance policies are important enough to avoid these factors. The various approaches envisaged in this regard, such as reactive maintenance (RM) or fixed preventive maintenance (PM) programs, do not provide satisfactory results and often fail to deal optimally with rare failures. Qiang Wan and Chunhui Liu (2018) therefore proposed the Poisson Time-Between-Events (TBE) method, which provides a structured way to monitor and manage maintenance needs by modeling the times between failures as exponentially distributed intervals, which is a basic characteristic of a homogeneous Poisson process. This method allows the dynamic determination of when maintenance is necessary, based on the observed failure rates.

To explain this method, the authors use the principles of the Poisson process, which means that the failures that will occur are random but with a constant mean rate. The exponential distribution is followed by the time between failures ($tTBE$):

$$f(t \mid \lambda) = \lambda e^{-\lambda t}, t > 0$$

where λ is the failure rate. This method helps to create some properties that will contribute to the specific situation. More specifically:

1. **Failure classification**:
    - **Type I failures**: This category includes failures caused by external factors, such as misuse of the operator, for example.
    - **Type II failures**: In this case, these are internal failures.
2. **Design of control charts**: To design the lower bound control (LCL) of the TBE Poisson control chart following the $tTBE$, they use the exponential distribution and obtain:

$$LCL = F^{-1}(a),$$

where F^{-1} is the inverse cumulative distribution function of the exponential distribution and α is the false alarm probability.

3. **Decision Rules:** To make the relevant decisions, the authors observe whether the $tTBE$ is above or below the lower control limit (LCL). For accuracy, they report:
    - **In-Control Region**: If $tTBE > LCL$, failures are infrequent, and only resets are required.
    - **Out-of-Control Region**: If $tTBE \leq LCL$, the failure rate has increased significantly, triggering RM to prevent further degradation.

As mentioned earlier, the Poisson method incorporates several maintenance mechanisms to assist it in the process. More specifically, two mechanisms are mentioned, which are:

1. **Reactive maintenance (RM)**: RM is a method that can deal with internal damage and restore the system to a healthy state, but it is quite costly. As previously mentioned, this mechanism is activated when $tTBE \leq LCL$.
2. **Preventive maintenance (PM):** This mechanism can deal with any error that occurs, whether type I or type II. In addition, the PM frequency is optimized in parallel with the control limits of the TBE diagram to balance cost and reliability.

The authors tested several examples and studies, where they were able to demonstrate the advantages of the method. Such as:

1. **Targeted Maintenance:** The method ensures RM is only performed when the failure rate indicates system degradation, reducing unnecessary interventions.
2. **Dynamic Monitoring:** By basing decisions on the exponential failure model, the TBE chart dynamically adapts to changes in failure patterns.
3. **Cost Savings:** Combining RM with optimized PM scheduling reduces total costs compared to traditional maintenance strategies.

The TBE Poisson method provides an innovative and efficient approach to maintenance in service facilities. By leveraging the exponential distribution of time between failures, it enables dynamic and cost-effective monitoring of facility performance. The integration of this method with preventive maintenance policies provides a strong framework for maintaining service reliability while minimizing costs. This makes it ideal for applications where balancing maintenance efficiency with system uptime is crucial.

3.2.2 TBE Poisson Method Using Cumulative Sum (CUSUM) Control Chart

The TBE Poisson control chart is a powerful tool for identifying shifts in the frequency of incidents, often reflecting changes in workplace conditions or levels of safety compliance. Also, TBE Poisson diagrams can allow near real-time monitoring, which is important for proactive safety management and early intervention. Since the Poisson distribution reflects the probability of many different events occurring within a given time frame, it is appropriate in this case. Because of this, the Poisson TBE approach is ideal for sectors such as

manufacturing, healthcare and construction, where event rates are often moderate but require careful observation. The use of the cumulative sum control charts (CUSUM) of this method is ideal in such cases, as it incorporates cumulative historical data and not just the most recent data points. In this way it allows for the early detection of subtle changes.

The cumulative sum control chart (CUSUM) alone focuses on monitoring by accumulating past information and improving the detection of even subtle changes in the average event rate. On the other hand, the Poisson distribution CUSUM chart focuses more on changes in the rate at which events occur. The cumulative sum control chart (CUSUM) alone focuses on monitoring by accumulating past information and improving the detection of even subtle changes in the average event rate. On the other hand, the Poisson distribution CUSUM chart focuses more on changes in the rate at which events occur. From the follow-up of Anna Schuh and Jaime A (2014), who from the cumulative deviations from the baseline, realized that the CUSUM diagram offers greater sensitivity compared to conventional diagrams, such as for example in relation to the Shewhart diagram. The CUSUM statistics, denoted $S_{Poisson,i}$, is calculated iteratively at each observation period i as follows:

$$S_{Poisson,i} = max(0, S_{Poisson,i-1} + X_i - k_{Poisson}),$$

where :

- X_i represents the observed number of incidents in time i,
- $k_{Poisson}$ is a reference parameter tailored to the process's in-control rate.

Based on the desired sensitivity they can calculate the Poisson's reference value k. Also, based on the data of phase I, they shall be able to calculate the mean incident rate ma and a target detection rate m_d. More specifically:

$$k_{Poisson} = \frac{m_d - m_a}{\ln(m_d) - \ln(m_a)},$$

where:

- m_a is the baseline mean incident rate, representing the process in a stable state, and
- m_d is the mean incident rate indicating an out-of-control state, such as a significant increase in incident frequency.

When the cumulative statistic $S_{Poisson,i}$ above a threshold $h_{Poisson}$, which is established depending on the required intended average run length (ARL), the CUSUM chart indicates an out-of-control situation. Also, when ARL gets a high value within the control this means less false alarms. A lower ARL value out of control facilitates faster detection of actual process changes.

According to the authors, three basic steps are needed for the operation of the CUSUM diagram with Poisson understanding. They are the following:

1. **Baseline setting (Phase I)**: This is the period where data are collected to establish a baseline of average event rate m_a, providing a stable reference against which future observations are compared. At the same time, the

$k_{Poisson}$ and $h_{Poisson}$ parameters are calculated in the same period to meet the ARL target criteria.

2. **Incident tracking (Phase II)**: Each update received by the CUSUM $S_{Poisson,i}$, statistic allows this chart to generate cumulative data on any shifts in the average incident rate.
3. **Signal generation**: When signal generation is reported, it means that $S_{Poisson,i}$, exceeds the $h_{Poisson}$ decision threshold, the diagram signals an out-of-control condition. In the case where the safety interventions are applied, the CUSUM statistic is reset to zero, placing the diagram in a position to monitor for further shifts, and as a result the process is stabilized.

Based on the above mentioned, but also from different studies and examples carried out by the authors, they report that CUSUM diagrams with Poisson distribution have several advantages. For example, a very important advantage mentioned is in real-time safety monitoring; these diagrams support proactive incident management. These charts allow safety managers to implement corrective measures before the incident rates increase further.

An effective method for monitoring safety is the TBE Poisson CUSUM diagram, especially in cases where events have a Poisson distribution. This approach detects fairly small changes in event rates earlier than conventional techniques, using aggregate data and comparing recent observations to a baseline. The TBE Poisson CUSUM Poisson diagram facilitates immediate responses for operations where safety is a priority, promoting continuous safety improvements. The approach is a useful tool for dynamic safety monitoring because of its sensitivity to changes in incident frequency, which allows for immediate reaction to new threats.

3.2.3 TBE Poisson Method Using Nonhomogeneous Poisson Processes

But this is not the case in many everyday processes, because the rates of events change over time due to factors such as system wear and tears or environmental changes. To address this problem, Sajid Ali (2021) use the non-homogeneous Poisson process (NHPP) based on his research. When conditions change, he prefer the NHPP method over the HPP method because it allows the event rate to vary over time. Also, as in most methods, by using NHPP models, he can generate TBE plots that are more accurate for detecting changes in event patterns.

Regarding the function of the NHPP model, the author defines it in terms of the intensity function λ(t), which describes how the event rate changes over time. A common model for λ(t) is the power law process (PLP), which is explained below:

$$\lambda(t) = \beta\lambda_0^{\beta} t^{\beta-1},$$

where:

- λ_0 is the baseline event rate, representing the rate when the process is stable.

- β is the shape parameter. If $\beta > 1$, the event rate increases over time (e.g., wear and tear). If $\beta < 1$, the event rate decreases (e.g., system improvement). If $\beta = 1$, the NHPP simplifies with a constant rate.

It provides an example of understanding the parameters it uses. More specifically, if the variable b takes the value 1.5, it means that events are more frequent over time, or it means that a machine is nearing the end of its life. On the other hand, if variable b takes the value 0.8, it means that events are less frequent, or more understandable, occur after the process has been maintained.

To construct a TBE diagram using NHPP, two conditions must be considered. These are:

1. **Understanding the behavior of events:** this is an important prerequisite because it is the beginning of the process of collecting data from phase I to calculate the variables λ_0 and b.
2. **Control limits:** It is necessary to define the control limits, which are based on the NHPP model. For the construction of a two-sided diagram the lower control limit (LCL) and the upper control limit (UCL) are:

$$LCL = F^{-1}(\alpha/2 \mid T_{i-1}),$$

$$UCL = F^{-1}(1 - \alpha/2 \mid T_{i-1}),$$

where F^{-1} is the inverse of the cumulative probability distribution of the waiting time. The positive thing in this case is that the control limits are adjusted after each event.

For the above methodology to work, the author outlines the four basic steps needed. These are:

1. **Start Monitoring:** This step is the initial step and takes place after the baseline data collection, where the monitoring of new events begins.
2. **Draw the data:** The second step calculates the waiting time between events and plots it against the dynamic control limits.
3. **Detection of signals:** in this step report in case a point is outside the control limits:
    a. Below LCL: The event rate has probably increased, indicating a possible deterioration of the process.
    b. Above UCL: The event rate has probably decreased, which may indicate an improvement**.**
4. **Take action:** At the end of the signal detection, it expands the process to detect any problem and correct it.

For a better understanding of the methodology the author uses an example of the use of the method. More specifically, it refers to an industrial machine that breaks down more and more often as the years go on. The small increases in the failure rate may not be noticed when a typical TBE chart is used. On the other hand, dynamic control limits on an NHPP-based chart immediately detect this increase, allowing for immediate repair before a failure occurs. In another example, staff training can reduce medication errors in a hospital setting. This reduced incident rate is served by an NHPP-based chart with β<1, which guarantees accurate monitoring without many false alarms.

https://doi.org/10.26219/heal.aueb.6034

In conclusion, the author reports that the TBE Poisson method, based on the NHPP, is a powerful tool for monitoring processes where event rates change over time. In processes where real-time process monitoring and early detection of changes are needed, such as in industry, it is a suitable method because of its accuracy and adaptability. In addition, the ability to detect changes early on helps to improve maintenance and reduce costs.

3.3 Weibull Time Between Events Control Charts

In the following analysis, the Weibull distribution will be used to explain several theories. Below we will see many different methods that seek to better explain changes in events for different situations. Furthermore, unlike exponential distribution, which assumes a constant rate of events, the Weibull distribution can accommodate increasing, decreasing or constant rates of risk, making it more applicable to a wide range of real-world processes.

3.3.1 TBE Weibull Method for Monitoring Process Performance

The Weibull distribution is widely used in time-between events (TBE) monitoring because it can model processes with varying event rates. So far, we have reported that the exponential distribution has a constant failure rate, while the Weibull distribution can assume a decrease, increase or maintenance of the failure rate over time. In this way this distribution is considered suitable for cases for monitoring systems that show deterioration, improvements or stabilization.

First, Fu-Kwun Wang and Berihun Bizuneh (2017) report the two basic parameters of the Weibull distribution. These are:

- The **parameter in figure** ($\boldsymbol{b}$) describes how the failure rate varies. More specifically, if parameter b takes values less than unity, it means that the failure rate decreases over time, but if it takes values greater than unity it means that the failure rate increases. While when β = 1 it means that the failure rate is constant.
- The **scale parameter** ($\boldsymbol{\theta}$) determines the typical time between events. With changes in the parameter θ directly affects the average time between failures.

Thus, by estimating these parameters, the Weibull TBE method can identify when a process becomes less reliable or when improvements occur.

The main objective of Weibull TBE is to detect shifts in the meantime between events, which depends on the scale parameter θ. The authors calculated the meaning in this way:

$$\mu_{TBE} = \theta\Gamma(1 + \frac{1}{\beta}),$$

where Γ is a mathematical function related to the shape parameter β. They also use control charts in the Weibull TBE method, which helps to detect changes in a process. One of the charts, which uses the cumulative sum of squares (CUSUM)

chart, which is suitable for small shifts. For a downward shift of θ, the CUSUM statistics are updated as follows:

$$C_{w,i}^{-} = min(0, C_{w,i-1}^{-} + y_i^{\beta} - k_w), \quad C_{w,0}^{-} = 0,$$

where k_w is a reference value based on in- and out-of-control scale parameters. For this chart, when the cumulative sum is above or below any of the control limits, it means the failure rate has increased and intervention is needed. On the other hand, they also use the exponentially weighted moving average (EWMA) chart, which they use in cases where they want to detect gradual changes. The mathematical equations they use are:

$$E_{w,i}^{-} = \lambda y_i^{\beta} + (1-\lambda)E_{w,i}^{-}, \qquad E_{w,0}^{-} = \mu_{in-control}$$

Here, λ controls how much influence recent data has. Finally, to manage a variety of displacements they use hybrid diagrams, which combine the advantages of the CUSUM and EWMA approaches. These diagrams are called MEC (Mixed EWMA-CUSUM) and are tuned to be more sensitive to incremental displacements, while MCE (Mixed CUSUM-EWMA) diagrams are better suited for the rapid detection of larger displacements.

In general, Weibull TBE control charts are effective in several areas. For example, it is used in production processes, in a reliability machine and hygienic care. Based on studies and examples carried out, they show that in a production machine when it starts to show more frequent failures, the TBE diagram can quickly signal the problem and can be repaired more quickly. On the other hand, in a healthcare environment, Weibull TBE charts help track the time between adverse events, such as medication errors.

The Weibull TBE approach has several advantages:

- Flexibility: A feature that makes it suitable for a wide range of applications.
- Sensitivity: To ensure early alerts for any problems, the charts are made to identify both little and significant movements.
- Practicality: By concentrating on variations in the average time between occurrences, the technique gives clear and actionable information.

A useful technique for tracking processes when event timings have a Weibull distribution is the Weibull TBE approach. It is essential for preserving quality and dependability because of its capacity to simulate fluctuating failure rates and identify changes early. This strategy offers a complete answer, using advanced control charts customized to certain requirements.

3.3.2 Adaptive Control Charts for Time-Between-Events Under Weibull Distributions

The distribution provides several conditions for extending Time-Between-Event (TBE) control charts, which allows monitoring of processes in which event rates

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change over time. In this case, researchers Liang Qu and Zhang Wu (2015) analyze further the parameter θ, which was also mentioned in the previous methodology for the Weibull distribution. The parameter θ allows the modelling of processes with dynamic risk rates, which is quite useful in cases where there is some damage or failure in the early stage, for example in a construction company.

To construct the functions, probability density function (PDF) and cumulative distribution function (CDF) are given as follows:

$$f_T(\,t \mid \lambda, \theta\,) = \lambda\theta(\lambda t)^{\theta-1} e^{-(\lambda t)^{\theta}},$$

$$F_T(t \mid \lambda, \theta) = 1 - e^{-(\lambda t)^{\theta}}, \qquad t \geq 0$$

Here, λ represents the event rate parameter. Furthermore, when the parameter θ of the equation takes value 1, this means that the risk rate remains constant, and this shows that they do not have a Weibull distribution but an exponential distribution. Whereas for values greater than 1 the hazard rate increases over time, modelling processes prone to wear failures, and conversely in cases where the parameter takes values less than 1 it reflects failures at the beginning of life with decreasing hazard rates. This flexibility of the parameter i allows the Weibull TBE control charts to capture different patterns of risk. The authors explained it better with different examples, such as that in healthcare with the Weibull model they can detect when a process transitions from an initial to a test phase, provided it is in a steady state.

Related to the construction of a Weibull TBE diagram, which involves setting control limits and optimizing key parameters to effectively monitor shifts in both the event rate (λ) and the risk rate shape parameter (θ). Initially, the principles of sequential analysis (SA) will be used to set the control limits, in which at each step the cumulative statistic Q_i is updated using the formula:

$$Q_{i+1} = Q_i + k - T_i,$$

$$with\ Q_0 = 0$$

Here, T_i is the observed time interval, and k is the reference parameter, chosen to balance sensitivity across a range of shifts. More specifically:

i. **Lower Control Limit (g):** Modified to guarantee that the target inspection interval d and the in-control Average Sampling Time (AST0) nearly correspond. This guarantees that the sample rate is neither too sparse nor frequent.
ii. **Upper Control Limit (h):** designed to reduce the possibility of false alarms by ensuring that the in-control ATS (ATS_0) reaches a certain level.

At the end of their research, the authors conclude that the Weibull Time-Between-Event (TBE) control chart with the incorporation of the parameter θ, as mentioned earlier in the methodology, allows them to make several significant improvements in process monitoring. In addition, the identification of constant

risk rates converts the TBE diagram to be able to be used in a large variety of different processes. Also, the way in which the control limits are calculated is quite practical, as it enables the chart to detect more quickly any changes in the process, and changes in the shifts in risk rates. For this reason, this diagram is quite a useful tool in sectors such as construction or health care.

3.3.3 Balanced Two-Sided CUSUM Chart for Monitoring Time-Between-Events

In processes where events occur periodically, monitoring the time between events (TBE) is crucial to detect shifts in failure rates (λ). For this reason, L. Qu and Z. Wu (2015) propose the basic two-sided CUSUM diagram, which provides separate detection for increases and decreases in L. However, these diagrams have limitations in effectively balancing the detection power at these shifts. Thus, the balanced two-sided CUSUM diagram was created, which will address these shortcomings. The way it deals with this is by incorporating an asymmetric loss function, which can optimize its parameters.

Based on L. Qu and Z. Wu (2015) the methodology for the balanced two-sided CUSUM diagram is based on the original CUSUM, but with some modifications that improve the basic performance. It tracks shifts in λ using two separate statistics: C^+, which tracks increases in λ, and C^-, which tracks decrease. These statistics are updated iteratively:

$$C_i^+ = max(0, C_{i-1}^+ + t_i - t_0 - k^+),$$

$$C_i^- = max(0, C_{i-1}^- + t_i - t_0 - k^-),$$

where t_i is the observed time between events, $t_0 = {}^1/_{\lambda_0}$ is the average TBE within control and k^+, k^- are reference parameters controlling sensitivity to shifts. Regarding control limits, which are denoted by H^+ and H^-, to consider a process under control, $C_i^+ \leq H^+$ and $C_i^- \leq H^-$ must be true, otherwise an out-of-control flag is triggered, which means that an intervening action must be taken.

The asymmetric loss function that uses the balanced two-sided CUSUM diagram to design the parameters helps to efficiently detect shifts in failure rates (λ). For accuracy, this function attempts to examine the effects that will exist when the shifts are upward or downward, but also the probability of these shifts occurring. The loss function combines the cost for upward and downward shifts with their detection performance, represented by the average signal time (ATS) and the probability distribution of the shifts. The aim is to minimize the specific loss function, so that in this way the chart is guided to the best choice of values for the reference parameters and control limits.

One of the advantages that the authors mention about two-sided CUSUM charts is that they can keep in balance the ability to detect between increases and decreases in failure rates. In addition, to prove this they have carried out

several simulation studies showing that it consistently achieves lower average losses (AL) over a wide range of process shifts compared to the basic CUSUM and T charts. In addition, these diagrams are suitable for situations where displacements cannot be predicted, as they are based on modelling to optimize performance.

Because the balanced two-sided CUSUM diagram is considered better than traditional control charts, it marks a substantial improvement around TBE monitoring. By balancing sensitivity to both upward and downward fluctuations, its new design maximizes the detection of shifts in failure rates (λ), contributing to more accurate and faster diagnosis of process anomalies. The diagram efficiently allocates the detection power by considering the different effects and probabilities of different types of displacements using an asymmetric loss function. This optimization is particularly useful in cases where the effects of upward variations, such as an increase in defect rates or failure incidents, are more severe, as it not only reduces false alarms but also improves the ability to detect significant shifts.

3.3.4 Cumulative Conforming Control Chart with Discrete Weibull Distribution

In this section, Sajid Ali and Tanzila Zafar (2019) will refer to an equally important diagram of the Weibull distribution which is the cumulative conformal control (CCC) diagram, which is appropriate for monitoring the time between events (TBE) in processes characterized by high quality and low defect rates. Because as we have mentioned in other sections there are several environments that often face challenges such as high false alarm rates or impractical control limits, for this reason they proposed the cumulative conforming control (CCC) chart. This chart, by incorporating the discrete Weibull distribution, overcomes these problems by focusing on the time or count between non-conforming events, making it an ideal solution for high performance processes. The discrete Weibull distribution offers versatility for a variety of applications by capturing situations where failure rates rise, fall, or stay the same, in contrast to more straightforward distributions like the exponential or geometric.

In the methodology section, the discrete Weibull distribution is defined by two parameters: the shape parameter (b) and the scale parameter (q). Regarding the shape parameter b, which determines the trend of the failure rate, when it takes values greater than unity it means that there is an increased failure rate, while when the values are less than unity it means a decreased failure rate. A constant failure rate will exist when value b is equal to unity. The parameter b is the same and the same conditions apply as in the previous sections, which mentioned the use of the Weibull distribution. What differs in this case is the scale parameter q, which determines the overall probability of observing longer or shorter intervals between events. Furthermore, when the parameter q takes small values the probability of shorter time intervals between events increases. So, in this way the probability mass function (PMF) of the discrete Weibull distribution is formed by:

$$P(Z = z) = q^{z^{\beta}} - q^{(z+1)^{\beta}}, z = 0,1,2,\ldots,$$

where z represents the time or number between events. The cumulative distribution function (CDF), which is useful for determining the probabilities up to a given point, is expressed as follows:

$$F(Z) = 1 - q^{(z+1)^{\beta}}.$$

In the continuation of the methodology, they use the parameters mentioned above to construct the lower and upper control limits (LCL and UCL). The limits are constructed with the following formulas:

$$LCL = \left[\frac{\ln\left(1-{}^{\alpha}/_{2}\right)}{\ln(q)}\right]^{1/\beta} \text{-1},$$

$$UCL = \left[\frac{\ln\left({}^{\alpha}/_{2}\right)}{\ln(q)}\right]^{1/\beta} \text{-1},$$

where α is the significance level controlling the false alarm rate. When the observed values fall below the lower control limit (LCL), it means that there is an increased rate of defects and corrective action should be taken In the opposite case, when the observed values are above the upper control limit (UCL), it means that there is a reduced rate of defects which implies an improvement of the process. When the latter case occurs, it means that it allows the chart to effectively detect shifts in the failure rate.

To demonstrate the effectiveness of the Weibull CCC discrete chart methodology, the authors have presented several examples and studies that they have carried out on this methodology. A typical example they used was the fires that took place in the forests of Greece during a specific period. With this example they demonstrated the dysfunctions of the traditional diagrams in identifying anomalies in the process. Whereas the discrete Weibull CCC chart was used to monitor the intervals between software failures, accurately identifying stable periods and highlighting potential problem areas. In addition, another use was in the workplace, where it was used to monitor accidents to assess worker safety by identifying trends in workplace incidents.

The discrete Weibull CCC chart offers several significant benefits. It is a versatile tool for monitoring processes with intricate failure dynamics because of its capacity to model different hazard rates. In contrast to more straightforward techniques, it effectively identifies both process progress and degradation, guaranteeing that even minute changes are quickly noticed. It is useful in real-world situations where event counts, as opposed to continuous observations, are the norm since it concentrates on discrete data. Its sturdy construction also strikes the ideal mix between accuracy and dependability by reducing false alarms while retaining high sensitivity.

In conclusion, the discrete Weibull CCC diagram can exploit the discrete Weibull distribution, enhances the ability to model a variety of failure stresses

and provides timely information about changes in the process. This method is useful in several sectors, such as public health or the working environment, to be able to intervene in the different anomalies that will occur in the process. Furthermore, it is a valuable tool for maintaining stability and quality in complex processes.

3.3.5 Comparison of Shewhart-Type TBE Control Charts for Weibull and Generalized Exponential Distributions

Time between events (TBE) control charts are indispensable tools for detecting rare events. Most charts are based on the exponential distribution but consider situations where the failure rate is constant. In situations where the failure rate varies, Sajid Ali and Ismail Shah (2020) will use more elastic distributions. The Weibull distribution and the generalized exponential (GE) distribution are suitable for this purpose. For this reason, in this section, Shewhart-type TBE control charts based on these two distributions are compared.

Related to the method, the authors will use Weibull and GE distributions to model TBE data where failure rates vary over time. For their study, they use the cumulative distribution function (CDF) of the Weibull distribution, which is:

$$F(x;\lambda,\beta) = (1 - \exp(-(\frac{x}{\lambda})^{\beta}),$$

where λ is the scale parameter and β is the shape parameter, as we have already mentioned from another study in the first section. In this case, however, it differs because the authors will use it in combination with the generalized exponential distribution (GE), which they use for the same purpose and its CDF is calculated as follows:

$$F(x;\lambda,\beta) = (1 - \exp(-\lambda x))^{\beta},$$

where λ is the rate parameter and β determines the shape. In addition, they use these distributions to calculate the control limits of the process. The lower control limit (LCL) and the upper control limit (UCL) are determined as follows:

$$LCL = \lambda^{-1}[-\ln(1 - {}^{a}/_{2})]^{1/\beta},$$

$$UCL = \lambda^{-1}[-\ln({}^{a}/_{2})]^{1/\beta},$$

where α sets the false alarm rate. When they have points that are below LCL, this means a possible increase in failure rates, on the other hand when they have

https://doi.org/10.26219/heal.aueb.6034

observations that are above the upper control limit of UCL it means fewer failures and possible process improvement.

It is necessary to check the performance of the Weibull and GE TBE control charts, and they do this either with the average run length (ARL), which calculates how quickly the chart detects process changes, or with the Expected Quadratic Loss (EQL), which combines detection accuracy and cost by evaluating how well the chart minimizes false signals over a range of shifts. According to studies and measurements of performance, the authors show that Weibull-based charts are better at detecting process improvements, this is because they are more sensitive to reducing failure rates. In addition, the authors conducted research and created data examples with real-world data to compare these charts. For example, GE diagrams were tested on earthquake-related data, where they were effective in detecting shifts in earthquake timing, demonstrating their superiority over Weibull diagrams in tracking natural events. In this way, the authors understand that the possibilities of each diagram change depending on the conditions and regions under consideration.

In conclusion, both diagrams are valuable tools for monitoring rare events in high quality processes. More specifically, Weibull charts are more effective for detecting process improvements, while on the other hand GE charts can provide balanced performance for detecting both increases and decreases in failure rates. The choice between the two charts depends on the areas one would like to address, or the specific needs of the process. The conclusion of the authors is that both charts are better than traditional charts, as they are more reliable and ensure greater accuracy.

https://doi.org/10.26219/heal.aueb.6034

3.4 Gamma Time Between Events Control Charts

In the following sections we will study methods using the Gamma distribution. The Gamma Time Between Events (TBE) control chart is an effective statistical tool for monitoring processes where the time intervals between events are of interest. Studies with several different diagrams, and combinations of them, will be used to explain several examples and situations.

3.4.1 Time Between Events Control Charts for Gamma Distribution

Traditional diagrams often assume that TBE data follows an exponential distribution, which simplifies calculations but limits flexibility in modeling processes with varying or skewed event rates. In contrast to the exponential distribution, the gamma distribution adapts to a wider range of TBE data, including processes with skewness, with the potential to provide a more robust alternative. The difference between gamma charts and other charts is that use the actual probability distribution of the data, instead of transforming the data to approximate normality. Based on Muhammad Taqi Shah et al. (2020), this way ensures greater accuracy and preserves the original properties of the data. With the methodology he will present and the examples he carried out, he tries to show that these charts ensure greater accuracy and preserve the original properties of the data.

First, the author analyzes the gamma concept to make the method that he uses more understandable. More specifically, the gamma distribution is defined by two parameters: the shape parameter (a) and the scale parameter (b), and its probability density function is:

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{a-1} e^{-x/\beta}, x > 0,$$

where $\Gamma(a)$ is the well-known mathematical gamma function. This distribution is appropriate for TBE data, because it models only positive values. The shape parameter (α) controls the sharpness of the distribution's peak, while the scale parameter (β) influences the spread. In addition, he reports with respect to the values of parameter a, that when the values are larger, the gamma distribution becomes more symmetrical, thus improving the accuracy of the control chart. In the continuation of the method, he reports the finding of control limits carried out based on the mean and variance of the gamma distribution. The types used to calculate the central line (CL), the upper control limit (UCL) and the lower control limit (LCL) are as follows:

$$CL = a_0\beta_0,$$

$$UCL = \alpha_0\beta_0 + k\beta_0\sqrt{\frac{\alpha_0}{n}},$$

$$LCL = \alpha_0\beta_0 - k\beta_0\sqrt{\frac{\alpha_0}{n}},$$

where α_0 and β_0 are the parameters within the control, n is the sample size and k defines the width of the control limits. When observations are noted which are outside the upper and lower control limits, this means that there is a shift in procedure. The average run length (ARL), or the average number of observations required to indicate a shift, is used to gauge the gamma-TBE chart's performance. While out-of-control ARL (ARL_1) gauges how rapidly the chart detects a change, in-control ARL (ARL_0) gauges how long the chart stays steady when the process is operating normally.

The authors then compared the gamma diagram with the exponential transformation (TE) diagram through simulations. In fact, the results of this study led to the conclusion that the gamma diagram had better performance, especially in cases of fine displacements. For example, when the scale parameter (β) shifted from 1.0 to 0.9 with a sample size of n=5, the ARL dropped from 369.46 (in-control) to 66.32 (out-of-control), reducing detection time by over 80%. Furthermore, as the sample size increased, the larger displacements were even better detected and the sensitivity of the C-TVE diagram was further improved. Unlike TE charts, which rely on data modifications that might introduce biases, gamma-TBE charts directly use the original data. This makes them more resilient, particularly for processes with skewed TBE data.

In conclusion, the TBE gamma chart is considerably more effective for monitoring TBE data than traditional charts. Based on the research results and the examples, TBE gamma charts can avoid data transformation, which means that he maintains the integrity of the original data and provide better sensitivity to process variations. All this makes this methodology a quite useful tool for the industrial sector, where the emphasis is on maintaining high quality and reliability.

3.4.2 Generalized Likelihood Ratio (GLR) Control Chart for Gamma-Distributed TBE Data

In industries such as semiconductors and pharmaceuticals, Time-Between-Events (TBE) control charts are commonly used to ensure high purity and reliability in processes. Often, exponential or transformed normal distributions are applied to these control charts. However, these distributions may not adequately represent the inherently skewed nature of TBE data. For this reason, Caterina Rizzo and Alessandro Di Bucchianico (2022), proposed Generalized Likelihood Ratio (GLR) control charts, which exploited the gamma distribution to better models which skewed TBE data. The creation and use of the Gamma GLR chart

https://doi.org/10.26219/heal.aueb.6034

are covered in this subsection, along with the benefits of using it to identify process changes.

The GLR control chart uses gamma distribution to detect shifts in the parameters of the TBE data. The gamma distribution, characterized by shape (α) and scale (β) parameters, has the probability density function:

$$f(x;\ \alpha,\beta) = \frac{x^{a-1}e^{-x/\beta}}{\Gamma(a)\beta^{\alpha}}\ \ ,x\ >\ 0$$

where $\Gamma(a)$ is the well-known mathematical gamma function. More specifically, this distribution is suitable for modeling positive and skewed data, with mean $\mu\ =\ \alpha\beta$ and variance $\sigma^2 = \alpha\beta^2$. The statistical likelihood ratio used by the GLR chart in this case is to detect changes in the parameters of the gamma distribution. According to the authors, the GLR test for a sample $X_1, X_2 \ldots, X_n$ evaluates the likelihood of the data under two hypotheses:

- **Null Hypothesis (H_0)**: The process is in control ($\alpha = \alpha_0\ ,\beta = \beta_0$).
- **Alternative Hypothesis (H_1)**: The process parameters have shifted ($\alpha = \alpha_0$ or $\beta = \beta_0$).

The likelihood ratio statistics are given by:

$$\Lambda = \ 2\left[\ ln\ L\left(\widehat{\alpha_1},\widehat{\beta_1};\ X\right) -\ ln\ L(\alpha_0,\beta_0;\ X)\right]$$

where $L(\cdot)$ is the likelihood function, and $\widehat{\alpha_1},\widehat{\beta_1}$ are the maximum likelihood estimates under H_1. If it obtains values exceeding a control limit, then the method considers the process to be out of control. The likelihood ratio statistic, which contrasts the probability of the data under the in-control parameters (α_0, β_0) with the likelihood with shifted parameters (α_1, β_1), is the basis for these limitations. Under the null hypothesis, the statistics have a known distribution, and Monte Carlo simulations are used to establish control limits that guarantee a certain in-control Average Run Length (ARL_0). Investigation and remedial action are prompted when observations surpass the upper control limit, which signifies a process change. These customized control limits reduce false alerts while increasing the chart's sensitivity to changes.

The authors then generated several simulation studies, which showed the Gamma GLR diagram to be superior to traditional methods such as Shewhart diagrams for TBE data with a gamma distribution. They also showed that this diagram is particularly effective in detecting both small and moderate shifts in process parameters. For example, one of the simulation studies showed that when the scale parameter (b) is reduced by 10%, the GLR diagram signals the shift faster than competing methods, with significantly lower Average Time to Signal (ATS).

As mentioned earlier, the GLR gamma diagram has been applied to high-purity processes such as semiconductor manufacturing and chemical production. It is useful in these industries because the shifts in the times between events detected by the GLR chart ensure that deviations are detected in a timely manner. They thus ensure that defects are prevented, and product quality is maintained. A specific

example they give to explain the previous sentence is that they used the chart for a pharmaceutical production, where shifts caused by equipment wear or environmental changes were detected quickly.

An effective way to monitor TBE data in processes with gamma distributed inter-event intervals is by using the Gamma GLR control chart. This is a better choice for high quality processes because of their high sensitivity to detect small and moderate changes. The GLR chart avoids the disadvantages of conventional techniques by directly using gamma distribution and offers a reliable framework for ensuring process stability and product quality.

3.4.3 Double Exponentially Weighted Moving Average (DEWMA) Control Chart for Gamma-Distributed TBE Data

As mentioned in all sections, Time-between-events (TBE) control charts are essential for monitoring processes where rare events, such as failures or defects, occur more often. A chart used in these cases is the Shewhart chart, which is effective but only for detecting large changes. Another case is the exponentially weighted moving average (EWMA) chart, which improves sensitivity to small changes by putting more emphasis on recent data. Based on this, the Double Exponentially Weighted Moving Average (DEWMA) chart improves on the previous chart by adding a second level of weighting. Based on Vasileios Alevizakos and Christos Koukouvinos (2019), in this way the chart manages to become more effective for processes where the TBE data follow a gamma distribution and where the detection of small changes in process behaviour is quite important.

The gamma distribution models the time between events in many real-world processes. The gamma distribution has two parameters, which are:

- The shape parameter (a), which controls the shape and skewness of the overall distribution.
- The scale parameter (b), which represents the average time between events.

For this distribution, the mean and variance are divided by $\mu = kh \; and \; \sigma^2 = kh^2$ respectively. Based on these types, they justify that the distribution is suitable for modelling skewed and only positive data. More specifically, the DEWMA control chart is suitable for monitoring changes in the scale parameter (b). This is because it is the b parameter whose shifts indicate immediate changes in the process. Thus, the diagram uses two weighted averages:

1. **First weighed average (Y_t):** average of recent observations with historical data.
2. **Second weighted average (Z_t):** Smooths the results of Y_t to improve trend detection.

And the types related to the DEWMA diagram are:

$$Y_t = kX_t + (1-k)Y_t - 1 \ ,$$

$$Z_t = kY_t + (1-k)Z_t - 1,$$

in this type of the variable X_t is the observed TBE value, k is the smoothing constant
$(0 < k \leq 1)$ and Y_t and Z_t are the intermediate and final DEWMA statistics. For the procedure the control limits need to be calculated, which are:

$$LCL_t = k\,h_0 - L\sqrt{Var(Z_t \mid IC)}$$

where in the formula the variable kh_0 is the within-control mean, where L denotes the width of the control limits and $Var(Z_t \mid IC)$ is adjusted for time-dependent variability. In cases where Z_t is below the lower control limit, it means a shift in the process.

The authors then carried out several simulation studies to prove their theory on the DEWMA diagram, which significantly outperforms traditional Shewhart and EWMA diagrams in detecting process shifts. In one of their examples, they state that when the parameter b decreases by 10%, in this case the DEWMA diagram signals the shift much faster, with a lower average run length (ARL). Furthermore, in the studies they also observe the difference between the two diagrams, since the Shewhart diagram is only suitable for large shifts, while the DEWMA chart excels at detecting small and moderate shifts, providing faster alerts with fewer false alarms. This has the effect of improving the EWMA diagram, because it manages to reduce the time for detecting shifts in gamma-distributed TBE data.

The authors add that the DEWMA diagram is quite practical in the industrial sector, and more specifically in cases where monitoring the time between events is critical. For example, in a reliability engine it is useful because it can and does track the time of equipment failures, this happens by detecting decreases in the variable b. In this case when this happens the diagram intervenes and helps in the planning of timely maintenance. Also, in the production section the DEWMA diagram is applied to monitor the time between defects in production lines. In which it prevents several quality problems due to the sensitivity to shifts, thus ensuring that potential issues are detected in time.

In conclusion, the DEWMA control chart is a powerful tool, which can be used in many areas to monitor TBE data with gamma distribution. As the authors mention in their comparisons, the combination of the two levels provides superior sensitivity to small and moderate shifts compared to Shewhart and EWMA charts. Finally, they conclude that this method and the ability of the diagram to quickly and accurately detect small trends make the DEWMA diagram a significant advance in TBE monitoring.

3.4.4 Triple Exponentially Weighted Moving Average (TEWMA) Control Chart for Gamma-Distributed TBE Data

The previously discussed Double Exponentially Weighted Moving Average (DEWMA) control chart method has been extended through the development of the Triple Exponentially Weighted Moving Average (TEWMA) control chart. This advanced method serves as a powerful tool for monitoring Time-Between-Events (TBE) data, particularly when such data follow a gamma distribution. While the DEWMA chart can be effective, it presents several limitations under certain conditions. In contrast, the TEWMA chart incorporates a third level of weighting, enhancing sensitivity and making it especially valuable for detecting small to moderate shifts in process behavior. According to Vasileios Alevizakos et al. (2020), who claim that this makes the TEWMA diagram ideal for applications where early detection of changes in TBE is critical.

Regarding the methodology, he will also use here the gamma distribution because it is quite effective for modeling TBE data, as it efficiently handles positive values and skewed distributions. As defined in the previous subchapter, the gamma distribution is defined by both the shape (a) and scale (b) parameters, where the mean value (μ) is given by μ=ab. Changes in time between events are best captured by shifts in the scale parameter b. The author continues the methodology by constructing the TEWMA diagram. The TEWMA diagram is a construction of both the EWMA and DEWMA diagrams, with the only change being the additional weighting level. This additional weighting level enables the chart to calculate the three retrospective statistics:

1. **Yt**: The EWMA statistic that averages the observed TBE values.
2. **Zt**: The DEWMA statistic that smooths the EWMA statistic.
3. **Wt**: The TEWMA statistic that applies an additional level of smoothing to Zt.

The equations are as follows:

$$Y_t = \lambda X_t + (1 - \lambda) Y_{t-1}$$

$$Z_t = \lambda X_t + (1 - \lambda) Z_{t-1}$$

$$W_t = \lambda X_t + (1 - \lambda) W_{t-1}$$

where X_t is the observed TBE value, λ is, the smoothing parameter ($0<\lambda\leq1$) and the initial values (Z_0, Y_0, W_0) are set to the expected mean value $k\theta_0$ under control conditions. It is also necessary to specify the control limits, which are defined as based on the variation of Wt:

$$UCL = k\theta_0 + L\sqrt{Var(W_t)}$$

$$LCL = k\theta_0 - L\sqrt{Var(W_t)}$$

where L controls the width of the control limits. Based on the equations and different investigations he has carried out, he observed that the TEWMA chart is particularly effective for continuous monitoring when the variance stabilizes, and the control limits converge over time.

The author then carried out several simulation studies where he came to some conclusions about the TEWMA diagram. More specifically, the TEWMA diagram is sensitive to both upward (b>b0) and downward (b<b0) shifts of the scale parameter. Furthermore, the results show that it outperforms the other charts in detecting small to moderate shifts. The distribution of the data in the studies carried out was not only with gamma distribution, but also with other distributions that had a fairly good performance. For example, in addition to the gamma distribution, the Weibull distribution was also used, which had quite good results.

This TEWMA diagram is quite useful in a wide range of industries. For example, it is used in aviation companies, because it can monitor the time between accidents to identify trends and detect safety problems early. In addition, the TEWMA diagram is a valuable tool for earthquake researchers, who use it to monitor the intervals between seismic events and to identify patterns of seismic activity. These applications are useful, because he shows that the diagram is necessary both for identifying errors and for valid decision-making.

In conclusion, the authors report that the TEWMA control chart is an important advance in TBE monitoring. The property of the chart is to provide unparalleled sensitivity to small and moderate displacements while maintaining robustness to different types of data, and it achieves this by incorporating three levels of exponential weighting. It is also a valuable tool for industries, based on studies and paradigms, as it can detect changes quickly and accurately.

3.4.5 Improved Shewhart-Type TBE Control Charts with Runs Rules

Based on this article and the chapters above, traditional control charts fail to provide important information in high performance processes where defects or failures occur infrequently, for example c-charts or u-charts. This is because the above diagrams often show zero, which makes it more difficult to detect changes. To address this problem, time-between-events (TBE) control charts, such as the tr chart, are used. More specifically, the tr chart is particularly effective when the TBE data follows a gamma distribution, because it tracks the time to the r-th event. The u and c charts have several limitations in detecting small shifts. N. Kumar, S. Chakraborti and A. C. Rakitzis(2017) propose two key improvements such as: incorporating running rules to improve sensitivity to small and moderate displacements and ARL-unbiased designs to ensure robust detection regardless of the direction of displacement.

The tr-chart mentioned by the authors, is designed to track the time Tr until the r-th event in a homogeneous Poisson process, which follows a gamma

https://doi.org/10.26219/heal.aueb.6034

distribution. With respect to the shape of the diagram, it will be necessary to define the probability density function, using the shape parameter r and the failure rate λ. It is defined as follows:

$$f_G(t \mid r, \lambda) = \frac{\lambda^r}{\Gamma(r)} t^{r-1} e^{-\lambda t}, t > 0, \lambda > 0.$$

The authors also set the limits of control that will be needed. The lower and upper control limit for the tr diagram is based on the within-control gamma distribution, ensuring that signals occur when the observed TBE deviates significantly from the expected time. For accuracy the control charts are defined as follows:

$$UCL = \frac{x^2_{2r,1-a_0/2}}{2\lambda_0}$$

$$LCL = \frac{x^2_{2r,a_0/2}}{2\lambda_0}$$

where in the type a_0 is the percentage of false alarm. In addition, the central control limit is also the expected average and is calculated as follows:

$$CL = \frac{x^2_{2r,0.5}}{2\lambda_0}$$

The probabilities of signals above the UCL and below the LCL are adjusted in the ARL-unbiased design, which the authors introduce to improve the detection performance. To maintain sensitivity to shifts in both directions, they use these adaptations to ensure that the average run length (ARL) is maximized when the process is under control ($\lambda = \lambda_0$). To improve the tr diagram they use the execution rules considering the patterns in the successive observations. Also, the authors describe the procedure for the control limits, more specifically, when a certain number of observations falls below the lower control limit, a signal is triggered.

Relative to the simulation results carried out by the authors, they show that the tr diagram outperforms standard tr diagrams in detecting small to moderate shifts in the failure rate (λ). Furthermore, to ensure reliable monitoring, they use the unbiased ARL designs, which address the issue of biased detection near the control condition. To clarify this, they also mention the comparison with other control charts, such as EWMA and CUSUM, as it provides better detection performance while maintaining simplicity, thus highlighting the practicality of the improved tr-chart.

This improved tr diagram has been applied in several areas. Initially, it was applied with great success in the field of medicine, where it was responsible for

https://doi.org/10.26219/heal.aueb.6034

detecting any errors that occurred and intervening where necessary. In addition, many researchers involved in natural disasters, for example earthquakes, use this chart to keep a watch on the time between events.

An important development in TBE monitoring is the enhanced Shewhart-type tr-chart. It retains the simple layout of conventional Shewhart charts while improving sensitivity to little changes by implementing ARL-unbiased designs and run rules. Because of these advancements, it is now a vital tool for sectors that need precise and trustworthy process monitoring, especially in settings with low failure or defect rates.

Chapter 4

Advanced and Non-Parametric Approaches

Advanced and non-parametric approaches to control mapping offer powerful alternatives to traditional parametric methods, especially when data distributions are unknown or complex. More specifically, First Passage Time (FPT) and Non-Parametric Methods will be used for the analysis of this section. In addition, several examples with real data will be used to better explain the module. Finally, this section explores advanced and non-parametric approaches, highlighting their advantages, implementation strategies and practical applications in process monitoring and reliability analysis.

4.1 First Passage Time (FPT) Time Between Events Control Charts

First, we start with the advanced method of the First Passage Time (FPT) Time Between Events (TBE) control chart, which is used to monitor the time intervals between events in a stochastic process. The difference between this method and others is that approaches based on FPT exploit the concept of first passage time - the time it takes a process to reach a predefined threshold for the first time. More specifically, this module will study the construction, applications, and benefits of FPT-based control charts for Time Between Events data.

4.1.1 Methodology and Comparative Analysis of First Passage Time Control Charts

In this chapter Sajid Ali (2020) will analyze First Passage Time (FPT) control charts, which are innovative tools designed to monitor time-between-event (TBE) data in complex processes where both time and size must be considered. Most traditional charts follow simple distributions such as exponential or gamma, in this case FPT charts follow a relatively more flexible framework using non-homogeneous Poisson processes (NHPP) with power law intensity. In this way, he makes the diagram particularly useful in cases where both time and damage thresholds are crucial for monitoring the health of the process, by allowing dynamic modelling of the intensity of events over time.

Regarding the methodology, the author for the FPT diagrams defines the TBE as the time required for the cumulative damage to exceed a critical threshold K. With this approach he manages to correct the cumulative damage after each time the threshold K is exceeded, thus allowing continuous modelling of the process. More specifically, for the generation of aggregate processes, the NHPP framework assumes an intensity function, which is as follows:

$$\lambda(t) \propto t^{\{\beta-1\}}$$

where b is the shape parameter. To model the size of each event, it uses the simplest distribution in these cases, the exponential one, which has a rate parameter θ. To explain the probability of cumulative loss exceeding K by time t, the first pass distribution uses the following equation:

$$P(Z \leq t) = 1 - exp\left(-exp(-K\theta)(\lambda t)^{\beta}\right),$$

where Z is the time to the critical point. The mean time to a critical point K is computed as:

$$E(Z) = \Gamma\left(1+\frac{1}{\beta}\right)\left(\frac{\exp(-K\theta)}{\lambda^{\beta}}\right)^{\frac{1}{\beta}}$$

The upper and lower control limits of the diagram are determined by the solution of the critical probabilities:

$$LCL=F\text{-}1(\alpha/2),$$

$$UCL=F\text{-}1(1-\alpha/2)$$

where α is the probability of a false alarm and F-1(p) is the inverse of the cumulative distribution. Furthermore, when a TBE value falls below the lower control limit it means that the process is out of control and indicates a deterioration of the process, while when a TBE value is above the upper control limit it indicates an improvement of the process.

On the other hand, Sajid Ali and Antonio Pievatolo (2018) proposed a different approach to the modelling of FPT diagrams. This approach is done through renewal reward processes, which helps to monitor the faults that occur in a process. More specifically, the failures reported by the authors separate them into two categories, the cumulative failure, which accumulates gradually over time, and the independent failure, which refers to more catastrophic events. First, they indicate how cumulative damage is modelled, which is as follows:

$$P(Z \leq t) = \sum_{n=0}^{\infty}\left[G^{(n)}(K) - G^{(n+1)}(K)\right]F^{(n+1)}(t),$$

where $G(m)$ is the cumulative damage distribution and $F(x)$ is the TBE distribution. Whereas for independent damage, it is calculated as follows:

$$P(Z \leq t) = \sum_{n=0}^{\infty} [G^n(K) - G^{n+1}(K)]\, F^{(n+1)}(t).$$

This method is quite useful for the practice procedures, because it can cope with any damage that occurs in the process.

In his article the author compares the proposed FPT charts with the existing FPT charts. In particular, he mentions the advantages of the FPT charts based on his methodology. These advantages are related to:

1. **Efficiency:** The proposed FPT charts can achieve lower ARL values both in cases with points above the upper control limit and in cases with points below the lower control limit, thus outperforming traditional models in detecting changes in process parameters**.**
2. **Flexibility**: The specific methodology mentioned in the NHPP with power-law intensity is ideal for a wider range of applications, as opposed to models that follow an exponential distribution in their data.

According to studies, FPT charts based on renewal rewards are suitable for processes where accumulated losses are a problem for the process.Traditional rate charts may be more effective for smaller thresholds (K), as they emphasise instantaneous changes in event size. On the contrary, in cases where the K variable takes higher values and there are several accumulated losses, the FPT charts based on the renewal reward show superior sensitivity. The above findings and comparisons are from the simulations that the writers made for different conditions.

After various studies conducted by the author, he reports that the FPT chart demonstrates superior performance in detecting both improvement and deterioration of the process compared to the traditional TBE and rate charts. Based on the simulations he performed; they proved that the power law intensity NHPP used to be ideal for calculating the changes in process variables and time. For example, FPT charts can quickly detect situations when monitoring production processes where machines either improve or degrade their reliability.

It then refers to different areas where this diagram can be used because of its flexibility. As he has already mentioned and given an example of this, an ideal sector is manufacturing, where he can monitor damage to machinery and intervene when necessary. It can also be ideally applied in the sports sector, more specifically in monitoring the impact of doping on players. On the other hand, FPT diagrams based on the renewal reward are useful for situations such as mining, because they detect time shifts between explosions or accidents, helping to ensure operational safety.

In conclusion, the FPT diagram is a quite useful tool in many fields, because it can and does offer an advanced solution to the monitoring of TBE data in systems where time and size jointly affect the health of the process. The ability of the chart to accurately detect displacement through the methodology reported by the author makes it ideal for industries seeking differentiated monitoring solutions. Furthermore, the FPT diagrams based on renewed reward further enhance the flexibility of this approach by incorporating cumulative and independent damage models, ensuring that a wide variety of different situations can be monitored.

4.1.2 Number-Between-Events (NBE) Chart as a Discrete FPT Method

Processes that include cumulative or stochastic events mean the time to exceed a critical threshold at which time-between-events (TBE) data monitoring is often focused. For this reason, many use the Number-Between-Events (NBE) control chart, which can be considered equivalent to the FPT method, meaning that the process of interest is modelled on the measurements of the conforming units. This method is quite useful in cases where the process is small, for example a process with a low production volume, where time-based monitoring is not feasible. The NBE diagram uses negative hypergeometric distribution, to monitor the number of fixed singles of a process until a non-compliant number of singles is created. A similar situation is created by FPT diagrams, which do something similar, such as tracking time until a critical point in the process is passed.

Regarding the methodology, Arne Johannssen et al. (2021) tried to create his own effective theory on this, based on the FPT methodology. More specifically, the author replaces the time variable used in the methodology in the FPT charts with the variable controlling the number of units. As mentioned in the introduction, based on the negative hypergeometric distribution, the procedure models the number of conforming units (Y) required to observe a given number of nonconforming units (r), and the following results:

$$P(Y = y) = (MN)(r - 1y - 1)(M - rN - y),$$

where:

- N: total population size.
- M: number of non-compliant units in the population.
- r: Target number of non-compliant units.
- Y: Number of conforming units observed before r is reached.

In addition, the variance of the process (Var(Y)) is calculated as follows:

$$E[Y] = M + 1r(N + 1),$$

$$Var[Y] = (M + 1)2(M + 2)r(N + 1)(N - M)(M + 1 - r).$$

Furthermore, the author state that the upper and lower control limits obtained are based on the control limits generated by the FPT charts, more specifically he is formulated as follows:

$$LCL = E[Y] - LVar[Y],$$

$$UCL = E[Y] + LVar[Y],$$

where L defines the width of the control limit. Where in this case the same things apply as for the FPT charts regarding the control limits. The key difference with FPT

https://doi.org/10.26219/heal.aueb.6034

charts is that in this case he uses discrete units of measurement instead of continuous time intervals, making the NBE chart ideal for settings with limited sizes.

The author reports that the two charts, the FPT and the NBE, have several similarities, especially in cases where he reports the lower bounds. Firstly, as already mentioned in both cases he focus on detecting shifts by monitoring the crossing of a critical threshold, in the case of FPT it is time and in the case of NBE it is a number. Also, FPT charts use NBE charts as a last resort in cases where they are in environments where time measurement is impractical.

The NBE diagram is a very useful tool in several areas. For example, it is a useful tool in the health care sector, where it is used to monitor adverse events in clinical studies with limited patient samples. As another example, in a production facility with limited production volume, the NBE chart detected shifts in the fraction of defective items more efficiently than time-based TBE charts. Its ability to adapt to finite sample sizes ensures early detection of process changes.

In summary, when there are cases where the sample size is limited or small enough, the Number-Between-Events (NBE) diagram is quite effective and useful. The use of the negative hypergeometric distribution, which contributes to the accurate monitoring of process shifts, makes the diagram quite valuable in fields such as hospitals or biomedical centers, where in these cases detecting the change in the process is quite critical. Additionally, the parallels between NBE and FPT charts show that both emphasize passing important thresholds, with NBE being especially well-suited for situations that call for unit-based analysis as opposed to time intervals.

4.2 Non-Parametric Methods Time Between Events Control Charts

The second part of the chapter will use non-parametric Time Between Events (TBE) control charts, which provide a flexible and powerful alternative to traditional parametric methods, eliminating the need for assumptions about the underlying distribution of event times. As we shall see, this component is useful in cases where the distribution of the data is unknown. It is also very useful in a few areas, and as we will study below, different diagrams will be used to describe the method and explore basic non-parametric techniques.

4.2.1 Nonparametric Time-Between-Events (TBE) Control Charts

In many cases where statistical process control (SPC) will be applied, several distributions fail to perform, and deviate significantly from standard assumptions, such as normality. In this way there is a weakening in traditional parametric diagrams, especially in processes with high variability. For this reason, Sajid Ali et al. (2016) propose in his research the non-parametric control charts, which are not so affected by high quality processes. This is because non-parametric control charts are less sensitive to the allocation assumptions. In addition, the usefulness of these charts includes their suitability in cases where system monitoring is needed for processes with data having skewed distributions. The author, using previous statistical techniques, managed to use these charts in scenarios with both single variable and two variable scenarios.

Next, the author reportσ that non-parametric control charts for TBE focus on deriving reliable statistical tests and monitoring schemes that are not linked to specific distribution assumptions. The main methodologies used are as follows:

1. **MAX and CUMAX Charts**: More specifically, for the maximum observed TBE values they used the MAX chart, introduced by Albers, as a monitoring statistic. For precision it is denoted as:

$$MAXn = max(X1, X2, \dots, Xn),$$

where Xi are the observed TBEs.
Furthermore, the cumulative MAX (CUMAX) chart was designed with this in place, with the aim of tracking the cumulative maximum overtime:

$$CU\ MAX_n = max_{\{1 \le i \le n\}}\left(\sum_{\{j=1\}}^{i} X_j\right)$$

improving the detection of gradual process changes.

2. **Tests based on ranking:** The author then indicate the method they use, which is the ranking-based tests, in which method he uses the ranks of TBEs instead of their raw values. For example, given n TBEs, the ranking statistic R_i of each observation X_i is calculated and the control statistic is:

$$W = \sum_{\{i=1\}}^{n} R_i$$

where W is compared to a control limit derived from the Wilcoxon distribution.

3. **Bivariate monitoring:** In his studies, author uses non-parametric bivariate diagrams to study two different qualitative characteristics in common. One way he does this is by combining the scores of the two characteristics, X and Y, into a common statistic:

$$\{Joint\ Rank\} = R_x + R_y,$$

and RY are the degrees of X and Y, respectively.

Based on the methodologies mentioned earlier, the author has used them in several examples and experiments to create more effective diagrams. Thus, based on the results, non-parametric diagrams offer several advantages compared to parametric diagrams. More specifically, these are:

- **Resilience to deviations of the distribution**: Unlike parametric charts, non-parametric charts perform consistently across a wide range of underlying data distributions, including those with heavy tails or multimodal characteristics.
- **Efficiency in high quality processes:** As previously mentioned by the author, a major advantage for non-parametric methods is that they show better performance compared to parametric methods in high quality processes.
- **Practical flexibility:** These diagrams are quite practical because they can be applied in many different cases, such as for continuous and discrete TBE scenarios.

An example mentioned by the author is that the MAX and CUMAX charts are ideal for businesses, because their flexibility allows professionals to configure and select a chart based on their business preferences.

In conclusion, the author reports that non-parametric TBE control charts are a good enough alternative to the way in which process monitoring is done. He wants to emphasize that these diagrams manage with his non-contextual methodologies to ensure effective detection of process changes, while maintaining flexibility in a variety of applications. Furthermore, the author believes that future research performed in this area could prove that he can be established in other environments, such as multivariate processes with varying time dependencies.

4.2.2 Nonparametric Adaptive EWMA Control Chart for Multivariate Process Monitoring

It is well known that traditional control charts are based on assumptions about the underlying data distributions, for cases involving the monitoring of multivariate processes. However, in these cases the diagrams present several problems, without succeeding in ensuring the desired result. Li Xue and Lisheng An (2024) wanted through their article and research to promote non-parametric diagrams, which are an alternative solution. More specifically, these charts are based on ranking-based statistics instead of raw data, making them more suitable for a wide range of applications. But in this article, the authors refer to the non-parametric adaptive EWMA control diagram, so that they can explain the shifts that take place in multivariate processes by adapting the process change.

Regarding the methodology for this chart, the authors explain that it uses ranks for each variable to eliminate the dependence on distributional assumptions. For a process with p quality characteristics, the ranking of an observation X_{jt} at time t is calculated marginally as R_{jt}. In this way, the exponentially weighted moving average (EWMA) uses the ranks for each variable, where it updates retrospectively as follows:

$$Z_{\{jt\}} = \lambda R_{\{jt\}} + (1 - \lambda) Z_{\{j(t-1)\}}$$

where λ is the smoothing parameter, which is used to determine the weight given to recent observations. Also, the parameter λ is varied based on the distance of the EWMA statistic from its expected value when the process is under control of mR, provided that the diagram is made more flexible. The adaptive smoothing parameter is expressed as follows:

$$\lambda t = \lambda_0 + k \cdot | Z_{jt} - \mu R |,$$

where λ_0 is the baseline smoothing factor and k is a resonance parameter controlling the degree of fit. They then go on to mention control limits, how they are formed, and when to consider a process out of control, which is the key element of the method. In particular, the control limits are determined by analyzing the behavior of Z_{jt} during the control phase of the process. This means, in a simple way, that the data collection is performed in a stationary process when it is under control. The lower control limit (LCL) and the upper control limit (UCL) are determined based on specific quantum elements of this empirical distribution. Furthermore, they state that a process is out of control when the following applies:

$$LCL_j \leq Z_{\{jt\}} \leq UCL_j$$

in which LCL_j and UCL_j are the lower and upper control limits respectively.

In the continuation of their research, the authors carried out several experiments and studies to explain and confirm their theory. For this reason, they give an example to make the use of the EWMA diagram easier to understand. More specifically, they provide a process with three variables, which is monitored over time, and is initially constant. Also, the data follows normal distribution centered around zero. Then, at t=50, they observe that the mean of one variable shift to simulate a disturbance. In this case, the EWMA chart accurately tracks the variation while keeping false alarms at lower levels than standard EWMA charts.

The authors then wanted to point out the differences between the specific diagrams. For this reason, they compare them with the traditional Shewhart diagrams and the EWMA parametric diagrams. The main observations on the differences they report include:

1. **Robustness to Distributional Assumptions**: Non-parametric charts are preferred in non-normal scenarios, where normality assumptions break down.
2. **Sensitivity to Small Shifts:** In most processes the Shewhart charts and parametric EWMA charts have difficulty in detecting the smallest changes, whereas the adaptive EWMA chart, with its dynamic weighting mechanism, effectively captures all types of variations.
3. **False alarm rates:** Compared to parametric EWMA charts, the adaptive non-parametric approach maintains a lower false alarm rate, especially when monitoring processes with unknown distributions.

In conclusion, the EWMA non-parametric adaptive control chart is a dynamic alternative for monitoring multivariable processes. The method used, such as exploiting orders to effectively detect changes in processes with complex distributions, makes it a very powerful tool. Furthermore, the ability to respond directly to process shifts, whether small or large, makes these diagrams a suitable choice for various industries. Finally, the authors mention that future studies on this topic could also focus on processes with time dependencies.

4.2.3 Modified One-Sided EWMA Control Chart for TBE Monitoring

As mentioned in the previous chapter, time-between-events (TBE) monitoring is a very useful tool, especially for high quality processes, to detect changes in process behavior. For this reason, several people have used traditional EWMA charts, although they effectively present a difficulty in detecting asymmetric shifts or process changes specific to TBE data. Non-parametric EWMA diagrams and in particular the one-sided non-parametric EWMA diagram can cope with these problems. The way that uses to deal with these problems is by enhancing the detection capability in scenarios with skewed events. Furthermore, XueLong Hu and YuLong Qiao (2021) point out that the methodology followed for the non-parametric one-sided EWMA charts is based on the traditional EWMA charts, but

with the addition of a one-sided limit to make it effective in the case of upward or downward variation from TBE.

Regarding the methodology, the difference from non-parametric EWMA diagrams is that in this case the authors prefer to emphasize the deviations in a specific direction. With this choice they manage to reduce false alarms and improve detection sensitivity. In case they want to focus on an upward change, the EWMA statistical upper control limit is set as follows:

$$Z_t^+ = max\left(0, \lambda X_t + (1-\lambda) Z_{\{t-1\}}\right)$$

In case they would like to focus on a downward change, the statistical lower control limit of the EWMA is set as follows:

$$Z_t^- = min\left(0, \lambda X_t + (1-\lambda) Z_{\{t-1\}}\right)$$

where λ is the smoothing parameter, X_t is observed TBE at time t and Z_{t-1} is the prior EWMA statistic. To show when the process is out of control, they use this formula:

$$Zt^+ > UCL$$

Or

$$Zt^- < LCL$$

Furthermore, using a reset mechanism tends to improve performance by incorporating past observations into the tracking statistics:

$$Z_t = max\left(\rho h_0, \lambda X_t + (1-\lambda) Z_{\{t-1\}}\right)$$

where ph_0represents the initial value, ensuring continuity in the monitoring process.

The authors have carried out several studies and examples to explain and prove the effectiveness of their method. They then provide an example with practical data to explain the situation. More specifically, they state that there is a context in which the time between machine failures is monitored, in which during the control phase, the TBEs follow a Gamma distribution with known parameters. A process shift reduces the average TBE by 25%, indicating a deterioration in reliability at a certain point. The point they want to prove in this case is that the non-parametric one-sided EWMA diagram can and successfully detects process shifts.

The authors then make a comparison with traditional EWMA and DEWMA diagrams, stating that the non-parametric one-sided EWMA diagrams have more advantages. Although the traditional and DEWMA diagrams are more effective at small displacements, because they have double control limits, the modified diagrams are more effective for detecting asymmetric displacements.

Furthermore, as the authors mention, they are easier to apply because they have simpler calculations than the other diagrams.

In general, an important advance for TBE monitoring is the one-sided non-parametric EWMA diagram, especially for cases with asymmetric shifts in high quality processes. Furthermore, based on comparisons, these diagrams have several advantages and are suitable for different scenarios. Finally, there may be several future studies on the application of the charts to multivariable control.

4.2.4 Nonparametric CUSUM Control Chart for Multivariate TBE and Amplitude Data

There are many cases where industries have processes that involve monitoring both the time and magnitude of events. A more specific example is that in healthcare it is often necessary to monitor the time between events (TBE) along with the amplitude of these events (TBEA). When traditional charts based on a specific distribution are used, which may be effective but cannot cope with the complex processes that will be encountered in everyday life. To address this problem, Zhen He and Yuan Gao (2021) designed a non-parametric CUSUM (Cumulative Sum) control chart method for monitoring multivariate TBEA data. This method is quite promising, since it can deal with processes where the distribution is unknown.

Regarding the methodology of the nonparametric multivariate TBEA (NMTBEA), the authors combine several methods to obtain the desired effect, such as categorization, log-linear modeling and a CUSUM procedure. These are done with the aim of the detection of shifts in the positional parameters of multivariate data. To explain their methodology, they explain each one separately:

1. **Data categorization**: The initial stage is data categorization, in which the TBE data are divided into intervals based on predefined thresholds. The use of categorization is made to convert any raw data that exists into a contingency table, with the prospect of formulating the behavior of the TBE variables.
2. **Log-linear modelling:** Then, once the contingency table has been created, a logarithmic model is applied to it. The application of this model shall be made with a view to estimating the expected frequencies of each category. For accuracy, the model they refer to is defined as follows:

$$ln(E_{ij}) = \mu + a_i + \beta_j + \gamma_{ij},$$

where E_{ij} represents the expected frequency for the i-th TBE category and the j-th amplitude category, μ is the overall average, α_i and β_j are the main effects for the TBE and amplitude categories, and γ_{ij} captures the interaction effects.

3. **CUSUM Statistics:** Finally, to complete the methodology, they use the residuals from the log-linear model to generate the CUSUM statistic. This CUSUM statistics, with its series, sum up the deviations over time:

$$S_t = max\left(0, S_{\{t-1\}} + R_t - k\right)$$

$$S_0 = 0$$

where $S_t$is the CUSUM statistic at time t, R_t represents the residuals at time t and k is a reference value controlling the sensitivity of the plot. A process shift is signaled when S_t exceeds a predefined control limit H. In addition, about the design of the control limits for the NMTBEA diagram, simulation shall be used to achieve a desired average path length within control (ARL).

The authors have tested several studies to be able to test these charts in comparison with other charts. They found that they have more advantages. An example they cite is that it achieves a lower average time to signal (ATS) for small displacements compared to parametric CUSUM diagrams with an exponential basis. Apart from the example, which they gave above, to prove that the NMTBEA diagram has several advantages, they made comparisons with other diagrams, such as:

- **QHCUSUM Chart:** In this case, this chart is suitable for univariate non-parametric control, which means that it cannot be processed in a multivariate control.
- **Chart JKEWMA:** The JKEWMA chart is suitable for multivariate non-parametric control, but it is not as sensitive to small shifts in position parameters as the NMTBEA chart.

The way it combines these two advantages, such as the fact that it is suitable for univariable and multivariable control and the fact that it is sensitive to small shifts, makes the NMTBEA diagram a very powerful tool.

Apart from the examples and studies, the authors want to emphasize the real-life sectors in which the diagrams will be useful. One industry that uses these specific charts is the automotive industry. More specifically, they used it to track the time it took and the size of defects to paint a car. In this way, it helps the owners of the car industry to adjust their costs in relation to the appropriate equipment they will need. On the other hand, it is not only industries that use the diagram but also for environmental monitoring purposes. More specifically, the diagram has been used to analyze the timing and intensity of pollution events at multiple monitoring stations. In this way, the diagram can provide a more detailed and accurate assessment of environmental changes.

In conclusion, the authors conclude that the non-parametric CUSUM control chart for multivariate TBEA data represents a significant advance in process monitoring. It is a diagram that combines many methods for its operation. It is a diagram that combines many methods for its operation. Furthermore, its combination of being able to cope with univariate and multivariate non-parametric control but also being sensitive to small process shifts makes it a valuable tool in relation to other parametric and non-parametric charts. Finally, it is a key tool for industries aiming at quality, reliability and innovation.

Chapter 5

Modern Time Between Events Techniques

In the last chapter on the different methods, we will deal with modern Time Between Events (TBE) techniques. Traditional methods are based on parametric or non-parametric methods, as already mentioned, but modern approaches incorporate adaptive algorithms and data-driven models to improve accuracy. We will study cases with dynamic environments where these methods are needed where event patterns evolve over time. Also, modern TBE techniques provide more powerful decision-making tools for several different fields.

5.1 GWMA-TBE and DEWMA-TBE Control Charts for Time-Between-Events Monitoring

As mentioned in the previous chapters, TBE monitoring is appropriate for several processes, such as reliability systems and high-quality production, which are characterized by rare occurrences. Vasileios Alevizakos and Christos Koukouvinos (2018) have analyzed the General Weighted Moving Average (GWMA) and Double Exponential Weighted Moving Average (DEWMA) plots, which are key tools in detecting shifts in TBE data. In this section, they want to analyze their advantages and use cases in modern TBE monitoring. In particular, the GWMA diagram is ideal for cases with non-stationary processes, and on the other hand DEWMA diagrams are ideal, based on the method they follow, for detecting subtle and progressive changes in TBE data.

To design the GWMA diagram, the authors use the traditional EWMA diagram with the addition of a flexible weighting scheme based on past observations. In this way, it allows the chart to keep the equilibrium with respect to the sensitivity to displacements, which is quite difficult for traditional charts. More specifically, the design of the graph uses the weighted average at time t as mentioned, as follows:

$$Z_t = \sum_{i=0}^{t} w_i \, X_{t-i}$$

where w_i are the weights satisfying $\sum_{i=0}^{t} w_i = 1$, and X_{t-i} represents past TBE observations. Concerning the control limits, which are determined based on the expected Z_t distribution during the in-control phase. For example, if the following distribution is normal, then the control limits are established as follows:

$$LCL = \mu_Z - k \cdot \sigma_Z,$$

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$$UCL = \mu_Z + k \cdot \sigma_Z$$

where μ_Z and σ_Z are the mean and standard deviation of Z_t, respectively, and k is a factor determining the significance level. The authors then use the following formulas to indicate when the process is out of control:

$$Z_t > UCL$$

Or

$$Z_t < LCL$$

Regarding the GWMA diagram, which is suitable for detecting small shifts in the TBE data, and as the authors mention is ideal in cases where there is moderate variability. Furthermore, this diagram includes a flexible weighting system, which justifies its advantage in cases where non-stationary processes exist. The dual smoothing mechanism used is formed as follows:

$$Z_t = \lambda_1 X_t + (1 - \lambda_1) Z_{\{t-1\}} + \lambda_2 \left(Z_{\{t-1\}} - Z_{\{t-2\}}\right)$$

where λ_1 and λ_2 are smoothing parameters, Xt is the current observation, and $Zt-1$, $Zt-2$ are past smoothed statistics. In this way, DEWMA charts are highly sensitive to small and gradual shifts in the TBE data, thus minimizing the delay in detecting changes.

Although both the GWMA-TBE and DEWMA-TBE charts are valuable tools for monitoring TBE processes, as the authors state, they are not used for the same business needs. To make this comparison more understandable, the authors have provided research and examples so that they can express the differences, which are as follows:

- **Sensitivity to Shifts:** The design of the DEWMA diagram allows it to be better in cases where they have small and gradual shifts. The design of the DEWMA diagram allows it to be better in cases where they have small and gradual shifts. On the contrary, GWMA diagrams are best used for moderate displacements, where they thus offer a balanced compromise between sensitivity and stability.
- **Adaptability:** GWMA diagrams are more flexible than DEWMA diagrams, which is why they are more adaptable to dynamic or non-stationary processes.
- **Implementation Complexity:** DEWMA diagrams may be quite effective in many areas, but they are quite difficult to calculate and implement. In contrast to GWMA diagrams, which are considerably simpler to implement and calculate, but require caution as they may require careful adjustment of weights for optimal performance.

In conclusion, both diagrams are useful tools, and each in its own way can contribute to TBE monitoring. More specifically, GWMA diagrams are ideal for dynamic environments with moderate changes, while DEWMA diagrams are more

useful in detecting small, gradual changes. Together, the two diagrams provide solutions for a wide range of applications in the field of reliability and quality control.

5.2 Homogeneously Weighted Moving Average (HWMA) Chart for TBE Monitoring

When high quality procedures exist, time between events (TBE) monitoring is critical, because failures or anomalies are quite rare. As an example, the traditional EWMA chart is used to monitor TBE and is also ideal in cases where there are small changes. On the other hand, the exponential weighting of the EWMA chart over time does not give the weighting needed to the cumulative information. For this reason, Muhammad Aslam et al. (2021), proposed the homogeneously weighted moving average (HWMA) chart, which is suitable to address this problem, since it offers an alternative approach by applying equal weights to all observations within a defined window. This uniformity in weighting enhances the ability to detect even moderate shifts. Also, the HWMA diagram is suitable for the combination of sensitivity and stability.

In the methodology of the HWMA chart, the authors state that the chart computes smoothed statistics based on a window of recent observations of constant size. More specifically, the HWMA diagram at time t is calculated as follows:

$$Z_t = \frac{1}{w}\sum_{i=0}^{w-1} X_{t-i}$$

where X_{t-i} represents the most recent TBE observations, and w is the window size. For the parameter w, it is necessary for its selection to have basic characteristics of the process, so that it does not present a problem in sensitivity. On the other hand, the control limits are formed because of the within-control distribution of Z_t. Also, the authors state that the procedure is out of control when it is in force:

$$Z_t > UCL$$

Or

$$Z_t < LCL$$

The limits UCL and LCL are typically determined through simulation or statistical analysis during the in-control phase.

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For a better understanding of the application of the diagram, the authors have chosen to describe it with an example, which relates to the monitoring between engine failures. In which they state that TBEs follow a constant distribution with a median of 50 hours, after which there is a shift, and from 50 hours the median decreases to 40 hours. To apply the HWMA diagram, they use a window size of $w = 10$. In this way, they can detect the shift in fewer observations, compared to the traditional EWMA chart. This example was used to emphasize the advantage of the diagram in detecting shifts much more quickly.

Based on the above example, the authors, in addition to highlighting the advantages of the HWMA diagram, wanted to compare it with other traditional diagrams, such as EWMA and DEWMA. More specifically, the advantages of the HWMA diagram in comparison to the others are:

- **Simpler Implementation:** The uniform system used in the diagram is quite easy to use, and quite easy to calculate.
- **Enhanced Detection for Moderate Shifts:** By incorporating equal contributions from all observations within the window, the HWMA chart effectively identifies moderate process changes that may be overlooked by exponentially weighted methods.
- **Noise Robustness:** When there is a constant window size the average helps to reduce the effect of random variability in the TBE data, ensuring consistent performance in noisy environments.

It seems that the HWMA chart has several advantages over other charts, but the calculation of the variable w plays a major role. In fact, careful consideration needs to be given to this variable, because it can affect the effectiveness of the chart.

The homogeneously weighted moving average (HWMA) diagram is an alternative solution to monitoring procedures for TBE data, especially in cases with moderate displacements. For practitioners looking to strike a compromise between sensitivity and stability, its uniform weighing method, simplicity, and efficacy make it a desirable choice. Its benefits in noise robustness and shift detection make it a useful addition to the TBE monitoring toolkit, even if it might not be as sensitive as EWMA charts for very little changes. The optimization of window size selection and the extension of the HWMA approach to multivariate and adaptive contexts may be the main areas of future research.

5.3 Simultaneous Monitoring of Magnitude and Time-Between-Events Data

For system control, it is not enough to monitor only the frequency of events to understand the system. For various processes, such as equipment maintenance or health care require more attention to how often events occur. For example, the two most important indicators that a production process needs are the time between machine failures and the cost of repairing the failures for an efficient process. The simultaneous monitoring of TBE and size, the combination of these two, is the reason for a complete picture of the process. Based on Ridwan A.

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Sanusi et al. (2020) this method with two factors is quite useful, since with one factor there is a higher probability that the problems will not be noticed, and therefore they will not have a reliable process.

So, they use the simultaneous monitoring approach, which involves the use of separate control charts for TBE, to examine the process to indicate whether it is under control or not. The main steps to achieve this are:

- **Monitoring TBE:** In this case, the authors will use the multi-valued diagrams, such as EWMA, GWMA or DEWMA, and will choose accordingly based on the needs of the process. More specifically, these diagrams contribute to the process of detecting any change in the process that may occur.
- **Monitoring Magnitude:** In this case, diagrams are needed, which will focus on changes in size, such as, for example, the increase in repair costs. For this reason, they use the traditional Shewhart or CUSUM charts, so that they can properly monitor the process.
- **Combining the Results:** For the smooth combination of the two charts and the decision on when to combine each chart, the authors set a decision rule. More specifically, they state that either the TBE or the magnitude diagram signals an out-of-control event, the process is flagged as needing attention. For better understanding they state that if the signal variable takes the value 1 it means that either the TBE or the size chart is signaling, and if it takes the value 0 it means that neither is signaling.
- **Setting Control Limits:** Simulations are then used to generate the control limits to estimate the expected behavior of the TBE and the size during the in-test phase. Furthermore, the control limits also help in this case to understand when the process is out of control.

In the continuation of their research, the authors included an example to explain their method and its usefulness. The example takes place in a factory, where the maintenance team that is available keeps track of the time that the faults occur and the cost of repairing them. The data from the monitoring is that on average there are failures every 50 hours, with repair costs of about $500. If the time between failures is reduced to 30 hours and repair costs increase to $700, the combined monitoring system will detect these changes more quickly than individual charts. With this example, the authors want to show that the TBE chart will detect the increased frequency of failures, while on the other hand the size chart will take care of the increase in repair costs. Thus, in this way, the combination of these two will give a quicker and faster sign of damage to the production process.

The combination of these two charts offers several advantages to the process. First, by combining these two, they have a combination of the outputs of two control charts, which makes it easier to identify patterns where the frequency and severity of events are linked. Also, with this method they use more charts, which means they can cover more needs, because each chart covers a need. On the other hand, this method may require more data or resources and will also be more complicated to calculate.

In conclusion, the authors, after research and several examples, report that the combination of these two methods is quite satisfactory, since they try to cover all

needs with this method. The method is quite useful in hygiene care or in general in cases where the frequency and severity of events are important.

5.4 Generally Weighted Moving Average (GWMA) Chart for Time-Between-Events Monitoring

In high quality processes most often use time-between-events (TBE) monitoring, where when shifts are detected, they can prevent costly inefficiencies. In these cases, they report that the traditional exponentially weighted moving average (EWMA) chart is used, which is quite common. But this chart has a disadvantage, which is that it does not provide the flexibility needed to adapt the sensitivity to the specific dynamics of a process. For this reason, N. Chakraborty et al. (2016) proposed the General Weighted Moving Average (GWMA) diagram, which is an improved version of the EWMA diagram. More specifically, what this chart achieves is to have greater adaptability and accuracy in detecting process shifts, which was lacking in the previous chart. In this way, it is a valuable tool in scenarios where the influence of recent versus past observations varies according to the expected nature of the process changes.

Regarding the methodology of the GWMA diagram, the authors incorporate a flexible weighting system to allow anyone to use it by emphasizing a certain observation over other observations. More specifically in the mathematical part, it is calculated as follows:

$$Z_t = \sum_{\{i=1\}}^{t} w_i X_i$$

where w_i represents the weight for the i-th observation, X_i is the corresponding TBE, and the weights are normalized such that their sum equals one. The weights mentioned earlier follow a uniform pattern in which all observations are treated equally, to give priority to recent data. Also, the authors have adapted the diagram in such a way that anyone using it can define customized weight distributions. Thus, in this way, each user can target a specific type of shift that takes place in the process. Where in the present procedure to be able to determine when the procedure is out of control, they will denote it by Z_t , and it is denoted as follows:

$$Z_t > UCL$$

Or

$$Z_t < LCL$$

This means that the process is out of control when it is below the lower control limit (LCL), or when it is above the upper control limit (UCL). In addition, the control limits can be calculated in more detail, using simulation to consider the specific weighting scheme applied.

To explain the methodology on the diagram, they use the most common example of engine failures, where they track the distance between them. This example reports that TBE follows a constant pattern with an average of 10 hours. They then use a GWMA chart that follows exponential weights, with the aim of detecting sudden increases in failure rates. Due to process degradation the TBE average decreases to 7 hours, and because the GWMA chart includes the adjusted weighting scheme, it detects this shift faster than a traditional moving average.

According to the authors, the GWMA diagram provides several advantages like those of the traditional EWMA diagram. Simply the difference of the GWMA diagram is that the flexibility allows the diagram to adapt to different processes, whether they refer to small shifts or large shifts that take place in the process. Furthermore, another advantage of the GWMA diagram is that with appropriate weighting it can outperform EWMA diagrams in scenarios where the selected weights are aligned with the nature of the process displacements.

In conclusion, the General Weighted Moving Average (GWMA) chart is a rather valuable tool for monitoring TBE. As mentioned by the authors, it is a suitable tool for users in cases where they want different types of shifts, whether they are small or large shifts. Its adaptability makes it particularly valuable for dynamic or complex processes where traditional methods may be difficult.

5.5 Progressive Mean Control Chart for Monitoring Time-Between-Events

In quality control procedures the monitoring of time between events (TBE) is a critical aspect, where events are rare. As much as they use traditional Shewhart or EWMA diagrams, they may be satisfactory up to a point, but they have difficulty in balancing sensitivity to small displacements with robustness against noise. For this purpose, Vasileios Alevizakos and Christos Koukouvinos(2019) proposed the Progressive Mean (PM) control chart to address this imbalance that arises with traditional charts. To achieve this, they use the cumulative average of TBE observations, which allows them to fit new data. In this way they manage to convert the chart into an efficient chart, in cases where long-term trends and gradual changes in the behavior of the process can be detected. They also try to focus on the ease of use of the method.

Then, regarding the methodology of the Progressive Mean control chart, what the chart tries to do is to calculate a follow-up statistic based on the cumulative average of the TBE data, updated with each new observation. For the design they define time t as follows:

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$$M_t = \frac{1}{t}\sum_{\{i=1\}}^{t} X_i$$

where X_i represents the i-th TBE observation. This equation helps the chart to ensure that the statistics reflect all available data, which in this way can make it sensitive to any change. Regarding the design of the control limits for the PM diagram, they are calculated based on the expected distribution resulting from Mt when in the control phase. When they determine the expected distribution, they use historical data or simulations to calculate the limits. In addition, to classify a process as out of control, the cumulative average Mt value needs to exceed the upper control limit (UCL) or be below the lower control limit (LCL). The authors mention that they try to constantly update the mean serum, allowing the diagram to dynamically adjust as more data is collected.

According to the authors, one of the strengths of the PM diagram is its resistance to noise. The diagram is more stable and reliable for long-term monitoring when the data is pooled over all observations, as random variations have less influence on the monitoring statistics. However, the chart may be less useful for detecting sharp, short-term changes because of its focus on aggregate data; techniques such as EWMA or GWMA, which place greater emphasis on recent observations, may be more useful in this regard.

To better explain the methodology used, the authors provide an example that they themselves used with real data. The example refers to the monitoring of patients' recovery times, in a hospital or in a healthcare environment in general. More specifically, what they observed is that when there was a change in the patient's treatment, this means that there would be an increase or decrease in their recovery time. In this case, the PM diagram tries to capture what is happening in the process with a view to suggesting a more optimal procedure. Thus, the chart helps both patients and doctors to save time on treatment, but also to increase efficiency.

The progressive mean control chart is a quite useful tool, as it offers many advantages compared to traditional charts. Basically, they report that the cumulative approach they use is quite effective and makes the process more ideal in cases where gradual changes are more critical than sudden changes. Furthermore, calculating the cumulative average does not require as many aggregate resources and does not make it complicated to implement. On the other hand, the PM charts show less sensitivity to short-term changes, which the authors compared with the EWMA and GWMA charts. They report that the EWMA and GWMA plots give priority to recent observations. This does not mean that the PM chart is not a useful tool, but it makes it less suitable in procedures where they need to detect sudden changes. Nevertheless, the advantages mentioned above by the authors are sufficient to make the charts valuable tools for a process.

In conclusion, the authors report that the progressive mean control chart provides a unique and effective approach for monitoring the time between events in processes with infrequent events. As they have shown above, the PM diagram is an effective tool, which is used in many different processes. Furthermore, its simplicity makes it a very simple and easy to use tool. Furthermore, the authors

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point out at the end of their study that the advantages of this diagram in combination with other diagrams, such as EWMA or GWMA, may in future studies prove to be a very useful combination.

Chapter 6

Computational Techniques for TBE

In the rest of the thesis we will study specific methods, in which we have carried out simulations so that we can observe and compare their results. As it appears above, we studied sparse methods dealing with Time Between Events data, but there were some methods that offered clear advantages over others in terms of detectability, adaptability to skewed distributions and practical applicability. Unlike other approaches that may struggle with non-normal data or have limited sensitivity to small shifts, these methods incorporate advanced statistical techniques to enhance performance. These methods improve detection efficiency through dynamic process analysis, weighted moving averages, and refined sampling strategies, thereby making them more effective for monitoring for real-world data. The point we want to emphasize is that by focusing on these seven methods, this study provides a focused but comprehensive evaluation of the most effective and practically viable approaches to monitoring TBE.

6.1 Improved Exponential EWMA Chart with Power Transformation

We will start with the method of the improved exponential power transformed EWMA diagram of Yang Zhang and Yanfen Shang (2022). This diagram is the improved version of the exponentially moving average, but it is also used when there is an increase in sensitivity to subtle shifts in the TBE average for early detection of process degradation in systems where events are rare.

Therefore, we perform a simulation with synthetic TBE data to test this methodology in practice based on what the authors report. There are steps based on the methodology in which steps must be performed to implement the simulation. The concrete steps are as follows:

1. Create observations with TBE data, with predefined values in mean and standard deviation (mean = 3 and standard deviation = 0.5). The specific values were chosen on the condition that they would be neither higher nor lower than data from everyday life.
2. Determination of a downward shift of the mean TBE after 50 observations, so that we can examine the localization of the process and that it is closer to real life data to show that there are changes over time.
3. According to the authors, for the transformation of the TBE data a power parameter β is needed, with the prospect of enhancing the sensitivity to relatively small downward shifts of the mean. For this reason, we set a power parameter β equal to 0.5 (β=0.5) in the simulation and take this value to effectively enhance the smallest mean shifts. Apart from this, it was necessary to recalculate the EWMA statistics, but this time with the transformed data and using the smoothing constant (λ=0.2), which was

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chosen as the specific value to have a balance between sensitivity and stability.

4. Finally, the control limits needed for the simulation were calculated based on the within-control average run length (ARL), as mentioned by the authors in their methodology. Based on the above, performing the simulation generates the following diagram:

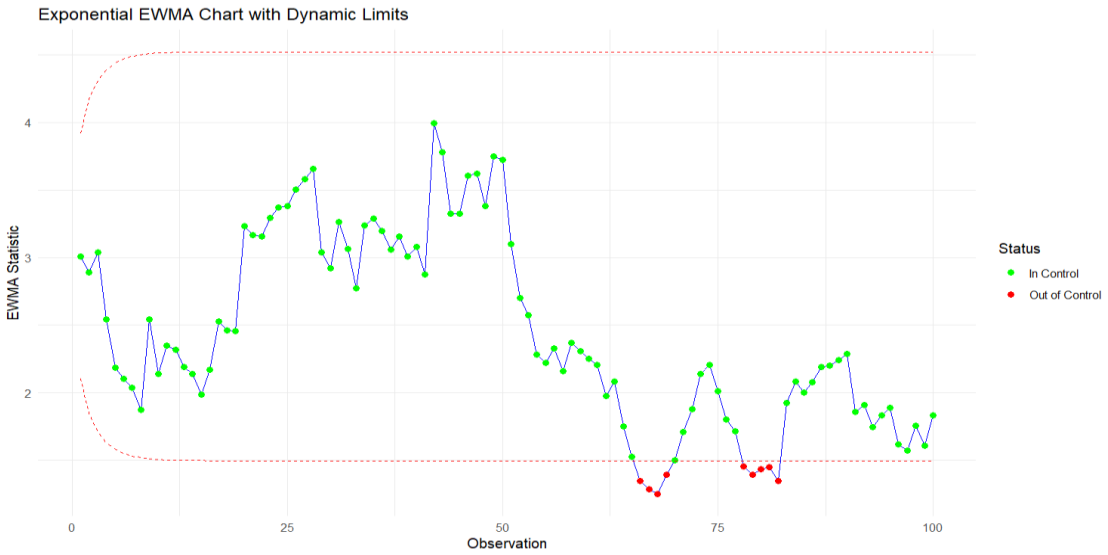

Figure 6.1: Improved Exponential EWMA chart with Power Transformation

In the diagram above, we see the blue line depicted, which is the EWMA statistic mentioned above. Also, the green and red dots shown in the chart are each observation that the EWMA statistics passed through. The green dots indicate observations within control, while the red dots indicate out-of-control points that exceed the control limits.

What makes the difference in this diagram is that while in the first 50 observations the process is in control, this means that no observation is out of control, after the 50th observation the process is out of control. This is because we notice that there are observations, which are below the lower control limit. With this change in the procedure, we can report that this chart is much more sensitive to small changes that can occur over time. Furthermore, we can support the conclusion of the authors, who argue that the improved EWMA diagram is much more effective than traditional diagrams in detecting small changes, and this is due to the use of power transformation.

To compare this, we will perform the same simulation but with the standard EWMA diagram. Below is the traditional EWMA diagram:

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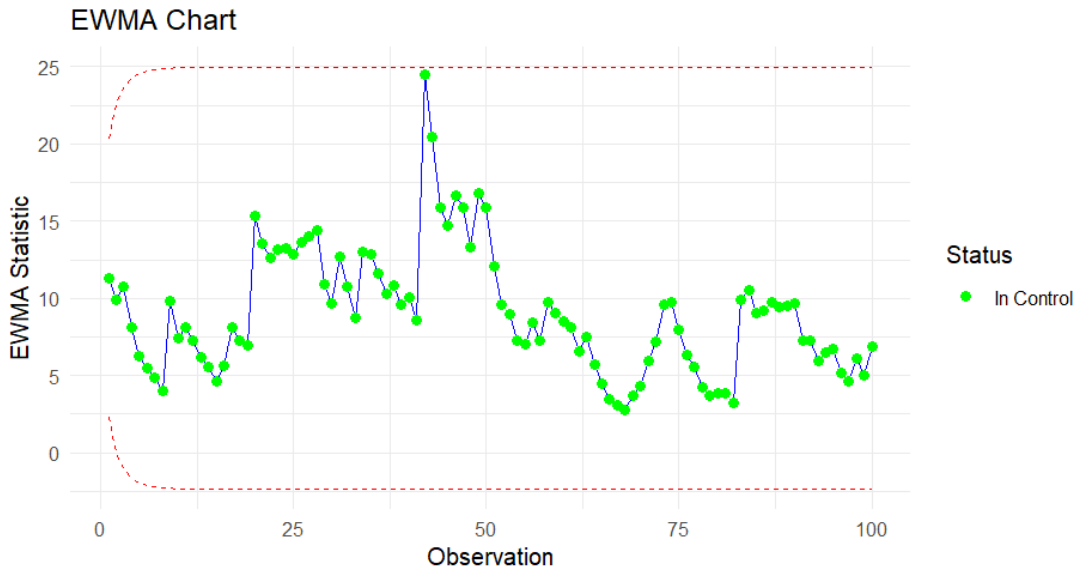

Figure 6.2: Standard EWMA chart

We note that in this case the EWMA statistics are within the control limits, without exceeding the upper or lower control limits of the graph. But this chart differs in the fact that it has a wider range of values for the EWMA statistics (0 to 20+), which proves to us that the traditional chart has larger variations in the process data. This makes it more difficult to detect small shifts from this method.

In conclusion, we conclude that the Improved EWMA Exponential Chart with power transformation is by far the best for monitoring processes where it is critical to detect smaller changes. The criterion that determines which is best is the circumstances and the requirements that exist, which means that if the need is to understand the larger variations it is better to use the traditional chart.

6.2 Generalized Group Runs (GGR) Control Chart

In the following we will analyze the method of Fang et al., which reports the Generalized Group Runs (GGR) Control Chart. For accuracy, we will perform a simulation with TBE data to demonstrate the fact that this method incorporates real-time data for adaptive monitoring. To perform the integration and track various time shifts between events using a combination of a TBE sub chart and a Generalized Group Conforming Run Length (GGCRL). Thus, we will describe below the steps needed to create the simulation of the method.

First, we need to generate time data between events (Y), which are generated using an exponential distribution with a fixed rate parameter λ=0.2. With the rate parameter we can calculate the average time between events, since it is derived from the formula $\theta 0=1/\lambda$, which is equal to 5 units based on the calculations. To be able to improve the detection of smaller shifts in the mean TBE, we preferred the value 3 in the variable r to help us in this detection.

The next very important step in the continuation of the method is to calculate the upper and lower control limits. For this reason, we will start by calculating the lower control limit (LCL), which we will use initially in the TBE subprogram, and for

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its calculation we will use the cumulative distribution function (CDF) of the Erlang distribution. Furthermore, to calculate the lower bound control we will use the false alarm rate α.

To create the diagram, we will need to first create a diagram that can record the observations and characterize them as compliant or non-compliant. To do this we use the TBE sub chart, in which any observation above the specified lower control limit is marked as conforming. On the other hand, observations depicted below the lower control limit are considered non-compliant. In fact, the specific diagram is illustrated below:

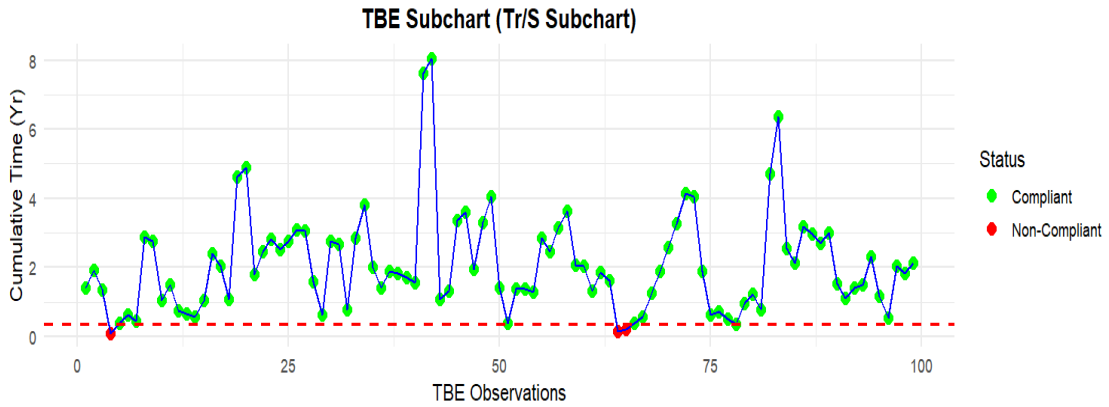

Figure 6.3: TBE Sub chart (Tr/S Sub chart)

Based on Figure 6.3, we can see that the green dots are the points that we call compliant, while the points that are below the lower control limit and marked with red dots are the non-compliant points. In more detail, the points mentioned above are the points that indicate that there may be a shift in the process and need further analysis.

After we have performed this chart, we continue with the Generalized Group Conforming Run Length (GGCRL) sub chart based on the methodology. The purpose of this plot is to track the conforming run lengths (CRL) for better motif identification, which indicates for the procedure the shift of the mean TBE based on the previous plot. Also, the CRLs are plotted in sequential runs. The diagram of the Group Conforming Run Length (GGCRL) is shown below:

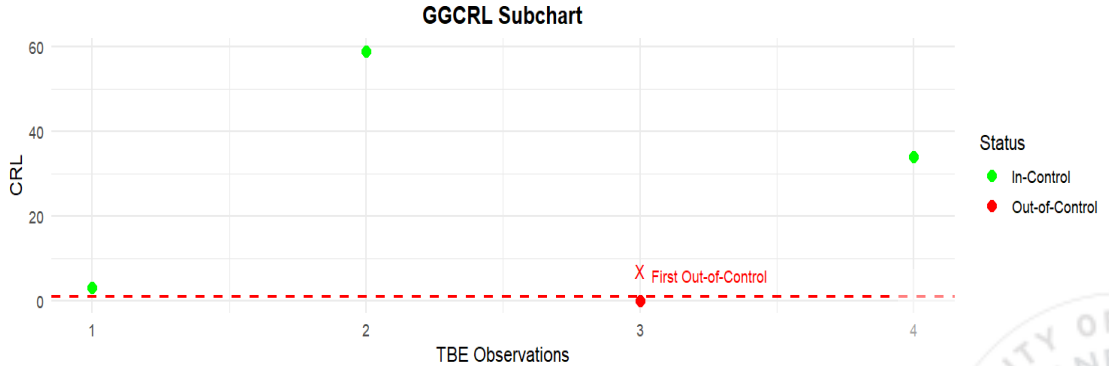

Figure 6.4: GGCRL Sub chart

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As shown in Figure 6.4, the horizontal line depicted is the boundary, which they call L, and any CRL below the line is considered critical and needs further consideration. In addition, we consider a point out of control when g=1 and any CRL is less than or equal to L, or when g>1. That we notice in the diagram is that there is only one point which is out of control, denoted by a red color, and which indicates a shift in the mean TBE. All other points are in control, denoted by green dots.

To compare this, we will run the same simulation with the same data, but with the Group Runs Sub chart in this example. The diagram is shown below:

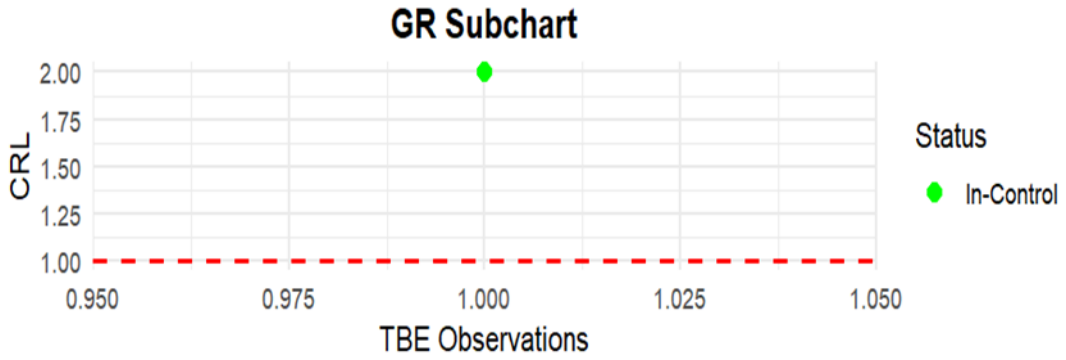

Figure 6.5: GR Sub chart

As shown in diagram 6.5 there is no red dot, only green. Which means that since there is only one green point, it indicates that most observations maintain sufficiently long compatible paths and that the process is under control. While in the Tr subchart there are red dots, which are non-compliant events, the GR subchart ensures that the compliant sequences are large enough to confirm the stability of the process.

In conclusion, we observe that using this method we draw the same conclusions as the author, which means that the GGR diagram provides a powerful mechanism for monitoring shifts in the process where events occur infrequently. The methodology used with the two layers ensures early detection of changes in the process, which is quite useful for industries, especially those with critical reliability. Furthermore, after comparing the two charts, it appears that a traditional chart in such situations cannot detect shifts in the process.

6.3 Weibull TBE Method for Monitoring Process Performance

The method we will use is the Weibull Time Between Events (TBE), which is a method used in processes that have varying event rates. What is mentioned by the authors of this method is that unlike exponential distribution, in this case it is possible to model systems with increasing, decreasing or constant failure rates. Thus, it is a key tool for industries or other organizations where they want to monitor process improvement or awareness over time.

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For this reason, we will perform data simulations in and out of control of the calculation of the exponentially weighted moving average (EWMA) and the cumulative sum (CUSUM) and the construction of control charts to detect changes in process behavior. The steps followed are as follows:

1. The Weibull scale parameter (θ) and the Weibull shape parameter (β) will be used to simulate data under control, which will take the values 10 and 1.2 respectively. The value we set for the parameter β is slightly increased, representing the least increased failure rate.
2. The next step, after defining parameters, is for the process to take a shift. In this case, the parameters will also accept a shift, in fact the Weibull scale parameter (θ) decreased to 6 after accepting a decrease in the time between failures, but the Weibull shape parameter (β) increased to 2.0 reflecting a larger increase in the failure rate.
3. The calculation of the EWMA statistics is quite important because it can detect shifts in the process, and it does this by combining past and current observations with a smoothing parameter (λ). Also, the EWMA statistics are updated continuously, and set a control limit based on three standard deviations above the control mean.
4. Similarly, it will be necessary to calculate the CUSUM statistics, which have a different property, which is to detect small shifts. Its initial value is zero and it continues by adding the deviation of each observation from a reference value.

Having analyzed the parameters and the values they need to take, we can now calculate the two charts, the CUSUM and EWMA charts, which will help us to detect changes in the failure rate or the average time between events. The analysis starts with the exponentially weighted moving average (EWMA) chart which is shown below:

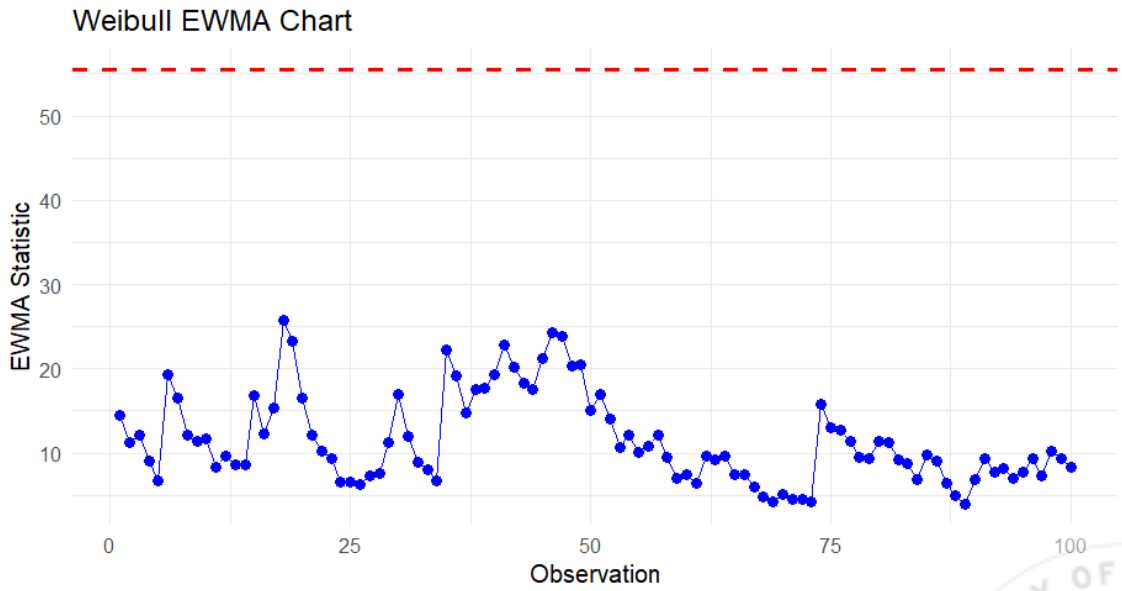

Figure 6.6: Weibull EWMA Chart

The blue points depicted in Figure 6.6 are periods of steady-state operation, with the EWMA statistics fluctuating around a constant value. Furthermore, we

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observe that these points remain below the upper control limit, which means that the process remains stable. On the other hand, the blue points depicted after the 50th observation, means that the process has a downward trend and the EWMA statistics are steadily decreasing. This trend indicates a gradual increase in the failure rate, which is consistent with the purpose of the graph to monitor progressive changes. Thus, the graph indicates, relatively slowly, the deterioration of the process and the propensity for some change in the process. We will then create the CUSUM chart, which is illustrated below:

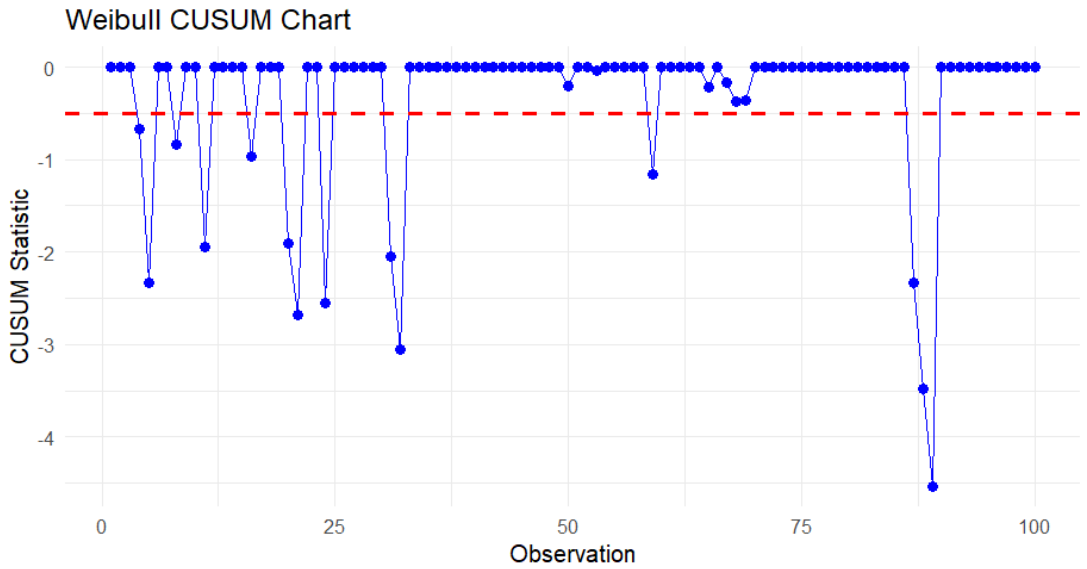

Figure 6.7: Weibull CUSUM Chart

In this case we find that there are many elements at the beginning of the process that are close to or just below the control limit, but this means that from the initial stages the process has shown several signs of instability or, more specifically, variability. However, we notice that the blue points, located after the 50th observation, have more extreme values, which means that they are far from the lower control limit of the graph.

To compare this, we will run the same simulation with the same data, but with the Weibull MEC chart in this example. The diagram is shown below:

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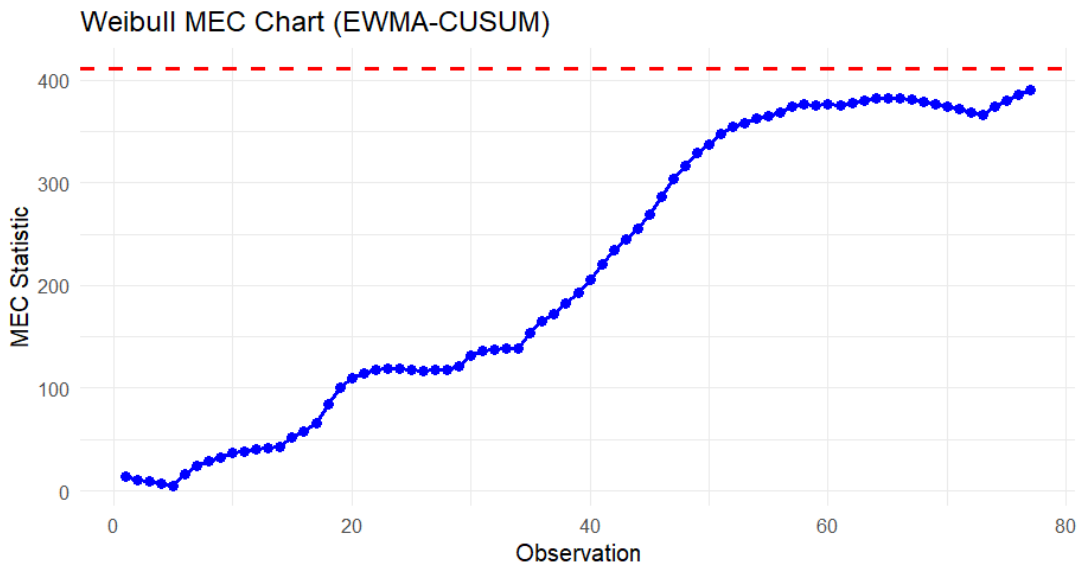

Figure 6.8: Weibull MEC chart

We note that in Figure 6.8 there is an increasing trend overtime compared to the other charts. At the beginning of the process, it appears to be upward, but near the 60th observation it seems to stabilize at a higher statistical level of MEC. Also, as in the previous plots, there is the red dotted line representing a control threshold, which is not exceeded in this plot. This means that the process is under control and there are no shifts. Compared to the other charts discussed in this section, the MEC chart provides an upward trend, but does not indicate smaller fluctuations. On the other hand, the EWMA chart shows the shifts but has a downward trend, while the CUSUM chart is suitable for detecting small shifts.

In conclusion, we note that both charts are useful tools for monitoring a process, but each for different purposes and conditions. In fact, the EWMA chart is significantly more suitable for detecting gradual shifts, as it shows a smooth decline in the statistics after a point, with several points crossing the control threshold. On the other hand, the CUSUM diagram is ideal for detecting sudden and smaller changes, with clear deviations and points that fall out of bounds after the transition point. Although we prefer charts that can detect small changes, in this case we will not choose the CUSUM chart because several green points below the control limit indicate poor process variability within the control. For this reason, we will choose the EWMA chart, which has smoother and slower detection of the shift, but is more reliable.

6.4 Generalized Multiple Dependent State (GMDS) Sampling

The next method is the Generalized Multiple Dependent State (GMDS) sampling approach for Time Between Events (TBE) data monitoring, in which we will perform a simulation to be able to verify the theory of Y. H. Dovoedo and S. Chakraborti (2011). For this method the authors used real time data to make the results more realistic. Also, the goal of this methodology is to enhance the fast detection of

shifts in event frequency. For this reason, we will perform the simulation to see if we get the same results. The steps of this method are as follows:

1. The value adjustment for the mid-process serum within control (θ0=5). The fitted value for the process mean within control (θ0=5). To be precise, the mean represents the average time between events when the process is operating normally. The value chosen is 5, which was chosen with the perspective of representing a stable process with a moderate frequency of events.
2. Adjustment of the values of the control parameters (k1=3, k2=2). These parameters are used to create external and internal control limits used in the diagram. In this case we will use the value 3 for the parameter κ1 for the external boundaries, to be able to detect the large changes that may occur but also to avoid any false alarms. On the other hand, parameter k2 which is for the inner limits will take the value 2 and is used to detect minor variations. These parameters and the values we set for them can replace the 3σ limits.
3. Determination of value in the recent number of samples (m=5). The specific one we set in the parameter is neither high enough nor low enough to consider a small history of recent data for decision making, which will help in the process.
4. The last parameter to be set is the threshold of the majority rule (k=3). The last parameter setting is the majority channel threshold (k=3). More specifically, the threshold mentioned above helps to prevent overreaction that can be generated to random noise but still detects consistent process changes.
5. The last step is the transformation used on TBE data derived from an exponential understanding. In particular, the formula $T^* = T^{1/3,6}$ is used to transform the data, which contributes to making the data more symmetrical and then to making the control limits easier to construct. The mean (μ_{T^*}) and standard deviation ($\sigma\sigma_{T^*}$) are calculated using the gamma function, ensuring accurate control limits.

After following the specific steps mentioned above, the methodology resulted in the following diagram:

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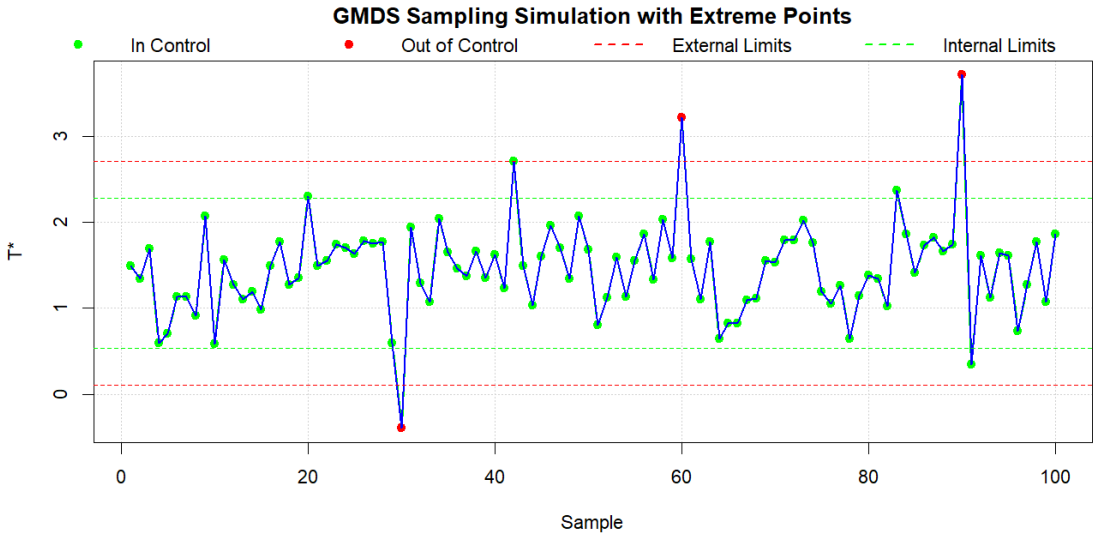

Figure 6.9: GMDS Sampling Simulation

As discussed above, the diagram shows the GMDS (Generalized Multivariate Detection System) sampling simulation, in which the x-axis is the sample index, and the y-axis represents the static control T^*. The diagram consists of the control limits, which are divided into internal and external control limits. The diagram consists of the control limits, which are divided into internal and external control limits. It appears that the largest space in the diagram is made up of the green dots, which indicates that there are no shifts and the process is under control. Then we notice that there are also red points, which are the points that exceed the outer control limits and indicate the large shifts that the process is undergoing. We see the red points most clearly near samples 60 to 95. Furthermore, we notice that there are also points that slightly exceeded the internal control limits, which show us that small displacements caused by the process can be detected through this diagram.

To compare this, we will run the same simulation with the same data, but with the Shewhart Type T-chart in this example. The diagram is shown below:

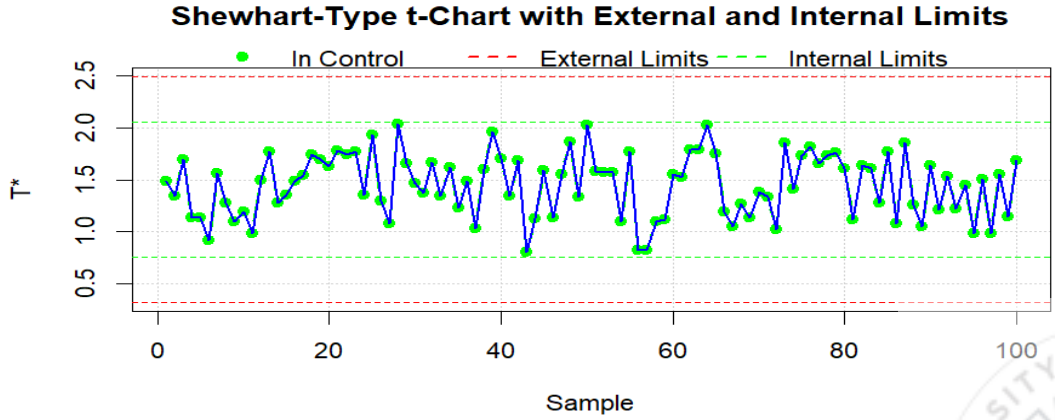

Figure 6.10: Shewhart-Type t-chart Simulation

As we observe in diagram 6.10 is the Shewhart diagram which is shown. To create it we used the same data, but also the same values for the parameters from the previous simulation for the GMDS diagram. We notice that in this case there are no values that exceed the limits. This means that we have only green dots, which show us that the process is under control. This diagram may be simpler in its interpretation and construction, but it is reduced in its ability to detect small shifts in the process. For this reason, the authors recommend the GMDS diagram when the detection of small changes in the process is required.

In conclusion, the diagram serves as an empirical representation of the GMDS's ability to monitor and detect outliers in a multivariate process. Also, the diagram emphasizes the definition of appropriate control limits, so that the process extremes can be seen. Furthermore, the GMDS methodology is a critical tool for monitoring the process, with the ability to detect changes, but also to keep the rest of the process stable. Finally, it appears from the comparison above that the GMDS diagram is preferable in flexibility and sensitivity, but the Shewhart diagram provides a faster and more direct approach.

6.5 Double Exponentially Weighted Moving Average (DEWMA) Control Chart for Gamma-Distributed TBE Data

Based on Shewhart and EWMA control methods and diagrams, Vasileios Alevizakos and Christos Koukouvinos (2019) created a new diagram which is the Double Exponentially Weighted Moving Average (DEWMA). This diagram is a quite useful tool designed to monitor the processes that take place between events (TBE), especially when the data of the process follows a gamma distribution. Furthermore, the methodology and design of this diagram is based on the Shewhart and EWMA diagrams but is upgraded by the addition of a second level of weighting to increase sensitivity to relatively small and moderate shifts that may occur in the process. The shifts in the process that the diagram must detect are usually wear and tear on the machines or any change in the production conditions of the process.

Regarding the simulation of the method, some defined steps are needed before the diagram is designed. The steps for the simulation are as follows:

1. Determination of the shape parameter a (a=2). First, before the parameter a we will set the size of the observations (n=100), which will be 100 observations so that it is a large enough size to show the performance of the graph. We will then set a to 2, which is a satisfactory value for representing a moderately skewed distribution and is close to real data.
2. Value for the scale parameter (b=10). The scale parameter (b) we will use in the procedure reflects the average time between events. Setting it to 10 is indicative for processes where we want these intervals to be relatively long, so that the performance of the process can be seen.
3. Determination of the smoothing constant value (k=0.3). This value is a value used most often in experiments that want to detect small or moderate variations.

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4. Calculating the mean (h0=a·b). In this case the mean is calculated using the values we have already determined above. Thus, in this way it is quite useful to help calculate the displacements.
5. Variance ($Var(Zt \mid IC)$): The variance is adjusted for time dependence in the DEWMA diagram. This adjustment is critical because the chart considers the correlations between successive observations, ensuring accurate control limits over time.
6. Calculation of the upper and lower control limits (LCL and UCL). To be able to distinguish when a process is out of control or in control, the control limits are needed. For this reason, they are calculated based on the mean value (h0) and a scaled standard deviation ($LVar(Zt)$).
7. Two levels of weighting (DEWMA statistics): in particular, the first weighted average (Yt) smoothed the individual observations, while the second (Zt) tries to enhance trend detection by reducing as much as possible the noise that can be generated.
8. Creating a diagram. Finally, all that remains is to create the chart after we have calculated all the values needed.

After studying all the steps involved in creating the chart and setting all the values needed, we can see the results. Below we are considering the specific chart:

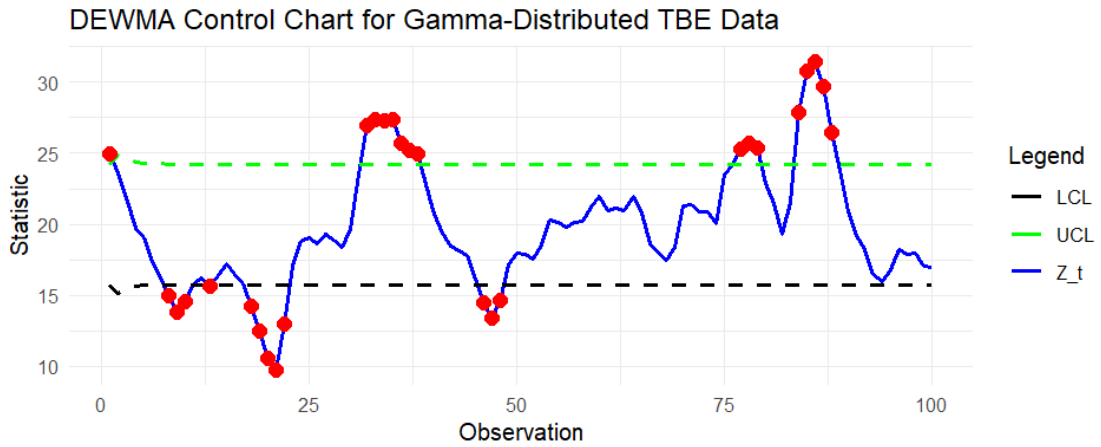

Figure 6.11: DEWMA Control Chart for Gamma-Distributed TEB Data

Figure 6.11 illustrates DEWMA which was designed for data following a gamma distribution. The purpose of this diagram is to identify the small changes that can occur over time. More specifically, we observe that the x-axis represents the number of observations we have generated in the simulation, and the y-axis contains the statistical values derived from the DEWMA methodology. The same is also represented by the blue line (Zt) with the y-axis, but in this case the blue line represents the DEWMA statistical values calculated for each observation. In addition, we note that the dotted lines depicted in Figure 11 are the upper and lower control limits, which are narrow enough boundaries to be able to detect small changes in the process. This is apparently accomplished since the small shifts in the process have been identified by red dots, which are the points that are above the upper control limit and the lower control limit.

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To compare this, we will run the same simulation with the same data, but with the EWMA for Gamma Distributed TBE data in this example. The diagram is shown below:

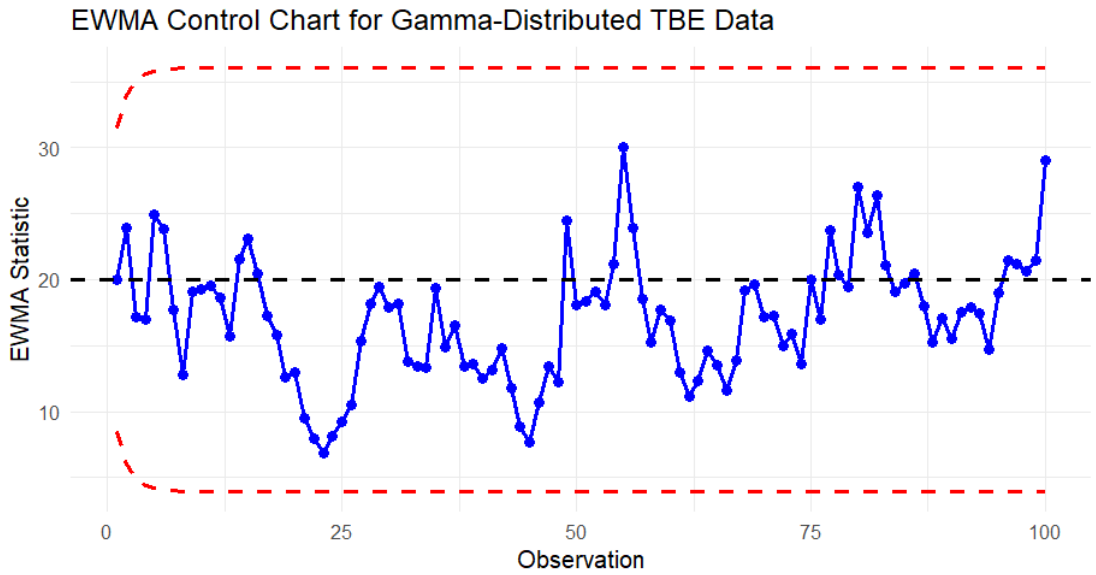

Figure 6.12: EWMA Control Chart for Gamma-Distributed TEB Data

In Figure 6.12 we observe that the EWMA Control Chart for Gamma-Distributed TEB Data is depicted. In which we used the same data and values for the parameters from the first simulation, to observe the behavior of the chart in such conditions. In the diagram the blue line represents the smoothed process over time, helping to identify trends or shifts rather than individual changes. Also, note that the upper and lower control limits of the process are depicted in the diagram, and the black dashed value represents the expected average of the process. This relates to the morphology of the graph, and in terms of interpretation we observe that at some points there are upward trends indicating a change in the process, but without exceeding the limits. Thus, we can conclude that the process is stable without extreme changes.

In conclusion, we can agree with the authors on the effectiveness of his method. This is because, as we have observed from simulation, it is a rather valuable tool to detect small and moderate changes in the process. Furthermore, it is worth noting the comparison between the diagrams, where the EWMA method does not detect small shifts in the process as the DEWMA diagram does.

6.6 GWMA-TBE and DEWMA-TBE Control Charts for Time-Between-Events Monitoring

The General Weighted Moving Average (GWMA) and Double Exponential Weighted Moving Average (DEWMA) control charts are among the most basic tools used for monitoring processes with TBE data. In general, Time-Between-Events (TBE) monitoring is a critical aspect of mechanical reliability and quality control.

These diagrams attempt to improve process reliability, and to detect any changes in the process that may occur over time. More specifically, the GWMA diagram is used to detect moderate shifts, while on the other hand the DEWMA diagram excels in detecting small and gradual changes with the help of the double smoothing mechanism.

In the continuation of our analysis, we will use the theory of Vasileios Alevizakos and Christos Koukouvinos (2018), regarding the GWMA and DEWMA diagrams on which we will perform a simulation to see if their conclusions can be valid. This can be done by following the steps of the methodology. The following are the steps of this methodology:

1. Determine number of observations (n=100). First, to start the process we set the distribution that the data will follow, which is the exponential distribution. This happens because exponential distribution is the appropriate one in TBE data modeling processes in high quality processes. In the observations section, we will set 100 observations to have a relatively large range and to be able to see the behavior of the graph.
2. Determination of the lambda parameter (λ=0,1). We set this value because an even lower λ would create longer time intervals between events, which would not help us to detect small shifts.
3. Separate parameters for GWMA and DEWMA. For accuracy the GWMA diagram uses the parameter weights, which in this case follow an exponential distribution and are used to give more importance to recent observations. This parameter shall also be normalized so that its sum is equal to 1 (∑w i=1). On the other hand, the parameter used in the DEWMA chart is lambda1 = 0.3 and lambda2 = 0.2, which can maintain the sensitivity of the chart, but also when they are relatively low their values can keep the chart more stable.
4. Calculation of upper and lower control limits (LCL and UCL). The same conditions apply to both charts regarding the creation of control limits. For accuracy, we will use the mean and standard deviation of the statistics during the within-test phase to create the control limits. In addition, the confidence intervals using the variable k are also used to calculate the control limits of the charts. This factor takes value 3, which is the most common value used in all diagrams, to balance sensitivity and false alarms.

After having carried out all the steps of the authors regarding the methodology, in the end we need to create the GWMA and DEWMA diagrams. Starting first with the GWMA diagram which is illustrated below.

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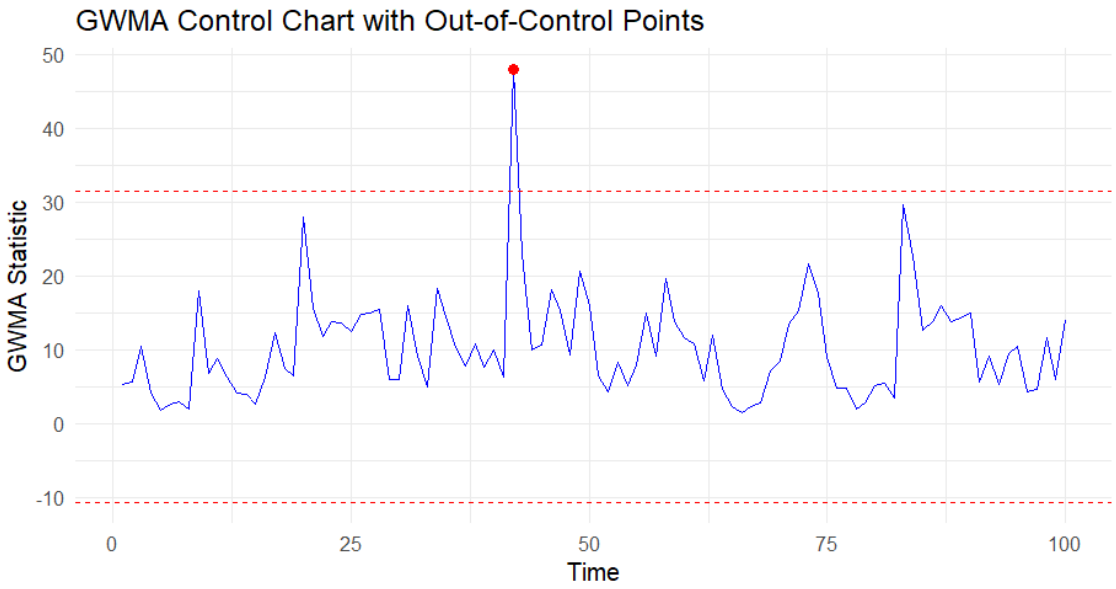

Figure 6.13: GWMA Control Chart

We note, based on diagram 6.13 which illustrates the GWMA diagram, that the diagram largely controls the process. Except for a point near time point 50, as shown in the red dot, which shows us that the process is out of control and has received a shift from an external factor. This point shows us that the diagram is suitable to detect the measured changes in the process. Furthermore, we understand that they are moderate because they are not alarmingly above the upper control limit of the process. The DEWMA control diagram is shown below:

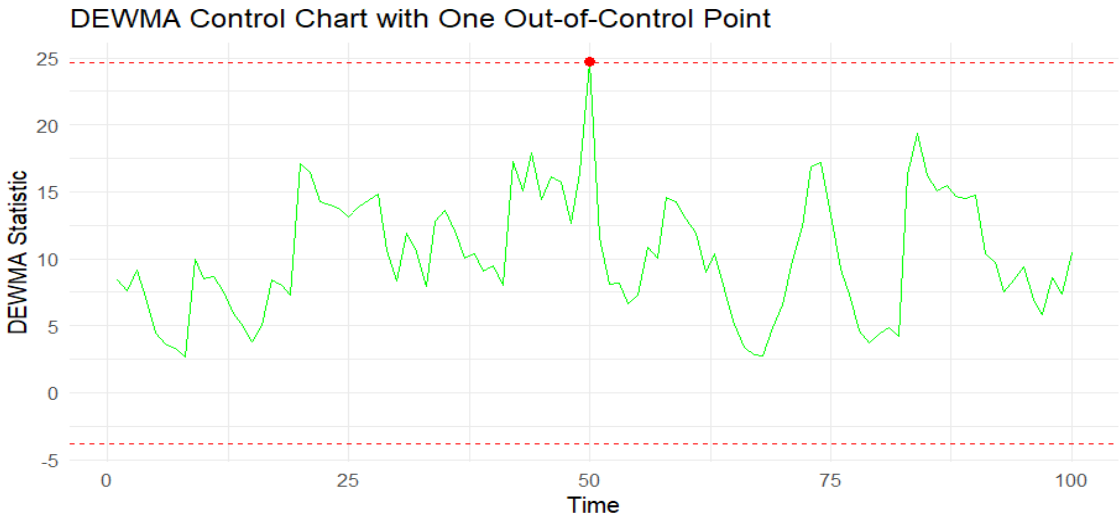

Figure 6.14: DEWMA Control Chart

In Figure 6.14, which illustrates the DEWMA control diagram for exponential understanding, we observe that in this diagram too, at the beginning the process is under control. Except for one point, which is depicted by a red dot, also in this case it is observed near time step 50. This shows that the double smoothing mechanism available ensures that even small and gradual changes in the process can be captured effectively. It is also evident that the adaptability of the chart is

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evident in its ability to closely follow the process trend while minimizing false alarms.

To compare this we will run the same simulation with the same data, but with the EWMA for Gamma Distributed TBE data in this example. The diagram is shown below:

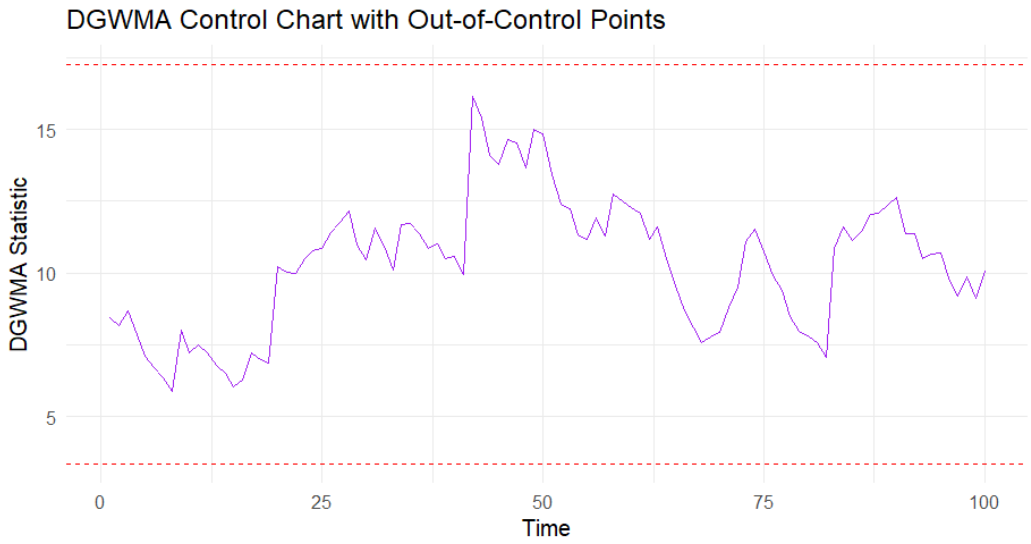

Figure 6.15: DGWMA Control Chart

Figure 6.15 shows a DGWMA (Dynamic Gaussian Weighted Moving Average) control chart that tracks the process variations over time. To describe the chart, observe that the x-axis is the time axis, and the y-axis is the DGWMA statistic. In addition, there is the purple line, which represents the values of the DGWMA statistics, and in this way shows the variations of the process. The last part of the diagram is the control limits, which are represented by red dotted lines, the upper and lower control limits. Moreover, we observe that this diagram has no points that exceed either the upper or the lower control limit, which means that the process is stable.

In conclusion, having done both charts, we can say that both are quite useful tools, but for different situations. This is because the DEWMA diagram is more sensitive to subtle process changes, making it ideal for scenarios where early detection of small changes is critical. On the other hand, the GWMA diagram provides a balanced approach, focusing on moderate changes while maintaining stability in dynamic environments. Also, there is the DGWMA diagram which provides notable changes, but without compromising the process as there are no out-of-bound points.

6.7 Methodology and Comparative Analysis of First Passage Time Control Charts

The control charts are currently used to monitor complex processes with TBE data using static distributions such as exponential or gamma. On the other hand, these methods may lack the flexibility needed to consider processes that are

more affected by time. For this reason, Sajid Ali (2020) proposes First Passage Time (FPT) control charts, which are a powerful tool in these cases. More specifically, these charts are used in cases where cumulative damage thresholds can determine the health of the process. To cope with such problems, the use of exponential distributions to model the sizes of the events is combined. Thus, in the following we will analyze the author's methodology using a simulation, in which we will follow all the steps mentioned in the theory.

Below, we will review the steps related to the First Passage Time (FPT) methodology to create the simulation. In this way we will be able to observe the results of the methodology, so that we can support the author's conclusions. Then here are the steps of the methodology, which are as follows:

1. Determination of total observations (n=100). We will set the number of observations to a relatively large size of 100 observations, so that we can create the FPT chart.
2. Parameter setting.
    - Initially, we will set the value of the variable b to 1.5, because it assumes a moderate increase in the intensity of the event. In addition, we set a value greater than one, so that we can have some relative damage to the process, such as, for example, machine wear.
    - Next, we have the rate variable (θ) for the exponential distribution, in which we will set it to the value of 0.1. We will not prefer higher values for the parameter in our simulation, because the higher the values the smaller the damage sizes we will have, which will not help us enough in the process.
    - Then, we have the intensity of the NHPP which is the variable λ0, which we set to 0.5. Relatively we set it to a medium value to ensure a moderate number of events that can occur in a process, without escaping from the real-world data.
    - Last, we will set the parameter K, which is the threshold for the cumulative damage, which we set with a value of 10, but this value can be changed depending on the conditions. In particular, the conditions we mentioned can be the resilience to the system or resistance to cumulative damage that it can have.
3. Create NHPP time intervals. After we have secured the values that the parameters will take, we will continue with the creation of the time intervals. The NHPP intensity is used to create time intervals, ensuring that time-dependent changes are incorporated.
4. Storage of FPT. To assist in the construction of the control diagram, in each simulation the time to exceed K is stored as the time of first crossing. We will repeat this procedure as many times as the observations we have made.
5. Calculation of control limits (LCL, UCL). To create the lower and upper control limits, we need to calculate the mean and standard deviation of the process. To create the mean, we will use the average FPT value for each individual simulation. On the other hand, to generate the standard deviation, the standard deviation of the FPT data is used, which was used

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to measure the dispersion. In this way we can calculate the lower and upper control limit, in which we use the following formulas:

$$LCL = \mu - z_{a/2} \cdot \sigma,$$

$$UCL = \mu + z_{a/2} \cdot \sigma$$

for the type we used $z_{a/2}$, which is the critical value from the standard normal distribution, to ensure a false alarm rate of α=0.05.

After following all the steps of this methodology and having found both the upper and lower control limit, we can construct the control chart. This control chart will help us to compare the author's results with the simulation, to see if they are relatively close. For this reason, the following diagram is shown below, which is:

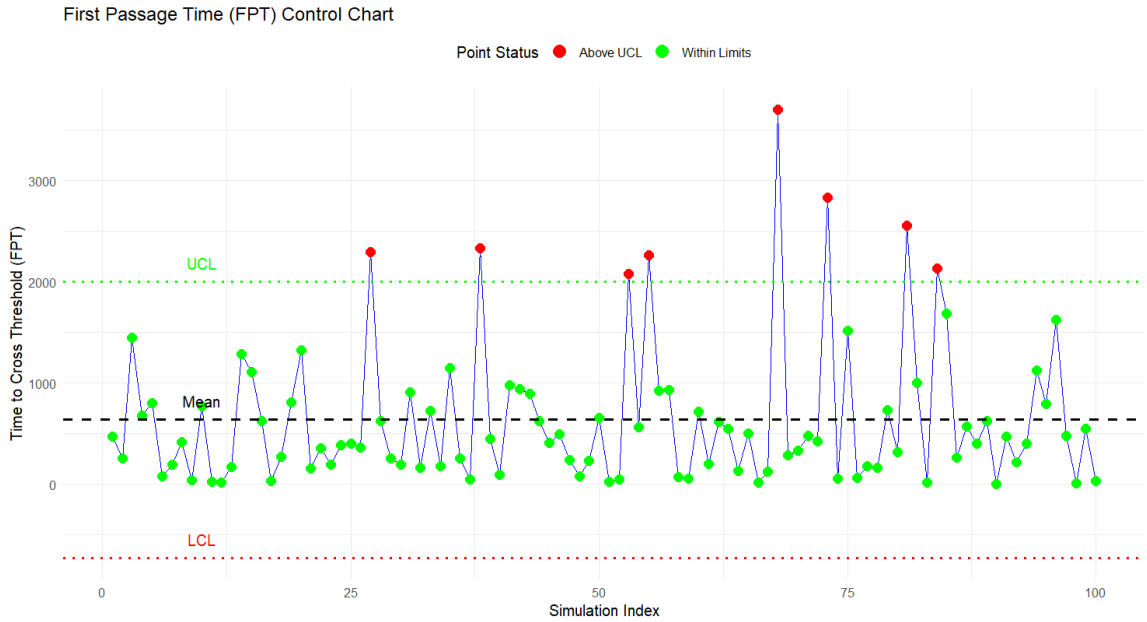

Figure 6.16: First Passage Time (FPT) Control Chart

Figure 6.16 shows First Passage Time (FPT) control line, which is contained by the green points, which are the points inside the process and means that the process is under control. Furthermore, the black line depicted in the diagram is the average value of the process, but there are also the control limits of the process, which in red is the lower control limit and in green the upper control limit. In addition, the x-axis is the simulation index, and the y-axis is the time to cross the threshold - FPT. Finally, related to the morphology of the graph, there are also red points, which are above the upper control limit, and this means that it also marks improvement or delayed cumulative failure. In general, our conclusion from this diagram is that the process is in control for most of its duration, but in cases where it is out of control it means that the process takes significantly longer than usual to reach the limit. This means that improvements are needed, such as reduced event intensity or slower damage accumulation.

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To compare this, we will run the same simulation with the same data, but with the Rate Control Chart in this example. The diagram is shown below:

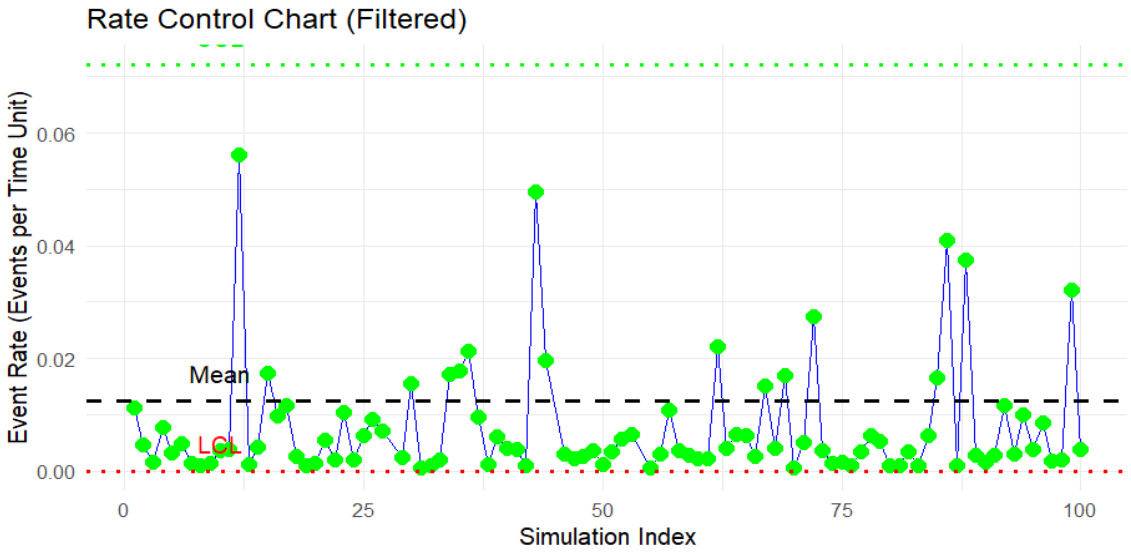

Figure 6.17: Rate Control Chart

Figure 6.17 shows the Rate chart, which consists of the simulation index on the x-axis and the event rate on the y-axis. Also, the red dashed line is the lower control limit (LCL), and the green dashed line is the upper control limit (UCL) of the chart. In addition to this, there is a black line in the center of the graph, which is the mean, representing the average event rate. Concerning the meaning of the graph, we notice that all values are in green color, which means that the process remains stable without exceeding the upper or lower control limit. In addition, we note that most prices are around the average price, apart from a few prices that have extreme variations without deviating from the limits.

In comparison, the two charts are quite similar, but also have differences. First, in the FPT chart we observe more variability in the data compared to the other chart, this is also evident from the dispersion between the boundaries in the FPT chart is much larger. Also, there are many similarities in the pattern of both charts, but in the FPT chart, as we have already mentioned, there are sharp fluctuations with points that exceed the upper control limit.

In conclusion, we can mention that this methodology is a quite useful tool and can be used in several sectors. For example, it can be used in the production sector, to monitor machine defects over time, or even in mining, where it can be used for safety issues such as monitoring the time between accidents. But to improve the process regardless of the sector, it is necessary that when the red marks (the points above the upper control limit and below the lower control limit) appear, there should be further research on the process to find the required improvement in the process.

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Chapter 7

Conclusions

The study of Time Between Events (TBE) plays a fundamental role in understanding and predicting the timing of events in various fields such as engineering, healthcare, economics, and social sciences. By using the methods mentioned above, we can infer the timing between events. We can obtain critical knowledge for better decision-making and better forecasting. For example, TBE analysis is used in engineering as mentioned above, and helps to predict equipment failures and optimize maintenance schedules. It is also very common to see it in healthcare situations, such as in hospitals where it is used for the frequency of patient visits. This means that it helps them prepare and have the appropriate supplies.

To make the analyses more correct, and to explain their use in these areas, it is necessary to mention the tools used, such as statistical process control charts. Some of them are the CUSUM (Cumulative Sum), EWMA (Exponentially Weighted Moving Average), Shewhart, and the combination of many of them, which play a key role in TBE analysis, providing a structured way of monitoring and identifying changes in the timing of events. As mentioned above each chart has a distinct utility where it helps us in different situations. For example, CUSUM diagrams are particularly effective in detecting small shifts in the meantime between events, while EWMA diagrams help to smooth out longer-term fluctuations. Despite the differences between the two charts, they have one common element in that they are both useful in the field of health care. In addition, both Shewhart charts offer a simple approach to identifying sudden deviations, which makes them useful in real-time monitoring scenarios where rapid intervention is required.

Having defined the tools to be used in this study, we need to divide the methods into categories. For this reason, we first categorize the methods into classical parametric models, which focus specifically on established distributions and methodologies. We then created the category of non-parametric techniques to address scenarios where traditional assumptions may not hold but also offer greater ease in creating more complex methods. Finally, we incorporated modern statistical methods and methods based on machine learning to improve the accuracy of predictions. For the selection of each method several different parameters need to be taken into consideration. Some of these parameters are the size of the data to be used to know which distribution to choose. Also, after knowing the distribution and the size of the data, we need to define the purpose of the analysis. In this case, if we define the purpose, we can have more information about which method is most appropriate for the process. For example, for a more structured basis for modeling events, exponential distributions, Weibull distributions, and gamma distributions are most often used. To be specific, these approaches are useful for simpler situations but are not as common because they are based on hypothetical scenarios rather than real data. On the other hand, a combination of distributions and methods provides a more flexible and accurate

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approach. For example, mixture models can combine different distributions, such as Weibull and exponential, to more effectively explain certain real-world events.

We note that there are several different studies and methods related to Time Between Events. This shows us that the continuous evolution of TBE methodologies highlights the importance of adapting to increasingly complex and data-driven environments. For example, research has reached a point where the methods help us in predictive maintenance or even in the various frauds they can detect. But that does not mean that there is no room for improvement in this area. For this reason, as a conclusion from the study, future research should focus on refining hybrid models but also keep a balance between interpretability with predictive power and incorporating real-time adaptive learning to enhance responsiveness in dynamic systems. With the evolution of these techniques and their interactions, we can further develop the potential of TBE analysis in solving practical challenges and optimizing decision-making in many areas.

Chapter 8

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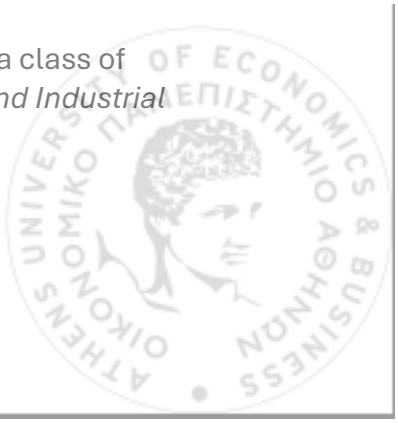

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