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The Double Half-Normal distribution and its extensions.

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Abstract: *We use the difference of two half-normal random variables as a device to generate a class of distributions whose members have Gaussian functional forms but possess the characteristic kink of the Laplace distribution and exhibit positive excess kurtosis. These distributions provide data-modeling tools alternative to the use of the Laplace distribution, but they also have applications wherever absolute values of normal random variables appear, for example in relation to the absolute error of estimators and predictors, but also in Brownian motion theory. We provide Monte Carlo evidence related to the performance of maximum likelihood estimation of the distribution.*

Key words: Double Half-Normal distribution, Skew Normal distribution, Laplace distribution, Double Exponential distribution, Maximum Likelihood, Brownian Motion.

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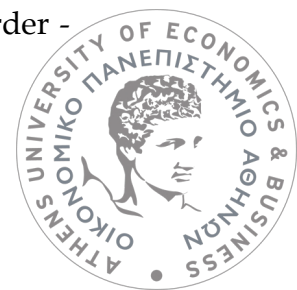
1. Introduction & motivation

There are many instances where researchers have found that the Laplace distribution provides a better approximation to the data than the Normal one, due to its heavier tails and tighter middle concentration which ends with the characteristic kink in its theoretical probability density function. Kotz et al. (2001) ch. 7-10 present an overview of relevant applications from the fields of engineering, finance, inventory management and quality control, astronomy, biological and environmental sciences. These applications employ also an Asymmetric Laplace distribution that exhibits skewness while maintaining the kink in the density (for which see Kotz et al. 2001 ch. 3 but also Hinkley and Revankar 1977). Turning to the extended normal family of distributions, the last decades have seen rich developments in the framework of the skew-normal (and related families), which was named and introduced as such by Azzalini (1985).

With the present paper we aim to provide an additional tool for modeling data, by presenting a class of univariate distributions whose structure is reminiscent of the skew-normal family, yet they can be considered as the "Laplace analogue" in the extended normal family, since they possess a kinked probability density function and tails heavier than the normal distribution.

We call the parent distribution "Double Half Normal" because its generating device is the difference of two Half-normal random variables (absolute values of zero-mean normals, and also identical to chi distributions with one degree of freedom), something that, apart from fitting data sets, provides added scope for applications of the distribution: When we study aggregated phenomena we apply addition of random variables, while we use multiplication to model interactive effects. Ratios and differences are used when comparative evaluations and the monitoring of thresholds are in order -



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ratios when the interest lies in relative magnitudes, differences when we are interested in levels. This paper contributes new results useful in this last case.

In the literature, some related work can be found in papers investigating absolute normal errors (e.g. Geary 1936), and especially Kamat (1953) who considered explicitly linear combinations of half-normal random variables but with his main focus being the derivation of joint raw and absolute moments, although he provided some results on their probability integrals.

Our work goes beyond the available results in that we provide characterization of the various distributions presented in terms of closed-form density and distribution functions, as well as their moment generating functions and theoretical moments. In Section 2 we treat the symmetric case. In Section 3 we generalize it to accommodate skewness and in Section 4 we extend it to represent correlation in the underlying variables. In Section 5 we go beyond the half-normal representation, which provides further flexibility in modeling data. In Section 6 we examine the performance of maximum likelihood in estimating the parameters of the more flexible member of this distribution family. Finally, Section 7 contains some theoretical applications of the Double Half Normal family of distributions that relate to the comparative assessment of estimators and predictors, as well as to Brownian motion theory. In the interests of brevity, all proofs have been relegated to a Technical Appendix which is available from the author upon request. All results have also been verified by computer calculations and/or Monte Carlo simulations.

2. The Double Half-Normal distribution

Definition. Let Z be a univariate, real-valued, continuous random variable. Z is said to follow the *Double Half-Normal distribution* with scale parameter $s>0$, $\mathrm{DHN}(s)$, if it has probability density function

https://doi.org/10.26219/heal.aueb.8168

$$f_{DHN}(z) = \frac{4}{s}\phi(z/s)\Phi(-|z|/s) \qquad [1]$$

and cumulative distribution function

$$F_{DHN}(z) = \begin{cases} 4\Phi_2\left(z/s,\, 0;\, -1/\sqrt{2}\right) & z \le 0 \\ 4\Phi_2\left(z/s,\, 0;\, 1/\sqrt{2}\right) - 1 & z \ge 0 \end{cases} \qquad [2]$$

where $\phi(\ )$, $\Phi(\ )$ are the standard normal probability density (PDF) and cumulative distribution functions (CDF) respectively, while $\Phi_2(\cdot,\cdot;\delta)$ is the bivariate standard normal CDF with correlation coefficient δ.

Both the CDF and the PDF of the Double Half-Normal are expressed using functional forms that relate to the univariate skew-normal distribution (whose CDF can be expressed as a bivariate normal integral with one of the integration limits set to zero, see for example Azzalini and Capitanio 2014, p. 34). It is the use of the absolute value that symmetrizes the PDF and produces the kink at its center, which also leads to a piece-wise distribution function. To contrast the DHN distribution with the skew-normal family, we note that the core abstract construction for the latter, as presented in Azallini & Capitanio (2014) ch.1, employs an odd function to "modulate" a symmetric density through its CDF, while for the DHN distribution we employ the absolute value which is an even function.

For unitary variance, the graphs of the Double Half Normal, Laplace and Normal are shown in Figure 1.

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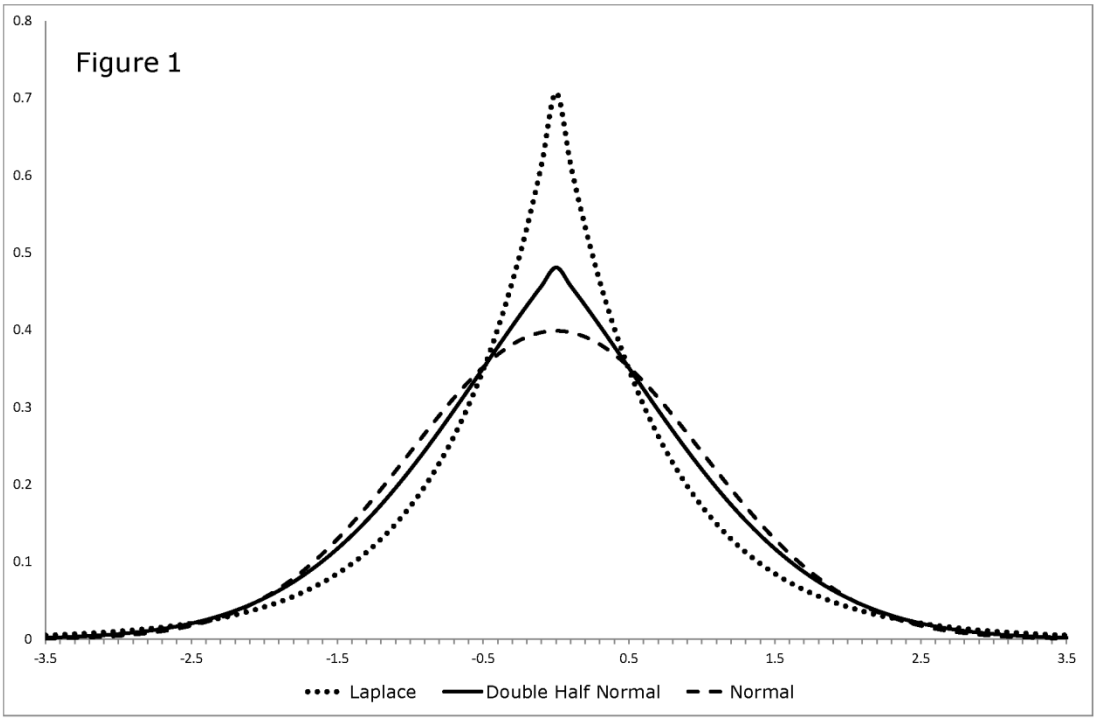
Figure 1: The Double Half Normal density for unitary variance.



Representations.

A) Let two independent standard half-normal random variables $H_1, H_2 \sim \mathrm{HN}(1)$ (so their mean is $\sqrt{2/\pi}$ and their variance $1-2/\pi$). Then their scaled difference follows the Double Half-Normal distribution with scale parameter $s=\sigma\sqrt{2}$,

$$\sigma(H_1 - H_2) = Z \sim \mathrm{DHN}(s),\ s=\sigma\sqrt{2} > 0 \qquad [3]$$

This representation provides also the name to the distribution, in analogy to the Laplace distribution which is alternatively called "Double Exponential"

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and can be derived as the distribution of the difference of two i.i.d. Exponential random variables.

B) The Double Half -normal distribution is also linked to the minimum of two i.i.d. Half-normal distributions. Specifically, let H_1, H_2 defined as previously. Then

$$\min\{\sigma H_1, \sigma H_2\} = |Z| \qquad [4]$$

with $Z \sim \text{DHN}(\sigma)$. This representation for the absolute value of a DHN random variable comes from the fact that the minimum m of two i.i.d. half normals with scale parameter σ has density $f(m) = (8/\sigma)\phi(m/\sigma)\Phi(-m/\sigma),\ m \geq 0$.

Moments. The moment generating function (MGF) can take alternative forms. From the first representation of the distribution, and given independence of the two half-normals, we have that

$$M_{DHN}(t) = M_{H_1}(\sigma t) \cdot M_{H_2}(-\sigma t) = 2e^{\frac{1}{2}\sigma^2 t^2}\Phi(\sigma t)\, 2e^{\frac{1}{2}\sigma^2 t^2}\Phi(-\sigma t)$$
$$\Rightarrow M_{DHN}(t) = 4\exp\left\{\tfrac{1}{2}s^2t^2\right\}\Phi\left(st/\sqrt{2}\right)\Phi\left(-st/\sqrt{2}\right) \qquad [5]$$

Alternatively, we can decompose the density in [1] for positive and negative values of the variable into a piece-wise density with skew-normal density kernels. These can be viewed as truncated/limited skew normal densities which have been studied in Jamalizadeh et al (2009). From this route, a "native" expression for the MGF is

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$$M_{DHN}(t) = 4\exp\left\{\tfrac{1}{2}s^2t^2\right\}\Big[\Phi\left(-st/\sqrt{2}\right) + \Phi_2\left(st/\sqrt{2}, -st; -1/\sqrt{2}\right) - \Phi_2\left(-st/\sqrt{2}\,st, -st; 1/\sqrt{2}\right)\Big] \quad [6]$$

We nevertheless note that the moments of the distribution (and of its extensions that will be presented in subsequent sections) are more conveniently derived by direct integration and using results found in Owen (1980), rather through the MGF.

The first and third moments of the Double Half-Normal distribution are zero, since it is symmetric. For the second and fourth we have

$$E\left(Z_{DHN}^2\right) = \mathrm{Var}\left(Z_{DHN}\right) = \frac{\pi-2}{\pi}s^2 \quad [7]$$

$$E\left(Z_{DHN}^4\right) = \left(\frac{3\pi-8}{\pi}\right)s^4 \quad [8]$$

The excess kurtosis coefficient is

$$\gamma_{2,DHN} = \frac{E\left(Z_{DHN}^4\right)}{\left[E\left(Z_{DHN}^2\right)\right]^2} - 3 = \frac{4(\pi-3)}{(\pi-2)^2} \approx 0.4345 \quad [9]$$

As should be expected from the graph of the densities, the excess kurtosis is positive but visibly smaller compared to the Laplace's distribution.

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3. Introducing asymmetry: the Asymmetric Double Half Normal distribution.

The Double Half-Normal distribution is the natural ceremonial flag of the family, but being a one-parameter distribution lacks flexibility. In this section we extend it to allow for skewness.

Definition. A continuous random variable Z is said to follow the *Asymmetric Double Half Normal distribution* with scale parameter $s>0$ and shape parameter $\lambda>0$, $\text{ADHN}(s,\lambda)$, if it has probability density function

$$f_{ADHN}(z)=\begin{cases}\frac{4}{s}\phi(z/s)\Phi(\lambda z/s) & z\le 0\\ \\ \frac{4}{s}\phi(z/s)\Phi(-\lambda^{-1} z/s) & z\ge 0\end{cases} \qquad [10]$$

We note that the form of the PDF is reminiscent of a known general method to introduce skewness based on a symmetric density as for example presented by Fernández and Steel (1998), i.e. by using inverse scaling factors for the positive and negative orthants. The ADHN density can be seen as the result of a variant of this method, since the inverse scaling factor is applied to a non-symmetric density, and it also changes sign.

Representation. With H_1, H_2 defined as before, we have that

$$\sigma_1 H_1 - \sigma_2 H_2 \;=\; Z \sim ADHN\left(s=\sqrt{\sigma_1^2+\sigma_2^2},\ \lambda=\sigma_1/\sigma_2\right) \qquad [11]$$

https://doi.org/10.26219/heal.aueb.8168

Essentially, this is why the parameter λ is constrained to be positive[1]. From this representation we obtain $s=\sigma_2\sqrt{\lambda^2+1}=\sigma_1\sqrt{\lambda^{-2}+1}$. As with the DHN distribution, this implies that the ADHN distribution can always be derived as the difference of two independent half normal random variables with different scale parameters, for all values of the parameter vector (s,λ) in $\mathbb{R}_+\times\mathbb{R}_+$.

Properties. Some properties of the ADHN density function are:

Property A. As $\lambda\to 0$ the distribution tends to a negative half-normal.

Property B. As $\lambda\to\infty$ the distribution tends to a half-normal.

Property C. If Z is an ADHN(s,λ) random variable, then $-Z$ is ADHN $\left(s,\lambda^{-1}\right)$.

Property D. The ADHN density is log-concave in z. This follows form the log-concavity of the skew-normal functional forms.

Distribution function. The distribution function of an ADHN random variable is

$$F_{ADHN}(z)=\begin{cases}4\Phi_2\left(z/s,\,0;\,-\delta\right) & z\le 0\\ 4\Phi_2\left(z/s,\,0;\,\lambda^{-1}\delta\right)\;-1 & z\ge 0\end{cases} \qquad [12]$$

where $\delta=\lambda/\sqrt{1+\lambda^2}$.

[1] We have calculated that for negative λ the density function requires a normalizing constant of $4/3$ instead of 4 in order to integrate to unity, while the distribution becomes bimodal, with a downward pointing kink that is a local minimum in between the two modes. We do not examine this case.

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Moments. The moment generating function for the asymmetric case can again have two alternative forms. The one stemming from the half-normal representation is

$$M_{ADHN}(t)=4\exp\left\{\frac{1}{2}s^2t^2\right\}\Phi(\delta st)\Phi\left(-\lambda^{-1}\delta st\right) \quad [13]$$

while the alternative form, derived directly from its density is

$$M_{ADHN}(t)=4\exp\left\{\frac{1}{2}s^2t^2\right\}\Big[\Phi\left(-\lambda^{-1}\delta st\right) + \Phi_2\left(\delta st, -st; -\delta\right) - \Phi_2\left(-\lambda^{-1}\delta st, -st; \lambda^{-1}\delta\right)\Big] \quad [14]$$

We obtain the moments

$$E\left(Z_{ADHN}\right) = \sqrt{\frac{2}{\pi}}\frac{\lambda-1}{\sqrt{1+\lambda^2}}s \quad [15]$$

$$E\left(Z_{ADHN}^2\right) = \left(1-\frac{4}{\pi}\frac{\lambda}{1+\lambda^2}\right)s^2 \Rightarrow \mathrm{Var}\left(Z_{ADHN}\right) = \frac{\pi-2}{\pi}s^2 \quad [16]$$

Note that unlike the skew-normal distribution, the variance does not depend on the shape parameter λ, and it is equal to the variance of the DHN distribution. Higher moments are

$$E\left(Z_{ADHN}^3\right) = \sqrt{\frac{2}{\pi}}(\lambda-1)\frac{2\left(1+\lambda^2\right)-\lambda}{\left(1+\lambda^2\right)^{3/2}}s^3 \quad [17]$$

https://doi.org/10.26219/heal.aueb.8168

$$E\left(Z_{ADHN}^4\right) = \left(3-\frac{16}{\pi}\frac{\lambda}{1+\lambda^2}\right)s^4 \qquad [18]$$

Skewness. Pearson's skewness coefficient is

$$\gamma_{1,ADHN} = \frac{\sqrt{2}(4-\pi)}{(\pi-2)^{3/2}}\frac{\left(\lambda^3-1\right)}{\left(1+\lambda^2\right)^{3/2}} \qquad [19]$$

It has the same range as the skewness coefficient of the skew-normal distribution, $(-0.995, 0.995)$, but it is negative for $0<\lambda<1$ and becomes positive for higher values of λ (while in the skew-normal case where the shape parameter is not constrained to be positive, the skewness sign changes with the sign of it).

Excess kurtosis. The excess kurtosis coefficient is

$$\gamma_{2,ADHN} = \frac{8(\pi-3)}{(\pi-2)^2}\frac{1+\lambda^4}{\left(1+\lambda^2\right)^2} \qquad [20]$$

with the maximum being ≈ 0.869. As with the skewness coefficient, the maximum excess kurtosis of the ADHN distribution is the same as that of the skew normal. But in the ADHN case, excess kurtosis never falls to zero, but obtains its positive minimum (which equals half the maximum) in the symmetric case where $\lambda=1$. The maximum itself is attained as the shape parameter goes either to zero or to infinity, i.e. when, in the half-normal representation, the imbalance between the two underlying scale parameters is maximized.

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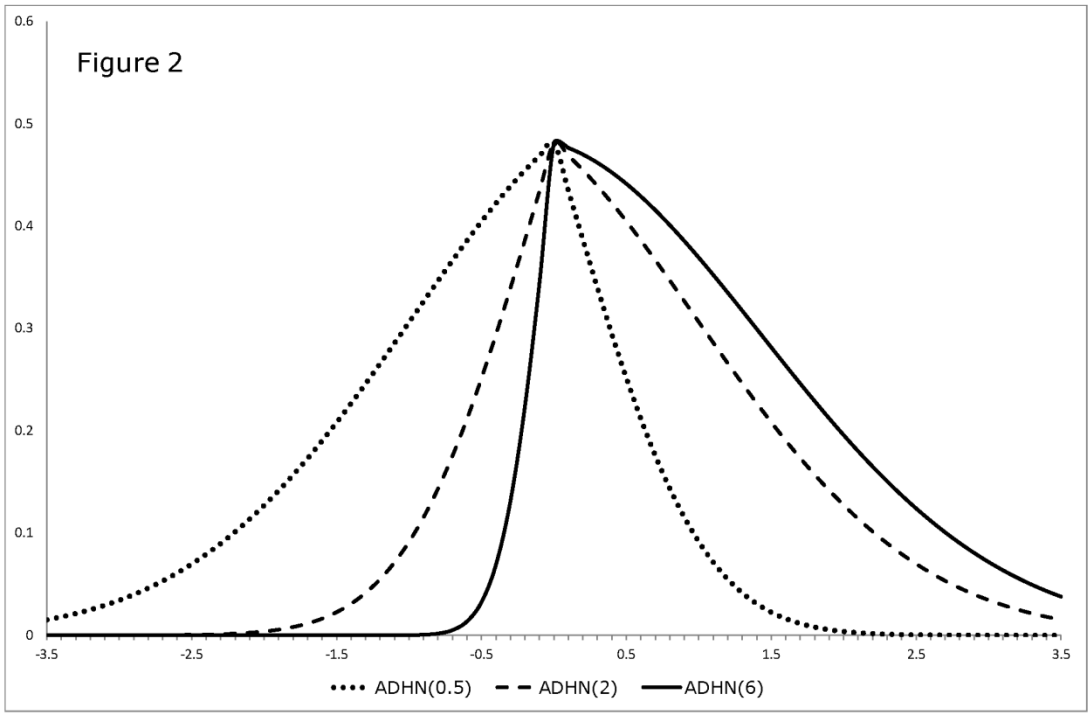
Figure 2 presents the graphs of ADHN densities for unitary variance and different shape parameters $\lambda = 0.5,\ 2,\ 6$.

Figure 2: ADHN densities for different shape parameters.



4. Allowing for underlying correlation.

In this section we extend the distribution so as to model the behavior of the difference of two *correlated* half-normal random variables which has useful applications as we will see in Section 7. At the same time, the distribution obtained can go beyond this representation providing added flexibility in modeling data (we discuss this further in Section 5).

Definition. A random variable Z is said to follow the *Correlated Asymmetric Double Half Normal distribution,* $\text{cADHN}(s_1, s_2, \lambda, \rho)$, with scale

https://doi.org/10.26219/heal.aueb.8168

parameters $s_1 > 0, s_2 > 0$, and shape parameters $\lambda > 0$, $|\rho| < 1$, if it has probability density function

$$f_{cADHN}(z) = \begin{cases} \frac{2}{s_1}\phi(z/s_1)\Phi(\lambda_1 z/s_1) + \frac{2}{s_2}\phi(z/s_2)\Phi(\lambda_2 z/s_2) & z \le 0 \\ \frac{2}{s_1}\phi(z/s_1)\Phi(-\lambda_3 z/s_1) + \frac{2}{s_2}\phi(z/s_2)\Phi(-\lambda_4 z/s_2) & z \ge 0 \end{cases} \quad [21]$$

where

$$\lambda_1 = \frac{\lambda - \rho}{\sqrt{1-\rho^2}}, \ \lambda_2 = \frac{\lambda + \rho}{\sqrt{1-\rho^2}}, \ \lambda_3 = \frac{\lambda^{-1} - \rho}{\sqrt{1-\rho^2}}, \ \lambda_4 = \frac{\lambda^{-1} + \rho}{\sqrt{1-\rho^2}}$$

Properties. Some properties of the cADHN distribution are:

Property cA. As $\lambda \to 0$ then, if $s_1 = s_2$ the distribution tends to a negative half-normal, otherwise it tends to a mixture of two equally weighted negative half-normals.

Property cB. As $\lambda \to \infty$, then, if $s_1 = s_2$ the distribution tends to a half-normal, otherwise it tends to a mixture of two equally weighted half-normals.

Property cC. If Z is a cADHN$(s_1, s_2, \lambda, \rho)$ random variable, then $-Z$ is cADHN $(s_1, s_2, \lambda^{-1}, \rho)$

Property cD. The cADHN density is log-concave in z.

Property cE. As $\rho \to 0$, then if $s_1 = s_2 = s$ the distribution tends to an ADHN(s, λ) random variable, otherwise it tends to a mixture of two equally weighted ADHN random variables.

Representations.

A) Let H_1, H_2 defined as before, but now assume further that the underlying normals follow a bivariate (zero-mean) normal distribution with

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correlation coefficient $|\rho|<1$. Then, H_1, H_2 follow jointly a bivariate half-normal distribution (see Kotz *et al.* 2000, p.327, eq. 46.210, or also Psarakis & Panaretos 2001)[2]. Then one can calculate that

$$\mathrm{Cov}(H_1,H_2)=\frac{2}{\pi}\left[\left(1-\rho^2\right)^{2/3}+\rho\cos^{-1}(-\rho)+\rho^2\sqrt{1-\rho^2}-1\right]-\rho>0$$

So under bivariate half normality, the correlation between the marginal half normals *will always be positive irrespective of whether the underlying normals are positively or negatively correlated*. In this framework, we obtain

$$\sigma_1 H_1-\sigma_2 H_2 \sim cADHN \qquad [22]$$

with parameters $s_1^2=\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2$, $s_2^2=\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2$ and $\lambda=\sigma_1/\sigma_2$, $|\rho|<1$. Note that in order for the representation to be valid, we must have $s_1^2=s_2^2-4\rho\lambda\sigma_2^2$. This implies the constraint $\mathrm{sign}\{s_2-s_1\}=\mathrm{sign}\{\rho\}$. Moreover, under this representation, the cADHN density becomes an even function in the parameter ρ.

Due to the existence of the parameter constraint, the half-normal representation does not fully characterize the cADHN distribution, but it is only one incarnation of it (it nevertheless gives it its name, being a representation of central interest). The cADHN distribution remains a proper distribution for all possible combinations of values of its parameters.

B) The distribution is linked to the minimum of two *independent* half-normals but with different scale parameters. Specifically

[2] We note in passing that unlike the univariate case, bivariate half-normality does not coincide with bivariate normality truncated from below at zero, when correlation is present.

https://doi.org/10.26219/heal.aueb.8168

$$\min\left\{\sigma_1 H_1, \sigma_2 H_2\right\} = |Z| \qquad [23]$$

where Z follows a cADHN distribution with parameters $s_1 = \sigma_1$, $s_2 = \sigma_2$ and $\lambda = 1$, $\rho = \left(\sigma_2^2 - \sigma_1^2\right) \Big/ \left(\sigma_2^2 + \sigma_1^2\right)$. This comes from the fact that the minimum $m \geq 0$ of two independent half-normals with scale parameters σ_1, σ_2 has density $f_m(m) = (4/\sigma_1)\phi(m/\sigma_1)\Phi(-m/\sigma_2) + (4/\sigma_2)\phi(m/\sigma_2)\Phi(-m/\sigma_1)$. Note that here ρ is not interpreted as a correlation coefficient.

Other representations could be formed for the cADHN random variable as a mixture of extrema of correlated bivariate normal random variables (see Roberts 1966, Cain 1994, and especially Loperfido 2002), but we do not pursue further this direction.

Distribution function. The cADHN distribution function is

$$F_{cADHN}(z) = \begin{cases} 2\Phi_2\left(z/s_1,\, 0;\, -\delta_1\right) + 2\Phi_2\left(z/s_2,\, 0;\, -\delta_2\right) & z \leq 0 \\ 2\Phi_2\left(z/s_1,\, 0;\, \delta_3\right) + 2\Phi_2\left(z/s_2,\, 0;\, \delta_4\right) - 1 & z \geq 0 \end{cases} \qquad [24]$$

with $\delta_i = \lambda_i \Big/ \sqrt{1+\lambda_i^2}$.

Moments. The MGF for the cADHN distribution is

$$\begin{aligned} M_{cADHN}(t) = {} & 2\exp\left\{s_1^2 t^2/2\right\} \cdot \left[\Phi_2\left(\delta_1 s_1 t,\, -s_1 t; -\delta_1\right) + \Phi\left(-\delta_3 s_1 t\right) - \Phi_2\left(-\delta_3 s_1 t,\, -s_1 t; \delta_3\right)\right] \\ & + 2\exp\left\{s_2^2 t^2/2\right\} \cdot \left[\Phi_2\left(\delta_2 s_2 t,\, -s_2 t; -\delta_2\right) + \Phi\left(-\delta_4 s_2 t\right) - \Phi_2\left(-\delta_4 s_2 t,\, -s_2 t; \delta_4\right)\right] \end{aligned} \qquad [25]$$

The expected value of the distribution is

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$$E\left(Z_{cADHN}\right)=\frac{\left(\lambda-1\right)}{\sqrt{2\pi}}\left[\frac{\left(1+\rho\right)s_1}{\sqrt{1+\lambda^2-2\rho\lambda}}+\frac{\left(1-\rho\right)s_2}{\sqrt{1+\lambda^2+2\rho\lambda}}\right] \quad [26]$$

Due to the restriction $|\rho|<1$ the term in the brackets is always real and positive, so the expected value exists always and its sign depends on $\lambda-1$.

The second raw moment is

$$E\left(Z_{cADHN}^2\right)=\left[\cos^{-1}\left(-\rho\right)-\frac{2\lambda-\rho\left(1+\lambda^2\right)}{1+\lambda^2-2\rho\lambda}\sqrt{1-\rho^2}\right]\frac{s_1^2}{\pi} + \left[\cos^{-1}\left(\rho\right)-\frac{2\lambda+\rho\left(1+\lambda^2\right)}{1+\lambda^2+2\rho\lambda}\sqrt{1-\rho^2}\right]\frac{s_2^2}{\pi} \quad [27]$$

Note that both expressions in brackets are always positive.

The 3rd raw moment is

$$E\left(Z_{cADHN}^3\right)=\frac{\left(\lambda-1\right)}{\sqrt{2\pi}}\left[\frac{\left(2-\rho\right)\left(1+\lambda^2\right)-\lambda\left(1+\rho\right)}{\left(1+\lambda^2-2\lambda\rho\right)^{3/2}}\left(1+\rho\right)^2 s_1^3 + \frac{\left(2+\rho\right)\left(1+\lambda^2\right)-\lambda\left(1-\rho\right)}{\left(1+\lambda^2+2\lambda\rho\right)^{3/2}}\left(1-\rho\right)^2 s_2^3\right] \quad [28]$$

Numerical explorations show that the cADHN distribution can have skewness coefficient outside the limits $\left(-0.995, 0.995\right)$ that characterize the ADHN and Skew Normal distributions. The same holds for excess kurtosis (when the distribution goes beyond the half-normal representation), which at times reaches the value 3 that is taken on by the Laplace distribution.

https://doi.org/10.26219/heal.aueb.8168

Finally the 4th raw moment of the distribution is

$$E\left(Z_{cADHN}^4\right)=\frac{s_1^4}{\pi}\left[3\cos^{-1}\left(-\rho\right)\ -\ \frac{\left(\psi_1-\psi_2\right)\sqrt{1-\rho^2}}{\left(1+\lambda^2-2\lambda\rho\right)^2}\right] \qquad [29]$$

$$+\ \frac{s_2^4}{\pi}\left[3\cos^{-1}\left(\rho\right)\ -\frac{\left(\psi_1+\psi_2\right)\sqrt{1-\rho^2}}{\left(1+\lambda^2+2\lambda\rho\right)^2}\right]$$

with $\psi_1=4\lambda\left(1+\lambda^2\right)\left(2+\rho^2\right),\ \psi_2=\rho\left[\left(1+\lambda^4\right)\left(5-2\rho^2\right)+18\lambda^2\right]$

5. Beyond the half-normal representation.

As already mentioned, the half-normal representation of the cADHN distribution does not fully characterize it, since it requires the parameter constraint $\text{sign}\left\{s_2-s_1\right\}=\text{sign}\left\{\rho\right\}$. We gain modeling flexibility if we abandon this connection and consider cases where the constraint does not hold (which we simply call the "unconstrained" case).

We provide a visual symmetric (which can be called the cDHN distribution) and asymmetric example for the "swap" case (Figures 3 and 4), where we compare a set of parameters that validates the half normal representation (specifically they correspond to $\sigma_1=\sigma_2=2,\ \sigma_1=2,\ \sigma_2=3$ for the symmetric and asymmetric cases respectively, with $\rho=0.5$ in both cases), to a set of parameters where we set $\rho=-0.5$ while keeping the s_1, s_2 parameters fixed to their previous values. Looking at the density of cADHN (eq. [21]), this amounts to swapping positions between the composite coefficients λ_1 and λ_2, and between λ_3 and λ_4 and consequently associate now s_1 with λ_2, λ_4 and s_2 with λ_1, λ_3.

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Figure 3: Constrained and Unconstrained cDHN densities.

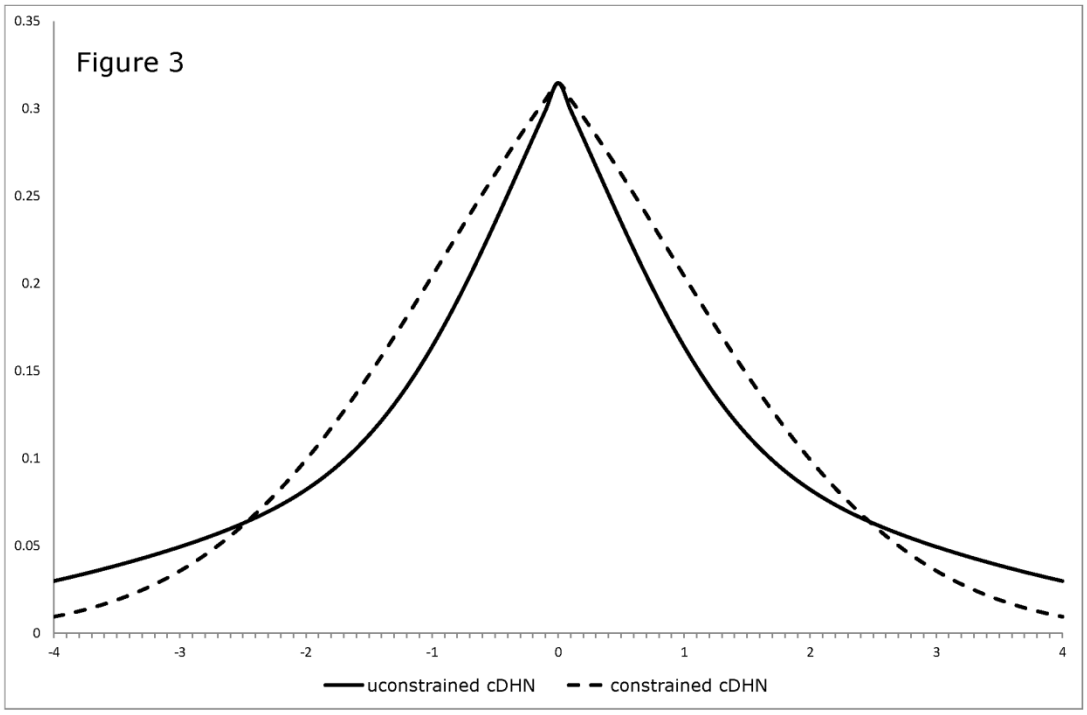
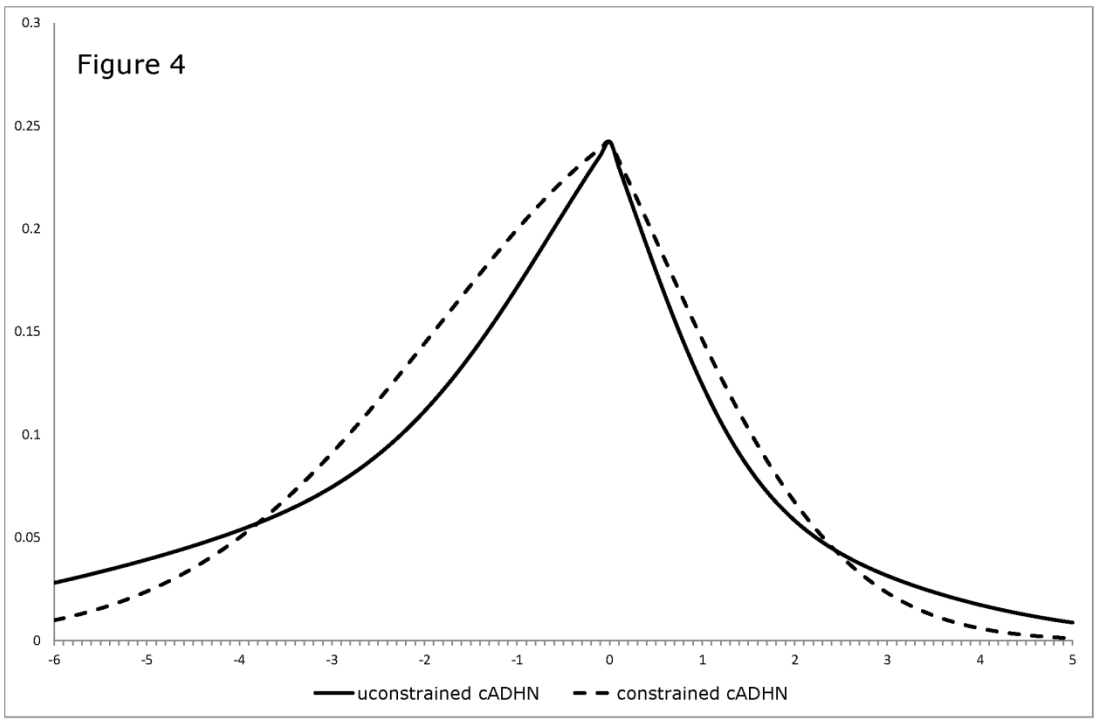


Figure 4: Constrained and unconstrained cADHN densities.



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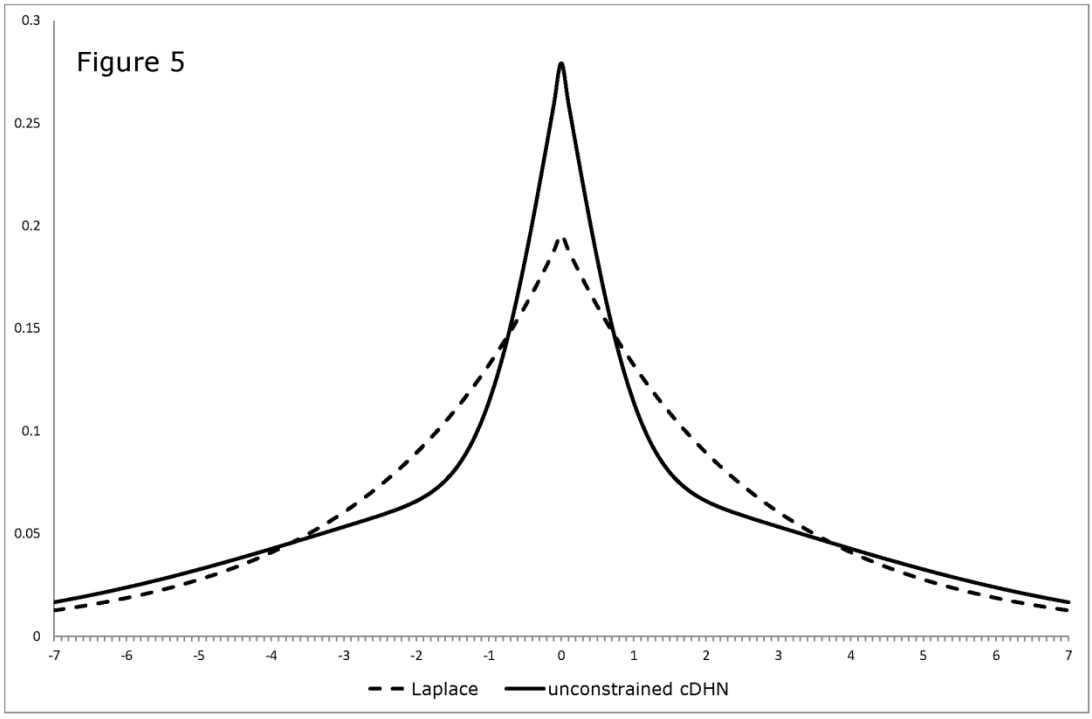
We see that the unconstrained density has a slimmer body and visibly heavier tails than the one arising from the half-normal representation. A higher value for the parameter ρ could even create a "shoulder" to the unconstrained distribution. In Figure 5, where we used the values $\rho = -0.5$, $s_1 = 2$, $s_2 = 5$, we see that the symmetric unconstrained density can have markedly different shapes than the Laplace one (for the same variance), while for other sets of parameters their density graphs can be very close.

Figure 5: Unconstrained cDHN and Laplace densities.



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6. Maximum Likelihood estimation of the cADHN distribution.

It has been found that maximum likelihood estimation with skew normal densities has certain peculiarities, related to the fact that the information matrix becomes singular when the shape parameter is equal to zero (see Azzalini and Capitanio 2014, ch. 3, for an exposition of the matter). This is linked to the consideration of a location parameter (which we have not included in our formulations). Regarding the distributions we examine, for the DHN distribution the shape parameter is fixed to ± 1 , while for the ADHN distribution a zero shape parameter is a boundary case, which would produce a sample of negative values only (see property A in Section 3). Faced with such a sample, there would be no reason to attempt to fit the ADHN distribution to the data. But for the cADHN case (eq. [21]), the shape parameters are functions of the actual parameters of the distribution, and any one of them can be zero (although only one of them at a time). Another issue here is that as we noted earlier, if the data come from the difference of two correlated half-normals, the true cADHN density becomes an even function in the parameter ρ and the parameter can be identified only with respect to its absolute magnitude. So its estimated sign should be backed by any available out-of-sample information, otherwise it could be an artifact of the estimation.

We investigated parameter estimation in the cADHN case, using maximum likelihood through a Monte Carlo study. We generated samples using the half-normal representation $Z = \sigma_1 H_1 - \sigma_2 H_2$ with parameter values $\sigma_1 = 2,\ \sigma_2 = 3,\ \rho = 0.6$, which lead to the following values for the cADHN distribution: $s_1 = 2.408,\ s_2 = 4.494,\ \lambda = 0.667,\ \rho = 0.6$.

We executed two runs: First we generated 10,000 i.i.d. samples each of size $n = 250$ ("MC-run A"). Then we pooled the same data, creating 1,000 samples each of size $n = 2500$, and estimated again ("MC-run B"). In this way

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we could see how the behavior of the MLE changed as the sample size grew. In MC-run A, in 11.2% of cases the MLE failed to converge, while in other cases it faced issues with the negative definiteness of the numerical Hessian matrix, falling back to using the outer product of the score in order to calculate standard errors. In contrast, in MC-run B the estimator converged in all cases and issues with the Hessian where encountered very few times. In Table 1 we summarize the estimation exercise, providing the main statistics of the empirical distribution of the ML estimator for both runs.

Table 1: Maximum Likelihood estimates of the cADHN distribution

Table 1: Maximum Likelihood estimates of the cADHN distribution								
Parameter	$\lambda =$	**0.667**	$s_1 =$	**2.408**	$s_2 =$	**4.494**	$\rho =$	**0.6**
MC run	**A**	**B**	**A**	**B**	**A**	**B**	**A**	**B**
Mean	0.831	0.704	2.472	2.402	3.709	4.584	0.760	0.663
Std. Error	(0.138)	(0.059)	(2.619)	(0.242)	(5.023)	(0.877)	(0.403)	(0.223)
Median	0.848	0.698	2.140	2.348	2.906	4.561	0.953	0.717
Minimum	0.496	0.586	0.876	1.963	0.000	1.119	-0.999	-0.444
Maximum	0.999	0.997	104.589	4.662	168.106	8.222	0.999	0.999
MC-run A: 10,000 samples of size 250. **MC-run B**: 1,000 samples of size 2,500 (same data).								

We see that the MLE has a positive bias in the MC-run A (smaller sample case), with the exception of the second scale parameter which it underestimates on average. The high standard errors are indicative of what happens at the sample-per-sample level: parameter s_1 appeared as statistically insignificant (at the 1% level) 10% of times, while for parameters s_2 and ρ the estimates emerged as statistically insignificant in 77% and 25% of the times,

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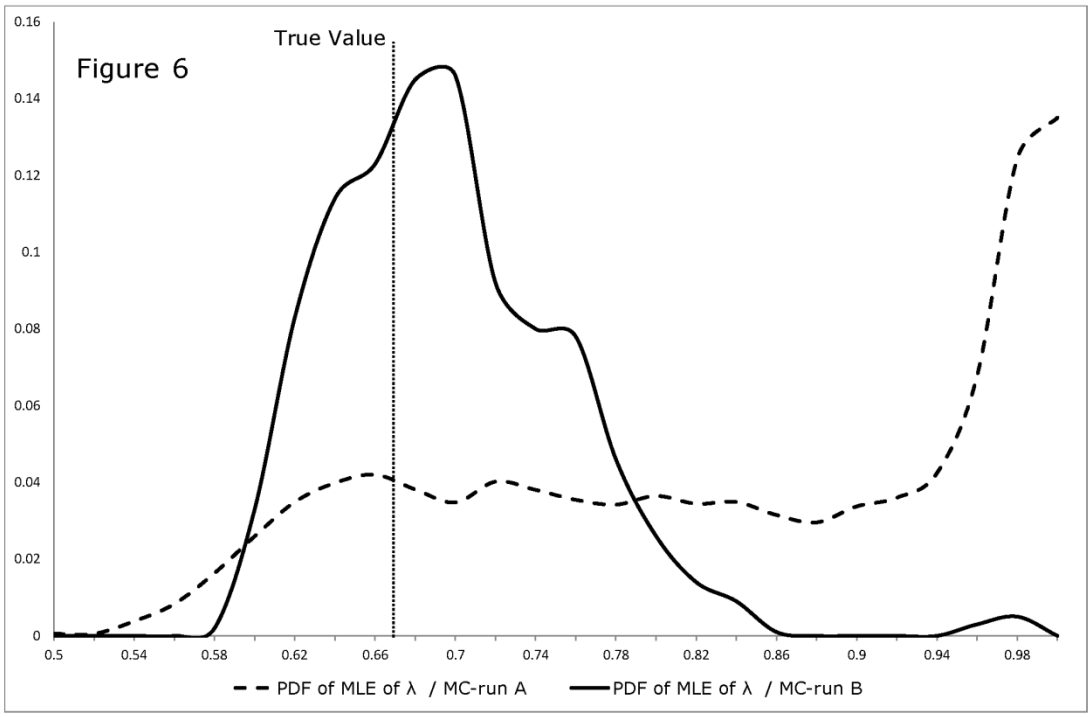
respectively. Moving to MC-run B, the performance of the ML estimator overall improved visibly, with bias almost eliminated and much smaller standard errors, while the very large estimates disappeared. Here only the parameter ρ appeared statistically insignificant (at 1% level) 33% of times, while for the other three the corresponding percentage was below 3%.

The empirical frequency distribution of the MLE was non-standard in both cases, although it changed considerably from the one MC-run to the other. We show in Figure 6 the distributions of the MLE for the shape parameter λ. We note that in the smaller sample case, MC-run A, the frequency graph exhibits an abrupt increase in frequency for values near unity. This implies that the corresponding samples did not support the existence of skewness (but note that the estimated value never exceeded unity).

Figure 6: Empirical Frequency of MLE estimator for parameter λ



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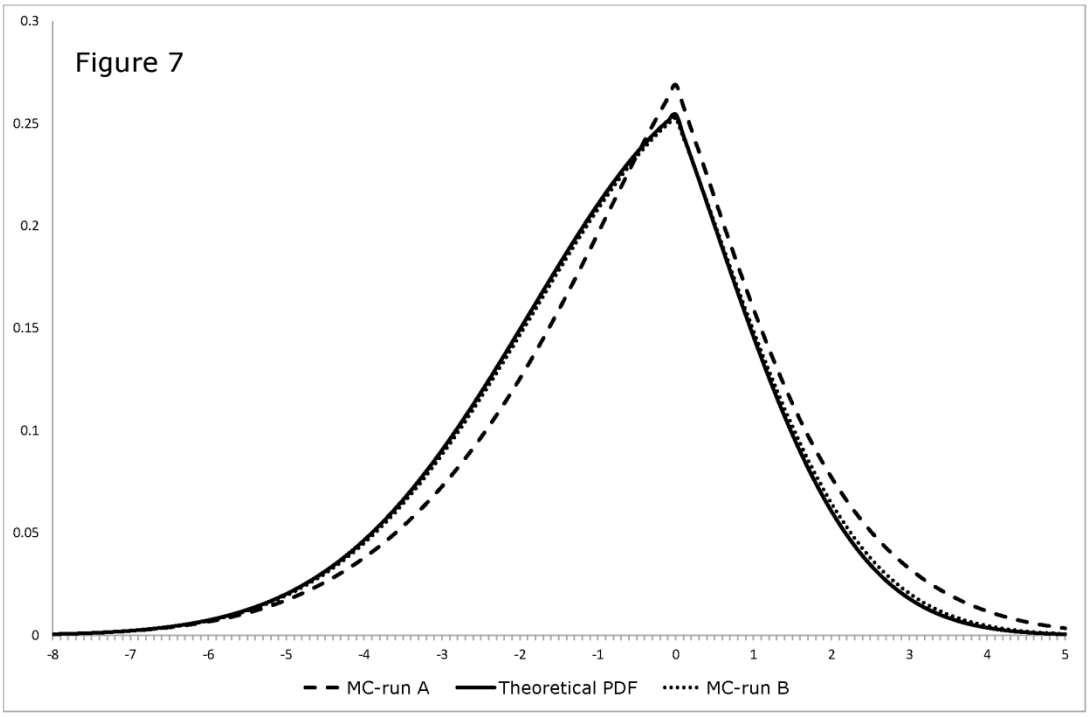
The moments and density graphs implied by the average parameter estimates are shown in Table 2 and Figure 7 respectively. In MC-run A, the main responsible for the probability mass displacement is the positively biased estimate of parameter λ. The density graph obtained from MC-run B is virtually indistinguishable from the theoretical density.

Table 2: Theoretical and estimated moments of the cADHN distribution

Table 2 : Theoretical and estimated moments of the cADHN distribution					
Moment	$E(Z)$	$E(Z^2)$	$\mathrm{Var}(Z)$	$E(Z^3)$	Skewness γ_1
Theoretical value	**-0.797**	**3.937**	**3.302**	**-10.772**	**-0.395**
MC-run A	-0.484	3.878	3.644	-7.329	-0.277
MC-run B	-0.746	3.910	3.353	-10.180	-0.368

Figure 7 : Theoretical and estimated cADHN densities.



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7. Some applications

Apart from the use of this class of distributions as an alternative data modeling option to the use of symmetric and asymmetric Laplace distributions mentioned in the Introduction, some theoretical applications of interest arise, and we end this paper by presenting them.

7.1. Selecting Estimators

Consider an unknown distribution parameter β and two estimators of it, $\hat{\beta}_1$ and $\hat{\beta}_2$, that are both asymptotically normal and consistent, with standard deviations $\sigma_1 < \sigma_2$, which leads to the same inequality for their mean squared errors, and so also for their mean absolute errors: "on average" the absolute error of $\hat{\beta}_1$ is smaller than that of $\hat{\beta}_2$, $E\left|\hat{\beta}_1 - \beta\right| < E\left|\hat{\beta}_2 - \beta\right|$. These absolute errors are asymptotically half normals, $\left|\hat{\beta}_1 - \beta\right| = \sigma_1 H_1$, $\left|\hat{\beta}_2 - \beta\right| = \sigma_2 H_2$ and so their difference follows the ADHN or the cADHN distribution (the latter if they are correlated and we have bivariate normality). Then, given $\sigma_1 < \sigma_2$, and using a known result for $\Phi_2(0, 0; -\delta)$ (see for example David 1953), we can calculate that, for the ADHN distribution

$$P\left(\left|\hat{\beta}_1 - \beta\right| \le \left|\hat{\beta}_2 - \beta\right|\right) = F_{ADHN}(0) = \frac{2}{\pi}\cos^{-1}\left(\frac{\sigma_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right) > 1/2 \qquad [30]$$

and for the cADHN distribution

$$P\left(\left|\hat{\beta}_1 - \beta\right| \le \left|\hat{\beta}_2 - \beta\right|\right) = F_{cADHN}(0) = \frac{1}{\pi}\cos^{-1}\left(\frac{\sigma_1^2 - \sigma_2^2}{s_1 s_2}\right) > 1/2 \qquad [31]$$

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In words, the probability that the lower-variance estimator has smaller absolute error is always greater than $1/2$, or alternatively expressed, it is always more probable than not that the absolute error of the lower-variance estimator is smaller. This is not implied by the expected-value inequality, and it is a distinct, probabilistic argument in favor of using the lower-variance estimator in single-sample inference: even if we don't get to have many samples, we will have higher probability of being closer to the true value by using the lower-variance estimator. Moreover, since we have available the distribution function, we can calculate the probabilities related to any amount of (positive or negative) difference between the two absolute errors. If we attempted such inference using the squared errors, first we would be dealing with wrong measurement units, and second, the distribution of their difference would be the difference of two independent (or correlated) Gamma distributions with different scale parameter (for the independent case, a recursive formula can be found in Moschopoulos 1985).

7.2 Forecasting

The previous results cover also the comparison of two predictors in terms of their absolute forecast errors. But in the context of forecasting, another application of the cADHN distribution arises: assume that we have a normally distributed weakly stationary process in mean deviation form $\{X_t\}$ with standard deviation σ_2, and an optimal linear predictor of it, g_x with standard deviation σ_1. Then we have that $\sigma_1 < \sigma_2$ and that their correlation coefficient equals $\rho = \sigma_1/\sigma_2$ (see for example Granger & Newbold 1986, pp 127-131). Moreover, the variable $|g_t| - |X_t|$ follows a cADHN distribution and we obtain that $P(|g_t| \leq |X_t|) > 1/2$.[3]

[3] The fact that in this case we have $\lambda = \rho$ does not create any special behavior.

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We know that optimal predictors under-predict on average and in absolute terms, due to them having lower variance than the series they predict, i.e. that $E|g_t| < E|X_t|$. The results we obtained from the ADHN and cADHN distributions tell us that under-prediction in absolute terms is more probable than not, for any given sample and for every single prediction. Here too, we can obtain a full range of probabilities connected to specific magnitudes of over- and under-prediction, an important information especially in one-off forecasts, which is a useful, and different piece of information than what we obtain by examining the absolute error, $|g_t - X_t|$, which only deals with the magnitude of the deviation.

7.3 Brownian motion theory

7.3.1. The arcsine law. A result related to the standard linear Brownian motion (or Wiener process) is that its maximizer over the $[0,1]$ interval is distributed according to the arcsine distribution (see for example Mörters and Peres, 2010, ch 5.4). This result, which can be extended to an arbitrary $[t_0, t_0 + h]$ interval, can be proven in a straightforward way using the ADHN distribution. Specifically, let $B(t)$ be a standard linear Brownian motion and its running maximum over $[t_0, t_0 + h]$ be $M(h) = \max_{t_0 \le u \le t_0 + h} B(u)$. $M(h)$ follows a half-normal distribution with scale parameter $\sqrt{h}$. The maximizer of $B(t)$ in $[t_0, t_0 + h]$ is a random variable almost surely uniquely determined by $M^* : B(M^*) = \max_{t_0 \le v \le t_0 + h} B(v)$. Its cumulative distribution function can be expressed as

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$$P\left(M^* \le v\right) = P\left[\max_{t_0 \le u \le v} B(u) > \max_{v \le u \le t_0+h} B(u)\right] = P\left[M\left(v - t_0\right) > M\left(t_0 + h - v\right)\right]$$

$$= P\left[M\left(t_0 + h - v\right) - M\left(v - t_0\right) < 0\right]$$

Since the two running maxima are distributed as (independent) half-normals, the last expression is the CDF of an ADHN random variable with parameters $\lambda = \sqrt{\left[h/\left(v - t_0\right)\right] - 1}$ and $s = \sqrt{h}$, evaluated at zero. From eq. [12] we then obtain

$$P\left(M^* \le v\right) = F_{ADHN}\left(0\right) = \frac{2}{\pi}\cos^{-1}\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right) = \frac{2}{\pi}\cos^{-1}\left(\sqrt{1 - \left[\left(v - t_0\right)/h\right]}\right)$$

and using the relation between arccos and arcsine for arguments in $[0,1]$ we arrive at $P\left(M^* \le v\right) = \left(2/\pi\right)\arcsin\left(\sqrt{\left(v - t_0\right)/h}\right)$ which is the CDF of the arcsine distribution supported in $\left[t_0,\, t_0 + h\right]$.

7.3.2. Asymptotics. The half-normality of the running maximum links the DHN family of distributions with the extreme value theory for Brownian motions. As an illustration assume that a phenomenon is modeled by a system of two Brownian motions which are cross-correlated with a fixed delay, i.e. the joint distribution of $\left(B_1\left(t + h\right), B_2\left(t\right)\right)'$ is bivariate normal with correlation coefficient $\left|\rho\right| < 1$. Then the difference of their running maxima $M_1\left(t + h\right) - M_2\left(t\right) = \sqrt{t + h}H_1 - \sqrt{t}H_2$ follows a cADHN distribution, with parameters $\lambda = \sqrt{1 + h/t}$, $s_1^2 = h + 2t\left(1 - \rho\lambda\right)$, $s_2^2 = h + 2t\left(1 + \rho\lambda\right)$, and ρ. We see that as we look further and further into the future, the shape parameter λ goes to unity, inducing symmetry of the distribution, but the scale parameters go to infinity. To obtain a non-degenarate asymptotic distribution we need to

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standardize by $t^{-1/2}$ and we have $t^{-1/2}\left[M_1(t+h)-M_2(t)\right]=\sqrt{1+h/t}H_1-H_2$ which follows a cADHN distribution that asymptotically becomes symmetric

$$\frac{1}{\sqrt{t}}\left[M_1(t+h)-M_2(t)\right] \xrightarrow[t\to\infty]{d} cADHN\left(s_1=\sqrt{2(1-\rho)},\ s_2=\sqrt{2(1+\rho)},\lambda=1,\ \rho\right)$$

with asymptotic variance

$$\mathrm{AVar}(Z)=2(1-\rho)\ +\ \frac{4}{\pi}\left(\rho\cos^{-1}(\rho)-\sqrt{1-\rho^2}\right).$$

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The Double Half Normal distribution and its extensions.

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TECHNICAL APPENDIX

A. Derivation of the cADHN, ADHN, and DHN probability density functions and cumulative distribution functions, eq. [21], [10] and [1], and [24], [12] and [2] of the main text.

We start at the end, by deriving the probability density function of the difference of two correlated Half-normal random variables that jointly follow a bivariate half-normal distribution, which gives the cADHN density. We then derive the corresponding distribution function without recourse to the underlying half-normal parameters and without imposing the parameter restriction discussed in the main text that is needed in order for the half-normal representation to be valid. In this way, we show that the cADHN distribution remains a proper one even when its parameters go beyond the half-normal representation. The other density and distribution functions presented in the main text can then immediately be derived as limiting cases of the PDF and CDF of the cADHN distribution for the appropriate parameter values.

A1. The distribution of the difference of two non-negative random variables.

https://doi.org/10.26219/heal.aueb.8168

We have two non-negative random variables, v, w and we consider their difference $z = v - w$.

The variable $z = v - w$ is not necessarily non-negative, while v and w are. The distribution function of z is defined as

$$\mathrm{F}_z(z) = \mathrm{P}(v - w \le z) = \mathrm{P}(v \le z + w).$$

This implies the necessary restriction $0 \le z + w$, since v is non-negative.

For $z \ge 0$ this inequality is always satisfied for all values in the domain of w. But if $z \le 0$ then a restriction on the domain of w arises, namely $0 \le z + w \Rightarrow -z \le w$. Therefore the distribution function of z will be piecewise.

For $z \ge 0$ we have

$$\mathrm{F}_z(z^+) = \int_{w=0}^{\infty} \int_{v=0}^{z+w} f_{v,w}(v,w)\, dv dw$$

where $f_{v,w}(v,w)$ is the joint pdf of v and w, while

$$f_z(z^+) = \frac{d}{dz}\mathrm{F}_z(z^+) = \int_{w=0}^{\infty} \left(\frac{\partial}{\partial z} \int_{v=0}^{z+w} f_{v,w}(v,w)\, dv \right) dw$$

$$\Rightarrow f_z(z^+) = \int_{w=0}^{\infty} f_{v,w}(z+w, w)\, dw$$

For $z \le 0$ we have

$$\mathrm{F}_z(z^-) = \int_{w=-z}^{\infty} \int_{v=0}^{z+w} f_{v,w}(v,w)\, dv dw$$

and so

$$\begin{aligned} f_z(z^-) = \frac{d}{dz}\mathrm{F}_z(z^-) &= \frac{d}{dz} \int_{w=-z}^{\infty} \int_{v=0}^{z+w} f_{v,w}(v,w)\, dv dw \\ &= -\int_{v=0}^{z-z} f_{v,w}(v,-z)\, dv \frac{\partial(-z)}{\partial z} + \int_{w=-z}^{\infty} \left(\frac{\partial}{\partial z} \int_{v=0}^{z+w} f_{v,w}(v,w)\, dv \right) dw \\ &= 0 + \int_{w=-z}^{\infty} f_{v,w}(z+w, w)\, dw \end{aligned}$$

Therefore the pdf of $z = v - w$ is

$$f_z(z) = \begin{cases} \int_{w=-z}^{\infty} f_{v,w}(z+w, w)\, dw & z \le 0 \\ \int_{w=0}^{\infty} f_{v,w}(z+w, w)\, dw & z \ge 0 \end{cases} \qquad [1]$$

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while its cumulative distribution function will be

$$F_z(z)=\begin{cases}\int_{-\infty}^{z} f_{z^-}(t)\,\mathrm{d}t & z\le 0\\ \int_{-\infty}^{0} f_{z^-}(t)\,\mathrm{d}t \;+\; \int_{0}^{z} f_{z^+}(t)\,\mathrm{d}t & z\ge 0\end{cases} \qquad [2]$$

A2. The joint pdf of u and ω.

Let the two non-negative random variables be half-normals,

$u \sim HN(\sigma_1)$ and $\omega \sim HN(\sigma_2)$,

The joint pdf of two correlated half-normal random variables can be written (see Kotz *et al.* 2000, p.327, eq. 46.210, or also Psarakis & Panaretos 2001) as:

$$f_{u,\omega}(u,\omega)=\left(\pi\sigma_1\,\sigma_2\sqrt{1-\rho^2}\right)^{-1}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{u}{\sigma_1}\right)^2+\left(\frac{\omega}{\sigma_2}\right)^2-\frac{2\rho u\omega}{\sigma_1\sigma_2}\right]\right\} \qquad [3]$$

$$+\left(\pi\sigma_1\,\sigma_2\sqrt{1-\rho^2}\right)^{-1}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{u}{\sigma_1}\right)^2+\left(\frac{\omega}{\sigma_2}\right)^2+\frac{2\rho u\omega}{\sigma_1\sigma_2}\right]\right\}$$

We substitute $z=u-\omega \Rightarrow u=z+\omega$ to get

$$f_{u,\omega}(u,\omega)=\left(\pi\sigma_1\,\sigma_2\sqrt{1-\rho^2}\right)^{-1}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{z+\omega}{\sigma_1}\right)^2+\left(\frac{\omega}{\sigma_2}\right)^2-\frac{2\rho(z+\omega)\omega}{\sigma_1\sigma_2}\right]\right\}$$

$$+\left(\pi\sigma_1\,\sigma_2\sqrt{1-\rho^2}\right)^{-1}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{z+\omega}{\sigma_1}\right)^2+\left(\frac{\omega}{\sigma_2}\right)^2+\frac{2\rho(z+\omega)\omega}{\sigma_1\sigma_2}\right]\right\}$$

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$$= \left(\pi\sigma_1\,\sigma_2\sqrt{1-\rho^2}\right)^{-1}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\frac{z^2}{\sigma_1^2}+\frac{2}{\sigma_1^2}z\omega+\frac{\omega^2}{\sigma_1^2}+\frac{\omega^2}{\sigma_2^2}-\frac{2\rho}{\sigma_1\sigma_2}z\omega-\frac{2\rho}{\sigma_1\sigma_2}\omega^2\right]\right\}$$
$$+\left(\pi\sigma_1\,\sigma_2\sqrt{1-\rho^2}\right)^{-1}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\frac{z^2}{\sigma_1^2}+\frac{2}{\sigma_1^2}z\omega+\frac{\omega^2}{\sigma_1^2}+\frac{\omega^2}{\sigma_2^2}+\frac{2\rho}{\sigma_1\sigma_2}z\omega+\frac{2\rho}{\sigma_1\sigma_2}\omega^2\right]\right\}$$

$$= \left(\pi\sigma_1\,\sigma_2\sqrt{1-\rho^2}\right)^{-1}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\frac{z^2}{\sigma_1^2}+\left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}-\frac{2\rho}{\sigma_1\sigma_2}\right)\omega^2+\left(\frac{2}{\sigma_1^2}-\frac{2\rho}{\sigma_1\sigma_2}\right)z\omega\right]\right\}$$
$$+\left(\pi\sigma_1\,\sigma_2\sqrt{1-\rho^2}\right)^{-1}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\frac{z^2}{\sigma_1^2}+\left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}+\frac{2\rho}{\sigma_1\sigma_2}\right)\omega^2+\left(\frac{2}{\sigma_1^2}+\frac{2\rho}{\sigma_1\sigma_2}\right)z\omega\right]\right\}$$

$$= \left(\pi\sigma_1\,\sigma_2\sqrt{1-\rho^2}\right)^{-1}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\frac{z^2}{\sigma_1^2}+\frac{\sigma_2^2+\sigma_1^2-2\rho\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2}\omega^2+\frac{2\sigma_2^2-2\rho\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2}z\omega\right]\right\}$$
$$+\left(\pi\sigma_1\,\sigma_2\sqrt{1-\rho^2}\right)^{-1}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\frac{z^2}{\sigma_1^2}+\frac{\sigma_2^2+\sigma_1^2+2\rho\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2}\omega^2+\frac{2\sigma_2^2+2\rho\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2}z\omega\right]\right\}$$

We define $s^2 \equiv \sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2$ and also $\hat{s}^2 \equiv \sigma_2^2+\sigma_1^2+2\rho\sigma_1\sigma_2$.

Taking out the term involving the *z* variable alone, we re-write the joint pdf as

$$f_{u,\omega}(u,\omega) = \left(\pi\sigma_1\,\sigma_2\sqrt{1-\rho^2}\right)^{-1}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\frac{z^2}{\sigma_1^2}\right\}$$
$$\times\left[\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left(\frac{s^2}{\sigma_1^2\sigma_2^2}\omega^2+\frac{2\sigma_2^2-2\rho\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2}z\omega\right)\right\}\right.$$
$$\left.+\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\frac{\hat{s}^2}{\sigma_1^2\sigma_2^2}\omega^2+\frac{2\sigma_2^2+2\rho\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2}z\omega\right]\right\}\right]$$

https://doi.org/10.26219/heal.aueb.8168

$$= \left(\pi\sigma_1\,\sigma_2\sqrt{1-\rho^2}\right)^{-1} \exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\frac{z^2}{\sigma_1^2}\right\}$$
$$\times\left[\exp\left\{-\frac{s^2}{2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}\omega^2 - \frac{\sigma_2^2-\rho\sigma_1\sigma_2}{\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z\omega\right\}\right. \qquad [4]$$
$$\left. + \exp\left\{-\frac{\hat{s}^2}{2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}\omega^2 - \frac{\sigma_2^2+\rho\sigma_1\sigma_2}{\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z\omega\right\}\right]$$

A3. The pdf of z .

For calcualtion purposes we write the pdf as

$$f_z(z) = \begin{cases} C\,\mathrm{I}_1 + C\,\mathrm{I}_2 & z \le 0 \\ \\ C\,\mathrm{I}_3 + C\,\mathrm{I}_4 & z \ge 0 \end{cases} \qquad [5]$$

where

$$C = \left(\pi\sigma_1\,\sigma_2\sqrt{1-\rho^2}\right)^{-1} \exp\left\{-\frac{z^2}{2\sigma_1^2\left(1-\rho^2\right)}\right\}$$

$$\mathrm{I}_1 = \int_{-z}^{\infty} \exp\left\{-\frac{s^2}{2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}\omega^2 - \frac{\sigma_2^2-\rho\sigma_1\sigma_2}{\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z\omega\right\} d\omega$$

$$\mathrm{I}_2 = \int_{-z}^{\infty} \exp\left\{-\frac{\hat{s}^2}{2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}\omega^2 - \frac{\sigma_2^2+\rho\sigma_1\sigma_2}{\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z\omega\right\} d\omega$$

$$\mathrm{I}_3 = \int_{0}^{\infty} \exp\left\{-\frac{s^2}{2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}\omega^2 - \frac{\sigma_2^2-\rho\sigma_1\sigma_2}{\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z\omega\right\} d\omega$$

$$\mathrm{I}_4 = \int_{0}^{\infty} \exp\left\{-\frac{\hat{s}^2}{2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}\omega^2 - \frac{\sigma_2^2+\rho\sigma_1\sigma_2}{\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z\omega\right\} d\omega.$$

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For the four integrals, we use the formulas provided in Gradshteyn and Ryzhik (2007) (p.336). The general formula is

$$\int_m^\infty \exp\left\{-\frac{1}{4\delta}\omega^2-\gamma\omega\right\}d\omega=\sqrt{\pi\delta}e^{\delta\gamma^2}\left[1-\operatorname{erf}\left(\gamma\sqrt{\delta}+\frac{m}{2\sqrt{\delta}}\right)\right]= \tag{6}$$

$$=\sqrt{\pi\delta}e^{\delta\gamma^2}\left[1-\operatorname{erf}\left(\frac{2\gamma\delta+m}{2\sqrt{\delta}}\right)\right]$$

A.3.1. Calculation of I_1 and $C\,\mathrm{I}_1$.

Matching coefficients between I_1 and eq. [6] we have

$$m=-z,\quad \frac{1}{4\delta_1}=\frac{s^2}{2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}\Rightarrow\ \delta_1=\frac{\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}{2s^2}\quad,\quad \gamma_1=\frac{\sigma_2^2-\rho\sigma_1\sigma_2}{\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z$$

$$\sqrt{\delta_1}=\frac{\sqrt{1-\rho^2}\sigma_1\sigma_2}{s\sqrt{2}},\quad \delta_1\gamma_1^2=\frac{\left(\sigma_2^2-\rho\sigma_1\sigma_2\right)^2}{2s^2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z^2,\quad \delta_1\gamma_1=\frac{\sigma_2^2-\rho\sigma_1\sigma_2}{2s^2}z$$

We first calculate the argument of the error function in [6]:

$$\frac{2\gamma_1\delta_1+m}{2\sqrt{\delta_1}}=\frac{2\frac{\sigma_2^2-\rho\sigma_1\sigma_2}{2s^2}z-z}{2\frac{\sqrt{1-\rho^2}\sigma_1\sigma_2}{s\sqrt{2}}}=\frac{\sigma_2^2-\rho\sigma_1\sigma_2-s^2}{\sqrt{2}\sqrt{1-\rho^2}\sigma_1\sigma_2}\left(\frac{z}{s}\right)=\frac{\sigma_2^2-\rho\sigma_1\sigma_2-\sigma_1^2-\sigma_2^2+2\rho\sigma_1\sigma_2}{\sqrt{2}\sqrt{1-\rho^2}\sigma_1\sigma_2}\left(\frac{z}{s}\right)$$

$$=\frac{\sigma_1^2-\rho\sigma_1\sigma_2}{\sqrt{2}\sqrt{1-\rho^2}\sigma_1\sigma_2}\left(\frac{-z}{s}\right).$$

Substituting all in I_1 we obtain

$$\mathrm{I}_1=\sqrt{\pi}\,\frac{\sqrt{1-\rho^2}\sigma_1\sigma_2}{s\sqrt{2}}\exp\left\{\frac{\left(\sigma_2^2-\rho\sigma_1\sigma_2\right)^2}{2s^2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z^2\right\}\left[1+\operatorname{erf}\left(\frac{\sigma_1^2-\rho\sigma_1\sigma_2}{\sqrt{2}\sqrt{1-\rho^2}\sigma_1\sigma_2}\left(\frac{z}{s}\right)\right)\right]$$

where we have used also the fact that the error function is odd.

The following relation holds between the error function and the standard normal cumulative distribution function $\Phi(x)$, $x \in \mathbb{R}$: $2\Phi(x)-1 = \text{erf}\left(\frac{x}{\sqrt{2}}\right)$.

In our case $\frac{x}{\sqrt{2}} = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sqrt{2}\sqrt{1-\rho^2}\sigma_1\sigma_2}\left(\frac{z}{s}\right) \Rightarrow x = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sqrt{1-\rho^2}\sigma_1\sigma_2}\left(\frac{z}{s}\right)$

and so

$$\mathrm{I}_1 = \sqrt{\pi}\frac{\sqrt{1-\rho^2}\sigma_1\sigma_2}{s\sqrt{2}}\exp\left\{\frac{\left(\sigma_2^2 - \rho\sigma_1\sigma_2\right)^2}{2s^2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z^2\right\}2\Phi\left(\lambda_1\left(\frac{z}{s}\right)\right), \quad \lambda_1 = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sqrt{1-\rho^2}\sigma_1\sigma_2}$$

Then we have

$$C\,\mathrm{I}_1 = \left(\pi\sigma_1\ \sigma_2\sqrt{1-\rho^2}\right)^{-1}\exp\left\{-\frac{z^2}{2\sigma_1^2\left(1-\rho^2\right)}\right\}\sqrt{\pi}\frac{\sqrt{1-\rho^2}\sigma_1\sigma_2}{s\sqrt{2}}\exp\left\{\frac{\left(\sigma_2^2 - \rho\sigma_1\sigma_2\right)^2}{2s^2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z^2\right\}2\Phi\left(\lambda_1\left(\frac{z}{s}\right)\right)$$

$$= \frac{1}{s\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2\sigma_1^2\left(1-\rho^2\right)} + \frac{\left(\sigma_2^2 - \rho\sigma_1\sigma_2\right)^2}{2s^2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z^2\right\}2\Phi\left(\lambda_1\left(\frac{z}{s}\right)\right)$$

$$= \frac{1}{s\sqrt{2\pi}}\exp\left\{\frac{-s^2\sigma_2^2 + \sigma_2^4 - 2\sigma_2^2\rho\sigma_1\sigma_2 + \rho^2\sigma_1^2\sigma_2^2}{2s^2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z^2\right\}2\Phi\left(\lambda_1\left(\frac{z}{s}\right)\right)$$

and since $s^2 \equiv \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$

$$= \frac{1}{s\sqrt{2\pi}}\exp\left\{\frac{-\sigma_1^2 - \sigma_2^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 + \rho^2\sigma_1^2}{2s^2\left(1-\rho^2\right)\sigma_1^2}z^2\right\}2\Phi\left(\lambda_1\left(\frac{z}{s}\right)\right)$$

$$= \frac{1}{s\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2s^2}\right\}2\Phi\left(\lambda_1\left(\frac{z}{s}\right)\right),$$

$$\Rightarrow C\,\mathrm{I}_1 = \frac{1}{s}\phi\left(\frac{z}{s}\right)2\Phi\left(\lambda_1\left(\frac{z}{s}\right)\right), \quad \lambda_1 = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sqrt{1-\rho^2}\sigma_1\sigma_2} = \frac{\left(\sigma_1/\sigma_2\right) - \rho}{\sqrt{1-\rho^2}} \qquad [7]$$

A.3.2. Calculation of I_2 and $C\,\mathrm{I}_2$.

Matching coefficients between I_2 and eq. [6] we have

$$m = -z, \quad \frac{1}{4\delta_2} = \frac{\hat{s}^2}{2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2} \Rightarrow \delta_2 = \frac{\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}{2\hat{s}^2}, \quad \gamma_2 = \frac{\sigma_2^2 + \rho\sigma_1\sigma_2}{\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z$$

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$$\sqrt{\delta_2}=\frac{\sqrt{1-\rho^2}\sigma_1\sigma_2}{\hat{s}\sqrt{2}},\quad \delta_2\gamma_2^2=\frac{\left(\sigma_2^2+\rho\sigma_1\sigma_2\right)^2}{2\hat{s}^2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z^2,\qquad \delta_2\gamma_2=\frac{\sigma_2^2+\rho\sigma_1\sigma_2}{2\hat{s}^2}z$$

We first calculate the argument of the error function:

$$\frac{2\gamma_2\delta_2+m}{2\sqrt{\delta_2}}=\frac{2\dfrac{\sigma_2^2+\rho\sigma_1\sigma_2}{2\hat{s}^2}z-z}{2\dfrac{\sqrt{1-\rho^2}\sigma_1\sigma_2}{\hat{s}\sqrt{2}}}=\frac{\sigma_2^2+\rho\sigma_1\sigma_2-\hat{s}^2}{\sqrt{2}\sqrt{1-\rho^2}\sigma_1\sigma_2}\left(\frac{z}{\hat{s}}\right)=\frac{\sigma_2^2+\rho\sigma_1\sigma_2-\sigma_1^2-\sigma_2^2-2\rho\sigma_1\sigma_2}{\sqrt{2}\sqrt{1-\rho^2}\sigma_1\sigma_2}\left(\frac{z}{\hat{s}}\right)$$

$$=\frac{\sigma_1^2+\rho\sigma_1\sigma_2}{\sqrt{2}\sqrt{1-\rho^2}\sigma_1\sigma_2}\left(\frac{-z}{\hat{s}}\right).$$

Substituting all in I_2 we obtain

$$\mathrm{I}_2=\sqrt{\pi}\frac{\sqrt{1-\rho^2}\sigma_1\sigma_2}{\hat{s}\sqrt{2}}\exp\left\{\frac{\left(\sigma_2^2+\rho\sigma_1\sigma_2\right)^2}{2\hat{s}^2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z^2\right\}\left[1+\operatorname{erf}\left(\frac{\sigma_1^2+\rho\sigma_1\sigma_2}{\sqrt{2}\sqrt{1-\rho^2}\sigma_1\sigma_2}\left(\frac{z}{\hat{s}}\right)\right)\right]$$

$$\Rightarrow \mathrm{I}_2=\sqrt{\pi}\frac{\sqrt{1-\rho^2}\sigma_1\sigma_2}{\hat{s}\sqrt{2}}\exp\left\{\frac{\left(\sigma_2^2+\rho\sigma_1\sigma_2\right)^2}{2\hat{s}^2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z^2\right\}2\Phi\left(\lambda_2\left(\frac{z}{\hat{s}}\right)\right),\ \lambda_2=\frac{\sigma_1^2+\rho\sigma_1\sigma_2}{\sqrt{1-\rho^2}\sigma_1\sigma_2}$$

Then

$$C\,\mathrm{I}_2=\left(\pi\sigma_1\ \sigma_2\sqrt{1-\rho^2}\right)^{-1}\exp\left\{-\frac{z^2}{2\sigma_1^2\left(1-\rho^2\right)}\right\}\sqrt{\pi}\frac{\sqrt{1-\rho^2}\sigma_1\sigma_2}{\hat{s}\sqrt{2}}\exp\left\{\frac{\left(\sigma_2^2+\rho\sigma_1\sigma_2\right)^2}{2\hat{s}^2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z^2\right\}2\Phi\left(\lambda_2\left(\frac{z}{\hat{s}}\right)\right)$$

$$=\frac{1}{\hat{s}\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2\sigma_1^2\left(1-\rho^2\right)}+\frac{\left(\sigma_2^2+\rho\sigma_1\sigma_2\right)^2}{2\hat{s}^2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z^2\right\}2\Phi\left(\lambda_2\left(\frac{z}{\hat{s}}\right)\right)$$

$$=\frac{1}{\hat{s}\sqrt{2\pi}}\exp\left\{\frac{-\hat{s}^2\sigma_2^2+\sigma_2^4+2\sigma_2^2\rho\sigma_1\sigma_2+\rho^2\sigma_1^2\sigma_2^2}{2\hat{s}^2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}z^2\right\}2\Phi\left(\lambda_2\left(\frac{z}{\hat{s}}\right)\right)$$

$$=\frac{1}{\hat{s}\sqrt{2\pi}}\exp\left\{\frac{-\hat{s}^2+\sigma_2^2+2\rho\sigma_1\sigma_2+\rho^2\sigma_1^2}{2\hat{s}^2\left(1-\rho^2\right)\sigma_1^2}z^2\right\}2\Phi\left(\lambda_2\left(\frac{z}{\hat{s}}\right)\right)$$

and since $\hat{s}^2\equiv\sigma_2^2+\sigma_1^2+2\rho\sigma_1\sigma_2$,

$$= \frac{1}{\hat{s}\sqrt{2\pi}} \exp\left\{ \frac{-\sigma_1^2 + \rho^2\sigma_1^2}{2\hat{s}^2\left(1-\rho^2\right)\sigma_1^2} z^2 \right\} 2\Phi\left(\lambda_2\left(\frac{z}{\hat{s}} \right) \right)$$

$$= \frac{1}{\hat{s}\sqrt{2\pi}} \exp\left\{ -\frac{z^2}{2\hat{s}^2} \right\} 2\Phi\left(\lambda_2\left(\frac{z}{\hat{s}} \right) \right)$$

$$\Rightarrow C\,\mathrm{I}_2 = \frac{1}{\hat{s}} \phi\left(\frac{z}{\hat{s}} \right) 2\Phi\left(\lambda_2\left(\frac{z}{\hat{s}} \right) \right), \ \lambda_2 = \frac{\sigma_1^2 + \rho\sigma_1\sigma_2}{\sqrt{1-\rho^2}\sigma_1\sigma_2} = \frac{\left(\sigma_1/\sigma_2\right)+\rho}{\sqrt{1-\rho^2}} \qquad [8]$$

A.3.3. Calculation of I_3 and $C\,\mathrm{I}_3$.

For integral I_3 we have $m=0$. Otherwise the individual coefficients are the same with integral I_1. Here the error function argument is

$$\gamma_1\sqrt{\delta_1} = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\left(1-\rho^2\right)\sigma_1^2\sigma_2^2} z \frac{\sqrt{1-\rho^2}\sigma_1\sigma_2}{s\sqrt{2}} = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sqrt{2}\sqrt{1-\rho^2}\sigma_1\sigma_2}\left(\frac{z}{s} \right)$$

Using also the relation with the standard normal cdf and substituting for all the coefficients we obtain

$$\mathrm{I}_3 = \sqrt{\pi}\frac{\sqrt{1-\rho^2}\sigma_1\sigma_2}{s\sqrt{2}} \exp\left\{ \frac{\left(\sigma_2^2 - \rho\sigma_1\sigma_2\right)^2}{2s^2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2} z^2 \right\} \left[2 - 2\Phi\left(\lambda_3\left(\frac{z}{s} \right) \right) \right] \lambda_3 = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sqrt{1-\rho^2}\sigma_1\sigma_2}$$

Then

$$C\,\mathrm{I}_3 = \left(\pi\sigma_1\ \sigma_2\sqrt{1-\rho^2} \right)^{-1} \exp\left\{ -\frac{z^2}{2\sigma_1^2\left(1-\rho^2\right)} \right\} \sqrt{\pi}\frac{\sqrt{1-\rho^2}\sigma_1\sigma_2}{s\sqrt{2}} \exp\left\{ \frac{\left(\sigma_2^2 - \rho\sigma_1\sigma_2\right)^2}{2s^2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2} z^2 \right\} \left[2 - 2\Phi\left(\lambda_3\left(\frac{z}{s} \right) \right) \right]$$

$$= \frac{1}{s\sqrt{2\pi}} \exp\left\{ -\frac{z^2}{2\sigma_1^2\left(1-\rho^2\right)} + \frac{\left(\sigma_2^2 - \rho\sigma_1\sigma_2\right)^2}{2s^2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2} z^2 \right\} \left[2 - 2\Phi\left(\lambda_3\left(\frac{z}{s} \right) \right) \right]$$

$$= \frac{2}{s}\phi\left(\frac{z}{s} \right) - \frac{1}{s}2\phi\left(\frac{z}{s} \right)\Phi\left(\lambda_3\left(\frac{z}{s} \right) \right) = \frac{2}{s}\phi\left(\frac{z}{s} \right)\left[1 - \Phi\left(\lambda_3\left(\frac{z}{s} \right) \right) \right]$$

$$\Rightarrow C\,\mathrm{I}_3 = \frac{1}{s}2\phi\left(\frac{z}{s} \right)\Phi\left(-\lambda_3\left(\frac{z}{s} \right) \right), \ \lambda_3 = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sqrt{1-\rho^2}\sigma_1\sigma_2} = \frac{\left(\sigma_2/\sigma_1\right)-\rho}{\sqrt{1-\rho^2}} \qquad [9]$$

A.3.4. Calculation of I_4 and $C\,\mathrm{I}_4$.

Analogously, for Integral I_4 we have $m=0$, otherwise the individual coefficients are the same with integral I_2. Here the error function argument is

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$$\sqrt{\delta_2}\gamma_2 = \frac{\sqrt{1-\rho^2}\sigma_1\sigma_2}{\hat{s}\sqrt{2}} \frac{\sigma_2^2+\rho\sigma_1\sigma_2}{\left(1-\rho^2\right)\sigma_1^2\sigma_2^2} z = \frac{\sigma_2^2+\rho\sigma_1\sigma_2}{\sqrt{2}\sqrt{1-\rho^2}\sigma_1\sigma_2}\left(\frac{z}{\hat{s}}\right)$$

and so

$$\mathrm{I}_4 = \sqrt{\pi}\frac{\sqrt{1-\rho^2}\sigma_1\sigma_2}{\hat{s}\sqrt{2}} \exp\left\{\frac{\left(\sigma_2^2+\rho\sigma_1\sigma_2\right)^2}{2\hat{s}^2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2} z^2\right\}\left[2-2\Phi\left(\lambda_4\left(\frac{z}{\hat{s}}\right)\right)\right],$$

$$\lambda_4 = \frac{\sigma_2^2+\rho\sigma_1\sigma_2}{\sqrt{1-\rho^2}\sigma_1\sigma_2}.$$

Then

$$C\,\mathrm{I}_4 = \left(\pi\sigma_1\ \sigma_2\sqrt{1-\rho^2}\right)^{-1}\exp\left\{-\frac{z^2}{2\sigma_1^2\left(1-\rho^2\right)}\right\}\sqrt{\pi}\frac{\sqrt{1-\rho^2}\sigma_1\sigma_2}{\hat{s}\sqrt{2}}\exp\left\{\frac{\left(\sigma_2^2+\rho\sigma_1\sigma_2\right)^2}{2\hat{s}^2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2} z^2\right\}\left[2-2\Phi\left(\lambda_4\left(\frac{z}{\hat{s}}\right)\right)\right]$$

$$C\,\mathrm{I}_4 = \frac{1}{\hat{s}\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2\sigma_1^2\left(1-\rho^2\right)}+\frac{\left(\sigma_2^2+\rho\sigma_1\sigma_2\right)^2}{2\hat{s}^2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2} z^2\right\}\left[2-2\Phi\left(\lambda_4\left(\frac{z}{\hat{s}}\right)\right)\right]$$

$$= \frac{2}{\hat{s}}\phi\left(\frac{z}{\hat{s}}\right) - \frac{1}{\hat{s}}2\phi\left(\frac{z}{\hat{s}}\right)\Phi\left(\lambda_4\left(\frac{z}{\hat{s}}\right)\right) = \frac{2}{\hat{s}}\phi\left(\frac{z}{\hat{s}}\right)\left[1-\Phi\left(\lambda_4\left(\frac{z}{\hat{s}}\right)\right)\right]$$

$$\Rightarrow C\,\mathrm{I}_4 = \frac{1}{\hat{s}}2\phi\left(\frac{z}{\hat{s}}\right)\Phi\left(-\lambda_4\left(\frac{z}{\hat{s}}\right)\right),\quad \lambda_4 = \frac{\sigma_2^2+\rho\sigma_1\sigma_2}{\sqrt{1-\rho^2}\sigma_1\sigma_2} = \frac{\left(\sigma_2/\sigma_1\right)+\rho}{\sqrt{1-\rho^2}} \qquad [10]$$

Inserting [7], [8], [9] and [10] into [5] we obtain

$$f_z(z) = \begin{cases} \frac{1}{s}2\phi(z/s)\Phi(\lambda_1 z/s) + \frac{1}{\hat{s}}2\phi(z/\hat{s})\Phi(\lambda_2 z/\hat{s}) & z \le 0 \\ \frac{1}{s}2\phi(z/s)\Phi(-\lambda_3 z/s) + \frac{1}{\hat{s}}2\phi(z/\hat{s})\Phi(-\lambda_4 z/\hat{s}) & z \ge 0 \end{cases} \qquad [11]$$

with

$$s^2 = \sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2\ ,\ \hat{s}^2 = \sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2\text{, and setting } \lambda \equiv \sigma_1/\sigma_2$$

$$\lambda_1 = \frac{\lambda-\rho}{\sqrt{1-\rho^2}}\ ,\ \lambda_2 = \frac{\lambda+\rho}{\sqrt{1-\rho^2}},\ \lambda_3 = \frac{\lambda^{-1}-\rho}{\sqrt{1-\rho^2}},\ \lambda_4 = \frac{\lambda^{-1}+\rho}{\sqrt{1-\rho^2}}$$

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which is the cADHN density with $s^2 = s_1^2$, $\hat{s}^2 = s_2^2$., **i.e. eq. [21] of the main text.**

A.3.5 : The densities of the ADHN and DHN distributions.

Setting $\rho = 0$, i.e. considering uncorrelated half-normals, we get $s_1 = s_2 = s$, $\lambda_1 = \lambda_2 = \lambda$, $\lambda_3 = \lambda_4 = \lambda^{-1}$. Then [11] becomes

$$f_z(z) = \begin{cases} \frac{4}{s}\phi(z/s)\Phi(\lambda z/s) & z \le 0 \\ \frac{4}{s}\phi(z/s)\Phi(-\lambda^{-1} z/s) & z \ge 0 \end{cases}$$

which is the ADHN density, **i.e. eq. [10] of the main text.**

Moreover, setting $\lambda = 1$, i.e. considering uncorrelated half-normals with the same scale parameter, we obtain

$$f_z(z) = \begin{cases} \frac{4}{s}\phi(z/s)\Phi(z/s) & z \le 0 \\ \frac{4}{s}\phi(z/s)\Phi(-z/s) & z \ge 0 \end{cases} \Rightarrow f_z(z) = \frac{4}{s}\phi(z/s)\Phi(-|z|/s)$$

which is the density of the DHN distribution, **i.e. eq. [1] of the main text.**

A4. The cdf of the cADHN distribution.

a. $z \le 0$

$$\mathrm{F}_z(z^-) = \int_{-\infty}^{z} \frac{1}{s} 2\phi(z/s)\Phi(\lambda_1 z/s)dz + \int_{-\infty}^{z} \frac{1}{\hat{s}} 2\phi(z/\hat{s})\Phi(\lambda_2 z/\hat{s})dz$$

$$= \int_{-\infty}^{z/s} 2\phi(z/s)\Phi(\lambda_1 z/s)d(z/s) + \int_{-\infty}^{z/\hat{s}} 2\phi(z/\hat{s})\Phi(\lambda_2 (z/\hat{s}))d(z/\hat{s})$$

$$\Rightarrow P(Z \le z \le 0) = \mathrm{F}_z(z^-) = G(z/s;\lambda_1) + G(z/\hat{s};\lambda_2) \qquad [12]$$

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where $G(\cdot;\cdot)$ denotes the cdf of the skew normal distribution, and we will use the expression of the skew-normal CDF in terms of the bivariate normal integral $G(z/s;\lambda) = \Phi_2\left(z/s, 0, \frac{-\lambda}{\sqrt{1+\lambda^2}}\right)$ (see for example Azzalini and Capitanio 2014, p. 34). So

$$P(Z \le z \le 0) = 2\Phi_2\left(z/s, 0; -\delta_1\right) + 2\Phi_2\left(z/\hat{s}, 0; -\delta_2\right) \qquad [13]$$

$$\delta_1 = \frac{\lambda_1}{\sqrt{1+\lambda_1^2}} \quad , \quad \delta_2 = \frac{\lambda_2}{\sqrt{1+\lambda_2^2}}$$

b. $z \ge 0$

$$P(0 \le Z \le z) = 2\Phi_2\left(0, 0; -\delta_1\right) + 2\Phi_2\left(0, 0; -\delta_2\right)$$

$$+ \int_0^{z/s} 2\phi(z/s)\Phi\left(\lambda_3 z/s\right)d\left(z/s\right) + \int_0^{z/\hat{s}} 2\phi(z/\hat{s})\Phi\left(\lambda_4\left(z/\hat{s}\right)\right)d\left(z/\hat{s}\right)$$

Using also a known result (eg David 1953)

$$P(0 \le Z \le z) = \left[1 - \frac{1}{\pi}\cos^{-1}\left(-\delta_1\right)\right] + \left[1 - \frac{1}{\pi}\cos^{-1}\left(-\delta_2\right)\right]$$

$$+ \left[2\Phi_2\left(z/s, 0; \delta_3\right) - 2\Phi_2\left(0, 0; \delta_3\right)\right] + \left[2\Phi_2\left(z/\hat{s}, 0; \delta_4\right) - 2\Phi_2\left(0, 0; \delta_4\right)\right]$$

with $\delta_3 = \frac{\lambda_3}{\sqrt{1+\lambda_3^2}}$, $\delta_4 = \frac{\lambda_4}{\sqrt{1+\lambda_4^2}}$

Continuing

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$$P(0 \le Z \le z) = 1 - 1 + \frac{1}{\pi}\cos^{-1}(\delta_1) + 1 - 1 + \frac{1}{\pi}\cos^{-1}(\delta_2)$$

$$+ \quad 2\Phi_2(z/s, 0; \delta_3) - \left[1 - \frac{1}{\pi}\cos^{-1}(\delta_3)\right]$$

$$+ \quad 2\Phi_2(z/\hat{s}, 0; \delta_4) - \left[1 - \frac{1}{\pi}\cos^{-1}(\delta_4)\right]$$

$$\Rightarrow P(0 \le Z \le z) = 2\Phi_2(z/s, 0; \delta_3) + 2\Phi_2(z/\hat{s}, 0; \delta_4) - 2$$

$$+ \frac{1}{\pi}\left[\cos^{-1}(\delta_1) + \cos^{-1}(\delta_2) + \cos^{-1}(\delta_3) + \cos^{-1}(\delta_4)\right]$$

Using the relation for summing two arccosines

$$P(0 \le Z \le z) = 2\Phi_2(z/s, 0; \delta_3) + 2\Phi_2(z/\hat{s}, 0; \delta_4) - 2$$

$$+ \frac{1}{\pi}\left[\cos^{-1}\left(\delta_1\delta_3 - \sqrt{(1-\delta_1^2)(1-\delta_3^2)}\right) + \cos^{-1}\left(\delta_2\delta_4 - \sqrt{(1-\delta_2^2)(1-\delta_4^2)}\right)\right]$$

We have

$$\delta_1\delta_3 - \sqrt{(1-\delta_1^2)(1-\delta_3^2)} = \frac{\lambda_1\lambda_3}{\sqrt{1+\lambda_1^2}\sqrt{1+\lambda_3^2}} - \sqrt{\left(1 - \frac{\lambda_1^2}{1+\lambda_1^2}\right)}\sqrt{\left(1 - \frac{\lambda_3^2}{1+\lambda_3^2}\right)}$$

$$= \frac{\frac{(\lambda - \rho)(\lambda^{-1} - \rho)}{1-\rho^2} - 1}{\frac{1+\lambda^2 - 2\lambda\rho}{1-\rho^2}} = \frac{\lambda(1 - \lambda\rho - \rho\lambda^{-1} + \rho^2 - 1 + \rho^2)}{1+\lambda^2 - 2\lambda\rho}$$

$$= \frac{(-\lambda^2\rho - \rho + 2\lambda\rho^2)}{1+\lambda^2 - 2\lambda\rho} = -\rho$$

Also

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$$\delta_2\delta_4 - \sqrt{\left(1-\delta_2^2\right)\left(1-\delta_4^2\right)} = \frac{\lambda_2\lambda_4}{\sqrt{1+\lambda_2^2}\sqrt{1+\lambda_4^2}} - \sqrt{\left(1-\frac{\lambda_2^2}{1+\lambda_2^2}\right)}\sqrt{\left(1-\frac{\lambda_4^2}{1+\lambda_4^2}\right)}$$

$$= \frac{\frac{\left(\lambda\ +\rho\right)\left(\lambda^{-1}+\rho\right)}{1-\rho^2}-1}{\frac{1+\lambda^2+2\lambda\rho}{\lambda\left(1-\rho^2\right)}} = \frac{\lambda\left(1+\lambda\rho+\rho\lambda^{-1}+\rho^2-1+\rho^2\right)}{1+\lambda^2+2\lambda\rho}$$

$$= \frac{\left(\lambda^2\rho+\rho+2\lambda\rho^2\right)}{1+\lambda^2+2\lambda\rho} = \rho$$

Inserting back we get

$$P(0 \le Z \le z) = 2\Phi_2\left(z/s, 0; \delta_3\right) + 2\Phi_2\left(z/\hat{s}, 0; \delta_4\right) - 2$$

$$+ \frac{1}{\pi}\left[\cos^{-1}\left(-\rho\right)+\cos^{-1}\left(\rho\right)\right]$$

$$= 2\Phi_2\left(z/s, 0; \delta_3\right) + 2\Phi_2\left(z/\hat{s}, 0; \delta_4\right) - 2$$

$$+ \frac{1}{\pi}\left[\pi-\cos^{-1}\left(\rho\right)+\cos^{-1}\left(\rho\right)\right]$$

$$P(0 \le Z \le z) = 2\Phi_2\left(z/s, 0; \delta_3\right) + 2\Phi_2\left(z/\hat{s}, 0; \delta_4\right) - 1$$

As $z \to \infty \Rightarrow P(0 \le Z \le z) \to 2\Phi(0) + 2\Phi(0) - 1 = 1$

as we need.

So the CDF of the cADHN distribution is

$$F_z(z) = \begin{cases} 2\Phi_2\left(z/s,\ 0;\ -\delta_1\right) + 2\Phi_2\left(z/\hat{s},\ 0;\ -\delta_2\right) & z \le 0 \\ 2\Phi_2\left(z/s,\ 0;\ \delta_3\right) + 2\Phi_2\left(z/\hat{s},\ 0;\ \delta_4\right) - 1 & z \ge 0 \end{cases} \qquad [14]$$

with

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$$\delta_1 = \frac{\lambda_1}{\sqrt{1+\lambda_1^2}} \quad , \quad \delta_2 = \frac{\lambda_2}{\sqrt{1+\lambda_2^2}}, \; \delta_3 = \frac{\lambda_3}{\sqrt{1+\lambda_3^2}} \quad , \quad \delta_4 = \frac{\lambda_4}{\sqrt{1+\lambda_4^2}}$$

$$\lambda_1 = \frac{\lambda-\rho}{\sqrt{1-\rho^2}} \quad , \; \lambda_2 = \frac{\lambda+\rho}{\sqrt{1-\rho^2}}, \; \lambda_3 = \frac{\lambda^{-1}-\rho}{\sqrt{1-\rho^2}}, \; \lambda_4 = \frac{\lambda^{-1}+\rho}{\sqrt{1-\rho^2}}$$

which is **eq. [24] of the main text.**

Now, in deriving the CDF we did not use the fact that the parameter λ is the ratio of two underlying scale parameters (the positivity constraint is part of the definition of the cADHN distribution), and also, we did not need to impose the constraint $\text{sign}\{\hat{s}-s\} = \text{sign}\{\rho\}$ which is needed for the half-normal representation. Therefore the cADHN distribution nests the half-normal representation.

A4.1. The CDFs of the ADHN and DHN distributions.

Setting $\rho = 0$ we get $s_1 = s_2 = s, \; \lambda_1 = \lambda_2 = \lambda, \; \lambda_3 = \lambda_4 = \lambda^{-1}$, and also

$$\delta_1 = \delta_2 = \frac{\lambda}{\sqrt{1+\lambda^2}} = \delta, \quad \delta_3 = \delta_4 = \frac{\lambda^{-1}}{\sqrt{1+\lambda^{-2}}} = \frac{1}{\sqrt{1+\lambda^2}} = \lambda^{-1}\delta .$$

So [14] becomes

$$F_z(z) = \begin{cases} 4\Phi_2\left(z/s, \, 0; \, -\delta\right) & z \le 0 \\ 4\Phi_2\left(z/s, \, 0; \, \lambda^{-1}\delta\right) - 1 & z \ge 0 \end{cases}$$

which is eq. [12] of the main text.

Moreover, setting $\lambda = 1$ we get $\delta = 1/\sqrt{2}$ and so

$$F_z(z) = \begin{cases} 4\Phi_2\left(z/s, \, 0; \, -1/\sqrt{2}\right) & z \le 0 \\ 4\Phi_2\left(z/s, \, 0; \, 1/\sqrt{2}\right) - 1 & z \ge 0 \end{cases}$$

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which is the CDF of the DHN distribution, i.e. **eq. [2] of the main text.**

B. Equations [4] and [23]

B1. Eq. [4] : $\min\{\sigma H_1, \sigma H_2\} = |Z|,\ Z \sim \text{DHN}(\sigma)$

The CDF of a Half-normal with scale parameter σ is $F_{HN}(x;\sigma) = 2\Phi(x/\sigma) - 1$. The CDF of the minimum $m = \min\{\sigma H_1, \sigma H_2\}$ of two independent half-normals with same scale parameter is therefore

$$F_m(m) = 1 - \left[1 - F_{HN}(m;\sigma)\right]^2 = 1 - \left(1 - 2\Phi(m/\sigma) + 1\right)^2 = 1 - 4\left[\Phi(-m/\sigma)\right]^2$$

Differentiating we obtain the density of the minimum,

$$f_m(m) = \frac{8}{\sigma}\phi(m/\sigma)\Phi(-m/\sigma), \quad m \geq 0 \ .$$

We turn to the absolute value of a DHN random variable $|Z|$. The density is symmetric around zero and we fold it at zero. We immediately have that

$$f_{DHN}(z) = \frac{4}{s}\phi(z/s)\Phi(-|z|/s) \Rightarrow f_{|Z|}(|z|) = \frac{8}{s}\phi(|z|/s)\Phi(-|z|/s),\ |z| \geq 0$$

Setting $s = \sigma$ the density becomes the same, and over the same domain, as the density of the minimum, which is what **eq. [4] of the main text** asserts.

B2. Eq. [23] : $\min\{\sigma_1 H_1, \sigma_2 H_2\} = |Z|$, where Z follows a cADHN distribution with parameters $s_1 = \sigma_1$, $s_2 = \sigma_2$ and $\lambda = 1$, $\rho = \left(\sigma_2^2 - \sigma_1^2\right)/\left(\sigma_2^2 + \sigma_1^2\right)$.

The CDF of the minimum $m = \min\{\sigma_1 H_1, \sigma_2 H_2\}$ of two independent half-normals with different scale parameters is

https://doi.org/10.26219/heal.aueb.8168

$$F_m(m) = 1 - \left[1 - F_{HN}(m;\sigma_1)\right]\left[1 - F_{HN}(m;\sigma_2)\right]$$

$$= F_{HN}(m;\sigma_1) + F_{HN}(m;\sigma_2) - F_{HN}(m;\sigma_1)F_{HN}(m;\sigma_2)$$

Differentiating to obtain the density we have

$$f_m(m) = f_{HN}(m;\sigma_1) + f_{HN}(m;\sigma_2) - f_{HN}(m;\sigma_1)F_{HN}(m;\sigma_2) - f_{HN}(m;\sigma_2)F_{HN}(m;\sigma_1)$$

Using the actual functional forms we have

$$f_m(m) = \frac{2}{\sigma_1}\phi(m/\sigma_1) + \frac{2}{\sigma_2}\phi(m/\sigma_2) - \frac{2}{\sigma_1}\phi(m/\sigma_1)\left[2\Phi(m/\sigma_2) - 1\right] - \frac{2}{\sigma_2}\phi(m/\sigma_2)\left[2\Phi(m/\sigma_1) - 1\right]$$

$$= \frac{4}{\sigma_1}\phi(m/\sigma_1)\Phi(-m/\sigma_2) + \frac{4}{\sigma_2}\phi(m/\sigma_2)\Phi(-m/\sigma_1),\ m \geq 0$$

We turn to the absolute value of a cADHN random variable. Its density is

$$f_{cADHN}(z) = \begin{cases} \frac{2}{s_1}\phi(z/s_1)\Phi(\lambda_1 z/s_1) + \frac{2}{s_2}\phi(z/s_2)\Phi(\lambda_2 z/s_2) & z \leq 0 \\ \frac{2}{s_1}\phi(z/s_1)\Phi(-\lambda_3 z/s_1) + \frac{2}{s_2}\phi(z/s_2)\Phi(-\lambda_4 z/s_2) & z \geq 0 \end{cases}$$

Setting $\lambda = 1$ this density becomes symmetric around zero, and also we get

$$\lambda_1 = \lambda_3 = \frac{1-\rho}{\sqrt{1-\rho^2}},\quad ,\ \lambda_2 = \lambda_4 = \frac{1+\rho}{\sqrt{1-\rho^2}},\ \lambda_3 = \frac{\lambda^{-1}-\rho}{\sqrt{1-\rho^2}},\ \lambda_4 = \frac{\lambda^{-1}+\rho}{\sqrt{1-\rho^2}}$$

Setting also $s_1 = \sigma_1, s_2 = \sigma_2$ we obtain

https://doi.org/10.26219/heal.aueb.8168

$$f_{cADHN}(z)=\begin{cases}\frac{2}{\sigma_1}\phi(z/\sigma_1)\Phi(\lambda_1 z/\sigma_1)+\frac{2}{\sigma_2}\phi(z/\sigma_2)\Phi(\lambda_2 z/\sigma_2) & z\le 0\\ \frac{2}{\sigma_1}\phi(z/\sigma_1)\Phi(-\lambda_1 z/\sigma_1)+\frac{2}{\sigma_2}\phi(z/\sigma_2)\Phi(-\lambda_2 z/\sigma_2) & z\ge 0\end{cases}$$

which can be written

$$f_{cADHN}(z)=\frac{2}{\sigma_1}\phi(z/\sigma_1)\Phi(-\lambda_1|z|/\sigma_1)+\frac{2}{\sigma_2}\phi(z/\sigma_2)\Phi(-\lambda_2|z|/\sigma_2)$$

This is a symmetric around zero density, so folding it at zero we obtain

$$f_{|Z|}(|z|)=\frac{4}{\sigma_1}\phi(|z|/\sigma_1)\Phi(-\lambda_1|z|/\sigma_1)+\frac{4}{\sigma_2}\phi(|z|/\sigma_2)\Phi(-\lambda_2|z|/\sigma_2)$$

For this density to be identical to the density of the minimum of the independent half normals we must have

$$\frac{\lambda_1}{\sigma_1}=\frac{1}{\sigma_1},\quad \frac{\lambda_2}{\sigma_2}=\frac{1}{\sigma_1}.$$

We have

$$\frac{\lambda_1}{\sigma_1}=\frac{1}{\sigma_2}\Rightarrow\frac{1-\rho}{\sigma_1\sqrt{1-\rho^2}}=\frac{1}{\sigma_2}\Rightarrow(\sigma_2/\sigma_1)^2(1-\rho)^2=1-\rho^2$$

$$\Rightarrow(\sigma_2/\sigma_1)^2-2(\sigma_2/\sigma_1)^2\rho+(\sigma_2/\sigma_1)^2\rho^2=1-\rho^2$$

$$\Rightarrow\left[(\sigma_2/\sigma_1)^2+1\right]\rho^2-2(\sigma_2/\sigma_1)^2\rho+\left[(\sigma_2/\sigma_1)^2-1\right]=0$$

The roots of this second degree polynomial in ρ are

https://doi.org/10.26219/heal.aueb.8168

$$\rho_{1,2} = \frac{2\left(\sigma_2/\sigma_1\right)^2 \pm \sqrt{4\left(\sigma_2/\sigma_1\right)^4 - 4\left[\left(\sigma_2/\sigma_1\right)^2 + 1\right]\left[\left(\sigma_2/\sigma_1\right)^2 - 1\right]}}{2\left[\left(\sigma_2/\sigma_1\right)^2 + 1\right]}$$

$$= \frac{2\left(\sigma_2/\sigma_1\right)^2 \pm \sqrt{4\left(\sigma_2/\sigma_1\right)^4 - 4\left[\left(\sigma_2/\sigma_1\right)^4 - 1\right]}}{2\left[\left(\sigma_2/\sigma_1\right)^2 + 1\right]} = \frac{2\left(\sigma_2/\sigma_1\right)^2 \pm \sqrt{4}}{2\left[\left(\sigma_2/\sigma_1\right)^2 + 1\right]}$$

$$= \frac{\left(\sigma_2/\sigma_1\right)^2 \pm 1}{\left[\left(\sigma_2/\sigma_1\right)^2 + 1\right]}$$

The one root is $\rho_1 = 1$. the second root is

$$\rho_2 = \frac{\left(\sigma_2/\sigma_1\right)^2 - 1}{\left(\sigma_2/\sigma_1\right)^2 + 1} = \frac{\sigma_2^2 - \sigma_1^2}{\sigma_2^2 + \sigma_1^2}$$

So the minimum of two independent half normals with different scale parameters σ_1, σ_2 is distributed as the absolute value of a cADHN random variable with parameters $s_1 = \sigma_1, s_2 = \sigma_2$ and $\lambda = 1$, $\rho = \left(\sigma_2^2 - \sigma_1^2\right) \Big/ \left(\sigma_2^2 + \sigma_1^2\right)$, which is what **eq. [23] of the main text,** asserts.

C. MOMENTS of the DHN family of distributions

The densities of the DHN, ADHN and cADHN distributions contain skew-normal functional forms, over restricted/truncated domains. Such distributions have been studied in Jamalizadeh *et al.* (2009) and based on their work we can immediately obtain the Moment Generating Functions for the DHN family. But directly integrating the densities is a much more efficient way to obtain the moments, since the solutions to the relevant integrals are available in closed form, for example in Owen (1980).

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The general expressions that we will need from this paper are the following (to the left the number of the expression in the original paper)

$$[1,011.2] \qquad \int_0^\infty x\phi(x)\Phi(ax)\,dx = \frac{1}{2\sqrt{2\pi}}\left[1+\frac{a}{\sqrt{1+a^2}}\right] \qquad [15]$$

$$[1,012.3] \qquad \int_0^\infty x^2\phi(x)\Phi(ax)\,dx = \frac{1}{4} + \frac{1}{2\pi}\arctan a + \frac{a}{2\pi\left(1+a^2\right)} \qquad [16]$$

$$[1,013.3] \qquad \int_0^\infty x^3\phi(x)\Phi(ax)\,dx = \frac{1}{2\sqrt{2\pi}}\left(2+\frac{2a^3+3a}{\left(1+a^2\right)^{3/2}}\right) \qquad [17]$$

$$[1,014] \qquad \int_0^\infty x^4\phi(x)\Phi(ax)\,dx = \frac{3}{4} + \frac{3}{2\pi}\arctan a + \frac{a\left(3a^2+5\right)}{2\pi\left(1+a^2\right)^2} \qquad [18]$$

C1. Moments of the DHN distribution (eq. [7], [8]. [9] of the main text)

We have

$$E\left(Z^k\right) = \int_{-\infty}^0 z^k\,(4/s)\,\phi(z/s)\,\Phi(z/s)\,\mathrm{d}z + \int_0^\infty z^k\,(4/s)\,\phi(z/s)\,\Phi(-z/s)\,\mathrm{d}z$$

Changing the variable of integration to $w = z/s$ we obtain

$$E\left(Z^k\right) = \int_{-\infty}^0 (ws)^k\,4\phi(w)\,\Phi(w)\,\mathrm{d}w + \int_0^\infty (ws)^k\,4\phi(w)\,\Phi(-w)\,\mathrm{d}w$$

$$= 4s^k\int_{-\infty}^0 w^k\phi(w)\,\Phi(w)\,\mathrm{d}w + 4s^k\int_0^\infty w^k\phi(w)\,\Phi(-w)\,\mathrm{d}w$$

and using basic properties of integration we have

$$\ldots = (-1)^k\,4s^k\int_0^\infty w^k\phi(w)\,\Phi(-w)\,\mathrm{d}w + 4s^k\int_0^\infty w^k\phi(w)\,\Phi(-w)\,\mathrm{d}w$$

$$= \left[(-1)^k+1\right]4s^k\int_0^\infty w^k\phi(w)\,\Phi(-w)\,\mathrm{d}w$$

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We can now apply the expressions from Owen(1980). Here $a=-1$.

We have

$$k=1:\ E\left(Z_{DHN}\right)=0$$

$$k=2:\ E\left(Z_{DHN}^2\right) = 8s^2\int_0^\infty w^2\phi(w)\Phi(-w)\mathrm{d}w = 8s^2\left[\frac{1}{4}+\frac{1}{2\pi}\arctan(-1)-\frac{1}{4\pi}\right]$$

$$= 8s^2\left[\frac{1}{4}-\frac{1}{2\pi}\frac{\pi}{4}-\frac{1}{4\pi}\right] = 2s^2\left[1-\frac{1}{2}-\frac{1}{\pi}\right]$$

$$= \left(\frac{\pi-2}{\pi}\right)s^2$$

$$k=3:\ E\left(Z_{DHN}^3\right)=0$$

$$k=4:\ E\left(Z_{DHN}^4\right) = 8s^4\int_0^\infty w^4\phi(w)\Phi(-w)\mathrm{d}w = 8s^4\left[\frac{3}{4}+\frac{3}{2\pi}\arctan(-1)-\frac{1}{\pi}\right]$$

$$= 8s^4\left[\frac{3}{4}-\frac{3}{2\pi}\frac{\pi}{4}-\frac{1}{\pi}\right] = 8s^4\left[\frac{3}{4}-\frac{3}{8}-\frac{1}{\pi}\right] = \left[6-3-\frac{8}{\pi}\right]s^4$$

$$= \left(\frac{3\pi-8}{\pi}\right)s^4$$

The excess kurtosis coefficient is

$$\gamma_{2,DHN} = \frac{E\left(Z_{DHN}^4\right)}{\left[E\left(Z_{DHN}^2\right)\right]^2}-3 = \frac{\left(\frac{3\pi-8}{\pi}\right)s^4}{\left(\frac{\pi-2}{\pi}\right)^2 s^4}-3 = \frac{(3\pi-8)\pi}{(\pi-2)^2}-3$$

$$= \frac{3\pi^2-8\pi-3\pi^2+12\pi-12}{(\pi-2)^2} = \frac{4(\pi-3)}{(\pi-2)^2}\approx 0.4345$$

The above are equations [7], [8], [9] of the main text.

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C2. Moments of the ADHN distribution (eq. [15], [16], [17], [18] of the main text).

We have

$$E\left(Z^k\right)=\int_{-\infty}^{0} z^k\,(4/s)\,\phi(z/s)\,\Phi(\lambda z/s)\,\mathrm{d}z+\int_{0}^{\infty} z^k\,(4/s)\,\phi(z/s)\,\Phi\left(-\lambda^{-1}z/s\right)\mathrm{d}z$$

Changing the variable of integration to $w=z/s$ we obtain

$$E\left(Z^k\right)=\int_{-\infty}^{0}(ws)^k\,4\phi(w)\,\Phi(\lambda w)\,\mathrm{d}w\;+\;\int_{0}^{\infty}(ws)^k\,4\phi(w)\,\Phi\left(-\lambda^{-1}w\right)\mathrm{d}w$$

$$=4s^k\int_{-\infty}^{0}w^k\phi(w)\,\Phi(\lambda w)\,\mathrm{d}w\;+\;4s^k\int_{0}^{\infty}w^k\phi(w)\,\Phi\left(-\lambda^{-1}w\right)\mathrm{d}w$$

$$=(-1)^k\,4s^k\int_{0}^{\infty}w^k\phi(w)\,\Phi(-\lambda w)\,\mathrm{d}w\;+\;4s^k\int_{0}^{\infty}w^k\phi(w)\,\Phi\left(-\lambda^{-1}w\right)\mathrm{d}w$$

For the first integral we have $a=-\lambda$ while for the second $a=-\lambda^{-1}$. Applying the expressions for the integrals

$$k=1:\;E\left(Z_{ADHN}\right)=-4s\int_{0}^{\infty}w\phi(w)\,\Phi(-\lambda w)\,\mathrm{d}w\;+\;4s\int_{0}^{\infty}w\phi(w)\,\Phi\left(-\lambda^{-1}w\right)\mathrm{d}w$$

$$=-4s\frac{1}{2\sqrt{2\pi}}\left[1-\frac{\lambda}{\sqrt{1+\lambda^2}}\right]\;+\;4s\frac{1}{2\sqrt{2\pi}}\left[1-\frac{\lambda^{-1}}{\sqrt{1+\lambda^{-2}}}\right]$$

$$=4s\frac{1}{2\sqrt{2\pi}}\left[-1+\frac{\lambda}{\sqrt{1+\lambda^2}}+1-\frac{\lambda^{-1}}{\sqrt{1+\lambda^{-2}}}\right]\;=\;2s\frac{1}{\sqrt{2\pi}}\left[\frac{\lambda}{\sqrt{1+\lambda^2}}-\frac{1}{\sqrt{1+\lambda^2}}\right]$$

$$=\sqrt{\frac{2}{\pi}}\frac{\lambda-1}{\sqrt{1+\lambda^2}}s$$

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$$k=2:\ E\left(Z^2_{ADHN}\right) = 4s^2\int_0^\infty w^2\phi(w)\Phi(-\lambda w)\,\mathrm{d}w + 4s^2\int_0^\infty w^k\phi(w)\Phi\left(-\lambda^{-1}w\right)\mathrm{d}w$$

$$= 4s^2\left[\int_0^\infty w^2\phi(w)\Phi(-\lambda w)\,\mathrm{d}w + \int_0^\infty w^2\phi(w)\Phi\left(-\lambda^{-1}w\right)\mathrm{d}w\right]$$

$$= 4s^2\left[\frac{1}{4} + \frac{1}{2\pi}\arctan(-\lambda) - \frac{\lambda}{2\pi\left(1+\lambda^2\right)} + \frac{1}{4} + \frac{1}{2\pi}\arctan\left(-\lambda^{-1}\right) - \frac{\lambda^{-1}}{2\pi\left(1+\lambda^{-2}\right)}\right]$$

$$= 4s^2\left[\frac{2}{4} - \frac{1}{2\pi}\left[\arctan(\lambda)+\arctan\left(\lambda^{-1}\right)\right] - \frac{\lambda}{2\pi\left(1+\lambda^2\right)} - \frac{\lambda}{2\pi\left(1+\lambda^2\right)}\right]$$

Since $\lambda>0$, we can apply the trigonometric identity $\arctan\left(\lambda^{-1}\right)=\frac{\pi}{2}-\arctan(\lambda)$. We obtain

$$E\left(Z^2_{ADHN}\right) = 4s^2\left[\frac{2}{4} - \frac{1}{2\pi}\left[\arctan(\lambda)+\frac{\pi}{2}-\arctan(\lambda)\right] - \frac{2\lambda}{2\pi\left(1+\lambda^2\right)}\right]$$

$$= 4s^2\left[\frac{2}{4} - \frac{1}{4} - \frac{\lambda}{\pi\left(1+\lambda^2\right)}\right] = \left(1-\frac{4}{\pi}\frac{\lambda}{\left(1+\lambda^2\right)}\right)s^2$$

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$$k=3:\ E\left(Z^3_{ADHN}\right) = -4s^3\int_0^\infty w^3\phi(w)\Phi(-\lambda w)\mathrm{d}w \ + \ 4s^3\int_0^\infty w^3\phi(w)\Phi\left(-\lambda^{-1}w\right)\mathrm{d}w$$

$$= -4s^3\frac{1}{2\sqrt{2\pi}}\left(2-\frac{2\lambda^3+3\lambda}{\left(1+\lambda^2\right)^{3/2}}\right) \ + \ 4s^3\frac{1}{2\sqrt{2\pi}}\left(2-\frac{2\lambda^{-3}+3\lambda^{-1}}{\left(1+\lambda^{-2}\right)^{3/2}}\right)$$

$$= 4s^3\frac{1}{2\sqrt{2\pi}}\left[-2+\frac{2\lambda^3+3\lambda}{\left(1+\lambda^2\right)^{3/2}}+2-\frac{2\lambda^{-3}+3\lambda^{-1}}{\left(1+\lambda^{-2}\right)^{3/2}}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{2\lambda^3+3\lambda}{\left(1+\lambda^2\right)^{3/2}} \ - \ \frac{\lambda^3\left(2\lambda^{-3}+3\lambda^{-1}\right)}{\left(1+\lambda^2\right)^{3/2}}\right]s^3 = \sqrt{\frac{2}{\pi}}\frac{2\lambda^3+3\lambda-2-3\lambda^2}{\left(1+\lambda^2\right)^{3/2}}s^3$$

$$= \sqrt{\frac{2}{\pi}}\frac{2\left(\lambda^3-1\right)-3\lambda\left(\lambda-1\right)}{\left(1+\lambda^2\right)^{3/2}}s^3 \ = \ \sqrt{\frac{2}{\pi}}\frac{2\left(\lambda-1\right)\left(\lambda^2+\lambda+1\right)-3\lambda\left(\lambda-1\right)}{\left(1+\lambda^2\right)^{3/2}}s^3$$

$$= \sqrt{\frac{2}{\pi}}\left(\lambda-1\right)\frac{2\left(1+\lambda^2\right)-\lambda}{\left(1+\lambda^2\right)^{3/2}}s^3$$

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$$k=4:\ E\left(Z_{ADHN}^{4}\right)=4s^{4}\int_{0}^{\infty}w^{4}\phi(w)\Phi(-\lambda w)\mathrm{d}w\ +\ 4s^{4}\int_{0}^{\infty}w^{3}\phi(w)\Phi\left(-\lambda^{-1}w\right)\mathrm{d}w$$

$$=4s^{4}\left[\frac{3}{4}-\frac{3}{2\pi}\arctan\lambda-\frac{\lambda\left(3\lambda^{2}+5\right)}{2\pi\left(1+\lambda^{2}\right)^{2}}\right]+4s^{4}\left[\frac{3}{4}-\frac{3}{2\pi}\arctan\lambda^{-1}-\frac{\lambda^{-1}\left(3\lambda^{-2}+5\right)}{2\pi\left(1+\lambda^{-2}\right)^{2}}\right]$$

$$=4s^{4}\left[\frac{6}{4}-\frac{3}{2\pi}\left[\arctan\lambda+\arctan\lambda^{-1}\right]-\frac{\lambda\left(3\lambda^{2}+5\right)}{2\pi\left(1+\lambda^{2}\right)^{2}}-\frac{\lambda^{-1}\left(3\lambda^{-2}+5\right)}{2\pi\left(1+\lambda^{-2}\right)^{2}}\right]$$

$$=4s^{4}\left[\frac{6}{4}-\frac{3}{2\pi}\left[\arctan\lambda+\frac{\pi}{2}-\arctan\lambda\right]-\frac{\lambda\left(3\lambda^{2}+5\right)}{2\pi\left(1+\lambda^{2}\right)^{2}}-\frac{\lambda^{4}\lambda^{-1}\left(3\lambda^{-2}+5\right)}{2\pi\left(1+\lambda^{2}\right)^{2}}\right]$$

$$=4s^{4}\left[\frac{6}{4}-\frac{3}{4}-\frac{\lambda\left(3\lambda^{2}+5\right)+\lambda^{3}\left(3\lambda^{-2}+5\right)}{2\pi\left(1+\lambda^{2}\right)^{2}}\right]$$

$$=4s^{4}\left[\frac{3}{4}-\frac{3\lambda^{3}+5\lambda+3\lambda+5\lambda^{3}}{2\pi\left(1+\lambda^{2}\right)^{2}}\right]=4s^{4}\left[\frac{3}{4}-\frac{8\lambda\left(1+\lambda^{2}\right)}{2\pi\left(1+\lambda^{2}\right)^{2}}\right]$$

$$=\left[3-\frac{16}{\pi}\frac{\lambda}{1+\lambda^{2}}\right]s^{4}$$

Variance of the ADHN distribution

$$\mathrm{Var}(Z)=E\left(Z^{2}\right)-\left[E(Z)\right]^{2}=\left(1-\frac{4}{\pi}\frac{\lambda}{\left(1+\lambda^{2}\right)}\right)s^{2}-\left(\sqrt{\frac{2}{\pi}}\frac{\lambda-1}{\sqrt{1+\lambda^{2}}}s\right)^{2}$$

$$=\left(1-\frac{4}{\pi}\frac{\lambda}{\left(1+\lambda^{2}\right)}\right)s^{2}-\frac{2}{\pi}\frac{(\lambda-1)^{2}}{\left(1+\lambda^{2}\right)}s^{2}=\left(1-\frac{4}{\pi}\frac{\lambda}{\left(1+\lambda^{2}\right)}-\frac{2}{\pi}\frac{(\lambda-1)^{2}}{\left(1+\lambda^{2}\right)}\right)s^{2}$$

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$$\Rightarrow \mathrm{Var}\left(Z_{ADHN}\right)=\left(1-\frac{4\lambda+2\lambda^2-4\lambda+2}{\pi\left(1+\lambda^2\right)}\right)s^2 = \left(1-\frac{2\left(1+\lambda^2\right)}{\pi\left(1+\lambda^2\right)}\right)s^2$$

$$= \frac{\pi-2}{\pi}s^2$$

The above are eq. [15], [16], [17], [18] of the main text.

Pearson's Skewness coefficient for the ADHN distribution (eq. [19] of the main text.

$$\gamma_1 = \frac{E\left[Z-E(Z)\right]^3}{\left[\mathrm{Var}(Z)\right]^{3/2}} = \frac{E\left(Z^3\right)-3E(Z)\mathrm{Var}(Z)-\left[E(Z)\right]^3}{\left[\mathrm{Var}(Z)\right]^{3/2}}$$

$$= \frac{\sqrt{\frac{2}{\pi}}(\lambda-1)\frac{2\left(1+\lambda^2\right)-\lambda}{\left(1+\lambda^2\right)^{3/2}}s^3-3\sqrt{\frac{2}{\pi}}\frac{\lambda-1}{\sqrt{1+\lambda^2}}s\frac{\pi-2}{\pi}s^2-\left[\sqrt{\frac{2}{\pi}}\frac{\lambda-1}{\sqrt{1+\lambda^2}}s\right]^3}{\left[\frac{\pi-2}{\pi}s^2\right]^{3/2}}$$

$$= \frac{\sqrt{\frac{2}{\pi}}(\lambda-1)\frac{2\left(1+\lambda^2\right)-\lambda}{\left(1+\lambda^2\right)^{3/2}}-\sqrt{\frac{2}{\pi}}\frac{\lambda-1}{\sqrt{1+\lambda^2}}\left[\frac{3\pi-6}{\pi}+\frac{2}{\pi}\frac{(\lambda-1)^2}{\left(1+\lambda^2\right)}\right]}{\left[\frac{\pi-2}{\pi}\right]^{3/2}}$$

$$= \frac{\sqrt{\frac{2}{\pi}}(\lambda-1)\frac{2\left(1+\lambda^2\right)-\lambda}{\left(1+\lambda^2\right)^{3/2}}-\frac{1}{\pi}\sqrt{\frac{2}{\pi}}\frac{\lambda-1}{\left(1+\lambda^2\right)^{3/2}}\left[(3\pi-6)\left(1+\lambda^2\right)+2(\lambda-1)^2\right]}{\left[\frac{\pi-2}{\pi}\right]^{3/2}}$$

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$$= \sqrt{\frac{2}{\pi}}(\lambda-1)\frac{\left[2\left(1+\lambda^2\right)-\lambda\right]-\frac{1}{\pi}\left[(3\pi-6)\left(1+\lambda^2\right)\ +\ 2(\lambda-1)^2\right]}{\left[\frac{\pi-2}{\pi}\right]^{3/2}\left(1+\lambda^2\right)^{3/2}}$$

$$= \sqrt{2}(\lambda-1)\frac{\pi\left[2\left(1+\lambda^2\right)-\lambda\right]-\left[(3\pi-6)\left(1+\lambda^2\right)\ +\ 2(\lambda-1)^2\right]}{(\pi-2)^{3/2}\left(1+\lambda^2\right)^{3/2}}$$

$$\sqrt{2}(\lambda-1)\frac{\pi\left[2\left(1+\lambda^2\right)-\lambda\right]-\left[(3\pi-6)\left(1+\lambda^2\right)\ +\ 2(\lambda-1)^2\right]}{(\pi-2)^{3/2}\left(1+\lambda^2\right)^{3/2}}$$

$$= \sqrt{2}(\lambda-1)\frac{2\pi\left(1+\lambda^2\right)-\lambda\pi-3\pi\left(1+\lambda^2\right)\ +\ 6\left(1+\lambda^2\right)\ -\ 2(\lambda-1)^2}{(\pi-2)^{3/2}\left(1+\lambda^2\right)^{3/2}}$$

$$= \sqrt{2}(\lambda-1)\frac{-\pi\left(1+\lambda^2+\lambda\right)\ +\ \left[6+6\lambda^2-2\lambda^2+4\lambda-2\right]}{(\pi-2)^{3/2}\left(1+\lambda^2\right)^{3/2}}$$

$$= \sqrt{2}(\lambda-1)\frac{-\pi\left(1+\lambda^2+\lambda\right)\ +\ 4\left[\lambda^2+\lambda+1\right]}{(\pi-2)^{3/2}\left(1+\lambda^2\right)^{3/2}} = \sqrt{2}(\lambda-1)\frac{\left(1+\lambda^2+\lambda\right)(4-\pi)}{(\pi-2)^{3/2}\left(1+\lambda^2\right)^{3/2}}$$

$$= \frac{\sqrt{2}(4-\pi)}{(\pi-2)^{3/2}}\frac{\lambda^3-1}{\left(1+\lambda^2\right)^{3/2}}$$

which is eq. [19] of the main text.

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Excess Kurtosis of the ADHN distribution (eq. [20] of the main text).

We first calculate the standardized fourth moment

$$\frac{E\left[Z-E(Z)\right]^4}{\left[\mathrm{Var}(Z)\right]^2} = \frac{E\left(Z^4\right)-4E(Z)E\left(Z^3\right)+6\left[E(Z)\right]^2E\left(Z^2\right)-3\left[E(Z)\right]^4}{\left[\mathrm{Var}(Z)\right]^2}$$

$$= \frac{\left[3 - \frac{16}{\pi}\frac{\lambda}{1+\lambda^2}\right]s^4 - 4\sqrt{\frac{2}{\pi}}\frac{\lambda-1}{\sqrt{1+\lambda^2}}s\sqrt{\frac{2}{\pi}}(\lambda-1)\frac{2\left(1+\lambda^2\right)-\lambda}{\left(1+\lambda^2\right)^{3/2}}s^3}{\left[\frac{\pi-2}{\pi}s^2\right]^2}$$

$$+ \frac{6\left[\sqrt{\frac{2}{\pi}}\frac{\lambda-1}{\sqrt{1+\lambda^2}}s\right]^2\left(1-\frac{4}{\pi}\frac{\lambda}{\left(1+\lambda^2\right)}\right)s^2 - 3\left[\sqrt{\frac{2}{\pi}}\frac{\lambda-1}{\sqrt{1+\lambda^2}}s\right]^4}{\left[\frac{\pi-2}{\pi}s^2\right]^2}$$

$$= \frac{\left[3 - \frac{16}{\pi}\frac{\lambda}{1+\lambda^2}\right] - 4\frac{2}{\pi}(\lambda-1)^2\frac{2\left(1+\lambda^2\right)-\lambda}{\left(1+\lambda^2\right)^2} + 6\frac{2}{\pi}\frac{(\lambda-1)^2}{1+\lambda^2}\left(1-\frac{4}{\pi}\frac{\lambda}{\left(1+\lambda^2\right)}\right) - 3\left(\frac{2}{\pi}\right)^2\frac{(\lambda-1)^4}{\left(1+\lambda^2\right)^2}}{\left[\frac{\pi-2}{\pi}\right]^2}$$

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$$= \frac{\left[3\pi^2 - \frac{16\lambda\pi}{1+\lambda^2}\right] - 8\pi(\lambda-1)^2\frac{2(1+\lambda^2)-\lambda}{(1+\lambda^2)^2} + 12\pi\frac{(\lambda-1)^2}{1+\lambda^2}\left(1-\frac{4}{\pi}\frac{\lambda}{(1+\lambda^2)}\right) - 12\frac{(\lambda-1)^4}{(1+\lambda^2)^2}}{(\pi-2)^2}$$

$$= \frac{\left[3\pi^2(1+\lambda^2)^2 - 16\lambda\pi(1+\lambda^2)\right] - 8\pi(\lambda-1)^2\left[2(1+\lambda^2)-\lambda\right] + 12(\lambda-1)^2\left[\pi(1+\lambda^2)-4\lambda\right] - 12(\lambda-1)^4}{(\pi-2)^2(1+\lambda^2)^2}$$

$$= \frac{3\pi^2(1+\lambda^2)^2 - 16\lambda\pi(1+\lambda^2) - 16\pi(\lambda-1)^2(1+\lambda^2) + 8\pi\lambda(\lambda-1)^2 + 12\pi(\lambda-1)^2(1+\lambda^2)}{(\pi-2)^2(1+\lambda^2)^2}$$

$$- \frac{12(\lambda-1)^2 4\lambda + 12(\lambda-1)^4}{(\pi-2)^2(1+\lambda^2)^2}$$

$$= \frac{3\pi^2(1+\lambda^2)^2 - 16\lambda\pi(1+\lambda^2) + 8\pi\lambda(\lambda-1)^2 - 4\pi(\lambda-1)^2(1+\lambda^2) - 12(\lambda^2-1)^2}{(\pi-2)^2(1+\lambda^2)^2}$$

$$= \frac{3\pi^2(1+\lambda^2)^2 - 16\lambda\pi(1+\lambda^2) + 8\pi\lambda(\lambda-1)^2 + 8\pi\lambda(\lambda-1)^2 - 4\pi(\lambda-1)^2(1+\lambda)^2 - 12(\lambda^2-1)^2}{(\pi-2)^2(1+\lambda^2)^2}$$

$$= \frac{3\pi^2(1+\lambda^2)^2 - 16\lambda\pi(1+\lambda^2) + 16\pi\lambda(\lambda-1)^2 - 4\pi(\lambda^2-1)^2 - 12(\lambda^2-1)^2}{(\pi-2)^2(1+\lambda^2)^2}$$

$$= \frac{3\pi^2(1+\lambda^2)^2 - 16\lambda\pi - 16\lambda^3\pi + 16\pi\lambda^3 - 32\pi\lambda^2 + 16\pi\lambda - 4(\pi+3)(\lambda^2-1)^2}{(\pi-2)^2(1+\lambda^2)^2}$$

$$= \frac{3\pi^2(1+\lambda^2)^2 - 32\pi\lambda^2 - 4(\pi+3)(\lambda^2-1)^2}{(\pi-2)^2(1+\lambda^2)^2}$$

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Now we subtract 3 to obtain the excess kurtosis coefficient

$$\gamma_2 = \frac{3\pi^2\left(1+\lambda^2\right)^2 - 32\pi\lambda^2 - 4\left(\pi+3\right)\left(\lambda^2-1\right)^2 - 3\left(\pi-2\right)^2\left(1+\lambda^2\right)^2}{\left(\pi-2\right)^2\left(1+\lambda^2\right)^2}$$

$$= \frac{\left[3\pi^2 - 3\left(\pi-2\right)^2\right]\left(1+\lambda^2\right)^2 - 32\pi\lambda^2 - 4\left(\pi+3\right)\left(\lambda^2-1\right)^2}{\left(\pi-2\right)^2\left(1+\lambda^2\right)^2}$$

$$= \frac{\left[12\pi-12\right]\left(1+2\lambda^2+\lambda^4\right) - 32\pi\lambda^2 - \left(4\pi+12\right)\left(\lambda^4-2\lambda^2+1\right)}{\left(\pi-2\right)^2\left(1+\lambda^2\right)^2}$$

$$= \frac{\lambda^4\left[12\pi-12-4\pi-12\right] + \lambda^2\left[24\pi-24-32\pi+8\pi+24\right] + 12\pi-12-4\pi-12}{\left(\pi-2\right)^2\left(1+\lambda^2\right)^2}$$

$$= \frac{\lambda^4 8\left[\pi-3\right] + 8\left(\pi-3\right)}{\left(\pi-2\right)^2\left(1+\lambda^2\right)^2}$$

$$\Rightarrow \gamma_{2.ADHN} = \frac{8\left(\pi-3\right)}{\left(\pi-2\right)^2}\frac{1+\lambda^4}{\left(1+\lambda^2\right)^2}$$

which is eq. [20] of the main text.

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C3. Moments of the cADHN distribution (eq. [26], [27], [28], [29] of the main text.

The density of the cADHN distribution is

$$f_{cADHN}(z)=\begin{cases}\frac{2}{s_1}\phi(z/s_1)\Phi(\lambda_1 z/s_1)+\frac{2}{s_2}\phi(z/s_2)\Phi(\lambda_2 z/s_2) & z\le 0\\ \frac{2}{s_1}\phi(z/s_1)\Phi(-\lambda_3 z/s_1)+\frac{2}{s_2}\phi(z/s_2)\Phi(-\lambda_4 z/s_2) & z\ge 0\end{cases}$$

So

$$E(Z^k)=\int_{-\infty}^{0} z^k\frac{2}{s_1}\phi(z/s_1)\Phi(\lambda_1 z/s_1)\mathrm{d}z + \int_{-\infty}^{0} z^k\frac{2}{s_2}\phi(z/s_2)\Phi(\lambda_2 z/s_2)\mathrm{d}z$$

$$+\int_{0}^{\infty} z^k\frac{2}{s_1}\phi(z/s_1)\Phi(-\lambda_3 z/s_1)\mathrm{d}z + \int_{0}^{\infty} z^k\frac{2}{s_2}\phi(z/s_2)\Phi(-\lambda_4 z/s_2)\mathrm{d}z$$

Using the same transformations as before the above becomes

$$E(Z^k)=2(-1)^k s_1^k\int_0^\infty w_1^k\phi(w_1)\Phi(-\lambda_1 w_1)\mathrm{d}w_1 + 2(-1)^k s_2^k\int_0^\infty w_2^k\phi(w_2)\Phi(-\lambda_2 w_2)\mathrm{d}w_2$$

$$+2s_1^k\int_0^\infty w_1^k\phi(w_1)\Phi(-\lambda_3 w_1)\mathrm{d}w_1 + 2s_2^k\int_0^\infty w_2^k\phi(w_2)\Phi(-\lambda_4 w_2)\mathrm{d}w_2$$

$$k=1:\ E(Z)=-\ 2s_1\frac{1}{2\sqrt{2\pi}}\left[1+\frac{-\lambda_1}{\sqrt{1+\lambda_1^2}}\right] - 2s_2\frac{1}{2\sqrt{2\pi}}\left[1+\frac{-\lambda_2}{\sqrt{1+\lambda_2^2}}\right]$$

$$+\ 2s_1\frac{1}{2\sqrt{2\pi}}\left[1+\frac{-\lambda_3}{\sqrt{1+\lambda_3^2}}\right] + 2s_2\frac{1}{2\sqrt{2\pi}}\left[1+\frac{-\lambda_4}{\sqrt{1+\lambda_4^2}}\right]$$

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$$= \frac{s_1}{\sqrt{2\pi}}\left[\frac{\lambda_1}{\sqrt{1+\lambda_1^2}} - \frac{\lambda_3}{\sqrt{1+\lambda_3^2}}\right] + \frac{s_2}{\sqrt{2\pi}}\left[\frac{\lambda_2}{\sqrt{1+\lambda_2^2}} - \frac{\lambda_4}{\sqrt{1+\lambda_4^2}}\right]$$

We have

$$\frac{\lambda_1}{\sqrt{1+\lambda_1^2}} - \frac{\lambda_3}{\sqrt{1+\lambda_3^2}} = \frac{\frac{\lambda-\rho}{\sqrt{1-\rho^2}}}{\sqrt{1+\frac{(\lambda-\rho)^2}{1-\rho^2}}} - \frac{\frac{\lambda^{-1}-\rho}{\sqrt{1-\rho^2}}}{\sqrt{1+\frac{(\lambda^{-1}-\rho)^2}{1-\rho^2}}}$$

$$= \frac{\lambda-\rho}{\sqrt{1+\lambda^2-2\lambda\rho}} - \frac{\lambda^{-1}-\rho}{\sqrt{1+\lambda^{-2}-2\lambda^{-1}\rho}}$$

$$= \frac{\lambda-\rho}{\sqrt{1+\lambda^2-2\lambda\rho}} - \frac{(\lambda^{-1}-\rho)\lambda}{\sqrt{1+\lambda^2-2\lambda\ \rho}} = \frac{\lambda-\rho-1+\lambda\rho}{\sqrt{1+\lambda^2-2\lambda\rho}}$$

$$= \frac{(\lambda-1)(1+\rho)}{\sqrt{1+\lambda^2-2\lambda\rho}}$$

Due to symmetry, we also obtain

$$\frac{\lambda_2}{\sqrt{1+\lambda_2^2}} - \frac{\lambda_4}{\sqrt{1+\lambda_4^2}} = \frac{\lambda+\rho}{\sqrt{1+\lambda^2+2\lambda\rho}} - \frac{(\lambda^{-1}+\rho)\lambda}{\sqrt{1+\lambda^2+2\lambda\rho}} = \frac{\lambda+\rho-1-\lambda\rho}{\sqrt{1+\lambda^2+2\lambda\rho}}$$

$$= \frac{(\lambda-1)(1-\rho)}{\sqrt{1+\lambda^2+2\lambda\rho}}$$

So

https://doi.org/10.26219/heal.aueb.8168

$$E\left(Z_{cADHN}\right) = \frac{s_1}{\sqrt{2\pi}}\left[\frac{(\lambda-1)(1+\rho)}{\sqrt{1+\lambda^2-2\lambda\rho}}\right] + \frac{s_2}{\sqrt{2\pi}}\left[\frac{(\lambda-1)(1-\rho)}{\sqrt{1+\lambda^2+2\lambda\rho}}\right]$$

$$= \frac{(\lambda-1)}{\sqrt{2\pi}}\left[\frac{(1+\rho)s_1}{\sqrt{1+\lambda^2-2\rho\lambda}} + \frac{(1-\rho)s_2}{\sqrt{1+\lambda^2+2\rho\lambda}}\right]$$

which is eq. [26] of the main text.

$$k=2:\ E\left(Z^2\right) = 2s_1^2\left[\frac{1}{4} + \frac{1}{2\pi}\arctan\left(-\lambda_1\right) - \frac{\lambda_1}{2\pi\left(1+\lambda_1^2\right)}\right]$$

$$+ 2s_1^2\left[\frac{1}{4} + \frac{1}{2\pi}\arctan\left(-\lambda_3\right) - \frac{\lambda_3}{2\pi\left(1+\lambda_3^2\right)}\right]$$

$$+ 2s_2^2\left[\frac{1}{4} + \frac{1}{2\pi}\arctan\left(-\lambda_2\right) - \frac{\lambda_2}{2\pi\left(1+\lambda_2^2\right)}\right]$$

$$+ 2s_2^2\left[\frac{1}{4} + \frac{1}{2\pi}\arctan\left(-\lambda_4\right) - \frac{\lambda_4}{2\pi\left(1+\lambda_4^2\right)}\right]$$

$$= \frac{s_1^2}{\pi}\left[\pi + \arctan\left(-\lambda_1\right) + \arctan\left(-\lambda_3\right) - \frac{\lambda_1}{\left(1+\lambda_1^2\right)} - \frac{\lambda_3}{\left(1+\lambda_3^2\right)}\right]$$

$$+ \frac{s_2^2}{\pi}\left[\pi + \arctan\left(-\lambda_2\right) + \arctan\left(-\lambda_4\right) - \frac{\lambda_2}{\left(1+\lambda_2^2\right)} - \frac{\lambda_4}{\left(1+\lambda_4^2\right)}\right]$$

We have

https://doi.org/10.26219/heal.aueb.8168

$$\arctan\left(-\lambda_1\right) + \arctan\left(-\lambda_3\right) = -\arctan\left(\lambda_1\right) - \arctan\left(\lambda_3\right)$$

$$= -\arcsin\left(\frac{\lambda_1}{\sqrt{1+\lambda_1^2}}\right) - \arcsin\left(\frac{\lambda_3}{\sqrt{1+\lambda_3^2}}\right) = -\arcsin\left(\delta_1\right) - \arcsin\left(\delta_3\right)$$

$$= -\left[\arccos\left(\sqrt{1-\delta_1^2}\right) + \arccos\left(\sqrt{1-\delta_3^2}\right)\right]$$

$$= -\arccos\left[\sqrt{\left(1-\delta_1^2\right)\left(1-\delta_3^2\right)} - \delta_1\,\delta_3\right]$$

We have calculated previously that $\delta_1\,\delta_3 - \sqrt{\left(1-\delta_1^2\right)\left(1-\delta_3^2\right)} = -\rho$. So we have that the arctan sum is equal to

$$-\arccos\left(\rho\right) = \cos^{-1}\left(-\rho\right) - \pi$$

Analogously, we obtain

$$\arctan\left(-\lambda_2\right) + \arctan\left(-\lambda_4\right) = -\cos^{-1}\left(-\rho\right) = \cos^{-1}\left(\rho\right) - \pi$$

Also,

$$\frac{-\lambda_1}{1+\lambda_1^2} = \frac{-\frac{\lambda-\rho}{\sqrt{1-\rho^2}}}{1+\frac{\left(\lambda-\rho\right)^2}{1-\rho^2}} = \sqrt{1-\rho^2}\,\frac{-\lambda+\rho}{1+\lambda^2-2\lambda\rho}$$

$$\frac{\lambda_3}{1+\lambda_3^2} = \frac{\frac{\lambda^{-1}-\rho}{\sqrt{1-\rho^2}}}{1+\frac{\left(\lambda^{-1}-\rho\right)^2}{1-\rho^2}} = \sqrt{1-\rho^2}\,\frac{\lambda^{-1}-\rho}{1+\lambda^{-2}-2\lambda^{-1}\rho} = \sqrt{1-\rho^2}\,\frac{\lambda-\lambda^2\rho}{1+\lambda^2-2\lambda\rho}$$

So

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$$\frac{-\lambda_1}{1+\lambda_1^2}-\frac{\lambda_3}{1+\lambda_3^2} = \sqrt{1-\rho^2}\,\frac{-\lambda+\rho}{1+\lambda^2-2\lambda\rho}-\sqrt{1-\rho^2}\,\frac{\lambda-\lambda^2\rho}{1+\lambda^2-2\lambda\rho}$$

$$= \sqrt{1-\rho^2}\,\frac{-\lambda+\rho-\lambda+\lambda^2\rho}{1+\lambda^2-2\lambda\rho} = -\sqrt{1-\rho^2}\,\frac{2\lambda-\rho\left(1+\lambda^2\right)}{1+\lambda^2-2\lambda\rho}$$

In an analogous manner

$$\frac{-\lambda_2}{1+\lambda_2^2} = \frac{-\frac{\lambda+\rho}{\sqrt{1-\rho^2}}}{1+\frac{(\lambda+\rho)^2}{1-\rho^2}} = \sqrt{1-\rho^2}\,\frac{-\lambda-\rho}{1+\lambda^2+2\lambda\rho}$$

$$\frac{\lambda_4}{1+\lambda_4^2} = \frac{\frac{\lambda^{-1}+\rho}{\sqrt{1-\rho^2}}}{1+\frac{\left(\lambda^{-1}+\rho\right)^2}{1-\rho^2}} = \sqrt{1-\rho^2}\,\frac{\lambda^{-1}+\rho}{1+\lambda^{-2}+2\lambda^{-1}\rho} = \sqrt{1-\rho^2}\,\frac{\lambda+\lambda^2\rho}{1+\lambda^2+2\lambda\rho}$$

So

$$\frac{-\lambda_2}{1+\lambda_2^2}-\frac{\lambda_4}{1+\lambda_4^2} = \sqrt{1-\rho^2}\,\frac{-\lambda-\rho}{1+\lambda^2+2\lambda\rho}-\sqrt{1-\rho^2}\,\frac{\lambda+\lambda^2\rho}{1+\lambda^2+2\lambda\rho}$$

$$= \sqrt{1-\rho^2}\,\frac{-\lambda-\rho-\lambda-\lambda^2\rho}{1+\lambda^2+2\lambda\rho} = -\sqrt{1-\rho^2}\,\frac{2\lambda+\rho\left(1+\lambda^2\right)}{1+\lambda^2+2\lambda\rho}$$

Bringing all results together, we obtain

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$$E\left(Z^2_{cADHN}\right) = \left[\cos^{-1}\left(-\rho\right) - \sqrt{1-\rho^2}\,\frac{2\lambda-\rho\left(1+\lambda^2\right)}{1+\lambda^2-2\lambda\rho}\right]\frac{s_1^2}{\pi}$$

$$+ \left[\cos^{-1}\left(\rho\right) - \sqrt{1-\rho^2}\,\frac{2\lambda+\rho\left(1+\lambda^2\right)}{1+\lambda^2+2\lambda\rho}\right]\frac{s_2^2}{\pi}$$

which is eq. [27] of the main text.

$$k=3:\ E\left(Z^3_{cADHN}\right) = -2\frac{s_1^3}{2\sqrt{2\pi}}\left(2+\frac{-2\lambda_1^3-3\lambda_1}{\left(1+\lambda_1^2\right)^{3/2}}\right) + 2\frac{s_1^3}{2\sqrt{2\pi}}\left(2+\frac{-2\lambda_3^3-3\lambda_3}{\left(1+\lambda_3^2\right)^{3/2}}\right)$$

$$-2\frac{s_2^3}{2\sqrt{2\pi}}\left(2+\frac{-2\lambda_2^3-3\lambda_2}{\left(1+\lambda_2^2\right)^{3/2}}\right) + 2\frac{s_2^3}{2\sqrt{2\pi}}\left(2+\frac{-2\lambda_4^3-3\lambda_4}{\left(1+\lambda_4^2\right)^{3/2}}\right)$$

$$= \frac{s_1^3}{\sqrt{2\pi}}\left(\frac{2\lambda_1^3+3\lambda_1}{\left(1+\lambda_1^2\right)^{3/2}} - \frac{2\lambda_3^3+3\lambda_3}{\left(1+\lambda_3^2\right)^{3/2}}\right) + \frac{s_2^3}{\sqrt{2\pi}}\left(\frac{2\lambda_2^3+3\lambda_2}{\left(1+\lambda_2^2\right)^{3/2}} - \frac{2\lambda_4^3+3\lambda_4}{\left(1+\lambda_4^2\right)^{3/2}}\right)$$

$$= \frac{s_1^3}{\sqrt{2\pi}}\left[2\left(\frac{\lambda_1^3}{\left(1+\lambda_1^2\right)^{3/2}} - \frac{\lambda_3^3}{\left(1+\lambda_3^2\right)^{3/2}}\right) + 3\left(\frac{\lambda_1}{\left(1+\lambda_1^2\right)^{3/2}} - \frac{\lambda_3}{\left(1+\lambda_3^2\right)^{3/2}}\right)\right]$$

$$+ \frac{s_2^3}{\sqrt{2\pi}}\left[2\left(\frac{\lambda_2^3}{\left(1+\lambda_2^2\right)^{3/2}} - \frac{\lambda_4^3}{\left(1+\lambda_4^2\right)^{3/2}}\right) + 3\left(\frac{\lambda_2}{\left(1+\lambda_2^2\right)^{3/2}} - \frac{\lambda_4}{\left(1+\lambda_4^2\right)^{3/2}}\right)\right]$$

Term by term we have

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$$\frac{\lambda_1^3}{\left(\sqrt{1+\lambda_1^2}\right)^3}-\frac{\lambda_3^3}{\left(\sqrt{1+\lambda_3^2}\right)^3}=\frac{(\lambda-\rho)^3-(1-\lambda\rho)^3}{\left(1+\lambda^2-2\lambda\rho\right)^{3/2}}$$

$$=\frac{(\lambda-\rho-1+\lambda\rho)\left[(\lambda-\rho)^2+(\lambda-\rho)(1-\lambda\rho)+(1-\lambda\rho)^2\right]}{\left(1+\lambda^2-2\lambda\rho\right)^{3/2}}$$

$$=\frac{(\lambda-1)(1+\rho)}{\left(1+\lambda^2-2\lambda\rho\right)^{3/2}}\left[(\lambda-\rho)^2+(\lambda-\rho)(1-\lambda\rho)+(1-\lambda\rho)^2\right]$$

For the second term in the 1st bracketed expression, we have

$$\frac{\lambda_1}{\left(1+\lambda_1^2\right)\sqrt{1+\lambda_1^2}}-\frac{\lambda_3}{\left(1+\lambda_3^2\right)\sqrt{1+\lambda_3^2}}=\frac{\frac{\lambda-\rho}{\sqrt{1-\rho^2}}}{\left(1+\lambda_1^2\right)\sqrt{1+\frac{(\lambda-\rho)^2}{1-\rho^2}}}-\frac{\frac{\lambda^{-1}-\rho}{\sqrt{1-\rho^2}}}{\left(1+\lambda_3^2\right)\sqrt{1+\frac{\left(\lambda^{-1}-\rho\right)^2}{1-\rho^2}}}$$

$$=\frac{\lambda-\rho}{\left(1+\lambda_1^2\right)\sqrt{1+\lambda^2-2\lambda\rho}}-\frac{\lambda^{-1}-\rho}{\left(1+\lambda_3^2\right)\sqrt{1+\lambda^{-2}-2\lambda^{-1}\rho}}$$

$$=\frac{\lambda-\rho}{\left(1+\lambda_1^2\right)\sqrt{1+\lambda^2-2\lambda\rho}}-\lambda^2\frac{\left(\lambda^{-1}-\rho\right)\lambda}{\left(1+\lambda_1^2\right)\sqrt{1+\lambda^2-2\lambda\ \rho}}$$

$$=\frac{(\lambda-\rho)-\lambda^3\left(\lambda^{-1}-\rho\right)}{\left(1+\lambda_1^2\right)\sqrt{1+\lambda^2-2\lambda\rho}}=\frac{\lambda-\rho-\lambda^2+\lambda^3\rho}{\left(1+\lambda^2-2\lambda\rho\right)^{3/2}}\left(1-\rho^2\right)$$

$$=\frac{\lambda(1-\lambda)-\rho\left(1-\lambda^3\right)}{\left(1+\lambda^2-2\lambda\rho\right)^{3/2}}\left(1-\rho^2\right)=\frac{\lambda(1-\lambda)-\rho(1-\lambda)\left(1+\lambda+\lambda^2\right)}{\left(1+\lambda^2-2\lambda\rho\right)^{3/2}}\left(1-\rho^2\right)$$

$$=\frac{(\lambda-1)(1+\rho)}{\left(1+\lambda^2-2\lambda\rho\right)^{3/2}}\left[\rho(1-\rho)\left(1+\lambda+\lambda^2\right)-\lambda(1-\rho)\right]$$

So the **first bracketed term** in the moment expression becomes

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$$\left[2\left(\frac{\lambda_1^3}{\left(1+\lambda_1^2\right)^{3/2}}-\frac{\lambda_3^3}{\left(1+\lambda_3^2\right)^{3/2}}\right)+3\left(\frac{\lambda_1}{\left(1+\lambda_1^2\right)^{3/2}}-\frac{\lambda_3}{\left(1+\lambda_3^2\right)^{3/2}}\right)\right]$$

$$= \frac{(\lambda-1)(1+\rho)}{\left(1+\lambda^2-2\lambda\rho\right)^{3/2}}\left[2(\lambda-\rho)^2+2(\lambda-\rho)(1-\lambda\rho)+2(1-\lambda\rho)^2\right.$$

$$\left. + \; 3\rho(1-\rho)\left(1+\lambda+\lambda^2\right)-3\lambda(1-\rho)\right]$$

Focusing on the expression in brackets, we have

$$\left[2(\lambda-\rho)^2+2(\lambda-\rho)(1-\lambda\rho)+2(1-\lambda\rho)^2\right.$$

$$\left. + \; 3\rho(1-\rho)\left(1+\lambda+\lambda^2\right)-3\lambda(1-\rho)\right]$$

$$= 2\lambda^2-4\lambda\rho+2\rho^2 \; + 2\lambda-2\lambda^2\rho-2\rho+2\lambda\rho^2 \; + \; 2-4\lambda\rho+2\lambda^2\rho^2$$
$$+ 3\rho+3\lambda\rho \; + 3\lambda^2\rho-3\rho^2-3\lambda\rho^2-3\lambda^2\rho^2 \; -3\lambda+3\lambda\rho$$

$$= 2\lambda^2-2\lambda\rho-\rho^2 \; -\lambda+\lambda^2\rho+\; \rho-\lambda\rho^2+2-\lambda^2\rho^2$$

$$= 2\left(1+\lambda^2\right)+\rho\left(1+\lambda^2\right)-\rho^2\left(1+\lambda^2\right)-\lambda\left(2\rho+1+\rho^2\right)$$

$$= \left(1+\lambda^2\right)(2-\rho)(1+\rho)-\lambda(1+\rho)^2$$

So finally

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$$\left[2\left(\frac{\lambda_1^3}{\left(1+\lambda_1^2\right)^{3/2}}-\frac{\lambda_3^3}{\left(1+\lambda_3^2\right)^{3/2}}\right)+3\left(\frac{\lambda_1}{\left(1+\lambda_1^2\right)^{3/2}}-\frac{\lambda_3}{\left(1+\lambda_3^2\right)^{3/2}}\right)\right]=$$

$$=\frac{(\lambda-1)(1+\rho)^2}{\left(1+\lambda^2-2\lambda\rho\right)^{3/2}}\left[\left(1+\lambda^2\right)(2-\rho)-\lambda(1+\rho)\right]$$

Moving to the **2nd bracketed term in the moment expression**, again term-by-term we have

$$\frac{\lambda_2^3}{\left(\sqrt{1+\lambda_2^2}\right)^3}-\frac{\lambda_4^3}{\left(\sqrt{1+\lambda_4^2}\right)^3}=\frac{(\lambda+\rho)^3-(1+\lambda\rho)^3}{\left(1+\lambda^2+2\lambda\rho\right)^{3/2}}$$

$$=\frac{(\lambda+\rho-1-\lambda\rho)\left[(\lambda+\rho)^2+(\lambda+\rho)(1+\lambda\rho)+(1+\lambda\rho)^2\right]}{\left(1+\lambda^2+2\lambda\rho\right)^{3/2}}$$

$$=\frac{(\lambda-1)(1-\rho)}{\left(1+\lambda^2+2\lambda\rho\right)^{3/2}}\left[(\lambda+\rho)^2+(\lambda+\rho)(1+\lambda\rho)+(1+\lambda\rho)^2\right]$$

For the second term in the 1st bracketed expression, we have

https://doi.org/10.26219/heal.aueb.8168

$$\frac{\lambda_2}{\left(1+\lambda_2^2\right)\sqrt{1+\lambda_2^2}}-\frac{\lambda_4}{\left(1+\lambda_4^2\right)\sqrt{1+\lambda_4^2}}=\frac{\frac{\lambda+\rho}{\sqrt{1-\rho^2}}}{\left(1+\lambda_2^2\right)\sqrt{1+\frac{(\lambda+\rho)^2}{1-\rho^2}}}-\frac{\frac{\lambda^{-1}+\rho}{\sqrt{1-\rho^2}}}{\left(1+\lambda_4^2\right)\sqrt{1+\frac{\left(\lambda^{-1}+\rho\right)^2}{1-\rho^2}}}$$

$$=\frac{\lambda+\rho}{\left(1+\lambda_2^2\right)\sqrt{1+\lambda^2+2\lambda\rho}}-\frac{\lambda^{-1}+\rho}{\left(1+\lambda_4^2\right)\sqrt{1+\lambda^{-2}+2\lambda^{-1}\rho}}$$

$$=\frac{\lambda+\rho}{\left(1+\lambda_2^2\right)\sqrt{1+\lambda^2+2\lambda\rho}}-\lambda^2\frac{\left(\lambda^{-1}+\rho\right)\lambda}{\left(1+\lambda_2^2\right)\sqrt{1+\lambda^2+2\lambda\ \rho}}$$

$$=\frac{(\lambda+\rho)-\lambda^3\left(\lambda^{-1}+\rho\right)}{\left(1+\lambda_1^2\right)\sqrt{1+\lambda^2+2\lambda\rho}}=\frac{\lambda+\rho-\lambda^2-\lambda^3\rho}{\left(1+\lambda^2+2\lambda\rho\right)^{3/2}}\left(1-\rho^2\right)$$

$$=\frac{\lambda(1-\lambda)+\rho\left(1-\lambda^3\right)}{\left(1+\lambda^2+2\lambda\rho\right)^{3/2}}\left(1-\rho^2\right)=\frac{\lambda(1-\lambda)+\rho(1-\lambda)\left(1+\lambda+\lambda^2\right)}{\left(1+\lambda^2+2\lambda\rho\right)^{3/2}}\left(1-\rho^2\right)$$

$$=\frac{-(\lambda-1)(1-\rho)}{\left(1+\lambda^2+2\lambda\rho\right)^{3/2}}\left[\rho(1+\rho)\left(1+\lambda+\lambda^2\right)+\lambda(1+\rho)\right]$$

So for the 2nd bracketed term we have

$$\left[2\left(\frac{\lambda_2^3}{\left(1+\lambda_2^2\right)^{3/2}}-\frac{\lambda_4^3}{\left(1+\lambda_4^2\right)^{3/2}}\right)+3\left(\frac{\lambda_2}{\left(1+\lambda_2^2\right)^{3/2}}-\frac{\lambda_4}{\left(1+\lambda_4^2\right)^{3/2}}\right)\right]$$

$$=\frac{(\lambda-1)(1-\rho)}{\left(1+\lambda^2-2\lambda\rho\right)^{3/2}}\left[2(\lambda+\rho)^2+2(\lambda+\rho)(1+\lambda\rho)+2(1+\lambda\rho)^2\right.$$

$$\left.-\ 3\rho(1+\rho)\left(1+\lambda+\lambda^2\right)-3\lambda(1+\rho)\right]$$

Focusing on the expression in brackets, we have

https://doi.org/10.26219/heal.aueb.8168

$$\Big[2(\lambda+\rho)^2+2(\lambda+\rho)(1+\lambda\rho)+2(1+\lambda\rho)^2$$

$$-\ 3\rho(1+\rho)\left(1+\lambda+\lambda^2\right)-3\lambda(1+\rho)\Big]$$

$$=2\lambda^2+4\lambda\rho+2\rho^2\ +\ 2\lambda+2\lambda^2\rho+2\rho+2\lambda\rho^2\ +\ 2+4\lambda\rho+2\lambda^2\rho^2$$
$$-\ 3\rho-3\lambda\rho\ -\ 3\lambda^2\rho-3\rho^2-3\lambda\rho^2-3\lambda^2\rho^2\ \ -3\lambda-3\lambda\rho$$

$$=2\lambda^2+2\lambda\rho-\rho^2\ -\lambda\ \ -\lambda\rho^2\ -\ \rho\ -\lambda\rho^2+2-\lambda^2\rho^2$$

$$=2\left(1+\lambda^2\right)-\rho\left(1+\lambda^2\right)-\rho^2\left(1+\lambda^2\right)-\lambda\left(-2\rho+1+\rho^2\right)$$

$$=\left(1+\lambda^2\right)(2+\rho)(1-\rho)-\lambda(1-\rho)^2$$

So finally

$$\left[2\left(\frac{\lambda_2^3}{\left(1+\lambda_2^2\right)^{3/2}}-\frac{\lambda_4^3}{\left(1+\lambda_4^2\right)^{3/2}}\right)+3\left(\frac{\lambda_2}{\left(1+\lambda_4^2\right)^{3/2}}-\frac{\lambda_2}{\left(1+\lambda_4^2\right)^{3/2}}\right)\right]=$$

$$=\ \frac{(\lambda-1)(1-\rho)^2}{\left(1+\lambda^2+2\lambda\rho\right)^{3/2}}\left[\left(1+\lambda^2\right)(2+\rho)-\lambda(1-\rho)\right]$$

Compacting, we obtain

$$E\left(Z_{cADHN}^3\right)=\frac{(\lambda-1)}{\sqrt{2\pi}}\Bigg[\frac{(2-\rho)\left(1+\lambda^2\right)-\lambda(1+\rho)}{\left(1+\lambda^2-2\lambda\rho\right)^{3/2}}(1+\rho)^2\,s_1^3$$

$$+\ \ \frac{(2+\rho)\left(1+\lambda^2\right)-\lambda(1-\rho)}{\left(1+\lambda^2+2\lambda\rho\right)^{3/2}}(1-\rho)^2\,s_2^3\Bigg]$$

which is eq. [28] of the main text.

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$$k=4: E\left(Z^4\right)=2s_1^4\int_0^{\infty} w_1^4\phi\left(w_1\right)\Phi\left(-\lambda_1 w_1\right)\mathrm{d}w_1 \quad + \quad 2s_2^4\int_0^{\infty} w_2^4\phi\left(w_2\right)\Phi\left(-\lambda_2 w_2\right)\mathrm{d}w_2$$

$$+\ 2s_1^4\int_0^{\infty} w_1^4\phi\left(w_1\right)\Phi\left(-\lambda_3 w_1\right)\mathrm{d}w_1 \quad + \quad 2s_2^4\int_0^{\infty} w_2^4\phi\left(w_2\right)\Phi\left(-\lambda_4 w_2\right)\mathrm{d}w_2$$

$$k=4: E\left(Z^4\right)=2s_1^4\left[\frac{3}{4}+\frac{3}{2\pi}\arctan\left(-\lambda_1\right)-\frac{\lambda_1\left(3\lambda_1^2+5\right)}{2\pi\left(1+\lambda_1^2\right)^2}\right]$$

$$+\ 2s_2^4\left[\frac{3}{4}+\frac{3}{2\pi}\arctan\left(-\lambda_2\right)-\frac{\lambda_2\left(3\lambda_2^2+5\right)}{2\pi\left(1+\lambda_2^2\right)^2}\right]$$

$$+\ 2s_1^4\left[\frac{3}{4}+\frac{3}{2\pi}\arctan\left(-\lambda_3\right)-\frac{\lambda_3\left(3\lambda_3^2+5\right)}{2\pi\left(1+\lambda_3^2\right)^2}\right]$$

$$+\ 2s_2^4\left[\frac{3}{4}+\frac{3}{2\pi}\arctan\left(-\lambda_4\right)-\frac{\lambda_4\left(3\lambda_4^2+5\right)}{2\pi\left(1+\lambda_4^2\right)^2}\right]$$

$$=2s_1^4\left[\frac{3}{2}+\frac{3}{2\pi}\arctan\left(-\lambda_1\right)+\frac{3}{2\pi}\arctan\left(-\lambda_3\right)-\frac{\lambda_1\left(3\lambda_1^2+5\right)}{2\pi\left(1+\lambda_1^2\right)^2}-\frac{\lambda_3\left(3\lambda_3^2+5\right)}{2\pi\left(1+\lambda_3^2\right)^2}\right]$$

$$+\ 2s_2^4\left[\frac{3}{2}+\frac{3}{2\pi}\arctan\left(-\lambda_2\right)+\frac{3}{2\pi}\arctan\left(-\lambda_4\right)-\frac{\lambda_2\left(3\lambda_2^2+5\right)}{2\pi\left(1+\lambda_2^2\right)^2}-\frac{\lambda_4\left(3\lambda_4^2+5\right)}{2\pi\left(1+\lambda_4^2\right)^2}\right]$$

https://doi.org/10.26219/heal.aueb.8168

$$= \frac{s_1^4}{\pi}\left[3\pi \; + \; 3\left[\arctan\left(-\lambda_1\right) + \arctan\left(-\lambda_3\right)\right] \; - \; \frac{\lambda_1\left(3\lambda_1^2+5\right)}{\left(1+\lambda_1^2\right)^2} - \; \frac{\lambda_3\left(3\lambda_3^2+5\right)}{\left(1+\lambda_3^2\right)^2}\right]$$

$$+ \; \frac{s_2^4}{\pi}\left[3\pi \; + \; 3\left[\arctan\left(-\lambda_2\right) + \arctan\left(-\lambda_4\right)\right] \; - \; \frac{\lambda_2\left(3\lambda_2^2+5\right)}{\left(1+\lambda_2^2\right)^2} - \; \frac{\lambda_4\left(3\lambda_4^2+5\right)}{\left(1+\lambda_4^2\right)^2}\right]$$

The arctan expressions have been calculated previously, so we have

$$E\left(Z_{cADHN}^4\right) = \frac{s_1^4}{\pi}\left[3\cos^{-1}\left(-\rho\right) \; - \; \frac{\lambda_1\left(3\lambda_1^2+5\right)}{\left(1+\lambda_1^2\right)^2} - \; \frac{\lambda_3\left(3\lambda_3^2+5\right)}{\left(1+\lambda_3^2\right)^2}\right]$$

$$+ \; \frac{s_2^4}{\pi}\left[3\cos^{-1}\left(\rho\right) \; - \; \frac{\lambda_2\left(3\lambda_2^2+5\right)}{\left(1+\lambda_2^2\right)^2} - \; \frac{\lambda_4\left(3\lambda_4^2+5\right)}{\left(1+\lambda_4^2\right)^2}\right]$$

For the remaining terms, we have

$$\frac{\lambda_1\left(3\lambda_1^2+5\right)}{\left(1+\lambda_1^2\right)^2} = \frac{3\left(\frac{\lambda-\rho}{\sqrt{1-\rho^2}}\right)^3 + 5\frac{\lambda-\rho}{\sqrt{1-\rho^2}}}{\left(1+\frac{\left(\lambda-\rho\right)^2}{1-\rho^2}\right)^2} = \frac{3\frac{\left(\lambda-\rho\right)^3}{\left(1-\rho^2\right)^{3/2}} + 5\frac{\lambda-\rho}{\left(1-\rho^2\right)^{1/2}}}{\left(\frac{1+\lambda^2-2\lambda\rho}{1-\rho^2}\right)^2}$$

$$= \frac{3\frac{\left(\lambda-\rho\right)^3}{\left(1-\rho^2\right)^{3/2}}\left(1-\rho^2\right)^2 + 5\frac{\lambda-\rho}{\left(1-\rho^2\right)^{1/2}}\left(1-\rho^2\right)^2}{\left(1+\lambda^2-2\lambda\rho\right)^2} = \frac{3\left(\lambda-\rho\right)^3\left(1-\rho^2\right)^{1/2} + 5\left(\lambda-\rho\right)\left(1-\rho^2\right)^{3/2}}{\left(1+\lambda^2-2\lambda\rho\right)^2}$$

$$= \frac{\sqrt{1-\rho^2}}{\left(1+\lambda^2-2\lambda\rho\right)^2}\left[3\left(\lambda-\rho\right)^3 + 5\left(1-\rho^2\right)\left(\lambda-\rho\right)\right]$$

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Also

$$\frac{\lambda_3\left(3\lambda_3^2+5\right)}{\left(1+\lambda_3^2\right)^2}=\frac{3\left(\frac{\lambda^{-1}-\rho}{\sqrt{1-\rho^2}}\right)^3+5\frac{\lambda^{-1}-\rho}{\sqrt{1-\rho^2}}}{\left(1+\frac{\left(\lambda^{-1}-\rho\right)^2}{1-\rho^2}\right)^2}=\frac{3\frac{\left(\lambda^{-1}-\rho\right)^3}{\left(1-\rho^2\right)^{3/2}}+5\frac{\lambda^{-1}-\rho}{\left(1-\rho^2\right)^{1/2}}}{\left(\frac{1+\lambda^2-2\lambda\rho}{1-\rho^2}\right)^2\lambda^{-4}}$$

$$=\frac{3\frac{\left(\lambda^{-1}-\rho\right)^3}{\left(1-\rho^2\right)^{3/2}}\left(1-\rho^2\right)^2+5\frac{\lambda^{-1}-\rho}{\left(1-\rho^2\right)^{1/2}}\left(1-\rho^2\right)^2}{\left(1+\lambda^2-2\lambda\rho\right)^2\lambda^{-4}}=\frac{3\left(\lambda^{-1}-\rho\right)^3\left(1-\rho^2\right)^{1/2}+5\left(\lambda^{-1}-\rho\right)\left(1-\rho^2\right)^{3/2}}{\left(1+\lambda^2-2\lambda\rho\right)^2\lambda^{-4}}$$

$$=\frac{\sqrt{1-\rho^2}}{\left(1+\lambda^2-2\lambda\rho\right)^2}\left[3\lambda^4\left(\lambda^{-1}-\rho\right)^3+5\lambda^4\left(1-\rho^2\right)\left(\lambda^{-1}-\rho\right)\right]$$

So

$$-\frac{\lambda_1\left(3\lambda_1^2+5\right)}{\left(1+\lambda_1^2\right)^2}-\frac{\lambda_3\left(3\lambda_3^2+5\right)}{\left(1+\lambda_3^2\right)^2}=$$

$$=-\frac{\sqrt{1-\rho^2}}{\left(1+\lambda^2-2\lambda\rho\right)^2}\left[3\left(\lambda-\rho\right)^3+5\left(1-\rho^2\right)\left(\lambda-\rho\right)+3\lambda^4\left(\lambda^{-1}-\rho\right)^3+5\lambda^4\left(1-\rho^2\right)\left(\lambda^{-1}-\rho\right)\right]$$

For the bracketed term we have

$$\left[3\left(\lambda-\rho\right)^3+5\left(1-\rho^2\right)\left(\lambda-\rho\right)+3\lambda^4\left(\lambda^{-1}-\rho\right)^3+5\lambda^4\left(1-\rho^2\right)\left(\lambda^{-1}-\rho\right)\right]$$

$$=3\left(\lambda-\rho\right)^3+5\left(1-\rho^2\right)\left(\lambda-\rho\right)+3\lambda\left(1-\lambda\rho\right)^3+5\lambda^3\left(1-\rho^2\right)\left(1-\lambda\rho\right)$$

$$=3\left(\lambda-\rho\right)^3+3\lambda\left(1-\lambda\rho\right)^3+5\left(1-\rho^2\right)\left(\lambda-\rho+\lambda^3\left(1-\lambda\rho\right)\right)$$

https://doi.org/10.26219/heal.aueb.8168

$$= 3(\lambda-\rho)\left(\lambda^2-2\lambda\rho+\rho^2\right)+3\lambda(1-\lambda\rho)\left(1-2\lambda\rho+\lambda^2\rho^2\right)+5\left(1-\rho^2\right)\left(\lambda\left(1+\lambda^2\right)-\rho\left(1+\lambda^4\right)\right)$$

$$= 3\lambda^3-6\lambda^2\rho+3\lambda\rho^2-3\lambda^2\rho+6\lambda\rho^2-3\rho^3\ +3\lambda-6\lambda^2\rho+3\lambda^3\rho^2\ -3\lambda^2\rho+6\lambda^3\rho^2-3\lambda^4\rho^3$$
$$+\ 5\left(1-\rho^2\right)\left(\lambda\left(1+\lambda^2\right)-\rho\left(1+\lambda^4\right)\right)$$

$$= 3\lambda^3-18\lambda^2\rho+9\lambda\rho^2-3\rho^3\ +3\lambda+9\lambda^3\rho^2\ -3\lambda^4\rho^3$$
$$+\ 5\left(1-\rho^2\right)\left(\lambda\left(1+\lambda^2\right)-\rho\left(1+\lambda^4\right)\right)$$

$$= 3\lambda\left(1+\lambda^2\right)-18\lambda^2\rho+9\lambda\rho^2\left(1+\lambda^2\right)-3\rho^3\left(1+\lambda^4\right)$$
$$+\ 5\left(1-\rho^2\right)\left(\lambda\left(1+\lambda^2\right)-\rho\left(1+\lambda^4\right)\right)$$

$$= \left(1+\lambda^2\right)\left(3\lambda+9\lambda\rho^2+5\lambda\left(1-\rho^2\right)\right)-\left(1+\lambda^4\right)\rho\left(3\rho^2+5\left(1-\rho^2\right)\right)-18\lambda^2\rho$$

$$= \left(1+\lambda^2\right)\left(8\lambda+4\lambda\rho^2\right)-\left(1+\lambda^4\right)\rho\left(5-2\rho^2\right)-18\lambda^2\rho$$

$$= 4\lambda\left(1+\lambda^2\right)\left(2+\rho^2\right)-\rho\left[\left(1+\lambda^4\right)\left(5-2\rho^2\right)+18\lambda^2\right]$$

Also,

$$\frac{\lambda_2\left(3\lambda_2^2+5\right)}{\left(1+\lambda_2^2\right)^2}=\frac{\sqrt{1-\rho^2}}{\left(1+\lambda^2+2\lambda\rho\right)^2}\left[3(\lambda+\rho)^3+5\left(1-\rho^2\right)(\lambda+\rho)\right]$$

and

$$\frac{\lambda_4\left(3\lambda_4^2+5\right)}{\left(1+\lambda_4^2\right)^2}=\frac{\sqrt{1-\rho^2}}{\left(1+\lambda^2+2\lambda\rho\right)^2}\left[3\lambda^4\left(\lambda^{-1}+\rho\right)^3+5\lambda^4\left(1-\rho^2\right)\left(\lambda^{-1}+\rho\right)\right]$$

So

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$$-\frac{\lambda_2\left(3\lambda_2^2+5\right)}{\left(1+\lambda_2^2\right)^2}-\frac{\lambda_4\left(3\lambda_4^2+5\right)}{\left(1+\lambda_4^2\right)^2}=$$

$$=-\frac{\sqrt{1-\rho^2}}{\left(1+\lambda^2+2\lambda\rho\right)^2}\left[3\left(\lambda+\rho\right)^3+5\left(1-\rho^2\right)\left(\lambda+\rho\right)+3\lambda\left(1+\lambda\rho\right)^3+5\lambda^3\left(1-\rho^2\right)\left(1+\lambda\rho\right)\right]$$

For the bracketed term we have

$$\left[3\left(\lambda+\rho\right)^3+5\left(1-\rho^2\right)\left(\lambda+\rho\right)+3\lambda\left(1+\lambda\rho\right)^3+5\lambda^3\left(1-\rho^2\right)\left(1+\lambda\rho\right)\right]$$

$$=3\left(\lambda+\rho\right)^3+3\lambda\left(1+\lambda\rho\right)^3+5\left(1-\rho^2\right)\left(\lambda+\rho+\lambda^3\left(1+\lambda\rho\right)\right)$$

$$=3\left(\lambda+\rho\right)\left(\lambda^2+2\lambda\rho+\rho^2\right)+3\lambda\left(1+\lambda\rho\right)\left(1+2\lambda\rho+\lambda^2\rho^2\right)+5\left(1-\rho^2\right)\left(\lambda\left(1+\lambda^2\right)+\rho\left(1+\lambda^4\right)\right)$$

$$=3\lambda^3+6\lambda^2\rho+3\lambda\rho^2+3\lambda^2\rho+6\lambda\rho^2+3\rho^3+3\lambda+6\lambda^2\rho+3\lambda^3\rho^2+3\lambda^2\rho+6\lambda^3\rho^2+3\lambda^4\rho^3$$
$$+5\left(1-\rho^2\right)\left(\lambda\left(1+\lambda^2\right)+\rho\left(1+\lambda^4\right)\right)$$

$$=3\lambda^3+18\lambda^2\rho+9\lambda\rho^2+3\rho^3+3\lambda+9\lambda^3\rho^2+3\lambda^4\rho^3$$
$$+5\left(1-\rho^2\right)\left(\lambda\left(1+\lambda^2\right)+\rho\left(1+\lambda^4\right)\right)$$

$$=3\lambda\left(1+\lambda^2\right)+18\lambda^2\rho+9\lambda\rho^2\left(1+\lambda^2\right)+3\rho^3\left(1+\lambda^4\right)$$
$$+5\left(1-\rho^2\right)\left(\lambda\left(1+\lambda^2\right)+\rho\left(1+\lambda^4\right)\right)$$

$$=\left(1+\lambda^2\right)\left(3\lambda+9\lambda\rho^2+5\lambda\left(1-\rho^2\right)\right)+\left(1+\lambda^4\right)\rho\left(3\rho^2+5\left(1-\rho^2\right)\right)+18\lambda^2\rho$$

$$=\left(1+\lambda^2\right)\left(8\lambda+4\lambda\rho^2\right)+\left(1+\lambda^4\right)\rho\left(5-2\rho^2\right)+18\lambda^2\rho$$

$$=4\lambda\left(1+\lambda^2\right)\left(2+\rho^2\right)+\rho\left[\left(1+\lambda^4\right)\left(5-2\rho^2\right)+18\lambda^2\right]$$

Combining all results we obtain

https://doi.org/10.26219/heal.aueb.8168

$$E\left(Z_{cADHN}^4\right)=\frac{s_1^4}{\pi}\left[3\cos^{-1}\left(-\rho\right)\ -\ \sqrt{1-\rho^2}\,\frac{4\lambda\left(1+\lambda^2\right)\left(2+\rho^2\right)-\rho\left[\left(1+\lambda^4\right)\left(5-2\rho^2\right)+18\lambda^2\right]}{\left(1+\lambda^2-2\lambda\rho\right)^2}\right]$$

$$+\ \frac{s_2^4}{\pi}\left[3\cos^{-1}\left(\rho\right)\ -\sqrt{1-\rho^2}\,\frac{4\lambda\left(1+\lambda^2\right)\left(2+\rho^2\right)+\rho\left[\left(1+\lambda^4\right)\left(5-2\rho^2\right)+18\lambda^2\right]}{\left(1+\lambda^2+2\lambda\rho\right)^2}\right]$$

or

$$E\left(Z_{cADHN}^4\right)=\frac{s_1^4}{\pi}\left[3\cos^{-1}\left(-\rho\right)\ -\ \frac{\left(\psi_1-\psi_2\right)\sqrt{1-\rho^2}}{\left(1+\lambda^2-2\lambda\rho\right)^2}\right]$$

$$+\ \frac{s_2^4}{\pi}\left[3\cos^{-1}\left(\rho\right)\ -\frac{\left(\psi_1+\psi_2\right)\sqrt{1-\rho^2}}{\left(1+\lambda^2+2\lambda\rho\right)^2}\right]$$

with

$$\psi_1=4\lambda\left(1+\lambda^2\right)\left(2+\rho^2\right),\ \psi_2=\rho\left[\left(1+\lambda^4\right)\left(5-2\rho^2\right)+18\lambda^2\right]$$

which is eq. [29] of the main text.

https://doi.org/10.26219/heal.aueb.8168

D. Equations [30] and [31] of the main text

D1. $P\left(\left|\hat{\beta}_1-\beta\right|\le\left|\hat{\beta}_2-\beta\right|\right)=F_{ADHN}\left(0\right)=\frac{2}{\pi}\cos^{-1}\left(\frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)>1/2$

We have

$$F_{ADHN}\left(0\right)=4\Phi_2\left(0,\,0;\,-\delta\right)=4\frac{1}{2}\left[1-\frac{\cos^{-1}\left(-\delta\right)}{\pi}\right]=\frac{2}{\pi}\left[\pi-\cos^{-1}\left(-\delta\right)\right]=\frac{2}{\pi}\cos^{-1}\left(\delta\right)$$

We have $\delta=\frac{\lambda}{\sqrt{1+\lambda^2}}=\frac{\sigma_1/\sigma_2}{\sqrt{1+\left(\sigma_1/\sigma_2\right)^2}}=\frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}}$

We examining the case

$$\sigma_1<\sigma_2\ \Rightarrow 2\sigma_1^2<\sigma_1^2+\sigma_2^2\Rightarrow\ \frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}}<\frac{1}{\sqrt{2}}$$

The arccosine is a decreasing function in its principal domain, so

$$\text{So }\frac{2}{\pi}\cos^{-1}\left(\delta\right)>\frac{2}{\pi}\cos^{-1}\left(1/\sqrt{2}\right)=\ \frac{2}{\pi}\frac{\pi}{4}\ =\frac{1}{2}$$

So

$P\left(\left|\hat{\beta}_1-\beta\right|\le\left|\hat{\beta}_2-\beta\right|\right)>1/2$ (eq. [30]).

D2. For equation [31]

$$P\left(\left|\hat{\beta}_1-\beta\right|\le\left|\hat{\beta}_2-\beta\right|\right)=F_{cADHN}\left(0\right)=\frac{1}{\pi}\cos^{-1}\left(\frac{\sigma_1^2-\sigma_2^2}{s_1s_2}\right)>1/2$$

We have

$$F_{cADHN}\left(0\right)=2\Phi_2\left(0,\,0;\,-\delta_1\right)+2\Phi_2\left(0,\,0;\,-\delta_2\right)$$

https://doi.org/10.26219/heal.aueb.8168

$$= 2\frac{1}{2}\left[1-\frac{\cos^{-1}(-\delta_1)}{\pi}\right]+2\frac{1}{2}\left[1-\frac{\cos^{-1}(-\delta_2)}{\pi}\right]=\frac{1}{\pi}\left[\pi-\cos^{-1}(-\delta_1)\right]+\frac{1}{\pi}\left[\pi-\cos^{-1}(-\delta_1)\right]$$

$$= \frac{1}{\pi}\left[\cos^{-1}(\delta_1)+\cos^{-1}(\delta_2)\right]$$

Summing the arccosines we have

$$\cos^{-1}(\delta_1)+\cos^{-1}(\delta_2)=\cos^{-1}\left(\delta_1\delta_2-\sqrt{\left(1-\delta_1^2\right)\left(1-\delta_2^2\right)}\right)$$

$$\delta_1\delta_2-\sqrt{\left(1-\delta_1^2\right)\left(1-\delta_2^2\right)}=\frac{\lambda_1}{\sqrt{1+\lambda_1^2}}\frac{\lambda_2}{\sqrt{1+\lambda_2^2}}-\sqrt{\left(1-\frac{\lambda_1^2}{1+\lambda_1^2}\right)\left(1-\frac{\lambda_2^2}{1+\lambda_2^2}\right)}$$

$$= \frac{\lambda_1\lambda_2-1}{\sqrt{\left(1+\lambda_1^2\right)\left(1+\lambda_2^2\right)}}=\frac{\frac{\lambda-\rho}{\sqrt{1-\rho^2}}\frac{\lambda+\rho}{\sqrt{1-\rho^2}}-1}{\sqrt{\left(1+\frac{(\lambda-\rho)^2}{1-\rho^2}\right)\left(1+\frac{(\lambda+\rho)^2}{1-\rho^2}\right)}}$$

$$= \frac{\frac{\lambda^2-\rho^2-1+\rho^2}{1-\rho^2}}{\frac{\sqrt{\left(1+\lambda^2-2\lambda\rho\right)\left(1+\lambda^2+2\lambda\rho\right)}}{1-\rho^2}}=\frac{\lambda^2-1}{\sqrt{\left(1+\lambda^2-2\lambda\rho\right)\left(1+\lambda^2+2\lambda\rho\right)}}$$

Since the distributiona here represents the difference of two half normals, we have

$\lambda=\frac{\sigma_1}{\sigma_2}$ and $s_1^2=\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2$, $s_2^2=\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2$.

Substituting we have

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$$\frac{\left(\sigma_1/\sigma_2\right)^2-1}{\sqrt{\left(1+\left(\sigma_1/\sigma_2\right)^2-2\left(\sigma_1/\sigma_2\right)\rho\right)\left(1+\left(\sigma_1/\sigma_2\right)^2+2\left(\sigma_1/\sigma_2\right)\rho\right)}}$$

$$=\frac{\frac{\sigma_1^2-\sigma_2^2}{\sigma_2^2}}{\sqrt{\frac{1}{\sigma_2^2}\left(\sigma_2^2+\sigma_1^2-2\sigma_1\sigma_2\rho\right)\frac{1}{\sigma_2^2}\left(\sigma_2^2+\sigma_1^2+2\sigma_1\sigma_2\rho\right)}}$$

$$=\frac{\sigma_1^2-\sigma_2^2}{s_1 s_2}$$

So

$$P\left(\left|\hat{\beta}_1-\beta\right|\le\left|\hat{\beta}_2-\beta\right|\right)=\frac{1}{\pi}\cos^{-1}\left(\frac{\sigma_1^2-\sigma_2^2}{s_1 s_2}\right)$$

Since we examine the case $\sigma_1<\sigma_2\Rightarrow\sigma_1^2-\sigma_2^2<0$

Therefore

$$\frac{1}{\pi}\cos^{-1}\left(\frac{\sigma_1^2-\sigma_2^2}{s_1 s_2}\right)>\frac{1}{\pi}\cos^{-1}(0)=\frac{1}{\pi}\frac{\pi}{2}=\frac{1}{2}$$

$$\Rightarrow P\left(\left|\hat{\beta}_1-\beta\right|\le\left|\hat{\beta}_2-\beta\right|\right)>\frac{1}{2}$$ (eq. [31]).

https://doi.org/10.26219/heal.aueb.8168

E. The normalizing constant of the ADHN distribution

In this section we show that when the parameter λ is positive, the normalizing constant of the ADHN distribution is 4, while when λ is negative the normalizing constant is 4/3.

We want to determine c such that

$$c\int_{-\infty}^{0}\frac{1}{s}\phi(z/s)\Phi(\lambda z/s)\mathrm{d}z + c\int_{0}^{\infty}\frac{1}{s}\phi(z/s)\Phi(-\lambda^{-1}z/s)\mathrm{d}z = 1$$

$$\Rightarrow \int_{-\infty}^{0}\frac{2}{s}\phi(z/s)\Phi(\lambda z/s)\mathrm{d}z + \int_{0}^{\infty}\frac{2}{s}\phi(z/s)\Phi(-\lambda^{-1}z/s)\mathrm{d}z = 2c^{-1}$$

Using the relation between the skew-normal CDF and the bivariate standard normal integral we have

$$\Rightarrow 2\Phi_2\left(0,\,0;\,\rho=\frac{-\lambda}{\sqrt{1+\lambda^2}}\right) + 1 - \int_{-\infty}^{0}\frac{2}{s}\phi(z/s)\Phi(-\lambda^{-1}z/s)\mathrm{d}z = 2c^{-1}$$

$$\Rightarrow 2\Phi_2\left(0,\,0;\,\rho=\frac{-\lambda}{\sqrt{1+\lambda^2}}\right) + 1 - 2\Phi_2\left(0,\,0;\,\rho=\frac{\lambda^{-1}}{\sqrt{1+\lambda^{-2}}}\right) = 2c^{-1}$$

Using another known result the above can be written

$$2\left[\frac{1}{2\pi}\arcsin\left(\frac{-\lambda}{\sqrt{1+\lambda^2}}\right)+\frac{1}{4}\right] + 1 - 2\left[\frac{1}{2\pi}\arcsin\left(\frac{\lambda^{-1}}{\sqrt{1+\lambda^{-2}}}\right)+\frac{1}{4}\right] = 2c^{-1}$$

$$\Rightarrow \frac{1}{2} - \frac{1}{\pi}\arcsin\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right) + 1 - \frac{1}{2} - \frac{1}{\pi}\arcsin\left(\frac{\lambda^{-1}}{\sqrt{1+\lambda^{-2}}}\right) = 2c^{-1}$$

$$\Rightarrow 1 - \frac{1}{\pi}\left[\arcsin\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)+\arcsin\left(\frac{\lambda^{-1}}{\sqrt{1+\lambda^{-2}}}\right)\right] = 2c^{-1}$$

$$\Rightarrow 1 - \frac{1}{\pi}\left[\arcsin\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)+\arcsin\left(\frac{\mathrm{sign}\lambda}{\sqrt{1+\lambda^2}}\right)\right] = 2c^{-1}$$

https://doi.org/10.26219/heal.aueb.8168

Using the relevant trigonometric identity,

$$\Rightarrow 1-\frac{2}{\pi}\left[\arctan\left(\frac{\frac{\lambda}{\sqrt{1+\lambda^2}}}{1+\sqrt{1-\frac{\lambda^2}{1+\lambda^2}}}\right)+\arctan\left(\frac{\frac{\text{sign}\lambda}{\sqrt{1+\lambda^2}}}{1+\sqrt{1-\frac{(\text{sign}\lambda)^2}{1+\lambda^2}}}\right)\right] = 2c^{-1}$$

$$\Rightarrow 1-\frac{2}{\pi}\left[\arctan\left(\frac{\lambda}{1+\sqrt{1+\lambda^2}}\right)+\arctan\left(\frac{\text{sign}\lambda}{|\lambda|+\sqrt{1+\lambda^2}}\right)\right] = 2c^{-1}$$

$$\Rightarrow 1-\frac{2}{\pi}\arctan\left(\frac{\frac{\lambda}{1+\sqrt{1+\lambda^2}}+\frac{\text{sign}\lambda}{|\lambda|+\sqrt{1+\lambda^2}}}{1-\frac{\lambda}{\left(1+\sqrt{1+\lambda^2}\right)}\frac{\text{sign}\lambda}{\left(|\lambda|+\sqrt{1+\lambda^2}\right)}}\right)=2c^{-1}$$

$$1-\frac{2}{\pi}\arctan\left(\frac{\frac{\lambda}{1+\sqrt{1+\lambda^2}}+\frac{\text{sign}\lambda}{|\lambda|+\sqrt{1+\lambda^2}}}{1-\frac{\lambda}{\left(1+\sqrt{1+\lambda^2}\right)}\frac{\text{sign}\lambda}{\left(|\lambda|+\sqrt{1+\lambda^2}\right)}}\right)=2c^{-1}$$

$$\Rightarrow 1-\frac{2}{\pi}\arctan\left(\frac{\lambda^2\text{sign}\lambda+\lambda\sqrt{1+\lambda^2}+\text{sign}\lambda\left(1+\sqrt{1+\lambda^2}\right)}{\left(1+\sqrt{1+\lambda^2}\right)\left(|\lambda|+\sqrt{1+\lambda^2}\right)-\lambda\text{sign}\lambda}\right) = 2c^{-1}$$

$$\Rightarrow 1-\frac{2}{\pi}\arctan\left(\frac{\lambda^2\text{sign}\lambda+\lambda\sqrt{1+\lambda^2}+\text{sign}\lambda\left(1+\sqrt{1+\lambda^2}\right)}{|\lambda|+\sqrt{1+\lambda^2}+|\lambda|\sqrt{1+\lambda^2}+1+\lambda^2-\lambda\text{sign}\lambda}\right) = 2c^{-1}$$

Case A : $\lambda > 0$

In this case the expression simplifies to

$$\Rightarrow 1-\frac{2}{\pi}\arctan\left(\frac{\lambda^2+\lambda\sqrt{1+\lambda^2}+1+\sqrt{1+\lambda^2}}{\sqrt{1+\lambda^2}+\lambda\sqrt{1+\lambda^2}+1+\lambda^2}\right) = 2c^{-1}$$

https://doi.org/10.26219/heal.aueb.8168

$$\Rightarrow 1-\frac{2}{\pi}\arctan(1) \quad = 2c^{-1} \;\Rightarrow 1-\frac{2}{\pi}\frac{\pi}{4} = 2c^{-1}$$

$$\Rightarrow \frac{1}{2} = 2c^{-1} \;\Rightarrow c = 4$$

which is the normalizing constant used in the main text.

Case B : $\lambda < 0$

Here the expression becomes

$$\Rightarrow 1-\frac{2}{\pi}\arctan\left(\frac{\lambda^2 \text{sign}\lambda + \lambda\sqrt{1+\lambda^2} + \text{sign}\lambda\left(1+\sqrt{1+\lambda^2}\right)}{|\lambda| + \sqrt{1+\lambda^2} + |\lambda|\sqrt{1+\lambda^2} + 1 + \lambda^2 - \lambda\,\text{sign}\lambda}\right) \quad = 2c^{-1}$$

$$\Rightarrow 1-\frac{2}{\pi}\arctan\left(\frac{-\lambda^2 + \lambda\sqrt{1+\lambda^2} - 1 - \sqrt{1+\lambda^2}}{-\lambda + \sqrt{1+\lambda^2} - \lambda\sqrt{1+\lambda^2} + 1 + \lambda^2 + \lambda}\right) \quad = 2c^{-1}$$

$$\Rightarrow 1-\frac{2}{\pi}\arctan\left(\frac{-\left(\lambda^2+1\right) + \sqrt{1+\lambda^2}\left(\lambda-1\right)}{\sqrt{1+\lambda^2} - \lambda\sqrt{1+\lambda^2} + 1 + \lambda^2}\right) \quad = 2c^{-1}$$

$$\Rightarrow 1-\frac{2}{\pi}\arctan\left(-\frac{\left(\lambda^2+1\right) + \sqrt{1+\lambda^2}\left(1-\lambda\right)}{1+\lambda^2+\sqrt{1+\lambda^2}\left(1-\lambda\right)}\right) \quad = 2c^{-1}$$

$$\Rightarrow 1-\frac{2}{\pi}\arctan(-1) \quad = 2c^{-1} \;\Rightarrow 1+\frac{2}{\pi}\frac{\pi}{4} \quad = 2c^{-1}$$

$$\Rightarrow \frac{3}{2} = 2c^{-1} \;\Rightarrow\; c = \frac{4}{3}$$

which is the normalizing constant mentioned in the relevant footnote in the main text.

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https://doi.org/10.26219/heal.aueb.8168

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