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Credit Risk Modeling

By

PALIAS EFSTRATIOS IRAKLIS

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DEDICATION

To my parents who stood by me all the time so far.

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VITA

I was born in Piraeus where I finished my school years. I graduated from the department of Statistics in the Athens University of Economics and Business and I continued with the Master Schedule on the same field.

I have an interest in music and I got a piano decree from a music school in Piraeus. I also joined a theatrical workshop and participate in theater plays now and then.

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ABSTRACT

In this thesis we conduct a study on the credit risk and attempt to build models to estimate the default probability of a bond as well as compute credit derivatives for the aforementioned risk.

To obtain the data we used the simulation method with the help of the R program.

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CHAPTER 1

INTRODUCTION TO CREDIT RISK AND CREDIT DERIVATIVES

In this chapter we are going to discuss about the basics of finance. Specifically we will focus on instruments of indebtedness known as **bonds** as well as the risk such instruments carry for the debtors. Furthermore, we are going to introduce the **credit default swaps** that allow a debtholder to transfer the aforementioned risk to another entity.

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1.1. BONDS

Firstly, we are going to present some definitions of financial terms that we are about to use extensively within the rest of our work.

Bond: An instrument of indebtedness of the bond issuer to the holders. It is a debt security under which the issuer owes the holders a debt and, depending on the terms of the bond, is obliged to pay them interest (the coupon) and/or to repay the principal at a later date, termed the **maturity** date.

Default: The failure to meet the legal obligations of a loan. It essentially means that a debtor has not paid a debt when they are required to have paid. The probability of a default occurring is often referred to as the **credit risk** and is the main subject of this thesis.

Credit Derivative: Any one of various instruments and techniques designed to separate and then transfer the **credit risk**, transferring it to an entity other than the lender or debt holder.

Credit Default Swap: A financial swap agreement that the seller of the CDS will compensate the buyer in the event of a loan default or other credit event. This is to say that the seller of the CDS insures the buyer against some reference loan defaulting. The buyer again of the CDS makes a series of payments (**spread**) to the seller and, in exchange, receives a payoff if the loan defaults.

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Now that we've put some light into the basic definitions, we will show how their prices are calculated.

Pricing a bond is vital for its holders because it can affect their economical status significantly. A major factor in determining the price is the **probability of default** of the entity (see section 1.2). This is natural because the bigger the probability is, the greater is the risk of default occurring and thus the level of the price increases to compensate for a potential loss. On the contrary, if the default probability is lower, the price goes down, as a result of the much lower risk associated with the bond.

Another major factor is the **yield curve**. The yield of a debt instrument is the overall rate of return available on the investment. Hence, the higher the yield is, the higher is the expected return. A typical yield curve has a slightly upward sloping shape, indicating that as we get closer to maturity, the yield rises.

Let Y(t) be the yield of a bond at time t and let T be the maturity date of that bond (t<T). The value of the bond at time t=0 (i.e. the time it is issued) is given by:

$$\mathbf{F \times exp(-y(T) \times T) = F \times B_0(T)}$$

Where F is the **principal** of the bond, i.e. the amount on which the issuer pays interest, and which, most commonly, has to be repaid at the end of the term.

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1.2. DEFAULTABLE BONDS

As mentioned in section 1.1 a bond carries the risk that the debtor will be unable to pay his debts by the time of maturity T. Being able to estimate the probability of that event occurring (default probability) is vital for both the debtor and the holder.

To calculate the aforementioned probability, we need to know the yield of a corporate bond, as well as the yield curve of a risk-free bond with the same maturity. Let us denote with **Y'(t)** the former and with **Y(t)** the latter. As given above, the value of the two bonds will be respectively:

$$\mathbf{F} \times \mathbf{exp}(-\mathbf{y}'(\mathbf{T}) \times \mathbf{T}) = \mathbf{F} \times \mathbf{B_0}'(\mathbf{T})$$

$$\mathbf{F} \times \mathbf{exp}(-\mathbf{y}(\mathbf{T}) \times \mathbf{T}) = \mathbf{F} \times \mathbf{B_0}(\mathbf{T})$$

Since the first bond carries the risk of default it is obvious that it will have a lower value than the second. Hence we can calculate their difference (the **spread**) by subtracting the two values, thus getting:

$$\mathbf{F} \times [\mathbf{B_0}(\mathbf{T}) - \mathbf{B_0}'(\mathbf{T})] = \mathbf{F} \times [\mathbf{exp}(-\mathbf{y}(\mathbf{T}) \times \mathbf{T}) - \mathbf{exp}(-\mathbf{y}'(\mathbf{T}) \times \mathbf{T})]$$

This is the present of the cost of default. It is easy to assume that the higher the value of the corporate bond, the lower that cost becomes.

The default probability can be estimated using the values presented above as follows:

$$\mathbf{P}(\mathbf{T}) = \frac{\mathbf{B_0}(\mathbf{T}) - \mathbf{B_0}'(\mathbf{T})}{\mathbf{B_0}'(\mathbf{T})}$$

However the above formula only holds true in the case we have no recovery. The recovery rate varies between 0 and 1. In that case the formula is altered based on that rate R. It then becomes:

$$\mathbf{P}(\mathbf{T}) = \frac{\mathbf{B_0}(\mathbf{T}) - \mathbf{B_0}'(\mathbf{T})}{(\mathbf{1} - \mathbf{R}) \times \mathbf{B_0}'(\mathbf{T})}$$

As expected, the higher the recovery rate R, the higher the probability of default.

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1.3. EFFECT OF DEFAULT IN YIELDS

A bond's default, as expected, can greatly alter its yield. This happens because if a bond has a high default probability, its issuer will have to increase the interest rate to attract investors to buy their bonds and thus compensate for the higher risk of a default event occurring. On the contrary, a lower default probability will attract more investors anyway, so the issuer can drop the interest rate because it's not as necessary as before.

In order to be able to distinguish bonds based on their default probability, certain agencies conduct evaluation of such bonds and rate them based on the aforementioned probability. Those agencies, as well as the exact ratings are described in the next section.

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1.4 CREDIT RATING

Since default of a bond is a highly undesirable event, as it can significantly affect its yield, it is quite natural to rank them based on evaluating their ability to pay back the debt as well as the likelihood of the debtor defaulting. Such a process is known as **credit rating** and it is done from **credit rating agencies** which collect and analyze information about the debtors.

The top 3 credit rating agencies are: **Moody's**, **Standard and Poor** and **Fitch Ratings**. They assign letters (such as AA+, B-) to represent the bond's quality, with **Standard and Poor's** and **Fitch Ratings** using only uppercase and **Moody's** using a slightly different notation. The highest possible rating is triple A and descends to triple C as the probability of default increases while the letter D represents default. A full table of those ratings and their explanations is shown below:

Moody's	Standard and Poor	Fitch Ratings	Rating Description
A	AAA	AAA	Prime
Aa1	AA+	AA+	High grade
Aa2	AA	AA	
Aa3	AA-	AA-	
A1	A+	A+	Upper medium grade
A2	A	A	
A3	A-	A-	
Baa1	BBB+	BBB+	Lower medium grade
Baa2	BBB	BBB	
Baa3	BBB-	BBB-	
Ba1	BB+	BB+	Non-investment rate speculative
Ba2	BB	BB	
Ba3	BB-	BB-	
B1	B+	B+	Highly speculative
B2	B	B	
B3	B-	B-	
Caa1	CCC+	CCC	Substantial risks
Caa2	CCC		Extremely speculative
Caa3	CCC-		Default imminent with little prospect for recovery
Ca	CC		
	C		
C	D	DDD	In default
-		DD	
		D	

As mentioned in section 1.3, credit ratings have a great impact on yields as the higher the risk of default, the greater the interest must be in order to attract people invest on them. Usually, high yield bonds are the ones that fall below the BBB- rating (or Baa3 for Moody's).

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CHAPTER 2

LÉVY PROCESSES

In this chapter we will introduce the group of stochastic processes known as Lévy processes. Such processes are widely used in finance in order to model the assets of an entity. First we are going to present their basic properties before analyzing the most notable of them separately.

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2.1. DEFINITION AND FUNDAMENTAL PROPERTIES

Firstly, we have to give a proper definition of the **stochastic process**. A stochastic process is a collection of random variables. It is similar to the arithmetic progression, but the elements are random i.e. not known beforehand. Also, unlike the progression, the values are collected on certain **time points**, instead of having a set position. For example, on a progression we can refer to the 6^{th} element (the element on the 6^{th} position), while on a stochastic process, we can only refer to the element at time t = 6.

A **Lévy process** is a stochastic process with independent and stationary increments. It represents the motion of a point whose successive displacements are random and independent, and statistically identical over different time intervals of the same length. A Lévy process may thus be viewed as the continuous-time analog of a random walk. A mathematical definition is given as follows:

2.1.1 Definition: A Lévy processes **X(t)** is a stochastic process that satisfies the following:

- X has right-continuous paths, almost surely.
- X(0) = 0, almost surely.
- X has stationary increments i.e. the distribution of the increment X(t+s) – X(t) depends only on the length of the interval (s) and not on t.
- X has independent increments i.e. if r < s < t < u, the increments X(s) – X(r) and X(u) – X(t) are independent random variables. In other words, increments over non-overlapping time intervals are independent.

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2.1.2 Fundamental Properties: As a result of the definition, Lévy processes also possess a set of useful properties:

i) Infinite divisibility: First we have to give the definition for random variables before generalizing it to Lévy processes.

Definition: Let X be a random variable with characteristic function φ_x and law P_x. Hence:

$$\boldsymbol{\varphi_x(u) = \int_{\mathbb{R}} e^{iux} P_x \, dx}$$

The law P_x of a random variable X is infinitely divisible if for all $n \in \mathbb{N}$ there exist i.i.d. random variables such that:

$$\boldsymbol{X = X_1^{(1/n)} + X_2^{(1/n)} + \ldots + X_n^{(1/n)}}$$

That is X is equal in distribution to the sum of $X_i^{1/n}$, i=1,…,n

Equivalently, the law of a random variable X is infinitely divisible if for all $n \in \mathbb{N}$ there exist i.i.d. random variables such that:

$$\boldsymbol{P_x = P_{x^{(1/n)}} \times P_{x^{(1/n)}} \times \ldots \times P_{x^{(1/n)}} \ (n \text{ times})}$$

As for the characteristic function of an infinitely divisible variable X, it can be written as follows:

$$\boldsymbol{\varphi_X(u) = [\varphi_{X^{(1/n)}}(u)]^n}$$

Now consider a Lévy process $L=(L_t)_{0 < t < T}$. For any $n \in \mathbb{N}$ and any $0 < t < T$ we have that:

$$\boldsymbol{L_t = L_{\frac{t}{n}} + \left(L_{\frac{2t}{n}} - L_{\frac{t}{n}}\right) + \ldots + \left(L_t - L_{\frac{(n-1)t}{n}}\right)}$$

The stationarity and independence of increments yield that the above terms form a sequence of i.i.d. random variables. Hence, we can conclude that the process L_t is infinitely divisible. Its characteristic function can be written similarly to the random variable's function.

ii) Random measures: A random measure is a **measure-valued random element** i.e. the realization of the concept of a random variable in more complicated spaces than a simple line.

Definition: Let (X,*A*) be a measurable space where X is a set of elements and *A* is a sigma-algebra of subsets of X. A random measure is a measure μ such that for every

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subset of X it holds that $\mu(X) < \infty$. A special case of the above is the probability measure, which assigns a number between 0 and 1 to an event X.

There is a class of stochastic processes based on the random measures defined above. Those are called point processes and we will give some examples below.

iii) Lévy - Khintchine formula: Let $\varphi_x(u)$ be the characteristic function of a random variable. Recall that the characteristic function completely defines the variable's distribution and vice-versa. Since a Lévy process is a collection of random variables, there exists a collection of characteristic functions for it. Let's denote them with $\boldsymbol{\varphi_t(\theta)}$ where t is the time point for which the random variable is collected. The Lévy - Khintchine formula gives us the form of the characteristic exponent, say $\psi(\theta)$:

$$\boldsymbol{\varphi_t(\theta) = \exp[-t \times \psi(\theta)]}$$

$$\boldsymbol{\psi(\theta) = i\alpha\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \times 1_{\{|x|<1\}})\pi(dx)}$$

Where $\alpha \in \mathbb{R}$, $\theta > 0$ and π is a measure concentrated on $\mathbb{R} \setminus \{0\}$ such that:

$$\boldsymbol{\int_{\mathbb{R}} (1 \wedge x^2)\pi(dx) < \infty}$$

Conversely, for every (α,θ,π) triplet there exists a unique Lévy process, whose characteristic exponent is given by the above formula.

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2.1.3 Examples: We will now present 3 processes that make use of the properties described above. Their simulation will be presented on the next chapter.

i) Brownian Motion: A continuous time stochastic process. A process B(t) is called Brownian Motion if it satisfies the following properties:

- B is continuous (not just right-continuous) for all t.
- B(0) = 0 almost surely.
- B has stationary increments (as in 2.1.1).
- B has independent increments (as in 2.1.1).
- B has Gaussian increments i.e. (B(t+s) – B(t)) follows a Normal(0,s) distribution.

Despite being a continuous process, it cannot be differentiated nowhere with respect to t. Also, its variation on the interval (0,t) is infinite, almost surely, while its squared variation equals t.

ii) Compound Poisson Process: A continuous time process with jumps that arrive randomly according to a **Poisson process**. Therefore, we should first define the Poisson Process.

The Poisson Process is a **counting** process. That means it measures how many events occur in a given time interval. Consequently, it is an increasing process whose values are the natural numbers. A process N(t) is called Poisson Process if it satisfies the following:

- N is right-continuous for all t.
- N(0) = 0 almost surely.
- N has stationary increments.
- N has independent increments.
- N has Poisson increments i.e. (N(t+s) – N(t)) follows a Poisson(λ×s) distribution.

Like the Poisson Process, the Compound Poisson Process has the parameter **λ**, which is called **rate**. It also has a second parameter which is the distribution of the jump sizes. In this thesis we are going to use the Normal distribution for the latter.

The Compound Poisson Process is thus given by the following formula:

$$\mathbf{Y(t)} = \sum_{\mathbf{i=0}}^{\mathbf{N(t)}} \mathbf{D_i}$$

Where D_i are Normally Distributed and N(t) is a Poisson process with rate λ.

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For any t > 0, the Compound Poisson Process simulates the N(t) which is how many events have occurred in the time interval (0,t) and then calculates the sum of that many normally distributed random variables, D_i.

iii) Gamma Process: A stochastic process whose increments are gamma distributed. It is a pure-jump increasing Lévy process. Just like the Gamma distribution, this process has two parameters, the shape **a**, and the rate **b**. Its starting value is always zero. Similarly to the Brownian Motion, it satisfies the following properties:

- G is right-continuous for all t.
- G(0) = 0 almost surely.
- G has stationary increments.
- G has independent increments.
- G has Gamma increments i.e. (G(t+s) – G(t)) follows a Gamma(a×s,b) distribution.

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2.2. STOCHASTIC INTEGRATION AND STOCHASTIC DIFFERENTIAL EQUATIONS

2.2.1 Itô integral over a Lévy process:

We would now like to see if it is possible to integrate over a Lévy process. Let's define the integral and try to interpret it as much as we can:

Let $f(0,\infty)$: $\Omega \to \mathbb{R}$ be a function that depends, in some ways, from the outcome of a Lévy process L_t and we want to integrate f, over that process. The integral will thus have the form:

$$\int_a^b \mathbf{f(t)dL_t}$$

The differential $\mathbf{dL_t}$ is similar to **dt**, but using the process change instead of the time change. Roughly speaking, it could be interpreted as $[L_{t+dt} - L_t]$ for a small $dt > 0$.

Let us now assume a partition $a = t_0 < t_1 < \ldots < t_n = b$ of the time interval [a,b]. The function f can be approximated by:

$$\mathbf{f(t) = \sum_{i=0}^{n-1} f(t_i) 1_{[t_i, t_{i+1})}(t)}$$

The Itô integral can be defined as a limit on L^2:

$$\mathbf{\int_a^b f(t)dL_t = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\left[L_{t_{i+1}} - L_{t_i}\right]}$$

There are some important points that must be stated regarding the above formula.

- The value of the process f at time t depends only on the values of L_s for $s<t$ and not $s \geq t$.
- The limit is defined on L^2, which is the space of all functions whose second moment's integral is finite.

The Itô integral possesses the following important properties:

- For two processes f_1, f_2 we have: $I(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 I(f_1) + \lambda_2 I(f_2)$ where $\lambda_1, \lambda_2 \in \mathbb{R}$
- $E\left[\int_a^b f(t)dL_t\right] = 0$ (the first moment)
- $E\left[\left|\int_a^b f(t)dL_t\right|^2\right] = E\left[\int_a^b |f(t)|^2 dt\right]$ (the second moment)

A stochastic integral finds applications in investments that depend on stock prices, for example, that follow such processes. We will present examples on chapter 3.

Using the above integral, we can also define a stochastic process, say I_t, as follows:

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$$\mathbf{I_t} \coloneqq \int_0^t \mathbf{f(t)dL_t}$$

This is the indefinite Itô integral.

Another major concept is the **Itô process**. A process is called Itô if it has the following form:

$$\mathbf{X_t = X_0 + \int_0^t u(s)ds + \int_0^t v(s)dL_s}$$

Where u,v are functions that satisfy:

$$\int_0^t v^2(s)ds < \infty \text{ and } \int_0^t u(s)ds < \infty \text{ a.s.}$$

The above process can be written in differential form as follows:

$$\mathbf{dX_t = udt + vdL_t}$$

Now that we have defined the Itô process we might wonder if a function (say **g**) of X_t can also be an Itô process and if it does which is its formula. Luckily, Itô's lemma exists to indeed prove that proposition:

Let X_t be an Itô process (see above) and let g(t,x) be a $C^{1,2}$ function. That means g has continuous first derivative with respect to t and continuous second derivative with respect to x. Define a new process as $Y_t \coloneqq g(t,X_t)$. Then Itô's lemma gives us the formula of Y_t as follows:

$$\mathbf{g(t,X_t) = g(0,X_0) + \int_0^t \left(\frac{\partial g}{\partial s} + u\frac{\partial g}{\partial x} + \frac{1}{2}v^2\frac{\partial^2 g}{\partial x^2}\right)ds + \int_0^t v\frac{\partial g}{\partial x}dL_t}$$

This formula can also be written in differential form:

$$\mathbf{dg(t,X_t) = \left(\frac{\partial g}{\partial s} + u\frac{\partial g}{\partial x} + \frac{1}{2}v^2\frac{\partial^2 g}{\partial x^2}\right)(t,X_t)dt + v\frac{dg}{dx}(t,X_t)dL_t}$$

These concepts are generalized in the next chapter where we talk about a special form of equations called stochastic differential equations.

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2.2.2: Stochastic Differential Equations: A stochastic differential equation (SDE) is a differential equation of which at least one term is a stochastic process. They are widely used in probability theory as well as in finance for modeling stock prices etc. In this section we will present their basics.

Every SDE takes the following form:

$$\mathbf{dX(t) = a\big(X(t)\big)dt + b\big(X(t)\big)dS(t)}$$

Where **X(t)** is the quantity of interest (e.g. the stock price) and **S(t)** is a stochastic process. As for **a** and **b**, they denote two functions of X(t), usually the **mean** and the **variance** respectively.

Note that from now on, we will be using the notation **X(t)** instead of $\mathbf{X_t}$!

In the above formula it is obvious that X(t) changes depending on time change dt, as well as the change of the process S(t) which cannot be known beforehand due to it being a stochastic process. If the latter is excluded (or set "close" to 0) we have an ordinary differential equation which has a unique solution.

The above SDE can also be written in integral form as follows:

$$\mathbf{X(t) - X(0) = \int_0^t a\big(X(t)\big)dt + \int_0^t b\big(X(t)\big)dS(t)}$$

There are two integrals on the right-hand side of this equation. The first is a well-known and easy-to-calculate Riemann Integral, while the second is a **stochastic integral**.

Let us now try to interpret the equation. The left-hand side is pretty straightforward as it represents the change of the quantity X between the time points 0 and t. As for the right-hand side there are two terms. The first one is being differentiated by **dt** so it is as simple as solving a Riemann integral. The second one, however, requires some deep thinking before it can be adequately understood.

As a first try, we give an example of the SDE. The X(t) could represent the location of a vehicle at time point t, which we would like to estimate by knowing the initial location X(0). It is obvious that it will depend on the speed of the vehicle during that time interval. This is what the Riemann integral represents. However, if we move a little closer to reality, we can deduce that there might be some random factors which can interfere with the vehicle's movement, such as an uphill, a rock slide on the road, or even a sudden storm that forces the driver to slow down. In other words, the actual speed of the vehicle at any given time is not entirely known. All those factors can be collectively referred to as **"noise"** and are represented by the stochastic integral of the equation's right-hand side.

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Taking this a little further, we can define a collection of all possible realizations of that "noise" and see how each one affects the final location of the vehicle X(t). Let's name that collection Ω and its elements (ω_1,ω2,ω3,…). Note that those elements are not necessarily independent. For example, a strong wind can be present in both ω_1 and ω_3. Each element can also be assigned with a probability of it occurring. Thus, we can get a little closer to estimating the final location X(t) by taking into account those events, as well as the probability of each one.

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2.2.3. Examples: In this section we will briefly present 3 processes that are based on SDEs and are widely used in credit risk modeling. Simulations and graphs of said processes will be presented in the next chapter.

i) Geometric Brownian Motion: A continuous time stochastic process the logarithm of which follows a **Brownian motion** with drift. It is widely used in finance for modeling stock prices, being the basis of the Black-Scholes model. A process S(t) is called Geometric Brownian Motion (GBM) if it satisfies the following stochastic differential equation:

$$\mathbf{dS(t) = \mu S(t)dt + \sigma S(t)dB(t)}$$

The equation can be solved with Itô's lemma giving us:

$$\mathbf{S(t) = S(0)exp\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B(t)}$$

It has the parameters which are μ and σ, while B(t) is a **Brownian Motion**. The μ and σ are called **percentage drift** and **percentage volatility** respectively.

From the above formula we can deduce that S depends on the process B(t) only through its final value.

The GBM tends to revert to 0 or ∞, depending on the parameters μ,σ. Specifically:

- If $\mathbf{\mu - \frac{\sigma^2}{2} < 0}$ the process tends to revert to 0.
- If $\mathbf{\mu - \frac{\sigma^2}{2} > 0}$ the process tends to infinity.

We can also replace the Brownian Motion in the above formulas with other Lévy process. In that case we have a more general form of the GBM. This gives us more flexibility about the models we can build.

ii) Ornstein-Uhlenbeck Process: A continuous time stochastic process that satisfies the following SDE:

$$\mathbf{dX(t) = \theta\big(\mu - X(t)\big)dt + \sigma dB(t)}$$

Using Itô's lemma we can get a solution of the above equation:

$$\mathbf{X(t) = \mu + exp(-\theta t)[X(0) - \mu] + \sigma exp(-\theta t)\int_0^T exp(-\theta s)dB(s)}$$

Like the GBM, it has the parameters **μ** and **σ**, but it also has a third parameter **θ**. The term **B(t)** denotes the **Brownian Motion**. This process has the notable property that as

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time passes, it approaches its mean μ. This approach is more intense as the parameter θ increases. Such a process is called a **mean-reverting process**.

From the above formula it becomes obvious that the process X depends on the process B(t) through its entire length and not just its final value, unlike the GBM.

As in the GBM, we can replace the Brownian Motion with any other Lévy process thus getting a more generalized form of the equation. However, this will also drastically alter the solution via Itô's lemma.

iii) Jump diffusion: A stochastic process that involves both **jumps** and **diffusion**. It has many applications including in option pricing which we will focus on this thesis. An example to the above would be the Ornstein-Uhlenbeck process (a diffusion process) but with jumps instead of being a continuous time process. In that case, the stochastic process of its SDE will be the process that yields the jump times.

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CHAPTER 3

SIMULATION OF LÉVY PROCESSES

In this chapter we will present the algorithm we follow in order to simulate the six processes described on chapter 2 and follow those steps to make simulations.

Firstly, we will show some plots that give a visual presentation of their flow. We will try to choose plots as representative as possible for each process. If needed, we will show at least two plots to see how the flow of the process changes depending on certain parameters.

Furthermore, we are going to simulate each process multiple times in order to estimate a mean of their final values or the actual distribution of their increments, whichever is more desirable for each one.

First we have the 3 processes from section 2.1:

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3.1. SIMULATION OF BROWNIAN MOTION

A Brownian Motion can be simulated with the following algorithm for a time interval (0,T).

1. Define a partition of the time interval as $0=t_0<t_1<\ldots<t_n=T$.
2. Set $B(t_0) = 0$
3. For i in 1:n :
    - Draw a number **n** from the Normal(0,1) distribution
    - Compute $\mathbf{B(t_i)} = B(t_{i-1}) + n\times(t_i - t_{i-1})$
4. End

A simulation example follows:

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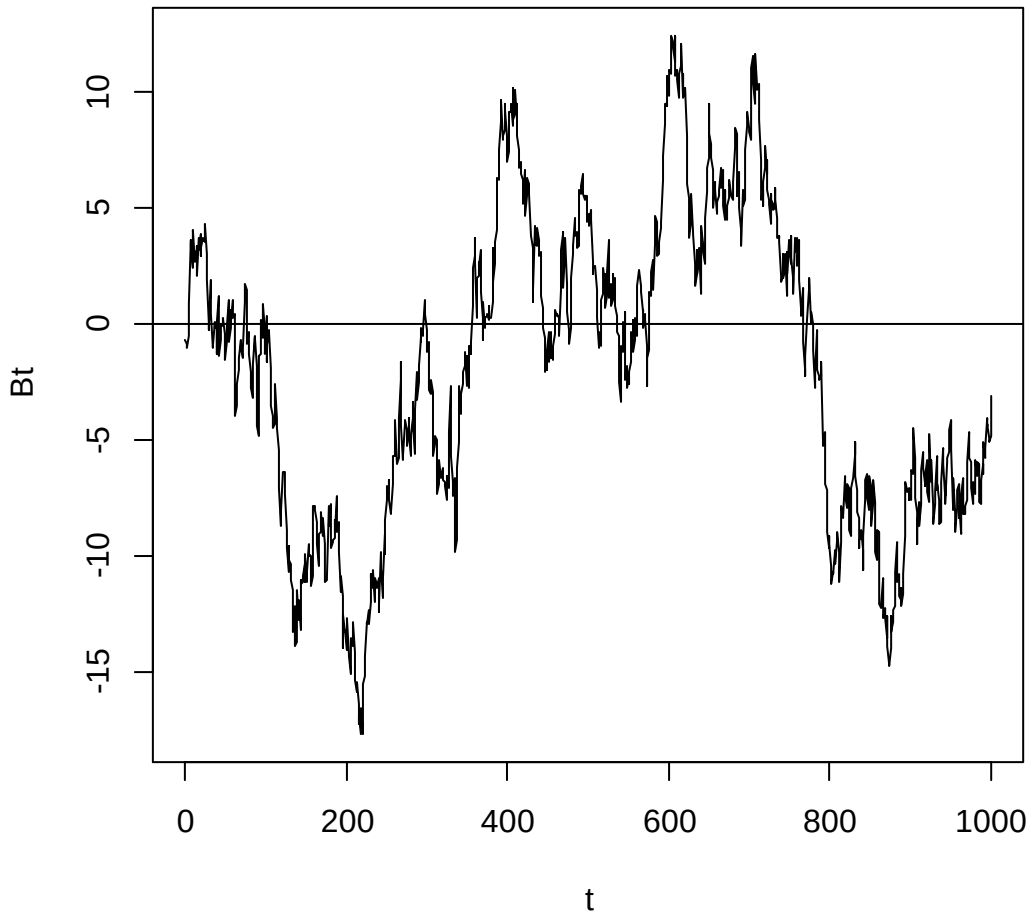
We will simulate 1000 values of a Brownian motion (from t=1,…,1000). As starting value, we set $B(t_0) = 0$ as in the properties. The final value sits at -2,969. All those values are plotted below:



Graph 1 Sample path of Brownian Motion with B(0)=0 and T=1000

We can deduce from the graph that the process does not appear to have an upward or downward trend. Instead it fluctuates around its initial value (0), which is denoted by the vertical line.

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There are no parameters of interest to estimate in this process.

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3.2. SIMULATION OF COMPOUND POISSON PROCESS

A Compound Poisson Process with rate **λ** can be simulated with the following algorithm (for a time horizon T):

1. Set $\mathbf{X(t_0)} = 0$, $t_0 = 0$
2. Set i=0
3. While t < T:
    - Set i = i+1
    - Draw a random number **u** from Uniform(0,1)
    - Compute $\mathbf{t_i} = -\log(u) / \lambda$ (the arrival time) and add it to t (the total time)
    - Draw a random number **N** from Normal(0,1). (the jump size)
    - Compute $B(t) = B(t-t_i) + N$
4. End

A simulation example follows:

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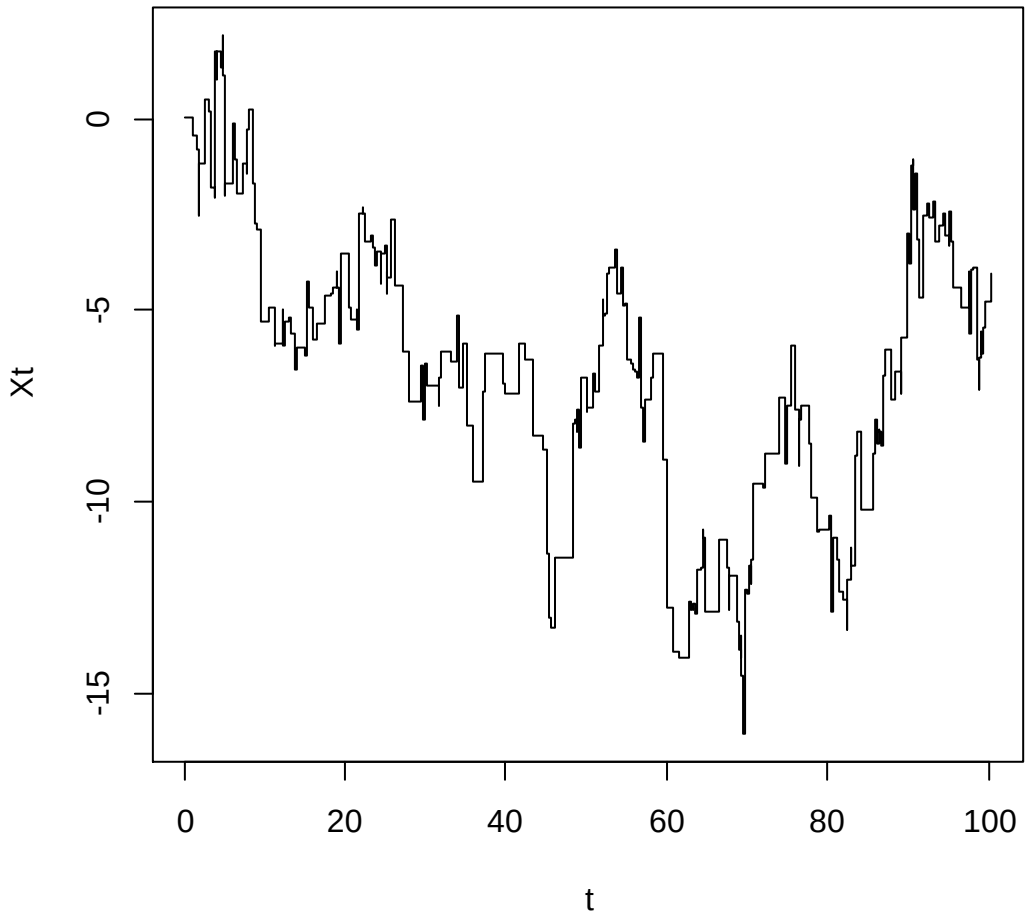
First we have a Compound Poisson Process with λ=3 and the time interval of (0,100).We will use a starting value of 0. The simulation yields 280 values which means that 280 events occurred during that time interval. Thus during the final time t=100, the simulation gives us the sum of 280 normally distributed random variables. The final value sits at -4,041. All those values are plotted below:



Graph 2 Sample path of Compound Poisson Process with λ=3, X(0)=0 and T=100

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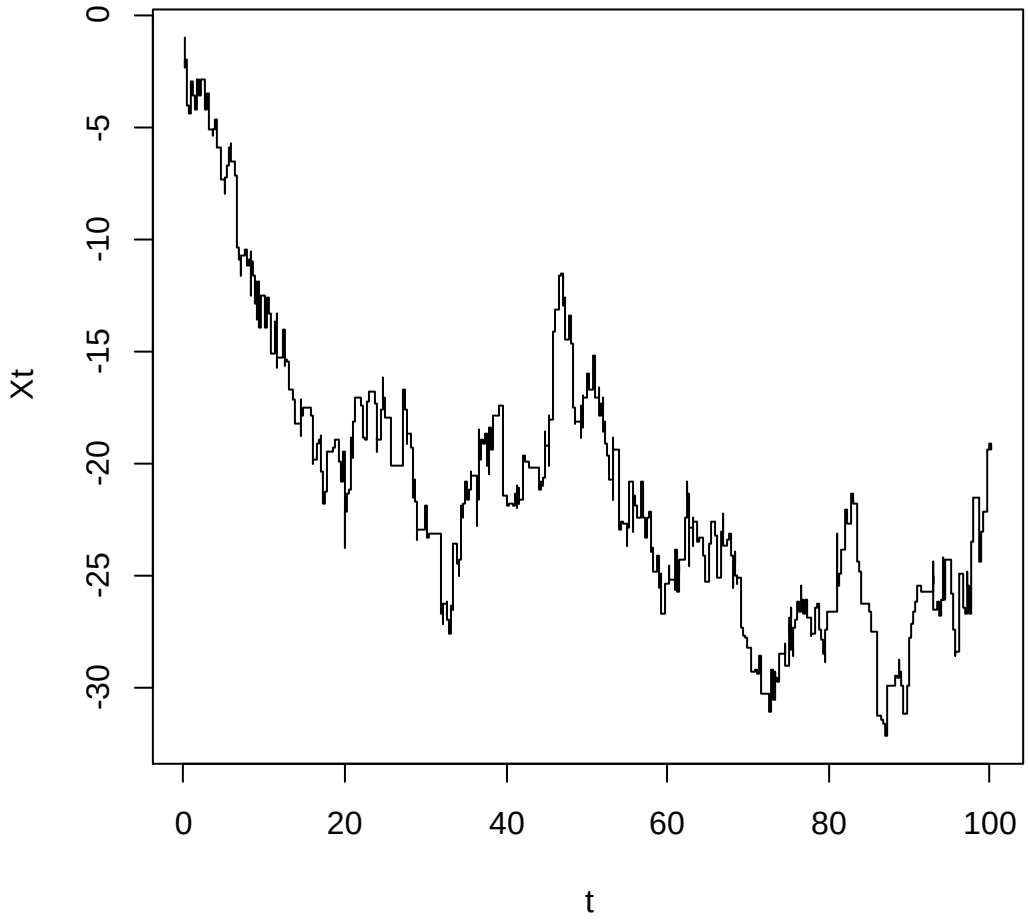
We will now increase the rate λ from 3 to 5 while keeping the time interval as well as the initial value the same as before. This time, the simulation yields 482 values, a number much higher than before. This is expected as the rate λ is higher meaning that more events are expected to occur during the same time interval. The final value sits at -19,371. All those values are plotted below:



Graph 3 Sample path of Compound Poisson Process with λ=3, X(0)=0 and T=100

https://doi.org/10.26219/heal.aueb.6265

The parameter of interest of this process would be the number of values yielded. In the following table we present a mean of those numbers for a set amount of simulations using the same parameters and time intervals as above:

Parameters	Number of simulations	Mean values yielded
$\lambda = 3$	10	303,3
	100	301,7
	1000	300,1
	10000	300,1
$\lambda = 5$	10	495,4
	100	497,8
	1000	501,8
	10000	500,9

We can easily deduce that the average number of values yielded is approximately equal to $\boldsymbol{\lambda \times T}$, where T is the time horizon which was 100.

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3.3. SIMULATION OF GAMMA PROCESS

A Gamma Process with parameters a,b can be simulated with the following algorithm for the time interval (0,T):

1. Define a partition of the time interval as $0=t_0<t_1<\ldots<t_n=T$.
2. Set $\mathbf{N(t_0)} = 0$
3. For i in 1:n :
    - Draw a random number **g** from Gamma(a,b)
    - Compute $\mathbf{N(t_i)} = N(t_{i-1}) + g \times dt$, where $dt=t_i-t_{i-1}$
4. End

A simulation example follows:

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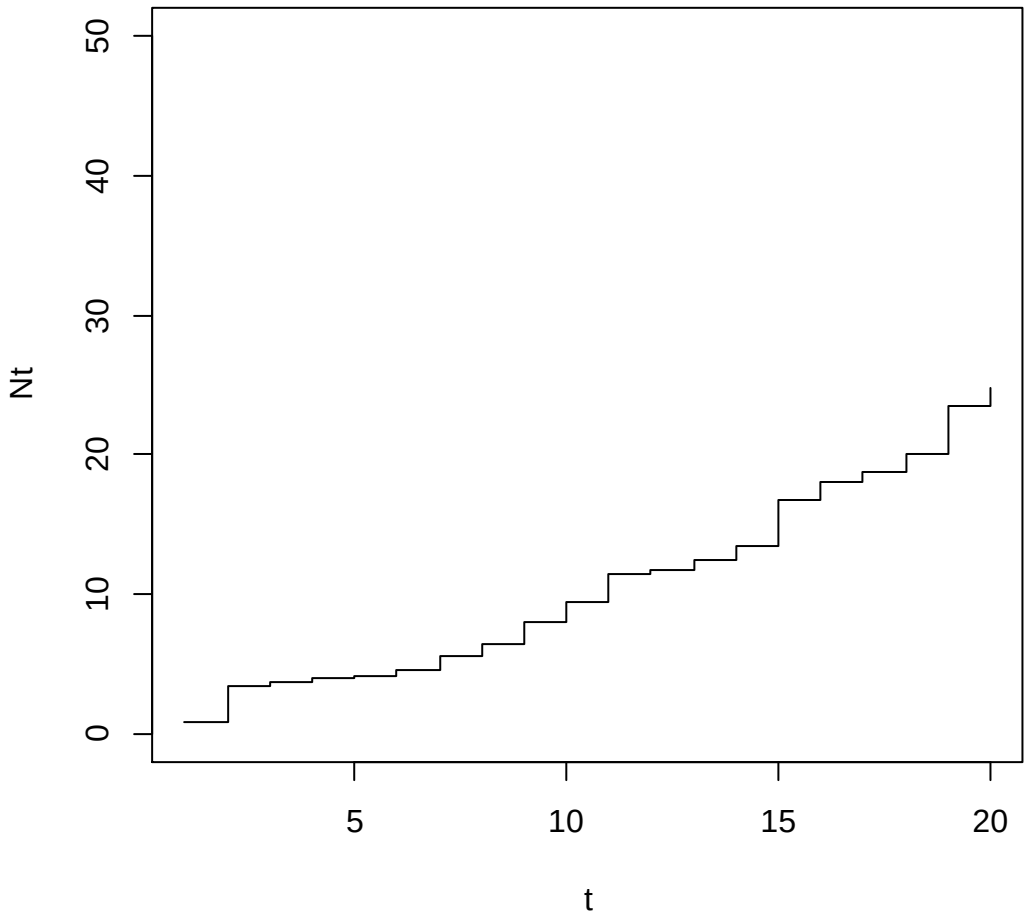
We are first going to simulate 20 values of the process with a = 1 and b = 1 for the time points t=1,…,20. We will use $N(t_0) = 0$ for the initial value, as in the definition. The final value sits at 24,706. All those values are plotted below:



Graph 4 Sample path of Gamma Process with a=1, b=1 and T=20

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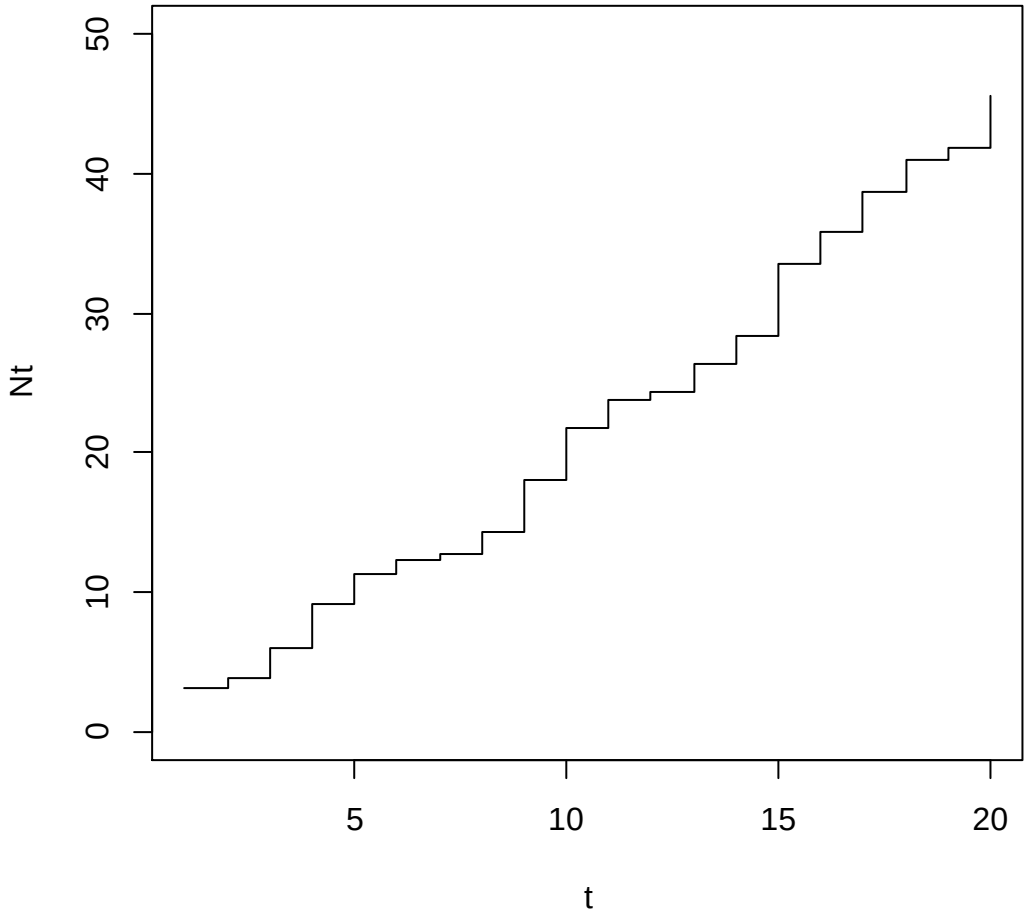
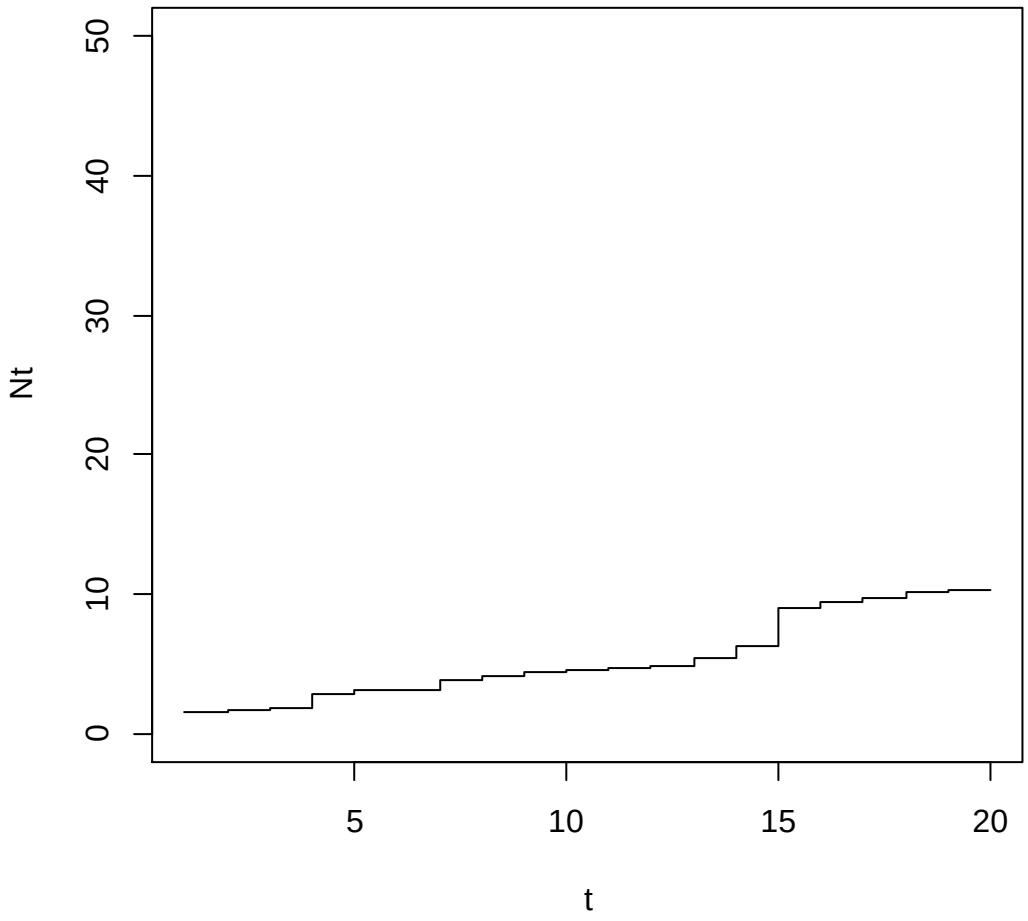
We will now double the shape parameter a from 1 to 2, while keeping the same b=1 as well as the same time points. The final value sits at 45,587 which is almost double the final value of the first simulation. All those values are plotted below:



Graph 5 Sample path of Gamma Process with a=2, b=1 and T=20

https://doi.org/10.26219/heal.aueb.6265

Next, we will double the rate parameter b from 1 to 2, while keeping the same a=1 as well as the same time interval. The final value sits at 10,343 which is almost half the final value of the first simulation. All those values are plotted below:



Graph 6 Sample path of Gamma Process with a=1, b=2 and T=20

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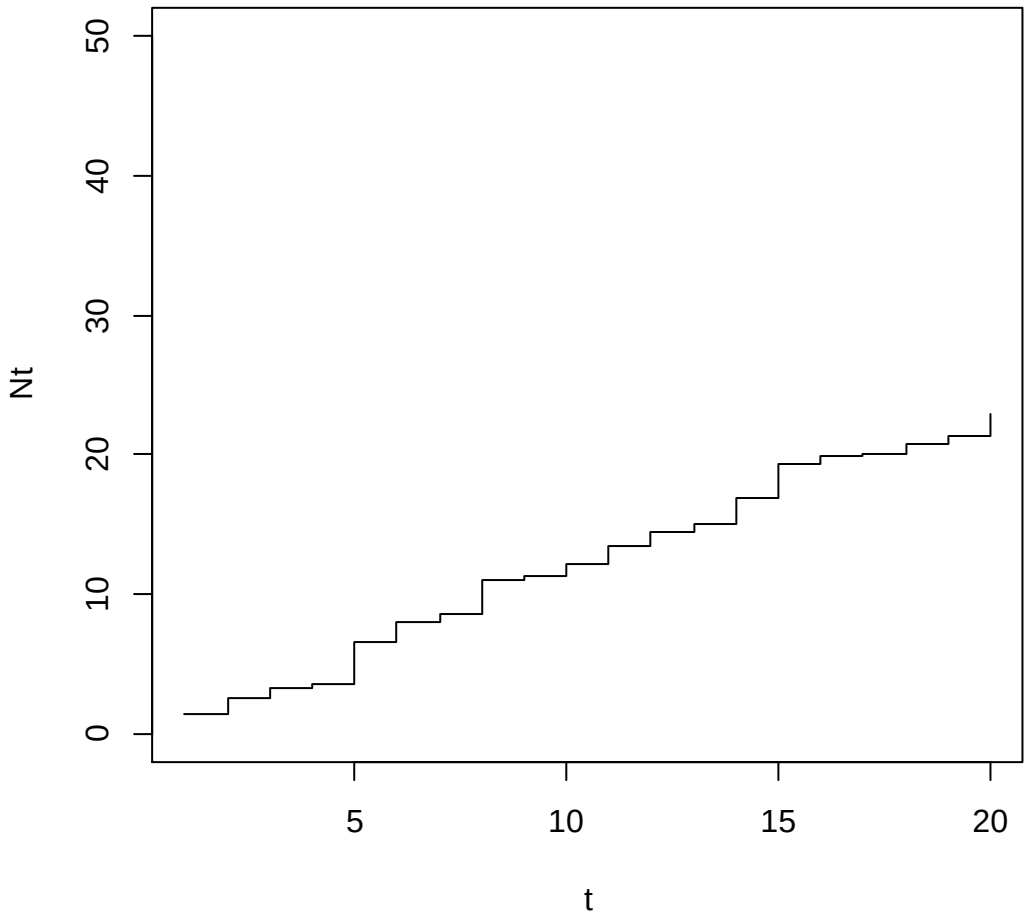
Finally we will double both parameters a and b from 1 to 2 while using the same time points. The final value sits at 25,854 which is very close to the final value of the first simulation. All those values are plotted below:



Graph 7 Sample path of Gamma Process with a=2, b=2 and T=20

https://doi.org/10.26219/heal.aueb.6265

The parameter of interest of this process would be the final value. In the following table we present a mean of those numbers for a set amount of simulations using the same parameters and time intervals as above:

Parameters	Number of simulations	Mean final values
a = 1 b = 1	10	18,95
	100	19,56
	1000	20,15
	10000	19,94
a = 2 b = 1	10	37,20
	100	39,67
	1000	40,36
	10000	40,03
a = 1 b = 2	10	9,27
	100	10,16
	1000	9,94
	10000	10,02
a = 2 b = 2	10	19,74
	100	20,10
	1000	20,21
	10000	20,01

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3.4. SIMULATION OF GEOMETRIC BROWNIAN MOTION

A Geometric Brownian Motion (GBM) with parameters μ,σ can be simulated with the following algorithm for a time interval (0,T).

1. Define a partition of the time interval as $0=t_0<t_1<\ldots<t_n=T$.
2. Choose an initial value $G(t_0)$
3. Simulate a Brownian Motion B(t) for those time points as described in 3.2
4. For i in 1:n :
    - Compute $\mathbf{G(t_i)} = G(t_{i-1}) + \exp(\mu - \sigma^2/2)\times dt + \sigma\times B(t_i)$, where $dt=t_i-t_{i-1}$
5. End

A simulation example follows:

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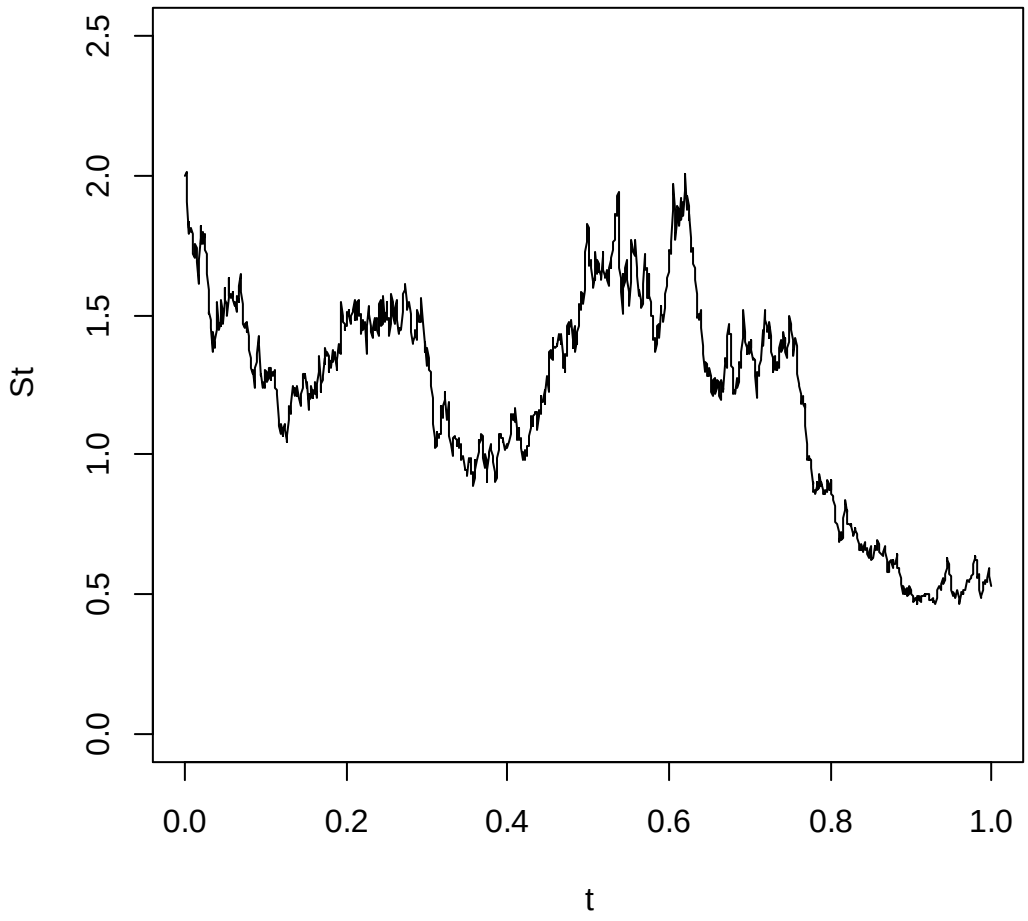
We are going to simulate 1000 values with parameters μ = 0,02 and σ = 1 and the time interval (0,1). We will choose $S(t_0) = 2$ for the initial value. The final value sits at 0,529. All those values are plotted below.



Graph 8 Sample path of Geometric Brownian Motion with μ=0.02, σ=1, S(0)=2 and N=1000

We see that the process tends to revert to 0 but it does so rather slowly. This was expected because $\boldsymbol{\mu - \frac{\sigma^2}{2} = -0,48} < 0$

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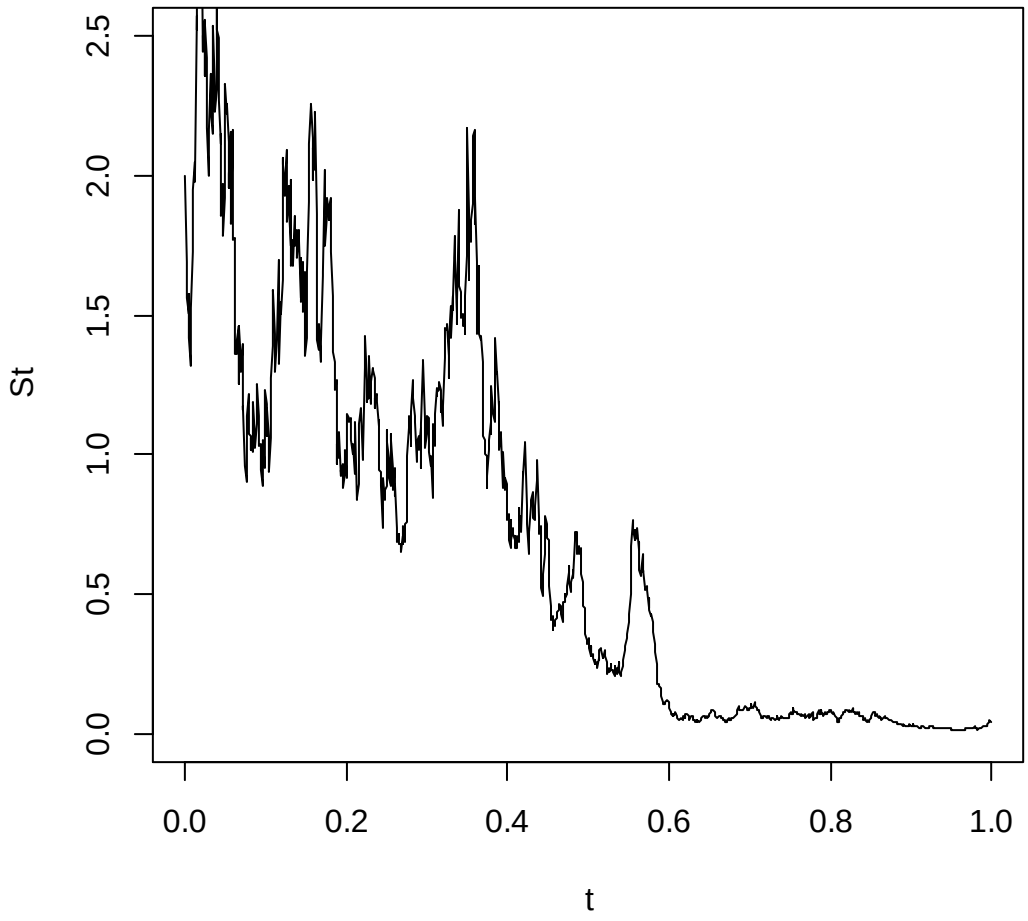
We will now increase the volatility σ from 1 to 3 while keeping all other parameters as well as the time interval the same. The final value sits at 0,403. All those values are plotted below:



Graph 9 Sample path of Geometric Brownian Motion with μ=0.02, σ=3, S(0)=2 and N=1000

We can easily figure that the process reverts to 0 much faster than before which is expected since we increased the volatility.

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Since the GBM reverts to either 0 or infinity, there are no parameters of interest to estimate. However, there are two formulas for the mean and variance of the process that can be checked. Those are the following:

- $\mathbf{E[X(t)] = X(0)\,exp(\mu t)}$
- $\mathbf{Var[X(t)] = X^2(0)e^{2\mu T}(e^{\sigma^2 t} - 1)}$

We would like to calculate each of them for t = T (i.e. the time horizon).

Setting X(0) = 2, μ = 0.02, t = 1 in the above formulas we get:

If σ = 1:

- $\mathbf{E[X(T)] = 2.04}$
- $\mathbf{Var[X(T)] = 4.16}$

If σ = 3:

- $\mathbf{E[X(T)] = 2.04}$
- $\mathbf{Var[X(T)] = 12410.45}$

Let us now try to estimate the above quantities by simulating the GBM multiple times. The following table presents the results:

Parameters	Number of simulations	Mean of final values	Variance of final values
σ = 1	10	2,33	6,03
	100	1,99	5,62
	1000	2,03	6,51
	10000	1,99	6,39
σ = 3	10	0,51	1,38
	100	1,37	81,61
	1000	1,39	149,23
	10000	1,35	219,03

We can see that once we increase the parameter σ, the variance rises drastically. As for the means, it seems they drop when σ increases but with such a high variance, it might not be wise to compare them. It should be noted, however, that for σ = 1, the means are close enough to the actual mean calculated above.

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3.5. SIMULATION OF ORNSTEIN - UHLENBECK PROCESS

An Ornstein-Uhlenbeck process with parameters **θ**, **μ**, **σ** can be simulated with the following algorithm for a time interval (0,T).

1. Define a partition of the time interval as $0=t_0<t_1<\ldots<t_n=T$
2. Choose an initial value $\mathbf{X(t_0)}$
3. For i in 1:n :
    - Draw a number **N** from Normal(0,1) distribution
    - Compute $\mathbf{X(t_i)} = X(t_{i-1}) + \theta\times(\mu - X(t_{i-1}))\times dt + \sigma\times N\times dt$, where $dt=t_i-t_{i-1}$
4. End

A simulation example follows:

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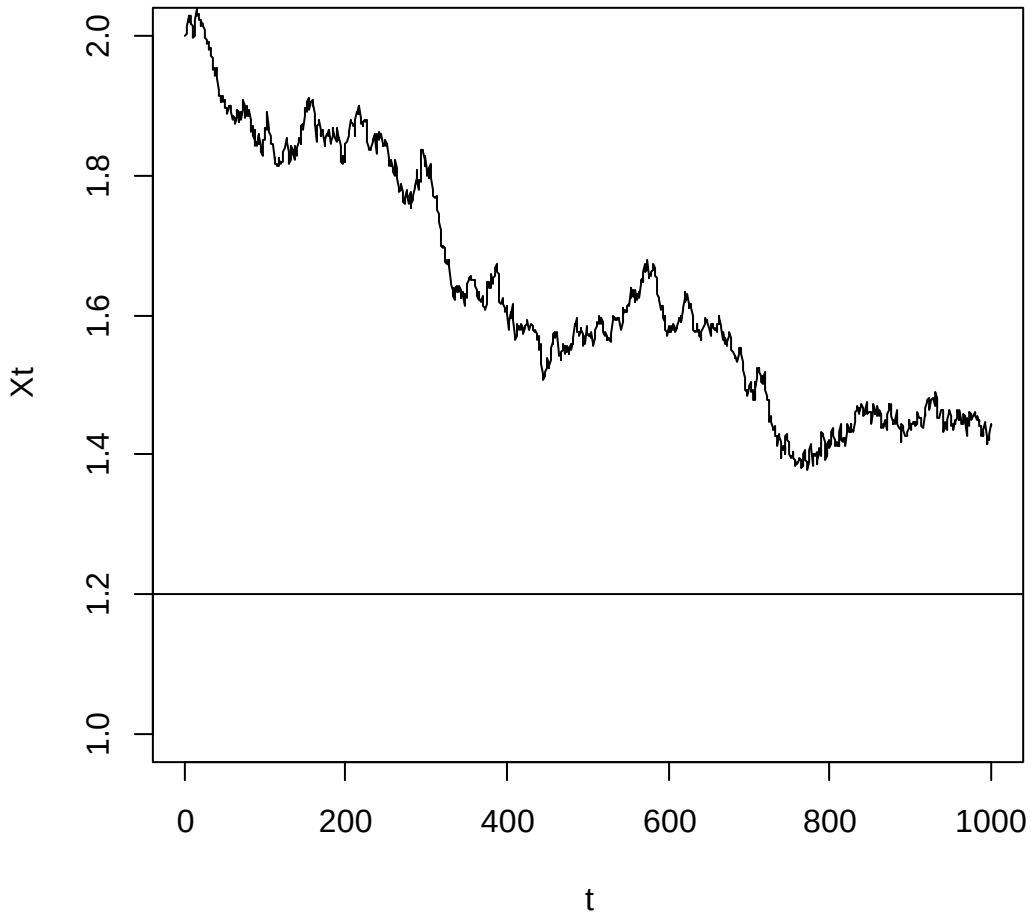
We are first going to simulate 1000 values of the process with $\mu = 1.2$, $\sigma = 0.3$ and $\theta = 1$ for the time points $t=1,\ldots,1000$. We will choose $X(t_0) = 2$ for the initial value. The final value sits at 1,439. All those values are plotted below:



Graph 10 Sample path of Ornstein-Uhlenbeck process with μ=1.2, σ=0.3, θ=1, X(0)=2 and T=1000

From the plot we can easily see that the process tends to revert to its mean μ which is 1.2, however the rate of it approaching that value is rather slow.

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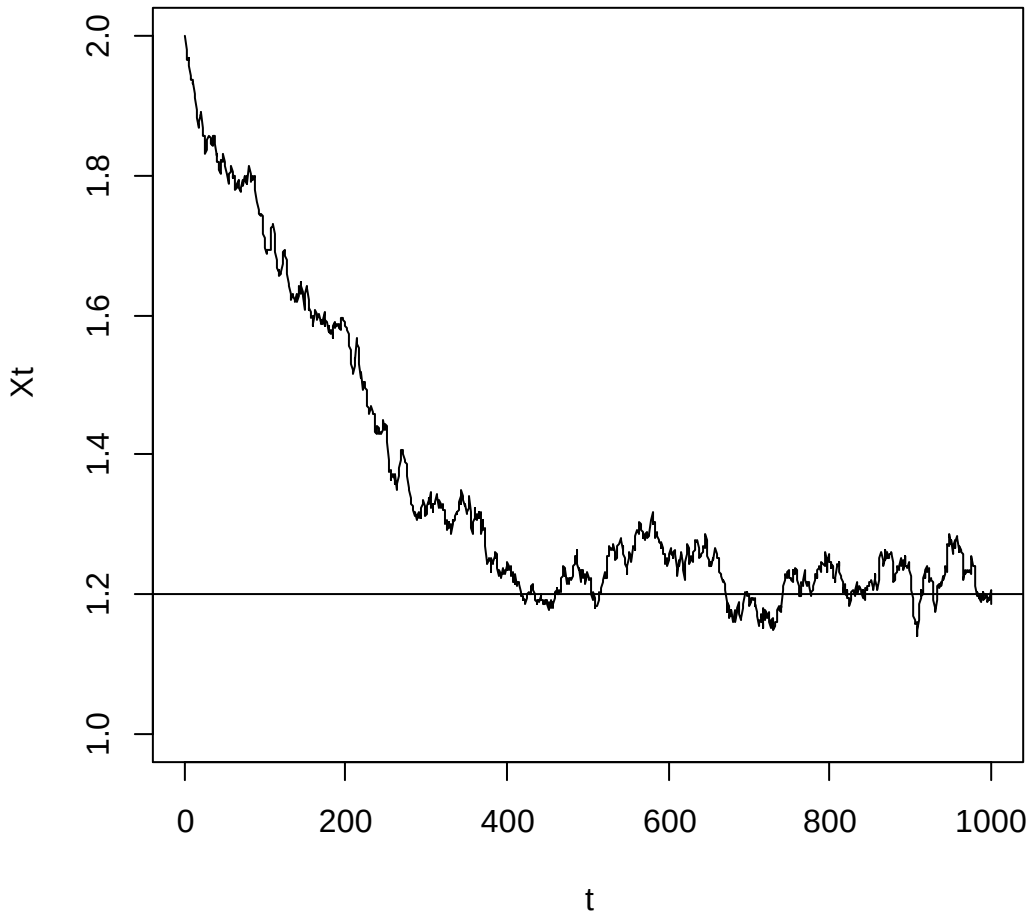
We will now increase the parameter θ from 1 to 3 while keeping all other parameters as well as the time points the same. The final value sits at 1,184. All those values are plotted below:



Graph 11 Sample path of Ornstein-Uhlenbeck process with μ=1.2, σ=0.3, θ=3, X(0)=2 and T=1000

We can see that by increasing the rate θ, the process now reverts to its mean μ much faster than before and once reaching it (approximately at time t = 400), it fluctuates around that value for the remainder of the simulation. This is similar to the GBM we presented before.

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The parameter of interest in this process is a bit more complicated the previous processes. Specifically, we would like to find how long it takes for the process to revert to its mean.

There are two main ways to approach this. One would be to calculate the probability of that occurring by simulating the process many times and computing the percentage of those reaching the mean. The other would be to find the first value of the process that reaches the mean and estimate a mean of how many observations it takes to get there each time. Since each approach offers a different point of view of the process, we will present both of them.

First we have the percentage of those reaching the mean:

Parameters	Number of simulations	% to reach μ
θ = 1	10	10,0 %
	100	15,0 %
	1000	11,7 %
	10000	12,6 %
θ = 3	10	90,0 %
	100	73,0 %
	1000	73,6 %
	10000	72,6 %

We can easily see how much the parameter θ influences the flow of the process.

Next we have the average number of observations required to reach the mean. In case a simulation does not reach the mean at all, that number is set to 1000:

Parameters	Number of simulations	Average number of observations to reach μ
θ= 1	10	997,2
	100	968,1
	1000	974,8
	10000	973,4
θ = 3	10	639,6
	100	737,2
	1000	759,1
	10000	764,3

Similarly, it becomes obvious that the higher the parameter θ, the less time it takes for the process to reach the long-term mean.

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3.6. SIMULATION OF JUMP DIFFUSION

For this simulation we will use the Ornstein-Ulhenbeck process but with a small twist. Instead of just simulating the increments for all the time points, we will also simulate the jump times ($T_1, T_2, \ldots T_m$) as well as their sizes. All in all, the algorithm will be the following for the time interval (0,T):

1. Define a partition of the time interval as $0=t_0<t_1<\ldots<t_n=T$
2. Choose an initial value $\mathbf{X(t_0)}$, $t_0 = 0$
3. For i in 1:m :
   - Simulate the Ornstein-Uhlenbeck process for the time interval (T_{i-1}, T_i), assuming $T_0 = 0$, using as starting value $\mathbf{X(T_{i-1})}$. The simulation will use the time points t_i that fall between T_{i-1} and T_i
   - Draw a number from the jump distribution, J_i (the jump size)
   - Add $\mathbf{J_i}$ to $\mathbf{X(T_i)}$
4. End

A simulation example follows:

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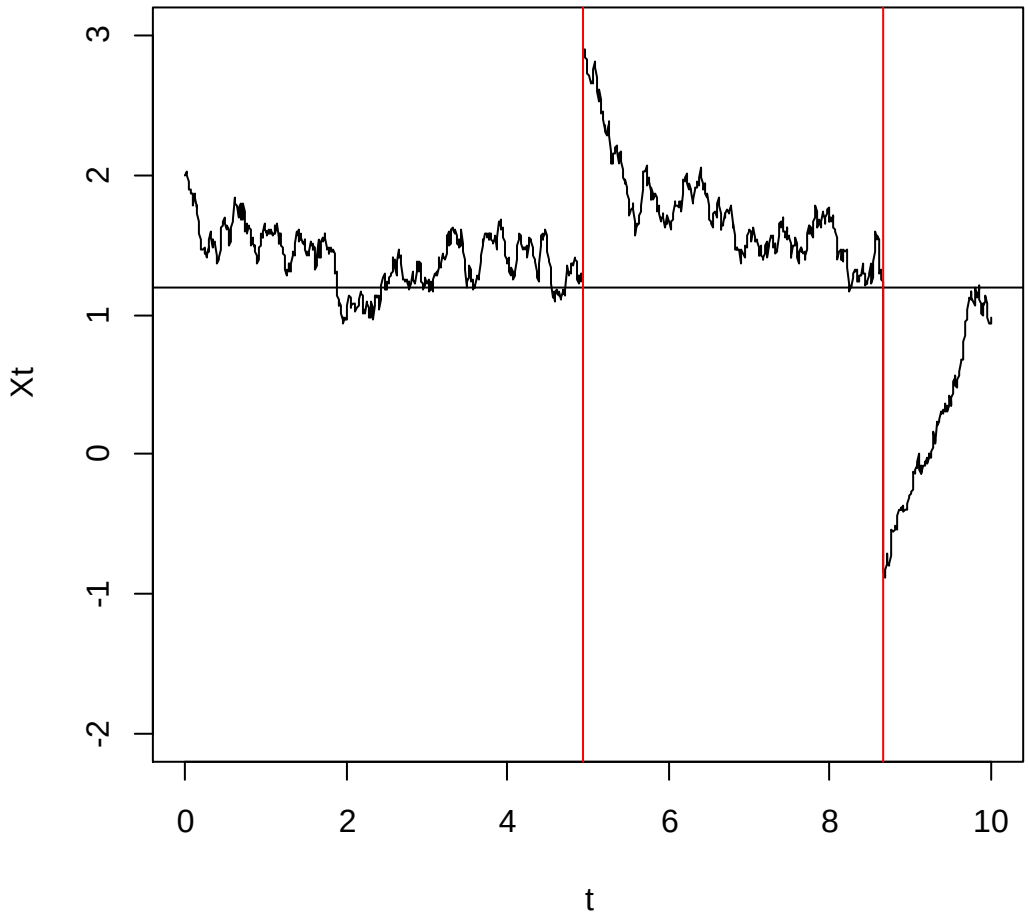
We will use the same parameters as in 3.3, that is $\mu = 1.2$, $\sigma = 0.3$ and $\theta = 1$ for the time points $t=1,\ldots,1000$. We will choose $X(t_0) = 2$ for the initial value. As for the jump times we will consider an exponential distribution with rate $\lambda = 0.3$. The jump sizes, on the other hand, will follow a Normal(0,1) distribution. The final value sits at 0,976. All those values are plotted below:



Graph 12 Sample path of Jump Diffusion with μ=1.2, σ=0.3, θ=3, X(0)=2 and T=1000. The jump times follow an Exp(0.3) distribution. The jump sizes follow a N(0,1) distribution

We can see that two jumps occurred in the time interval (0,10), denoted by the red vertical lines. A table of the jumps times and sizes is shown below. The process slowly tends to revert to its mean μ, which is 1.2, just like in section 3.3. Every time a jump sends the process away from μ, it reverts to it quite rapidly and fluctuates around it until the next jump.

Jump time (T_i)	Jump size (J_i)
4,932	+ 0,782
8,651	- 1,057

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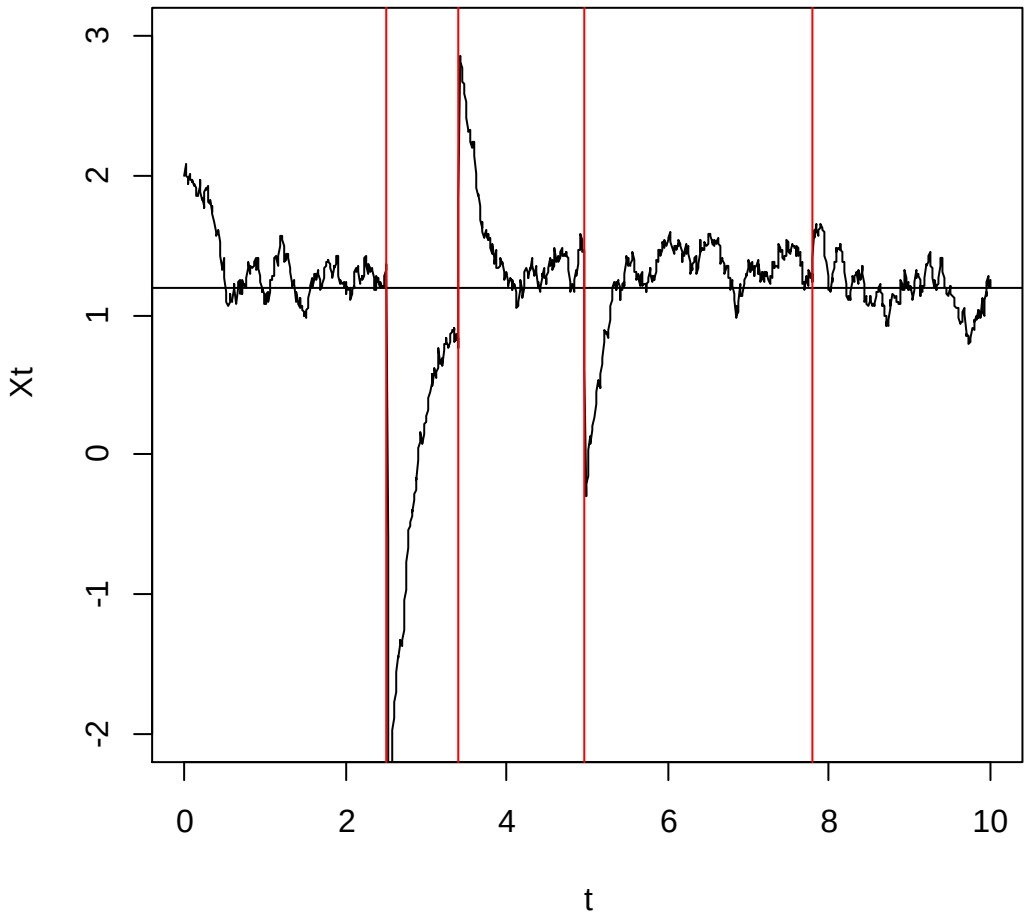
We will now increase the parameter θ from 1 to 3 while keeping all other parameters as well as the time points the same. The final value sits at 1,259. All those values are plotted below:



Graph 13 Sample path of Jump Diffusion with μ=1.2, σ=0.3, θ=3, X(0)=2 and T=1000. The jump times follow an Exp(0.3) distribution. The jump sizes follow a N(0,1) distribution

We can see that four jumps occurred during the time interval (0,10), denoted by the red vertical lines. A table of the jumps times and sizes is shown below. The process, once again tends to revert to μ but it does so much faster than before. This happens due to the increase of θ. Just like before, after every jump, the process rapidly reverts close to μ.

Jump time (T_i)	Jump size (J_i)
2,518	- 2,011
3,407	+ 1,021
4,966	- 0,874
7,793	+ 0,197

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The parameters of interest are the same as the Ornstein-Uhlenbeck process, but they must be estimated for every jump that happens, instead of just once. Also there is a significant difficulty given the fact that after a jump sends the process away, there might be a second one before the process can revert to its mean. Taking that into account, we won't conduct estimations for this process.

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CHAPTER 4

LÉVY PROCESSES APPLICATION IN CREDIT RISK MODELING

In this chapter we will try to model the probability of default on an entity. There are many ways to approach this and we will focus the two most widely used. Those are the **firm-value models** and the **intensity models**.

The former, simply models the asset value and tries to estimate the probability of it falling below a threshold L. Such models are presented in section 4.2.

The latter, focuses on directly estimating and illustrating the default probability, giving a rough representation of the intensity of the risk at any time. Such models are presented in section 4.3.

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4.1. PROBABILITY OF DEFAULT AND ITS EFFECT ON YIELDS

As we already mentioned in chapter 1 the default probability is a factor that can greatly affect the price of a bond. Generally, the higher the default probability, the more the price drops to compensate for potential losses for the holder. In this chapter we will also try to estimate the price based on the default probability.

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4.2. FIRM-VALUE MODELS

As we mentioned, a firm value model can be used to estimate the asset value's flow and determine whether it falls below a threshold L, in which case we have a **default**. We will use the processes described in chapter 2 to build those models.

There are two ways of approaching this. The first acknowledges as default the situation where only the **final** asset value (i.e. at maturity) sits below L. The second is much stricter and says that default occurs **any time** the asset value falls below L. Consequently, the probability of default using the second approach is much higher. Since each one offers its own applications, we will present both of them.

We will denote the asset value at time t with $\mathbf{V_t}$, $(0 < t < T)$ and examine each of the processes to see which of them are suitable for estimating default probabilities.

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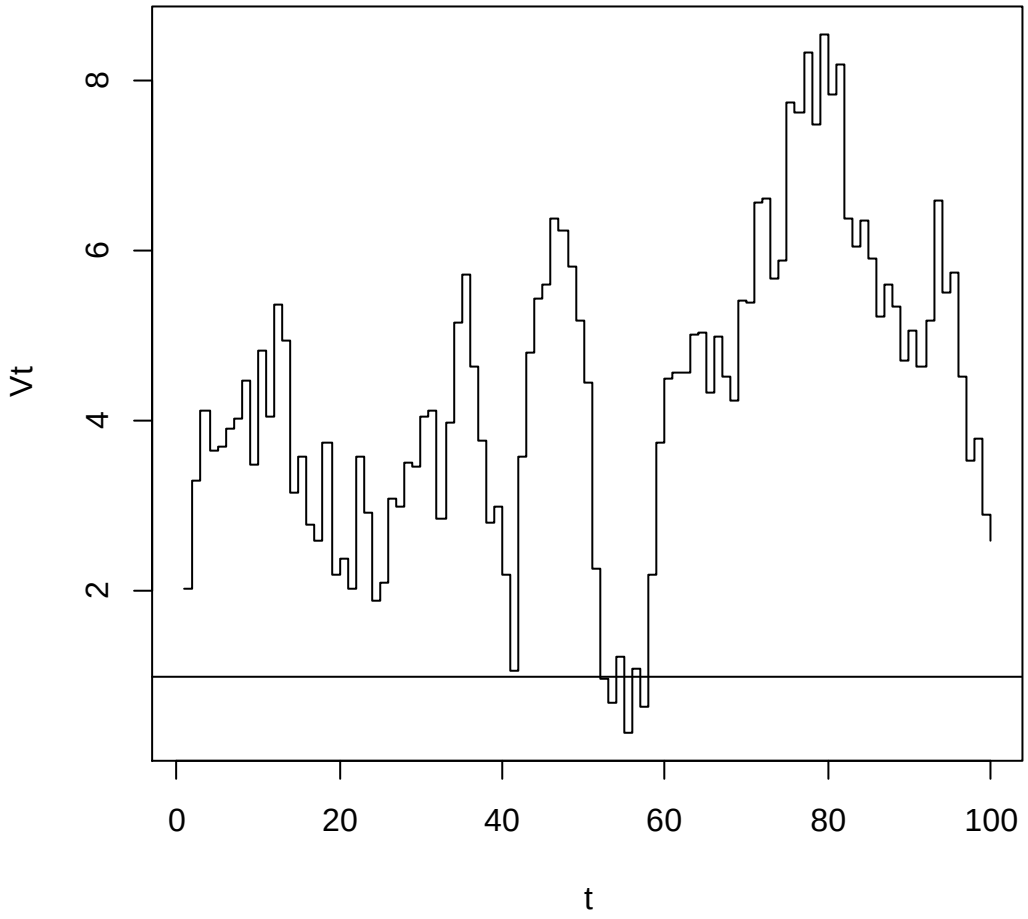
i) The **Brownian motion** (see 2.1.3) can be used for simulations because it does not have a trend. Assuming a threshold L, a possible outcome would be the following:



Graph 14 Sample path of Brownian Motion with X(0)=2, using as default threshold L=1

Here the starting value is 2 and the threshold L is set to be 1. We can see that the process falls below L around the time t = 50 but returns above it shortly after and does not drop below L until the end.

Based on the two methods of determining default, we can deduce that according to the first, the debtor does not default, as the final value sits above 1, while the second acknowledges a default as some values are below L.

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Below is a table that records the probability of the processes falling below L at any given time (second row) or at final time (third row):

Number of simulations	% of falling below L at the final time	% falling below L at any time
10	30,0 %	90,0 %
100	43,0%	94,0%
1000	45,1%	86,2%
10000	46,3%	87,4%

By comparing the probabilities, it becomes obvious that the second approach yields a much greater default probability due to it being stricter than the first.

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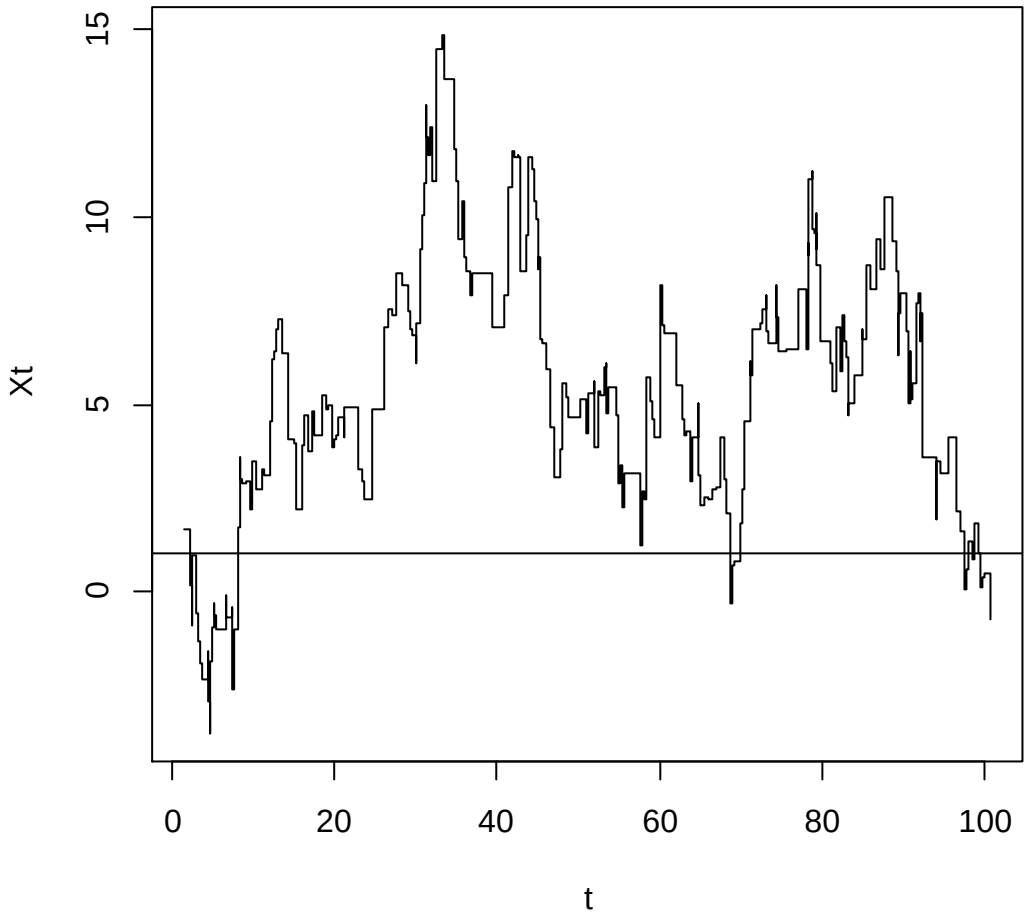
ii) The **Compound Poisson Process** itself, is not ideal for modeling the asset value since its initial value is always 0. However, we could get around that limitation by adding the initial asset value (V_0) to every value observed, and technically starting from the initial asset value, instead of 0.

With that in mind, a possible flow of the process could be the following:



Graph 15 Sample path of Compound Poisson process with λ=3, X(0)=0 and T=100, with all its values increased by 2, and using as default threshold L=1

Below is a table that records the probability of the processes falling below L at any given time (second row) or at final time (third row):

Number of simulations	% of falling below L at the final time	% falling below L at any time
10	30,0 %	90,0 %
100	53,0 %	71,0 %
1000	44,6 %	88,5 %
10000	45,2 %	87,5 %

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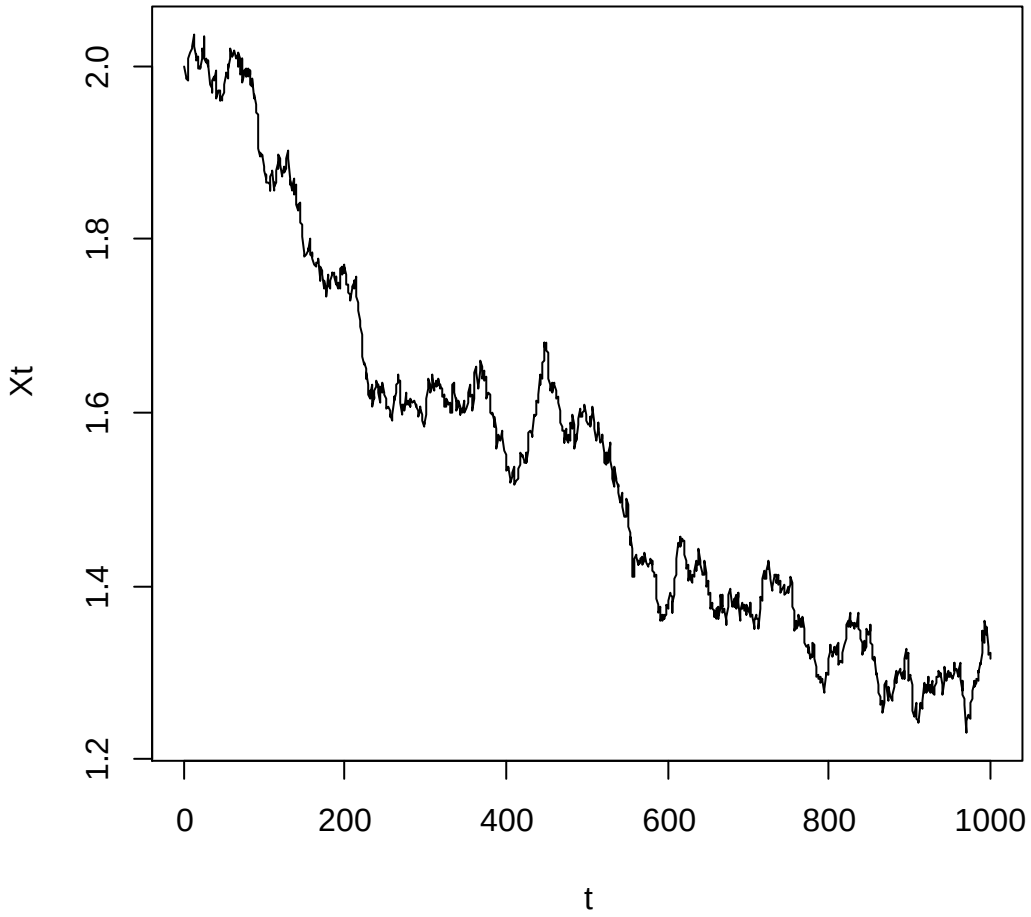
iii) The **Gamma Process** is a strictly increasing process and therefore it cannot be used to model the asset value.

iv) The **Geometric Brownian Motion** is a process that tends to revert to 0. Hence, it is suitable to model the asset value only when it is certain the value reaches "close" to 0 in the long-term and offers little flexibility.

v) The **Ornstein-Uhlenbeck Process** is also a mean-reverting process and like the GBM it is only useful when the asset value tends to approach a certain value in the long-term.

Since the process tends to revert to its mean, then fluctuates around it, one reasonable choice for the threshold L, would be a value below that mean (or above, if we start lower than the mean). For example we could choose the L to be $0.9\times\mu$.

With that in mind, a possible flow of the process would be the following:



Graph 16 Sample path of Ornstein-Uhlenbeck process with μ=1.2, σ=0.3, θ=3, X(0)=2 and T=1000, using as default threshold L=1.08

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We can see that although the process reaches its mean, which is 1,2, it does not fall below L, which is 1,2×0,9 = 1,08.

Below is a table that records the probability of the processes falling below L at any given time (second row) or at final time (third row):

Number of simulations	% of falling below L at the final time	% falling below L at any time
10	0,0 %	10,0 %
100	2,0 %	2,0 %
1000	2,2 %	4,2 %
10000	1,9 %	3,9 %

We can see that only a low percent of simulations actually fall below 1,08. This might happen because of the low σ parameter, which is responsible for the low variation once the process reaches μ. Let's try increasing σ to see how much influence it exerts.

Below is the same table but for σ = 0,8 instead of σ = 0,3:

Number of simulations	% of falling below L at the final time	% falling below L at any time
10	10,0 %	40,0 %
100	31,0 %	53,0 %
1000	22,0 %	45,5 %
10000	21,8 %	45,6 %

We can see that by increasing σ, a much higher percent of simulations yield default.

vi) The **Jump Diffusion** is similar to the Ornstein-Uhlenbeck Process but it contains jumps. Therefore, it offers much utility when there are "**shocks**" in the financial market every now and then. A "**shock**" is an unexpected or unpredictable event that affects economy, either positively or negatively.

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4.3. INTENSITY MODELS

Intensity models focus directly on modeling the survival probability. The basic idea is that at any time t, there is a probability that the debtor defaults, which depends on the overall "health". The "health" depends on the **default intensity** (say λ) which represents how possible it is to default at any given time. Let's give a proper definition of this new term.

Let's assume the debtor has "survived" up to time t and let's denote the default time with τ. The intensity of default can be seen as the following:

$$\boldsymbol{\lambda_t = \lim_{h \to 0} \frac{P(t < \tau < t + h \mid \tau > t)}{h}}$$

As for λ there are two cases for it. The first one assumes that λ is a deterministic and the other assumes that λ is stochastic and hence it is not known at any given time. We will now present two types of models, one for each of the above cases.

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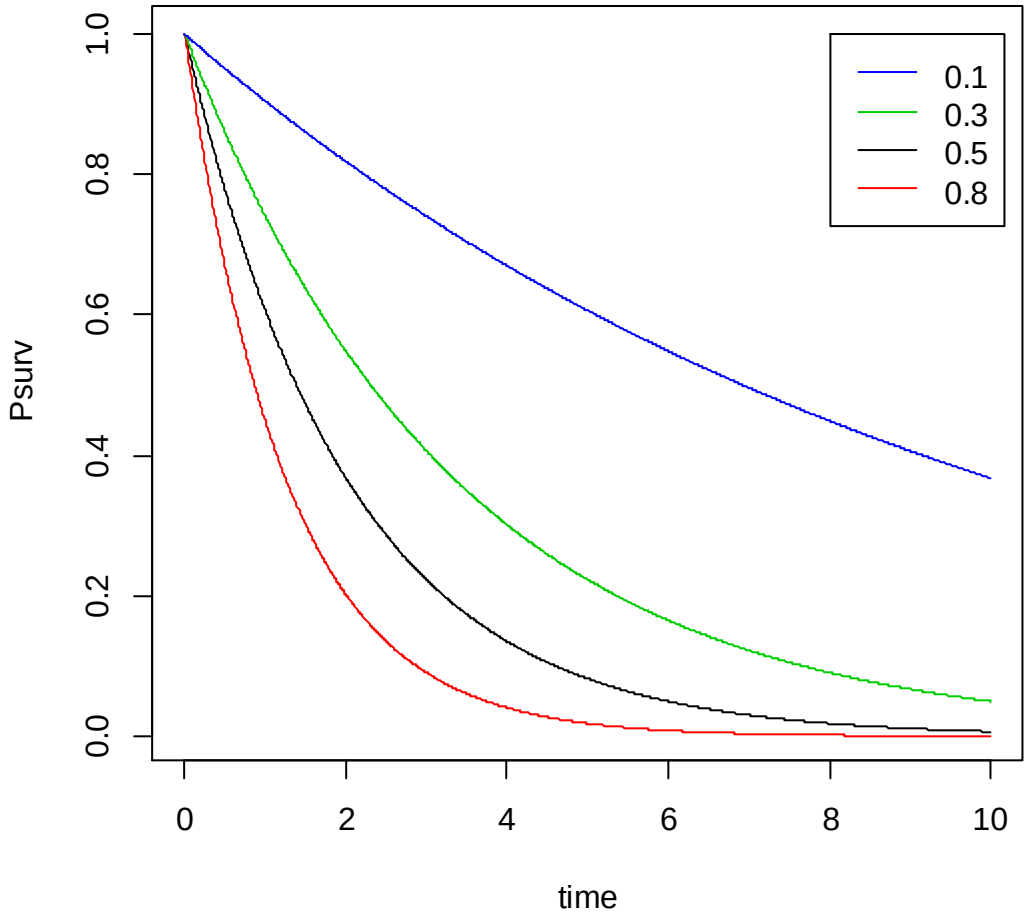
4.3.1 Jarrow and Turnbull model:

A widely used intensity model was proposed by Jarrow and Turnbull using the Poisson process. According to it, the survival probability up to time t equals:

$$\mathbf{P_{Surv}(t) = exp\ (-\lambda t)}$$

It follows from the formula that the higher the λ, the lower the survival probability. Also, the said probability decreases as time passes, which is expected of course.

If λ is constant, we can easily estimate the survival probabilities for any time t. The following graph illustrates the survival probabilities for various values of λ.



Graph 17 Survival probabilities from Jarrow and Turnbull model using the Homogenous Poisson process (HP) for certain values of λ

We can easily see that the higher λ is, the lower the probability of survival becomes. Some of those values are also presented in the following table:

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Parameter λ	Probability of Survival %			
	t=0	t=2	t=5	t=10
0,1	100,00	81,87	60,65	36,79
0,3	100,00	54,88	22,31	4,98
0,5	100,00	36,79	8,21	0,67
0,8	100,00	20,19	1,83	0,03

However, it is natural to assume that λ is not constant but rather fluctuates as time passes. For example we could have the following values for λ:

λ	from time	to time
0,5	0	2
0,1	2	5
0,7	5	8
0,4	8	10

With the above table, we can build the graph for the survival probabilities.

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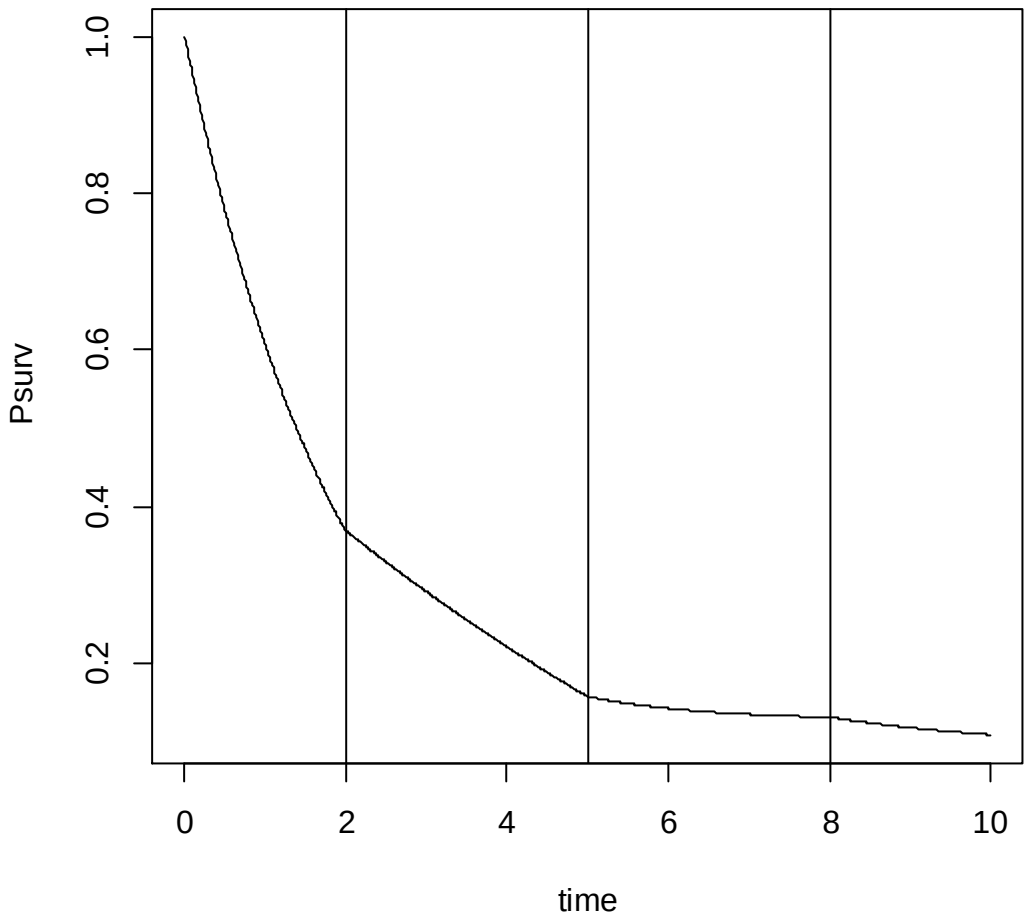

Graph 18 Survival probabilities of Jarrow and Turnbull model using the Inhomogenous Poisson process (IHP)

The vertical lines denote the time points where the parameter λ changes. We can see that there is a noticeable change of the rate the line drops between those time points due to the different λ.

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4.3.2 Cox model:

The Jarrow and Turnbull model, presented above assume λ to be deterministic, i.e. known beforehand. However, a more general category of such models can be build, by assuming λ to be stochastic. Those are called Cox models and are referred to as **doubly stochastic** models because we have two sources of randomness. The first one is the stochastic behavior of the parameter λ and the second is the Poisson process itself, which determines the default time.

On such models, the probability of survival up to time t has the following formula:

$$\mathbf{P_{Surv}(t) = E\left[exp\left(-\int_0^t \lambda_s \, ds\right)\right]}$$

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4.4. YIELDS

In the final section, we will try to estimate the expected price, P, of a defaultable bond, based on the survival probability. It is given by the following formula:

$$\mathbf{P} = \sum_{\mathbf{i=1}}^{\mathbf{N}} \frac{\mathbf{C} \times \mathbf{p_i}}{(\mathbf{1} + \mathbf{r})^{\mathbf{i}}}$$

- **C** is the bond's coupon
- $\mathbf{p_i}$ is the survival probability
- **r** is the bond's yield (usually 2%)
- **N** is the number of time points we use

In order to assess the effect of default, we compare the discounted payoff of two bonds for the same discount factor **r**, one with riskless coupon payments and one with risky coupon payments for a given survival probability modeled as above. We note that the risky bond for given discount factor r, has smaller price than the riskless bond. Since the price of the bond decreases as its yield increases, the yield of the defaultable bond is expected to be larger than the price of a non-defaultable bond, and the difference between the two yields is called **spread**.

In this thesis, due to the lack of real data, we assume the r to be known, and working with the above formula, we calculate the P on the left-hand side. We do this twice, once with the normal p_i (the risky bond) and once assuming each of them to be 1 (the risk-free bond) and we are interested in the **difference between those two prices**.

We would like to estimate the above for both intensity models and firm-value models. The first is quite simple, as the survival probability is known at any given time. However, the second needs a lot of work, as the only way to estimate the probability is to calculate how many processes do not default (see chapter 3). It also has to be done at many different time horizons so as to obtain as many p_i as possible.

With that being said, let's start with the simpler intensity models. As mentioned in section 4.3, there are two types of such models. The first assumes the parameter λ (intensity of default) to be known, while the second assumes it to be stochastic.

We will use the model in 4.3.1 with the same values for λ. As for the time points, we will use each of them from 1 to 10. The risk free price is 898,2585.

λ	Price	Difference from risk free
0.1	548,587	349,671
0.3	254,516	643,742
0.5	145,882	752,376
0.8	78,715	819,543

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If λ changes over time, we adjust the survival probability at time points 1 to 10 and conduct the same calculations with the above formula.

For the Cox model (section 4.3.2) λ is stochastic. However we can still estimate the expected amount by noting its value at times 1 to 10 and calculating the default probability for the said λ. Since λ is always positive the choice of processes it follows are limited. One possible process would be the Geometric Brownian Motion (GBM). However, other processes like the Ornstein-Uhlenbeck can also prove useful, as long as we select an ideal starting value and variance so that they don't fall below 0.

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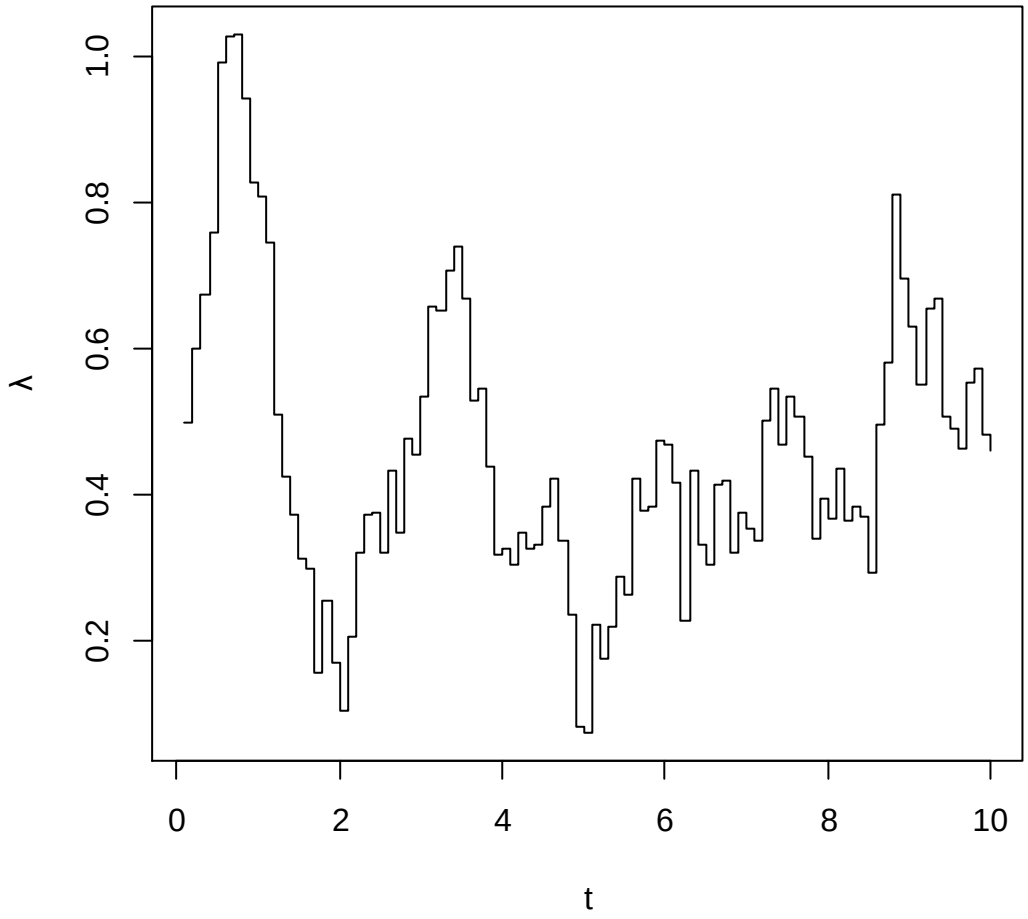
Let's use the Brownian Motion for its simulation with parameters μ = 0, σ = 0,5 and use as starting value 2. The process flow is depicted below.



Graph 19 Sample path of the default intensity λ, following the Browian Motion

Time	λ
1	0,81
2	0,10
3	0,54
4	0,32
5	0,07
6	0,46
7	0,36
8	0,37
9	0,63
10	0,46

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We can see that the flow of λ is quite erratic, due to it being stochastic. In order to calculate the survival probability up to time 10, we need to integrate the curve. Since we don't have a closed form of the function of λ, one thing we can do is using the **Monte-Carlo** approximation. According to it, we simulate (many!) points whose x,y coordinates lie between 0-10 and 0-1 respectively and find how many of them fall below the curve.

Using 100000 points, only 45467, or 45,467 % of them fall under the curve, which means the integral is estimated to be 4,5467. Finally, using the formula for the survival probability given in 4.3.2 we get:

$$\mathbf{P_{surv}(10) = 0{,}012}$$

However, it would be much more helpful, if we could calculate the survival probability for all times from 1 to 10. If so, we must integrate the curve for ten intervals, i.e. 0-1, 1-2, 2-3, etc. Using 100000 points for each interval we get the following result:

Time interval	Points falling below the line	Survival probability for that interval	Cumulative survival probability up to time t
0-1	83566	0,433	0,433
1-2	28085	0,755	0,327
2-3	43035	0,650	0,212
3-4	52109	0,594	0,126
4-5	27993	0,756	0,095
5-6	34818	0,706	0,067
6-7	34998	0,705	0,047
7-8	45701	0,633	0,030
8-9	51838	0,595	0,018
9-10	40218	0,669	0,012

Finally, using the formula for the expected amount paid, we get:

Price	Difference from risk free
129,195	769,063

If we conduct another simulation, the results would change, but the overall method of calculating the expected amount is the same.

Let us now try to estimate the survival probabilities as well as the expected amount paid for the firm value models.

In order to get a respectable approach of the survival probability we have to use the methods described in chapter 3. That is, we simulate a good number of the said processes, and calculate the percentage of those falling below a threshold L, either at the final time, or at any time. However, this has to be done for all time horizons from

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1 to 10, which greatly complicates our work. This is essentially a generalization of what we did in chapter 3.

For consistency with the aforementioned chapter, we will use the same parameters.

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i) We will start with the **Brownian motion**. By simulating 10000 processes, using as starting value $X_0 = 2$ and as threshold L=0.4, we get the following:

Time horizon T	Survival Probability at time T	Survival Probability up to time T
100	56,86 %	17,73 %
200	54,73 %	12,57 %
300	54,33 %	10,44 %
400	53,14 %	8,96 %
500	53,79 %	8,06 %
600	52,71 %	7,33 %
700	52,89 %	6,81 %
800	52,28 %	6,41 %
900	52,01 %	6,01 %
1000	52,02 %	5,76 %

The second row does not offer much information as the probabilities are close to each other regardless of the time T, except of the first ones where the survival probability is slightly higher due to the starting value of 2. The third row, is much more informative, and shows us how quickly the survival probability diminishes as time passes. Therefore, it's only natural that we use the third row for our computations.

Price	Difference from risk free
82,54226	815,7162

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ii) The next process is the **Compound Poisson Process**, with all its values increased by 2. Using $\lambda = 3$ and as threshold $L = 1$, we have the following:

Time horizon T	Survival Probability at time T	Survival Probability up to time T
10	62,66 %	38,45 %
20	59,03 %	27,58 %
30	57,59 %	22,57 %
40	56,70 %	19,73 %
50	55,17 %	17,43 %
60	55,65 %	15,89 %
70	54,77 %	14,80 %
80	54,46 %	14,00 %
90	54,46 %	13,03 %
100	54,08 %	12,43 %

Just like the Brownian motion, the second row's values are close to each other. The third row's probabilities however, drop as the time horizon increases. The drop rate is much higher on the first times than on the subsequent times.

Price	Difference from risk free
179,5306	718,7279

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iii) The third process is the **Ornstein Uhlenbeck process**. Using $\mu = 1.2$, $\sigma = 0.3$ and $\theta = 1$ and as threshold L = 1.08, we have the following:

Time horizon T	Survival Probability at time T	Survival Probability up to time T
100	100,00 %	100,00 %
200	100,00 %	100,00 %
300	100,00 %	100,00 %
400	99,92 %	99,86 %
500	99,40 %	98,59 %
600	98,18 %	95,51 %
700	96,43 %	90,72 %
800	94,18 %	84,22 %
900	91,98 %	77,45 %
1000	90,19 %	70,33 %

Unlike the previous two processes, the probabilities of both the second and third columns drop continuously over time with the third row dropping much faster, for obvious reasons.

Price	Difference from risk free
828,1096	70,14885

We can see that the difference is very low compared to the previous examples, just because the survival probability is much greater.

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CONCLUSION

Credit risk is an important part of finance. A reasonable question in credit risk theory is the efficient modeling of the probability of default for the debtor which in turn affects the prices of defaultable bonds and their yields. Taking into account the large number of derivative assets, whose underline is essentially the ability of debtors to honor their obligations, the correct modeling of the probability of default is of paramount importance. This thesis has focused on the modeling of default probabilities, using techniques from the theory of Lévy processes. We presented two general classes of models for predicting default events, firm value models and intensity based models, and created R routines for the simulation of said models. Using the simulations, we can acquire a feeling of the effect of the multiple parameters involved in these models. The algorithms can be used for pricing or hedging purposes.

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