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ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ
ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

SCHOOL OF INFORMATION SCIENCES
& TECHNOLOGY

DEPARTMENT OF STATISTICS
POSTGRADUATE PROGRAM

ARIMA Model Based Seasonal Adjustment Methods and Application

By

Sofia P. Kapella

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ΣΧΟΛΗ ΤΩΝ ΕΠΙΣΤΗΜΩΝ & ΤΕΧΝΟΛΟΓΙΑΣ ΤΗΣ ΠΛΗΡΟΦΟΡΙΑΣ

ΤΜΗΜΑ ΣΤΑΤΙΣΤΙΚΗΣ
ΜΕΤΑΠΤΥΧΙΑΚΟ ΠΡΟΓΡΑΜΜΑ

Εποχιακή Προσαρμογή βασισμένη σε μοντέλο ARIMA Μέθοδοι και Εφαρμογή

Σοφία Π. Καπέλλα

ΔΙΑΤΡΙΒΗ

Που υποβλήθηκε στο τμήμα στατιστικής του Οικονομικού Πανεπιστημίου Αθηνών ως μέρος των απαιτήσεων για την απόκτηση Μεταπτυχιακού Διπλώματος Ειδίκευσης στην Στατιστική (Πλήρους Φοίτησης)

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VITA

I was born on May 18th, 1991 in Athens. In 2009 graduated from Varvakio high school and I entered the Department of Mathematics at the National and Kapodistrian University of Athens. In 2017, I got my bachelor degree and I enrolled the Master of Science in Statistics at Athens University of Economics and Business.

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ABSTRACT

This thesis deals with seasonal adjustment using the Arima Model Based method (AMB) in which the observed time series is interpreted as the sum of three unobserved components, the seasonal, the trend and the rest of the series. An ARIMA model is estimated for the time series but also for each component. We address the issues of model specification and present the evaluation of the unobserved components as a signal extraction problem. The signal is always the component of interest. The optimal estimators are achieved with 'Minimum Mean Squared Error'. A brief overview of the basic concepts and definitions is provided at the beginning. Next, we show how the AMB method works and at the end an application is briefly presented in JDemetra, a software introduced by Eurostat.

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ΠΕΡΙΛΗΨΗ

Η διατριβή πραγματεύεται την εποχική προσαρμογή με τη μέθοδο ARIMA Model Based, στην οποία η παρατηρούμενη χρονολογική σειρά ερμηνεύονται ως το άθροισμα τριών μη παρατηρημένων συνιστωσών, την εποχική, της τάσης και της υπόλοιπης σειράς. Ένα μοντέλο ARIMA εκτιμάται για την παρατηρούμενη χρονολογική σειρά αλλά και για κάθε συνιστώσα. Παρουσιάζουμε τις υποθέσεις του μοντέλου και την εκτίμηση των μη παρατηρούμενων συνιστωσών ως πρόβλημα εξόρυξης σήματος. Το σήμα είναι κάθε φορά η συνιστώσα που μας ενδιαφέρει. Οι βέλτιστοι εκτιμητές επιτυγχάνονται με το Έλάχιστο Μέσο Τετραγωνικό Σφάλμα'. Στην αρχή παρέχεται μια σύντομη ανασκόπηση των βασικών εννοιών και ορισμών. Στη συνέχεια, παρουσιάζουμε πώς λειτουργεί η μέθοδος AMB και στο τέλος παρουσιάζεται συνοπτικά μια εφαρμογή στο *JDemetra*, ένα λογισμικό που εισήγαγε η Eurostat.

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Chapter 1

Introduction

The very basic intuition behind the concept of seasonal adjustment is the decomposition of an observed time series into seasonal, trend and irregular components, the latter being a residual effect. In essence, it consists in the removal of the seasonal variation from a time series. Since neither the seasonally adjusted (SA) series, (i.e. trend and irregular component), nor the seasonal component are directly observed, both can be seen as *unobserved components (UC)* of the series, and seasonal adjustment becomes a problem of UC estimation, where each component is defined by the frequency of the associated variation. Because the SA series is supposed to provide a cleaner signal of the underlying evolution of the variable, seasonal adjustment can also be viewed as a *signal extraction problem* in a ”signal plus noise” decomposition of the series, where the signal is the component of interest and the noise the rest of the series.

More specifically, if X_t denotes the observed series, the simplest formulation could be

$$X_t = T_t + S_t + N_t \tag{1.1}$$

where the variables T_t, S_t, N_t denote the unobserved components, trend, seasonal

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and irregular component respectively. In the early days, the components were often specified to follow deterministic models that could be estimated by simple regression (see 1.3 below). We shall follow the convention: a Deterministic Model denotes a model that yields forecasts with zero error when the model parameters are known. Stochastic Models will provide forecasts with nonzero random errors even when the parameters are known. For example, a deterministic trend component T_t could be specified as the linear trend

$$T_t = a + bt, \tag{1.2}$$

and the seasonal component S_t could be modelled with dummy variables, as in

$$S_t = \sum_j c_j d_{jt}, \tag{1.3}$$

where $d_{jt} = 1$ when t corresponds to the j^{th} period of the year, and $d_{jt} = 0$ otherwise. An equivalent formulation can be expressed in terms of deterministic sine-cosine functions.

Gradual realization that seasonality evolves in time (an obvious example is the weather, one of the basic causes of seasonality) lead to the replacement of deterministic models by the so-called Moving Average methods, which can be seen as approximating the trend by local polynomials (Kendall, 1976) and the seasonal by local cosine functions (Box, Hillmer, & Tiao, 1979). A Linear Filter will simply denote a linear combination of the series X_t, as in

$$\widehat{X}_{it} = c_{-k_1}X_{t-k_1} + ... + c_{-1}X_{t-1} + c_0X_t + c_1X_{t+1} + ... + c_{k_2}X_{t+k_2}, \tag{1.4}$$

where $\widehat{X}_{it}$ is an MA filter and the weights c_j could be found in such a way as to capture the relevant variation associated. See, e.g. the definition of Wiener-Kolmogorov filters in 3.15.

Since the moving features can be seen as the result of randomness, a natural way to think about the components is in the frequency domain (see section 2.4)

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. Broadly, if a component is designed to capture the series variation for a specific frequency region, the moving average filter to estimate the component can be seen as a "band-pass" filter, that should have a close to "1" gain in that region, and a "0" gain for other frequencies. Filters have been constructed in an ad-hoc manner to display that "band-pass" structure. These filters are fixed, and independent of the time series under analysis. In the area of data treatment for policy and monitoring of the economy, where seasonal adjustment is the most frequent application, massive use is made of $X11$-type filters (Shiskin, 1965). Figure 1a and 1b display the gain of the X11 filter, i.e., the way X11 filters the frequencies of the series spectrum, for a quarterly and monthly series, respectively. When the gain is 1, the frequency is fully transmitted; when the gain is zero, the frequency is ignored. If applied to a series with the spectrum of Figure 1c, the filter removes the variation around the seasonal frequencies, and provides a SA series with the spectrum of Figure 1d.

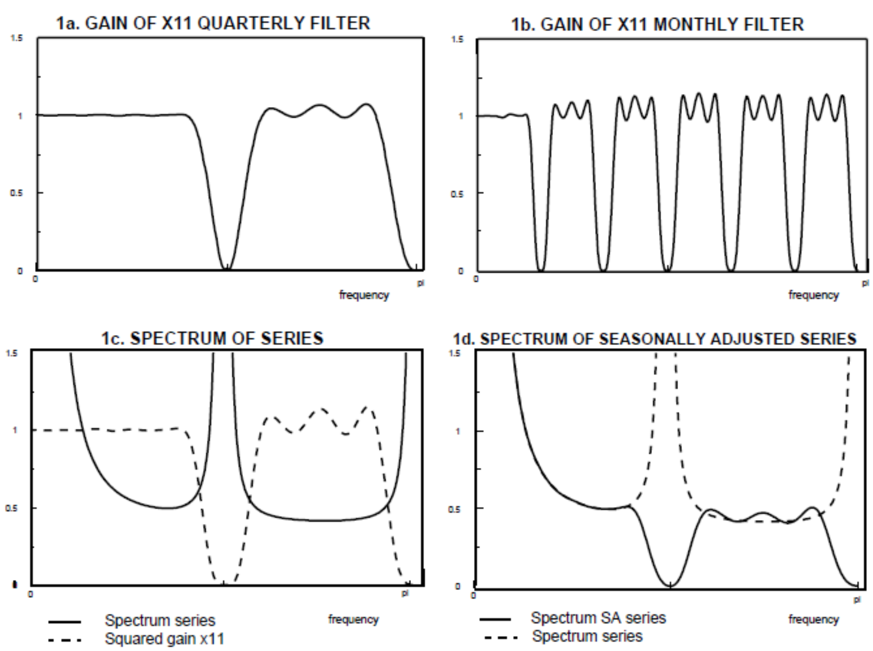

Figure 1.1: X11 Filter

Figures are from (Gómez & Maravall, 2001).

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Over time, however, application of ”ad-hoc” filtering has evidenced some serious limitations. An important one is the fact that, due to its fixed character, for some series the component may be overestimated, while for other series, it may be underestimated.

For a series containing a highly stochastic seasonal component, as evidenced by the width of the seasonal peaks in the series spectrum of Figure 2a, the width of the dips in the squared gain of the X11 filter seem too narrow. Application of the filter to the series yields a SA series with the spectrum of Figure 2b. The underadjustment causes the ackwards peaks for frequencies that are in the neighbourhood of the seasonal ones. On the other hand, for a series containing a close to deterministic seasonal component, as evidenced by the narrow peaks in the spectrum of Figure 2c, the width of the dips in the filter gain are too wide and, as seen in Figure 2d, X11 removes variance that is not a associated with the seasonal peaks of the series. In this case the result is overadjustment.

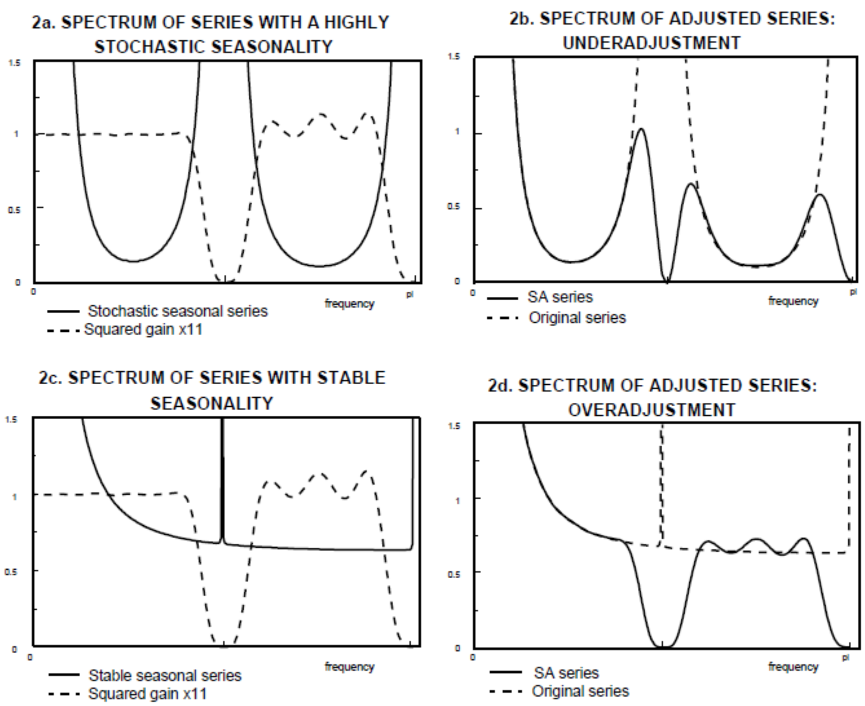

Figure 1.2: Under-Over adjustment by X11 Filter

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FigureS are from (Gómez & Maravall, 2001).

To overcome this limitation, and in the context of seasonal adjustment, an alternative approach was suggested (around 1980) whereby the filter adapted to the particular structure of the series, as captured by an estimated ARIMA model (see details about ARIMA procedure in section 2.5). Particularly, it was found that stochastic trends and stochastic seasonality induced by roots of the autoregressive polynomial on the unit circle were more appropriate to capture such a phenomenon. The approach, known as the ARIMA-model-based (AMB) approach (see in chapter 3), consists of two steps.

- First, an ARIMA model is estimated for the observed series.
- Second, signal extraction techniques are used to estimate the components with filters that are, in some well-defined way, optimal.

Finally, in chapter 4 we give an illustration example, using *JDemetra+*, a software which was introduced by Eurostat to improve access to above SA methods for non-specialists. It offer a user-friendly interface to the two SA algorithms: TRAMO/SEATS and X12-ARIMA (has been generated by X11) and facilitates the comparison of the output from those two algorithms. X-12-ARIMA developed at the U.S. Census Bureau and TRAMO/SEATS (”Time Series Regression with ARIMA Noise, Missing Observations, and Outliers”/”Signal Extraction in ARIMA Time Series”) developed by Victor Gómez and Agustín Maravall, from the Banco de España. Both methods are divided into two main parts. The first part is called pre-adjustment and removes deterministic effects from the series by means of a regression model with ARIMA noise. The second part is the decomposition of the time series that aims to estimate and remove a seasonal component from the time series. Although TRAMO/SEATS and X12-ARIMA use a very similar approach in the first part to estimate the same model on the processing step, they

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differ completely in the decomposition step. We will illustrate only the second part SEATS. Further information about the *JDemetra+* software and its functionalities is contained in the *JDemetra+ Reference Manual (2017)*.

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Chapter 2

Brief review of basic concepts and definitions

2.1 Stochastic Processes and Stationarity

The starting point is the concept of a Stochastic Process. A stochastic process is a family of random variables $\{X_t, t \in T\}$. In time series analysis the index set T is a set of time points, very often $\{\pm 1, \pm 2, \pm 3, ...\}$, $\{1, 2, 3, ...\}$, $(-\infty, \infty)$ (Brockwell, Davis, & Fienberg, 1991). For our purposes, a time series is a stochastic process $\{X_t\}_{-\infty \leq t \leq \infty}$ from which only $(X_{t_1}, X_{t_2}, ..., X_{t_T})$ is observed. The distribution of $\{X_t\}_{-\infty \leq t \leq \infty}$ is completely specified by the finite dimensional distribution, i.e. of $(X_{t_1}, X_{t_2}, ..., X_{t_T})$.

From an applied perspective, the two most important added assumptions are that the process is stationary and the joint distribution of $(X_1, X_2, ..., X_t)$ is a multivariate normal distribution. The first implies that $(X_1, X_2, ..., X_t)$=$(X_{1+k}, X_{2+k}, ..., X_{t+k},)$ for any value of t, where k is an integer; that is, the joint distribution remains unchanged if all time periods are moved a constant number of periods. In particular, the marginal

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distribution remains invariant. This implies constant moments independent of t,

$$E(X_t) = \mu_X; \quad V(X_t) = V_X \tag{2.1}$$

where E and V denote the expectation and the variance operators, and μ_X and V_X are constants that do not depend on t. In practice, thus, stationarity implies a constant mean level and bounded deviations from it. It is a very strong requirement and few actual economic series will satisfy it. Its usefulness comes from the fact that relatively simple transformations of the nonstationary series will render it stationary. For economic series, it is usually the case that constant variance can be achieved through the log/level transformation combined with proper outlier correction, and constant mean can be achieved by *differencing*.

2.2 Differencing

Denote by B the backward operator, such that

$$B^j X_t = X_{t-j} \quad j = 0, 1, 2, ...$$

and let X_t denote a quarterly observed series. If X_t is a deterministic linear trend, as in $X_t = a + bt$, then

$$(1 - B)X_t = b, \tag{2.2}$$

$$(1 - B)^2 X_t = 0. \tag{2.3}$$

In general, it can easily be seen that $(1 - B)^d$ will reduce a polynomial of degree d to a constant. Obviously, $(1 - B^4)X_t$ will also cancel a constant (or reduce the linear trend to a constant); but it will also cancel other deterministic periodic functions, such as for example, one that repeats itself every four quarters. To find the set of functions that are cancelled with the transformations $(1 - B^4)X_t$, we have to find the solution of the homogeneous difference equation

$$(1 - B^4)X_t = X_t - X_{t-4} = 0, \tag{2.4}$$

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with characteristic equation $r^4 - 1 = 0$,. The solution is given by

$$r = \sqrt[4]{1},$$

that is, the four roots of the unit circle. The four roots are

$$r_1 = 1, \quad r_2 = -1, \quad r_3 = i, \quad r_4 = -i. \tag{2.5}$$

The first two roots are real and the last two are complex conjugates, with modulus 1 and frequency $w = \pi/2$ (frequencies will always be expressed in radians).
In the case of the seasonal differencing operator $(1 - B^{12})$ applied to the monthly time series the characteristic equation $r^{12} - 1 = 0$ has the solution given by

$$r = \sqrt[12]{1},$$

two real and ten complex conjugates with modulus 1 and frequencies $\pi/6, \pi/3, \pi/2, 2\pi/3, 5\pi/6$ and π.
Complex conjugate roots generate periodic movements of the type

$$r_t = Acos(wt + B) \tag{2.6}$$

where A denotes the amplitude, B denotes the phase (the angle at t=0) and w the frequency (the number of full circles that are completed in one unit of time). The period of function 2.6, to be denoted τ , is the number of units of time it takes for a full circle to be completed, and is related to the frequency w by the expression

$$\tau = \frac{2\pi}{w} \tag{2.7}$$

From (2.5), the general solution of $(1 - B^4)X_t = 0$ can be expressed in short as

$$X_t = c_0 + \sum_{j=1}^{2} c_j cos\left(j\frac{\pi}{2}t + d_j\right), \tag{2.8}$$

where c_0 is a constant associated with zero frequency root $B = 1$, and $d_2 = 0$. Considering (2.7), the first term in the sum of (2.8) will be associated with a period

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of $\tau = 4$ quarters and will represent thus a seasonal component with a once-a-year frequency; the second term has a period of $\tau = 2$ quarters, and hence will represent a seasonal component with a twice-a-year frequency. Noticing that the characteristic equation can be rewritten as $(B^{-1})^4 - 1 = 0$, (2.5) implies the factorization

$$1 - B^4 = (1 - B)(1 + B)(1 + B^2).$$

The factor $(1 - B)$ is associated with the constant and the zero frequency, the factor $(1+B)$ with the twice-a-year seasonality with frequency $w = \pi$, and the factor $(1+B^2)$ with the once-a-year seasonality with frequency $w = \pi/2$. The product of these last two factors yields the annual aggregation $1 + B + B^2 + B^3$. Hence the transformation $(1 + B + B^2 + B^3)X_t$ will remove a deterministic seasonal component in X_t.

For the most-often-found case in which stationarity is achieved through the differencing $(1 - B)(1 - B^4)$, the factorization

$$(1 - B)(1 - B^4) = (1 - B)^2(1 + B + B^2 + B^3)$$

directly shows that the solution to

$$(1 - B)(1 - B^4) = 0$$

will be of the type:

$$X_t = a + bt + \sum_{j=1}^{2} c_j cos\left(j\frac{\pi}{2}t + d_j\right), \tag{2.9}$$

with $d_2 = 0$. Thus the differencing will remove the same cosine (seasonal) functions as before, plus the local linear trend $(a + bt)$. For the case $(1 - B)^2(1 - B^4)$, the factorization $(1 - B)^2(1 - B^4) = (1 - B)^3(1 + B + B^2 + B^3)$ shows that the cancelled trend will now be a second order polynomial in t, the rest remaining unchanged. For quarterly series, higher order differencing is never encountered in practice.

For a final and important remark, let D denote, in general, the complete differencing applied to the series X_t so as to achieve stationarity. When specifying the

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ARIMA model for X_t, we shall not be stating that $DX_t = 0$ (as, for example, in (2.3),) but that

$$DX_t = Z_t,$$

where Z_t is a zero-mean, stationary stochastic process with relatively small variance. Thus every period the solution of $DX_t = 0$ will be perturbed by the stochastic input Z_t (Brockwell et al., 1991; Gómez & Maravall, 2001). In terms of expression (2.9), what this perturbation implies is that seasonality is described by a stochastic model, which allows the seasonal pattern to evolve in time: this resembles a behavior like the one induced by (2.9), but with coefficients being non-constant in time. This gradual evolution of the coefficients provides the model with an adaptive behavior that will be associated with the ”moving” features of the trend and seasonal components.

2.3 Linear stationary process, Wold representation and Autocorrelation Function

If X_t denotes again the observed variable and $Y_t = DX_t$ its stationary transformation, under assumptions that were mentioned in section 2.1, the variable $(Y_1, Y_2, ..., Y_T)$ will have a proper multivariate normal distribution. One important property of this distribution is that the expectation of some (unobserved) variable linearly related to Y_t, conditional on $(Y_1, Y_2, ..., Y_T)$, will be a linear function of $Y_1, Y_2, ..., Y_T$. Thus conditional expectations will directly provide linear filters. An additional important property is that, because the first two moments fully characterize the distribution, stationarity in mean and variance will imply stationarity of the process. In particular, stationarity will be implied by the constant mean and variance condition (2.1), plus the condition that

$$Cov(X_{t+s}, X_t) = \gamma_X(s) \tag{2.10}$$

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for $s = 0, \pm 1, \pm 2, ...$ Note that the function (2.10) calls autocovariance function (ACVF). The autocorrelation function (ACF) of $\{X_t\}$ was defined similarly as the function $\rho(.)$ whose value at lag s is $\frac{\gamma_j}{\gamma_0}$. If the following conditions on the ACF:

1. $\rho_0 = 1$
2. $\rho_j = \rho_{-j}$
3. $|\rho_j| < 1$ for $j \neq 0$
4. $\rho_j \to 0$ as $j \to \infty$
5. $\sum_{j=0}^{\infty} |\rho_j| < \infty$,

are satisfied, then a zero-mean, finite variance, normally distributed process is stationary.

Hence the covariance between Y_t and Y_{t-s} should depend on their relative distance s, not on the value of t. Therefore,

$$(Y_1, Y_2, ..., Y_T) \sim N(\mu, \Sigma),$$

where μ is a vector of constant means, and Σ is the variance-covariance matrix

$$\Sigma = \begin{vmatrix} V_Y & \gamma_1 & \gamma_2 & ... & \gamma_{T-1} \\ \gamma_1 & V_Y & \gamma_1 & ... & \gamma_{T-2} \\ ... & ... & ... & ... & \\ ... & ... & ... & V_Y & \gamma_1 \\ \gamma_{T-1} & ... & & & V_z \end{vmatrix} \qquad (V_Y = \gamma_0),$$

a positive definite symmetric matrix. Let F denote the forward operator, $F = B^{-1}$, such that

$$F^j Y_t = Y_{t+j}, \quad j = 0, 1, 2, ...$$

a more parsimonious representation of the 2^{nd}-order moments of the stationary process Y_t is given by the Autocovariance Generating Function (AGF)

$$\gamma(B, F) = \gamma_0 + \sum_{j=1}^{\infty} \gamma_j (B^j + F^j). \qquad (2.11)$$

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To transform this function in to a scale-free function, we divide by the variance γ_0, and obtain the Autocorrelation Generating Function (ACF),

$$\rho(B,F) = \rho_0 + \sum_{j=1}^{\infty} \rho_j (B^j + F^j) \tag{2.12}$$

where $\rho_j = \gamma_j/\gamma_0$. Under the normality assumption, a complete realization of the stochastic process will be fully characterized by μ_Y, V_Y and $\rho(B,F)$. When $\rho_j = 0$ for all $j \neq 0$, the process will be denoted a White Noise process. Therefore, a white noise process is a sequence of normally identically independently distributed random variables. The ACVF (or ACF) is the basic tool in the so-called "Time Domain Analysis" of a time series.

To start the modelling procedure, a general result on linear time series processes will provide us with an analytical representation of the process that will prove very useful. This is the so-called Wold representation. If Y_t denote a linear stationary stochastic process with no deterministic component, then Y_t can be expressed as the one-sided moving average:

$$\begin{aligned} X_t &= \sum_{j=0}^{\infty} \psi_j Z_{t-j} = \Psi(B) Z_t, \\ \Psi(B) &= \sum_{j=0}^{\infty} \psi_j B^j \end{aligned} \tag{2.13}$$

where $Z_t \sim WN(0, V_{Z_t})$, and $\psi_0 = 1, \quad \sum_{j=0}^{\infty} \psi_j^2 < \infty$. The last condition reflecting a sufficient condition for convergence of the polynomial $\Psi(B)$. Given the ψ_j coefficients, Z_t represents the one-period ahead forecast error of Y_t, that is

$$Z_t = Y_t - \widehat{Y}_{t|t-1},$$

where $\widehat{Y}_{t|t-1}$ is the forecast of Y_t made at period $t-1$. Since Z_t represents what is new in Y_t, that is, what is not contained in its past $(Y_{t-1}, Y_{t-2}, Y_{t-3}, ...)$ it will be referred to as the *Innovation of the process.* The representation of Y_t in terms of its innovations, given by (2.13), is unique, and is usually referred to as the Wold representation.

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A useful corollary is that if $\gamma(B,F)$ represents the AGF of the process Y_t, then

$$\gamma(B,F) = \Psi(B)\Psi(F)V_Z. \tag{2.14}$$

In particular, for the variance,

$$V_Y = (1+\psi_1^2+\psi_2^2+...)V_Z. \tag{2.15}$$

2.4 The spectrum

The spectrum is the basic tool in the so-called "Frequency Domain Approach" to time series analysis. It represents an alternative way to look and interpret the information contained in the second-order moments of the series.

Consider the AGF of the stationary process X_t, given by (2.11), where B is a complex number of unit modulus, which can be expressed as e^{iw}. Replacing B and F by their complex representation, (2.11) becomes the function

$$f(w) = \gamma_0 + \sum_{j=1}^{\infty} \gamma_j (e^{-iwj} + e^{iwj}),$$

or, using the identity

$$e^{-iwj} + e^{iwj} = 2cos(jw),$$

and dividing by 2π one obtains

$$f(w) = \frac{1}{2\pi}\left[\gamma_0 + 2\sum_{j=1}^{\infty}\gamma_j cos(jw)\right]. \tag{2.16}$$

The move from (2.10) to (2.16) is the so-called Fourier cosine transform of the ACVF $\gamma_X(.)$, and is denoted the Power Spectrum. Replacing the ACVF by the ACF (i.e., dividing by the variance γ_0), we obtain the Spectral Density Function

$$f^*(w) = \frac{1}{2\pi}\left[1 + 2\sum_{j=1}^{\infty}\rho_j cos(jw)\right]. \tag{2.17}$$

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It is easily seen that $f(w)$ -or $f^*(w)$- are periodic functions, and hence the range of frequencies can be restricted to $(-\pi, \pi)$, or to $(0, 2\pi)$. Moreover, given that the cosine function is symmetric around zero, we only need to consider the range $(0, \pi)$. From (2.16), knowing the ACVF of a process, the power spectrum is trivially obtained. Alternatively, knowledge of the power spectrum permits us to derive the ACVF by means of the inverse Fourier transform, given by

$$\gamma_k = \int_{-\pi}^{\pi} f(w) cos(wk) dw.$$

Thus, for $k = 0$,

$$\gamma_0 = \int_{-\pi}^{\pi} f(w) dw, \tag{2.18}$$

which shows that the integral of the power spectrum is the variance of the process. Therefore, the area under the spectrum for the interval dw is the contribution to the variance of the series that corresponds to the range of frequencies dw. Roughly, the power spectrum can be seen as a decomposition of the variance by frequency. For the rest of the paper, in order to simplify the notation, power spectra will be expressed in units of 2π, and, because of the symmetry condition, only the range $w \in [0, \pi]$ will be considered. This function will be referred simply as the Spectrum.

2.5 ARIMA Models

A wide class of stationary processes is the **autoregressive-moving average (ARMA) process**. The process $X_t, t \in \mathbb{Z}$ is said to be an **ARMA(p,q) process** if X_t is stationary and if for every t,

$$X_t - \phi_1 X_{t-1} - ... - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + ... + \theta_q Z_{t-q}, \quad Z_t \sim N(0, V_{Z_t}).$$

The above equation can be written in the more compact form

$$\phi(B) X_t = \theta(B) Z_t, \quad , t \in \mathbb{Z}, \tag{2.19}$$

where ϕ and θ are the p^{th} and q^{th} degree complex polynomials,
$\phi(z) = 1 + \phi_1 z + \phi_2 z^2 + ... + \phi_p z^p$, $\theta(z) = 1 + \theta_1 z + ... + \theta_q z^q$, and B is a backwards operator, $B^j X_t = X_{t-j}$.
For the general $ARMA(p, q)$ process if $\phi(z) = 1 + \phi_1 z + \phi_2 z^2 + ... + \phi_p z^p \neq 0$ for all complex z with $|z| = 1$ (unit circle), then

$$\chi(z) = \frac{1}{\phi(z)} = \sum_{j=-\infty}^{\infty} \chi_j z^j, \quad 1 - \delta < |z| < 1 + \delta \quad with \sum_{j=-\infty}^{\infty} |\chi_j| < \infty \tag{2.20}$$

and the unique stationary solution given from

$$X_t = \chi(B)\theta(B)Z_t = \psi(B)Z_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \quad with \quad \psi(B) = \chi(B)\theta(B). \tag{2.21}$$

If there exist constants ψ_j such that $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ for all t then the X_t is called **a causal function of Z_t** . Equivalently, X_t is a causal $ARMA(p, q)$ process if ,

$$\phi(z) \neq 0 \quad for \quad all \quad |z| \leq 1. \tag{2.22}$$

An $ARMA(p, q)$ process X_t is **invertible** if there exist constants π_j such that $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and $Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$ for all t. Equivalently,

$$\theta(z) \neq 0 \quad for \quad all \quad |z| \leq 1. \tag{2.23}$$

At this point it is specified that the Autocovariance Generating Function (ACGF) of an ARMA(p, q) process. As we saw by (2.20) and (2.21) any $ARMA$ $\phi(B)X_t = \theta(B)a_t$ for which $\phi(z) \neq 0$ when $|z| = 1$ can be written be in the form (2.19) with

$$\psi(z) = \frac{\theta(B)}{\phi(B)}, \qquad r^{-1} < |z| < r$$

for some $r > 1$. Hence, setting $B = z, F = z^{-1}$, the (2.14) is

$$\gamma(z) = V_{Zt} \frac{\theta(z)\theta(z^{-1})}{\phi(z)\phi(z^{-1})}, \qquad r^{-1} < |z| < r \tag{2.24}$$

In fact, by writting $z = e^{-iw}$, (2.24) is the spectrum of X_t,

$$f_X(w) = V_{Z_t} \frac{|\theta(e^{-iw})|^2}{|\phi(e^{-iw})|^2}, \quad -\pi \leq w \leq \pi \tag{2.25}$$

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and the integral of $f_X(w)$ over $0 \leq w \leq 2\pi$ is equal to $2\pi V_X$ (Brockwell, Davis, & Calder, 2002).

Finally, if we fit an $ARMA(p,q)$ model $\phi(B)Y_t = \theta(B)a_t$ to the differenced series $Y_t = (1-B^s)X_t$, then the model for the original series is $\phi(B)(1-B^s)X_t = \theta(B)a_t$. This is a special case of the general seasonal ARIMA (SARIMA) model defined as follows.
If d and D are nonnegative integers, then X_t is a ***seasonal ARIMA(p, d, q)x(P, D, Q)$_s$ process with period s*** if the differenced series $Y_t = (1-B)^d(1-B^s)^D X_t$ is a causal ARMA process defined by

$$\phi(B)\Phi(B^s)Y_t = \theta(B)\Theta(B^s)Z_t, \quad Z_t \sim N(0, V_{Z_t}) \tag{2.26}$$

where B is a backwards operator, $B^s s_t = s_{t-s}$, $\phi(B) = 1 + \phi_1 B + \phi_2 B^2 + ... + \phi_p B^p$, $\Phi(B^s) = 1 + \Phi_1 B + \Phi_2 B^2 + ... + \Phi_P B^{sP}$, $\theta(B) = 1 + \theta_1 B + ... + \theta_q B^q$, and $\Theta(B) = 1 + \Theta_1 B + ... + \Theta_Q B^Q$.

Note that a unit MA root causes a zero in the spectrum, and a unit AR root causes a point of ∞. Given that a spectrum with points of ∞ has a nonconvergent integral, and that no standardization can provide a proper spectral density, the term spectrum is usually replaced by **Pseudo-spectrum** (Hatanaka & Suzuki, 1963). For our purposes, however, the points of ∞ pose no serious problem, and the pseudo-spectrum can be used in much the same way as the stationary spectrum. In particular, if, for the nonstationary series, we use the nonconvergent representation (2.13), compute the function (2.13) through (2.14) and, in the line of Hatanaka and Suzuki, refer to this function as the ”pseudo-AGF”, the pseudo-spectrum is the Fourier transform of the pseudo-AGF. Bearing in mind that, when referring to nonstationary series, the term ”pseudo-spectrum” would be more appropriate, in order to avoid excess notation, we shall simply use the term spectrum in all cases.

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It is worth noting that, if a series contains an important component for a certain frequency w_0, its spectrum should reveal a peak around that frequency. Given that a possible definition of a trend is a cyclical component with period $\tau = \infty$, the spectral peak in this case should occur at the frequency $w = 0$. The seasonal component, in turn, captures the spectral peaks at seasonal frequencies. Finally, the irregular component captures the rest including peaks at all other frequencies.

For example, we assume the model $(1 - B)(1 - B^4)X_t = \theta(B)a_t$. Then X_t is a SARIMA process, the AR roots are given in (2.5) and the figure 2.1 shows the spectrum of a particular case of model and its spectral decomposition in to trend, seasonal, and irregular components. The trend captures the peak around $w = 0$, and the seasonal component the peaks around the seasonal frequencies.

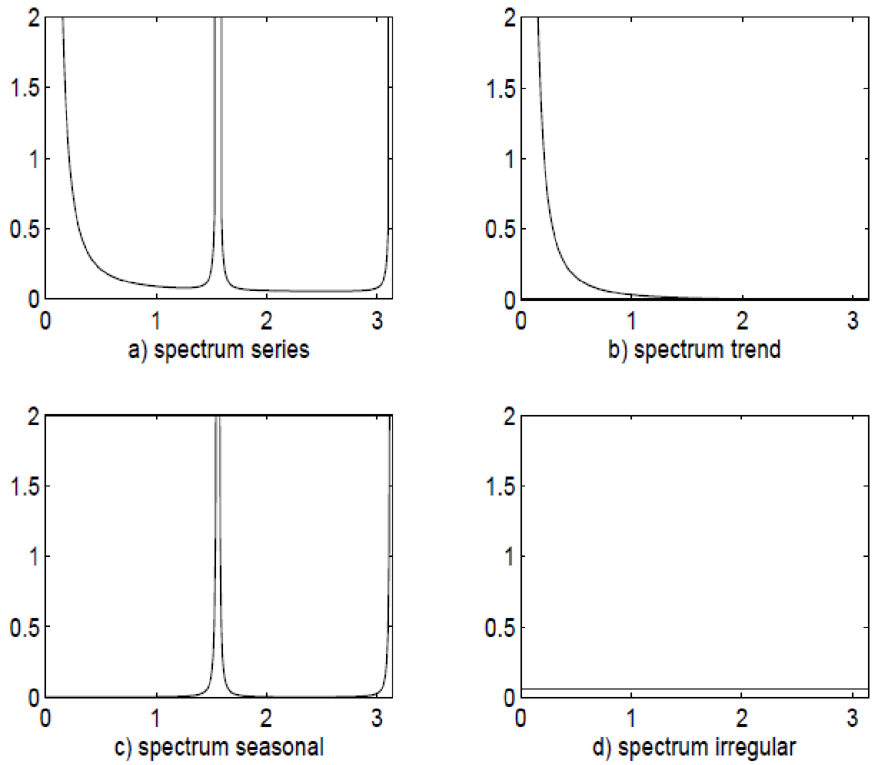

Figure 2.1: Spectral AMB Decomposition

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Figures are from (Gómez & Maravall, 2001).

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Chapter 3

The Arima Model Based seasonal adjusted approach

3.1 The seasonal ARIMA Model

Seasonal adjustment in ARIMA-model-based (AMB) method involves the decomposition of X_t for the observed time series at time t, in three unobserved components, seasonal S_t, trend T_t and the rest of the series, the noise N_t. First, a model for X_t is estimated. Then deriving appropriate models for the components that are compatible with the overall one are specified. We assume that trading day and other deterministic effect have been removed from X_t and then X_t follows a **seasonal** $\boldsymbol{ARIMA(p,d,q)x(P,D,Q)_s}$ **process with period s**,

$$\phi(B)\Phi(B^s)(1-B)^d(1-B^s)^D X_t = \theta(B)\Theta(B^s)Z_t, \quad Z_t \sim N(0, V_{Z_t}) \tag{3.1}$$

where B is a backwards operator ($B^s s_t = s_{t-s}$), and by stating $z = B^{-1}$,

$\phi(z) = 1 + \phi_1 z + \phi_2 z^2 + ... + \phi_p z^p$,

$\Phi(z^s) = 1 + \Phi_1 z + \Phi_2 z^2 + ... + \Phi_P z^{sP}$,

$\theta(z) = 1 + \theta_1 z + ... + \theta_q z^q$, and

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$\Theta(z) = 1 + \Theta_1 z + ... + \Theta_Q z^Q$.
$(1-z)^d$ and $(1-z^s)^D$ denotes the differences taken on X_t in order to achieve stationarity (Brockwell et al., 2002). Most often is assumed $D = 1$. More compact (3.1) can be written as:

$$\phi^*(B)X_t = \theta^*(B)Z_t \tag{3.2}$$

where $\phi^*(B) = \phi(B)\Phi(B^s)(1-B)^d(1-B^s)^D$ and $\theta^*(B) = \theta(B)\Theta(B^s)$ can be estimated from the data.

The autoregressive polynomial $\phi^*(B)$ is allowed to have unit roots, which are typically estimated with considerable precision. For example, unit roots in $\phi^*(B)$ would be present if the series were to contain a nonstationary trend component, or if the series had been underdifferenced. They can also appear as nonstationary seasonal harmonics.

3.2 Decomposition of the overall model into models for the components

We also assume that X_t can be represented as

$$X_t = S_t + T_t + N_t \tag{3.3}$$

such that each of the components follows an ARIMA model,

$$\begin{aligned} \phi_S(B)S_t &= \theta_S(B)Z_{St}, \quad Z_{St} \sim WN(0, V_{s_t}) \\ \phi_T(B)T_t &= \theta_T(B)Z_{Tt}, \quad Z_{Tt} \sim WN(0, V_{T_t}) \\ \phi_N(B)N_t &= \theta_N(B)Z_{Nt}, \quad Z_{Nt} \sim WN(0, V_{N_t}) \end{aligned} \tag{3.4}$$

The following conditions occur in AMB approach. Each of the pairs of polynomials $\{\phi_S(B), \theta_S(B)\}$, $\{\phi_T(B), \theta_T(B)\}$,$\{\phi_N(B), \theta_N(B)\}$ have their zeros lying on or outside

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the unit circle and have no common zeros. Further, the autoregressive polynomial, $\phi_N(B)$, is required that have no common zeros, with either $\phi_S(B)$ or $\phi_T(B)$, because otherwise it would imply the existence of additional seasonal and trend components that could then be absorbed into S_t and T_t. $\theta_i(B)$ are moving average polynomials and $Z_{S_t}, Z_{T_t}, Z_{N_t}$ are independent. If p_i, q_i are the degrees of $\phi_i(B), \theta_i(B)$, $i = S, T$ it is assumed that $q_i \leq p_i$. When $q \leq q$ in the overall model, N_t will be white noise; when $q > p$, N_t will be an $MA(q-p)$, and hence colored noise (Burman, 1980; Maravall, n.d.).

The model consisting of equations (3.3), (3.4), together with those assumptions will be referred to as an Unobserved Component ARIMA model (UCARIMA) and one of its directions is exactly the AMB approach. The other one is the Structural Time Series (STS) and starts by directly specifying the models for the components. For the rest of the paper only the AMB method will be dicsussed.

The components have not been precisely defined yet. To define the polynomials, $\phi_i(B), \theta_i(B)$ we consider the spectrum. The spectrum of X_t and of the components are respectively:

$$
\begin{aligned}
f(w) &= V_{Z_t} \frac{\theta(e^{iw})\theta(e^{-iw})}{\phi(e^{iw})\phi(e^{-iw})} \\
f_S(w) &= V_{S_t} \frac{\theta_S(e^{iw})\theta_S(e^{-iw})}{\phi_S(e^{iw})\phi_S(e^{-iw})} \\
f_T(w) &= V_{T_t} \frac{\theta_T(e^{iw})\theta_T(e^{-iw})}{\phi_T(e^{iw})\phi_T(e^{-iw})} \\
f_N(w) &= V_{N_t} \frac{\theta_N(e^{iw})\theta_N(e^{-iw})}{\phi_N(e^{iw})\phi_N(e^{-iw})}
\end{aligned}
\tag{3.5}
$$

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Then the independence of variances gives that

$$f_X(w) = f_S(w) + f_T(w) + f_N(w) \tag{3.6}$$

and it becomes

$$V_{Z_t}\frac{\theta(e^{iw})\theta(e^{-iw})}{\phi(e^{iw})\phi(e^{-iw})} = V_{S_t}\frac{\theta_S(e^{iw})\theta_S(e^{-iw})}{\phi_S(e^{iw})\phi_S(e^{-iw})} + V_{T_t}\frac{\theta_T(e^{iw})\theta_T(e^{-iw})}{\phi_T(e^{iw})\phi_T(e^{-iw})} + V_{N_t}\frac{\theta_N(e^{iw})\theta_N(e^{-iw})}{\phi_N(e^{iw})\phi_N(e^{-iw})}$$

$$\begin{aligned} V_{Z_t}\frac{\theta(e^{iw})\theta(e^{-iw})}{\phi(e^{iw})\phi(e^{-iw})} &= V_{S_t}\frac{\theta_S(e^{iw})\theta_S(e^{-iw})\phi_T(e^{iw})\phi_T(e^{-iw})\phi_N(e^{iw})\phi_N(e^{-iw})}{\phi_S(e^{iw})\phi_S(e^{-iw})\phi_T(e^{iw})\phi_T(e^{-iw})\phi_N(e^{iw})\phi_N(e^{-iw})} \\ &+ V_{T_t}\frac{\theta_T(e^{iw})\theta_T(e^{-iw})\phi_S(e^{iw})\phi_S(e^{-iw})\phi_N(e^{iw})\phi_N(e^{-iw})}{\phi_S(e^{iw})\phi_S(e^{-iw})\phi_T(e^{iw})\phi_T(e^{-iw})\phi_N(e^{iw})\phi_N(e^{-iw})} \\ &+ V_{N_t}\frac{\theta_N(e^{iw})\theta_N(e^{-iw})\phi_S(e^{iw})\phi_S(e^{-iw})\phi_T(e^{iw})\phi_T(e^{-iw})}{\phi_S(e^{iw})\phi_S(e^{-iw})\phi_T(e^{iw})\phi_T(e^{-iw})\phi_N(e^{iw})\phi_N(e^{-iw})} \end{aligned}$$

The consistency between the overall model and the ones for the components implies the two constraints:

$$\phi^*(B) = \phi_S(B)\phi_T(B)\phi_N(B), \tag{3.7}$$

$$\theta^*(B)Z_t = \phi_T(B)\phi_N(B)\theta_S(B)Z_{St} + \phi_S(B)\phi_N(B)\theta_T(B)Z_{Tt} + \phi_S(B)\phi_T(B)\theta_N(B)Z_{Nt}. \tag{3.8}$$

(Hillmer & Tiao, 1982; Gómez & Maravall, 2001)

3.3 Obtaining the overall ARIMA and the components models

At this point we estimate overall model and wish to specify component models which satisfy (3.7) and (3.8) In this section we present the methodology for the latter in more detail.

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Consider (3.7) and assumption that AR polynomials share no common root, factorization of $\phi^*(B)$ directly yields the polynomials $\phi_i(B)$ of the unobserved components. The different roots may be allocated to the different components according to the behavior they induce in the series. Thus, the AR polynomials of the components are identified and can be obtained from the AR polynomial in the model for the observed series (Maravall, n.d.).

The AR polynomial $\phi_S(B) = 1 + B + ... + B^{s-1}$ contains the non-stationary seasonal roots, i.e. roots with modulo one or almost one, and frequency w such as: (a) $f_s(w)$ is infinite at the $w = \frac{2k\pi}{s}$ for $k = 1, ...[s/2]$, (b) $f_s(w)$ has relative minimum at w=0 and near the frequencies $w = \frac{(2k-1)\pi}{s}$
$\phi_T(B) = (1 - B)^{d+1}$ contains non-stationary trend roots, i.e. $f_T(w)$ is infinite at $w = 0$ and very large for small w
$\phi_N(B)$, the irregular polynomial, contains the rest. (Hillmer & Tiao, 1982)

In particular, SEATS assigns the roots of AR polynomial to the components as follow:

- Roots of $(1 - B)^d$ are assigned to trend component.
- Roots of $(1 - B^s)^D = (1 - B)^D(1 + B + ... + B^{s-1})^D$ are assigned to the trend component, i.e. root of $(1 - B)^D$, and to the seasonal component, root of $(1 + B + ... + B^{s-1})^D$.
- When the modulus of the inverse of a real positive root is greater than k, then the root is assigned to the trend component, where k can be entered by the user. Otherwise it is integrated in the irregular component.
- Real negative inverse roots of $\phi(B)$ associated with the seasonal two-period cycle and complex roots for which the argument (angular frequency) is close enough to the seasonal frequency are assigned to the seasonal component if

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their moduli are greater than k. Otherwise they are assigned to the irregular component. Closeness is controlled by the ϵ parameter, which also entered by the user.

- If D, seasonal differencing order, is present and $B\Phi < 0$, $B\Phi$ is the estimate of the seasonal autoregressive parameter, the real positive inverse root is assigned to the trend component and the other $(s-1)$ inverse roots are assigned to the seasonal component. When $D = 0$, the root is assigned to the seasonal when $B\Phi < -0.2$ and/or the overall test for seasonality indicates presence of seasonality. Otherwise it goes to the irregular component. Also, when $B\Phi > 0$ roots are assigned to the irregular component.

Note that, in SEATS default values for p, q, P, Q are in the range $0 \leq p, q \leq 3, 0 \leq P, Q \leq 2$. This is done sequentially (for fixed regular polynomials, the seasonal ones are obtained, and viceversa), and the final orders of the polynomials are chosen according to the BIC criterion, with some possible constraints aimed at increasing parsimony and favoring ”balanced” models, i.e. similar AR and MA orders (Gomez & Maravall, 1996).

Then, consider all possible model specifications (under UCARIMA framework) that satisfy the identity (3.8), and have nonnegative component spectra. For a given model for the observed series, they form the set of *admissible* decompositions. Different admissible decompositions have different model specifications for the components, and therefore will display different properties. The component is a tool that we devise, we can choose, among the different admissible decompositions, the one that presents certain desirable features (Hillmer & Tiao, 1982).

For some ARIMA models and some value of the parameters not all component spectra are non-negative for the $0 \leq w \leq \pi$. The decomposition in those cases is termed non-admissible. Using a program (e.g. SEATS) the model will be replaced

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with a relative close decomposable one, so that an admissible decomposition is always provided and tested.

For $\theta_i(B)$, the discussion will be clearer if we look at a particular example, namely the quarterly Airline Model

$$(1-B)(1-B^4)X_t = (1-\theta_1 B)(1-\theta_4 B^4)Z_t, \quad Z_t \sim WN(0, V_{Zt}). \tag{3.9}$$

It is a model similar to the so-called *Airline Model* of Box and Jenkins (Box, Jenkins, Reinsel, & Ljung, 2015). Assume we wish to decompose X_t into a trend T_t, a seasonal S_t and an irregular component N_t, as in (3.3). The AR part of (3.9) can be rewritten $(1-B)(1-B^4) = (1-B)^2(1+B+B^2+B^3) = (1-B)^2(1+B)(1+B^2)$, so $\phi_T(B) = (1-B)^2$, since $\phi_T(1) = 0$ (i.e. B=z and z=1 has zero frequency which means that period tends to infinity) and $\phi_S(B) = (1+B)(1+B^2)$, since $\phi_S(i) = \phi_S(-i) = \phi_S(-1) = 0$ (i.e. $z = i$, $z = -i$, $z = -1$, associated with frequencies $\pi, \pi/2$, which are the seasonal frequencies). The irregular component can be assumed to be white noise. Then the components follow the models, $(1-B)^2 T_t = \theta_T(B) Z_{Tt}$, $(1+B)(1+B^2)S_t = \theta_S(B) Z_{St}$, and $N_t = V_{Nt}$ respectively. Thus, letting $\theta(B) = (1-\theta_1 B)(1-\theta_4 B^4)$, consistency with the overall model (i.e. equation 3.8) implies

$$\theta(B)Z_t = (1+B)(1+B^2)\theta_T(B)Z_{Tt} + (1-B)^2\theta_S(B)(Z_{St} + (1-B)(1-B^4)Z_{Nt} \tag{3.10}$$

Since the left hand side (l.h.s.) of (3.10) is an MA of order 5, we can set $\theta_T(B)$ and $\theta_S(B)$ to be of order 2 and 3, respectively, so that the three terms in the r.h.s. of (3.8) are also of order 5. The component models will then be of the type:

$$\begin{aligned} (1-B)^2 T_t &= (1+\theta_{T,1}B + \theta_{T,2}B^2)Z_{Tt}, \\ (1+B)(1+B^2)S_t &= (1+\theta_{S,1}B + \theta_{S,2}B^2 + \theta_{S,3}B^3)Z_{St} \\ N_t &= Z_{Nt}. \end{aligned} \tag{3.11}$$

Equating the covariances of the l.h.s. and the r.h.s. of (3.10), a system of 6 equations is obtained. These equations express the relationship between the parameters of the

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overall model and the unknown parameters in the component models. Since the number of the latter is 8 $(\theta_{T,1}, \theta_{T,2}, \theta_{S,1}, \theta_{S,2}, \theta_{S,3}, V_{St}, V_{Tt}, V_{Nt})$, there is an infinite number of structures of the type (3.11) that are compatible with the same model (3.9). In order to select one, additional information has to be incorporated.

In the decomposition of model (3.9) into three orthogonal components as in (3.11), the sum of the components spectra should be equal to the spectrum for the observed series, (3.6). The idea of choosing a decomposition, within the set of admissible decompositions, is the one that provides the smoothest components. Components from which no additive noise can be extracted were first proposed by Box, Hillmer and Tiao (Box et al., 2015, 1979); they have been termed "canonical components". The canonical component has the important property that any other admissible component can be seen as the canonical one plus superimposed (orthogonal) noise. The canonical component is uniquely obtained by simply subtracting from any admissible component spectrum its minimum. Moreover if an admissible decomposition exists, the canonical requirement identifies the component. As Hillmer and Tiao (Hillmer & Tiao, 1982) show, the canonical condition also minimizes the variance of the component p-innovation Z_{Tt}, since Z_{Tt} is the source of the stochastic variability, the canonical component can be seen as the closest to a deterministic component that is compatible with the stochastic structure of the series. If the spectrum of a trend component should be monotonically decreasing in w, its minimum will be obtained for $w = \pi$. If this minimum is larger than zero, further white noise could still be removed from the trend. Therefore, the noise-free condition implies $f_T(\pi) = 0$, which is equivalent, in the time domain, to the presence of the root $B = -1$ in the polynomial $\theta_T(B)$.

As with the trend, any admissible seasonal component can be seen as the canonical one with superimposed noise, and hence, if an admissible component is available, the canonical one can be trivially obtained. Finally, if, in the decomposition of (3.9)

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the canonical trend and seasonal components are specified, then the variance of the irregular (white-noise) component is maximized.

In the time domain analysis, the canonical requirement has general constraints among the coefficients. A canonical trend implies $\theta_T(-1) = 1 - \theta_{T,1} + \theta_{T,2} = 0$, and if, for example, the spectral zero of the canonical seasonal component occurs at $w = 0$, then $\theta_S(1) = 1 + \theta_{S,1} + \theta_{S,2} + \theta_{S,3} = 0$. The two constraints, added to the system of 6 covariance equations associated with (3.10), provide now a system of 8 equations which can be solved for the 8 unknown parameters, and hence the model becomes identified.

3.4 Estimating the values of the components

The foregoing does not allow exact calculation of S_t, T_t and N_t from X_t although it does tell us what models they follow, which allows us to use *signal extraction theory* to estimate them. When considering estimation of a component, it will prove convenient to rewrite the X_t as the sum of two components:

$$X_t = s_t + n_t, \tag{3.12}$$

where s_t denotes the component of interest (the 'signal'), and n_t denotes the sum of the other components (the non-signal or 'noise'). The two components follow the models:

$$\begin{aligned} \phi_s(B)s_t &= \theta_s(B)Z_{st}, \quad Z_{st} \sim WN(0, V_s) \\ \phi_n(B)n_t &= \theta_n(B)Z_{nt}, \quad Z_{nt} \sim WN(0, V_n) \end{aligned} \tag{3.13}$$

The model for the observed series is given by (3.2), and the aggregation relationships become $\phi^*(B) = \phi_s(B)\phi_n(B)$, and $\theta^*(B)Z_t = \theta_s(B)\phi_n(B)Z_{st} + \theta_n(B)\phi_s(B)Z_{nt}$.

We proceed to obtain the estimators $\widehat{s_t}$ and $\widehat{n_t}$ given a realization of X_t, such that $E(s_t - \widehat{s_t}|X_T)^2$ is minimized, i.e., the Minimum Mean Squared Error Estimator

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(MMSE) of s_t. Under the joint normality assumption, $\widehat{s_t}$ is also equal to the conditional expectations $E(s_t|X_T)$, and hence, a linear function of the elements in X_T. In AMB procedure these estimators can be obtained using the Wiener-Kolmogorov (WK) filter, (see below 3.15).

3.4.1 Signal extraction from a complete realization X_t

Start with the case of a complete realization X_t, extending from $t = -\infty$ to $t = \infty$. Denote this realization by X. Assume, first, the case of stationary series (and hence stationary components), and write the models (3.2), (3.13) in their MA expression as $X_t = \psi(B)Z_t, s_t = \psi_s(B)Z_{st}, n_t = \psi_n(B)Z_{nt}$, where $\psi(B) = \frac{\theta^*(B)}{\phi^*(B)}, \psi_s(B) = \frac{\theta_s(B)}{\phi_s(B)}$, and $\psi_n(B) = \frac{\theta_n(B)}{\phi_n(B)}$. Using well-known results (Cleveland & Tiao, 1976; Bell & Hillmer, 1984), the MMSE estimators of the 'signal' is given by

$$\widehat{s_t} = s_{t|\infty} = E(s_t|X) = v(B,F)X_t, \tag{3.14}$$

where the v-polynomial represent the symmetric filter

$$v(B,F) = \frac{V_{Zst}}{V_{Zt}} \frac{\psi_s(B)\psi_s(F)}{\psi(B)\psi(F)} = v_0 + \sum_{j=1}^{\infty} v_j(B^j + F^j) \tag{3.15}$$

The filter $v(B,F)$ is the so-called Wiener-Kolmogorov (WK) filter and the estimator $\widehat{s_t}$ is called *final or historical estimator*. Assume it can be safely truncated after L periods, so that we can write the historical estimator as

$$\widehat{s_t} = v_0 X_t + \sum_{j=1}^{L} v_j(B^j + F^j)X_t \tag{3.16}$$

Replacing the ψ-polynomials by their rational expressions, after cancellation of roots, it is obtained that

$$v(B,F) = k_s \frac{\theta_s(B)\theta_s(F)\phi_n(B)\phi_n(F)}{\theta^*(B)\theta^*(F)} \tag{3.17}$$

where $k_s = \frac{V_{Zst}}{V_{Zt}}$.

It is seen that no AR roots appear in the denominator of the filter, which under

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assumption that $\theta_i(B), i = s, n$ share no unit root in common, will always converge. In fact, expression (3.14) also yields the optimal estimator of s_t in the (unit roots) non-stationary case. Direct inspection of $v(B, F)$ shows that the filter is centered at t, symmetric and convergent in B and F. In particular, the filter will be finite when the overall model (3.2) is a finite AR process. The desired filter coefficients can be obtained by noting that the filter is precisely the ACGF of the ARIMA model

$$\theta^*(B)Z_t = \theta_s(B)\phi_n(B)a_t, \quad Var(a_t) = k_s \tag{3.18}$$

The assumption that $\theta_i(B), i = s, n$ share no unit root in common guarantees stationarity; as for invertibility, the filter that yields s_t will be noninvertible when n_t is nonstationary(Maravall, n.d.).

3.4.2 The Model for the Estimator

In order to analyze the estimator, it will prove helpful to obtain the model that express the component of interest as function of the innovation Z_t. From (3.14), (3.17), and (3.2), it is obtained that

$$\phi_s(B)\hat{s}_t = \theta_s(B)a_s(F)Z_t \tag{3.19}$$

where

$$a_s(F) = k_s \frac{\theta_s(F)\phi_n(F)}{\theta^*(F)} \tag{3.20}$$

It is obviously that (3.4) and (3.19), share the same AR polynomial. The stationarity - inducing transformation is the same for both, and component and estimator have the same order of integration. Moreover the two models (3.4) and (3.19) share the same polynomials in the operator B.

The basic difference between the two models is the presence of the polynomial $a_s(F)$ in the model for the estimator. This forward filter expresses the two-sided

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character of the WK filter, that is, the dependence of the final estimator $\widehat{s_t}$ on innovations posterior to period t (this dependence goes to zero as the time distance increases). Furthermore, they will display different variances, covariances (for the stationary transformation), and different spectra. The spectrum of a component is similar to that of its estimator, except for the dips displayed by the latter at the frequencies for which the other components present spectral peaks. From (3.14) and (3.17) the spectrum of $\widehat{s_t}$ is equal to

$$f_{\widehat{s_t}}(w) = R^2(w) f_x(w), \tag{3.21}$$

where,

$$R(w) = \frac{f_s(w)}{f_x(w)} = \frac{1}{1 + \frac{1}{r(w)}}, \tag{3.22}$$

where $r(w) = \frac{f_s(w)}{f_n(w)}$ represents the signal-to-noise ratio and (3.21) can be written as

$$f_{\widehat{s_t}}(w) = \left(\frac{f_s(w)}{f_x(w)} \right) f_s(w). \tag{3.23}$$

The function (3.22) is also referred to as the gain of the filter. For each w it computes the signal-to-noise ratio . If the ratio is high then the contribution of that frequency in the estimation of the signal will also be high. For example, if the trend is the signal, then $R(0) = 1$, and the frequency $w = 0$ will only be used for trend estimation. Since the noise, in this case, contains seasonal nonstationarity, for the seasonal frequencies, $R(w) = 0$, so that these frequencies are ignored in computing the trend. The associated spectral zeroes in $f_{\widehat{st}}(w)$ explain the dips in the estimator's spectra; they also imply that model (3.19) is noninvertible. This noninvertibility of the estimator is also evident from the unit seasonal roots of $\phi_n(B)$ in $a_s(F)$, which appear in the MA part of model (3.14). The variances of the theoretical component are larger of those of its estimator , since $\frac{f_s(w)}{f_x(w)} \leq 1$. Relatively more stochastic components will imply smaller underestimation and hence the estimator displays a bias towards stability.

3.4.3 Signal extraction from a finite realization X_T

The previous statements of the estimators were centered on the final (or historical) estimators. For a finite (long enough) realization, they can be assumed to characterize the estimators for the central observations of the series, but the periods close to beginning or the end, the filter cannot be computed and some *preliminary estimator* has to be used. As before, let X and X_T denote the complete and finite realization of the time series, respectively, and let s_t the component of interest ('signal'). To project s_t on the finite realization X_T, since $X_T \subset X_t$, the preliminary estimator can be expressed as

$$\widehat{s}_{t|T} = E(s_t|X_T) = E(E(s_t|X)|X_T) = E(\widehat{s}_t|X_T), \tag{3.24}$$

which implies that $\widehat{s}_{t|T}$ can be expressed as

$$\widehat{s}_{t|T} = v(B, F)X_{t|T}, \tag{3.25}$$

where $v(B, F)$ is the WK filter given by (3.17), B and F operate on t, and $X_{t|T} = E(X_t|X_T)$.

Explaining the above preliminary estimator is obtained by replacing observations not yet available with forecasts. For example, $\widehat{s}_{t|t-1} = ... + v_1 X_{t-1} + v_0 \widehat{X}_{t|t-1} + v_1 \widehat{X}_{t+1|t-1} + ...$, where $\widehat{X}_{t|t-1}$ and $\widehat{X}_{t|t-1}$ are obtained as ARIMA forecasts. In Burman - Wilson (Burman, J. P. (1980), "Seasonal adjustment by signal extraction") algorithm is proved that only a few forecasts and backcasts are needed, typically, about two years. As new observations become available the estimator of s_t is revised. The *revision* the preliminary estimator will undergo until it becomes the historical one is the difference $(\widehat{s}_t - \widehat{s}_{t|T})$ or, subtracting (3.25) from (3.16),

$$r_{t|T} = \sum_{j=1}^{t+L-T} v_{T-t+j} \widehat{\epsilon}_{T+j|T}, \tag{3.26}$$

that is, the revision is a linear combination of the forecast errors. Large revisions are unquestionably an undesirable feature of a preliminary estimator, and expression

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(3.26) shows the close relationship between forecast error and revision: the better we can forecast the observed series, the smaller the revision in the preliminary estimator of the signal will be.

In practice, for central years of the series the final estimator is valid, e.g. $\widehat{s}_{T-100|T}$, $\widehat{s}_{T-j|T}$ is the preliminary estimator for j covering a few years and $\widehat{s}_{T|T}$ is the concurrent estimator. Let the component of interest be the seasonally adjusted series. Each one of these estimators is the output of a different model. Therefore, SA series is a mixture of realizations with different underlying models. Thus, SA series is available at some point in time is nonlinear and this is a reason to avoid using it in modeling.

3.4.4 Error in the estimator of a component

Point estimators or forecasts of the components are of limited interest unless some information is provided about their precision. This information is unavailable when ad-hoc filters are applied. The model-based approach provides the framework for obtaining the standard error of the components estimators. In particular, in AMB approach, they can be computed, using the procedure in (Hillmer, 1985). More generally, the model-based method allows us to derive, under our assumptions, the full distribution of the different estimation errors (associated with the final or with some preliminary estimator).

Because of the stochastic nature of s_t, its historical estimation error $\widehat{s}_t$ contains an error $e_t = s_t - \widehat{s}_t$, to be denoted *historical estimation error*. It is straightforward to see that the AGF of the historical estimation error, is equal to the AGF of the stationary ARMA model

$$\theta(B)X_t = \theta_s(B)\theta_n(B)b_t, \quad b_t \sim WN\left(0, \frac{V_s V_n}{V_Z}\right) \tag{3.27}$$

Stationarity of (3.27) implies that component and estimator are cointegrated. Therefore the distribution of e_t is easily obtained. For the concurrent estimator, the one of

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most applied relevance, let $r_t = \widehat{s}_T - \widehat{s}_{T|T}$ denote the revision error. The *preliminary estimation error*, $e_t|T$, is

$$e_{t|T} = s_T - \widehat{s}_{T|T} = (s_T - \widehat{s}_T) + (\widehat{s}_T - \widehat{s}_{T|T}) = e_T + r_T.$$

From the orthogonality of e_t, and $e_{t|T}$, the variance and ACF of the total estimation error $e_t|T$ are, then, straightforward to obtain. Notice that the fact that e_T, and r_T have finite variance implies that the theoretical component, s_t, its historical estimator, $\widehat{s}_t$, and its preliminary estimator or forecast, $s_{t|T}$ are all pairwise cointegrated.

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Chapter 4

An application in JDemetra+

This chapter illustrates the ARIMA model based seasonal adjustment method. The method is aplied to the time series of Denmark, (a series chosen arbitrarily), by using the *JDemetra+* software. This software is developped and published by EUROSTAT. *JDemetra+* software, enables the user to perform seasonal adjustment and other operations. Two seasonal adjustment methods are implemented, X12-ARIMA/X13ARIMA-SEATS and TRAMO/SEATS. We are interested in the second method, performed by TRAMO/SEATS. TRAMO is the regression part (it use RSAfull method gives the choice between fixing the ARIMA model structure to $(0,1,1)x(0,1,1)$ or searching for the ARIMA model using automatic model identification procedure), which removes deterministic effects, outliers, and identifies calendar effects (trading day, leap year, easter effect). Then it estimates a SARIMA model which is the starting point for the seasonal decomposition part performed by SEATS. This latter us the actual focus of our presentation.

The dataset is the "Volume index of production - construction" and is accessible from Eurostat's site. It includes the volume index of Denmark's production from 01/01/99 to 01/04/15. Time series are monthly so there are 197 observations. There

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are some missing values. In the *Main results*, there is a figure that includes a chart of the original series, trend and seasonally adjusted series together with the forecasts for the next year for each of these series. *JDemetra+* removes the last observations from the series and calculates a one-period-ahead out-of-sample forecast of the series. The forercasted values are then compared with the actual values. The user may decide how many of the last observations will be considered (one, two, or three) in this procedure. Here is three, the default.

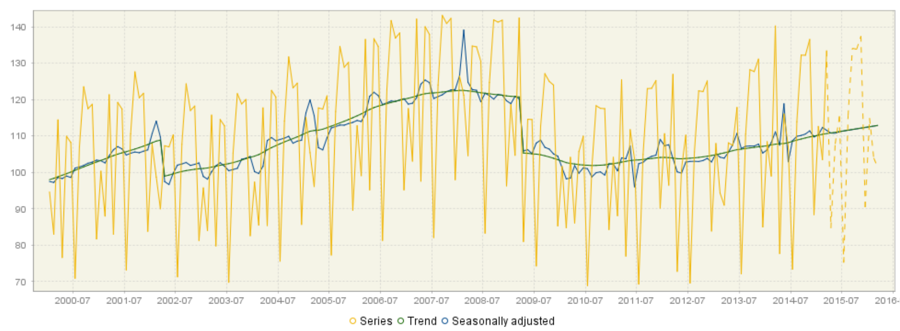

Figure 4.1: The original series, trend and the seasonally adjusted series together with the forecast.

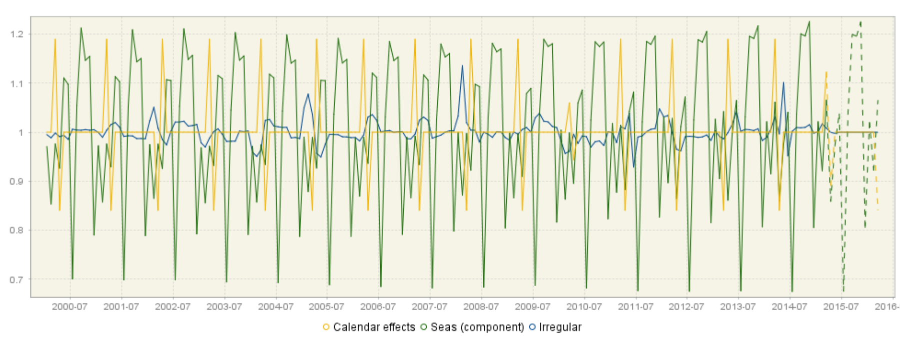

Figure 4.2: The decomposition results: calendar effects, the seasonal component and the irregular component. Seas (component) stands for the seasonal component.

The second graph from the Charts section presents the calendar effects (they are estimated by TRAMO and will not bother us), the seasonal component and

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the irregular component. In *JDemetra+* the irregular component is assumed to be white noise component with maximal variance; therefore its forecasts are, in general, zero. The lack of certain movements (seasonal and/or irregular) is manifested by a horizontal line with values equal to zero.

In the *Pre-processing* → *Arima* section is presented the theoritical spectrum of the estimated model from TRAMO (see figure 4.4).

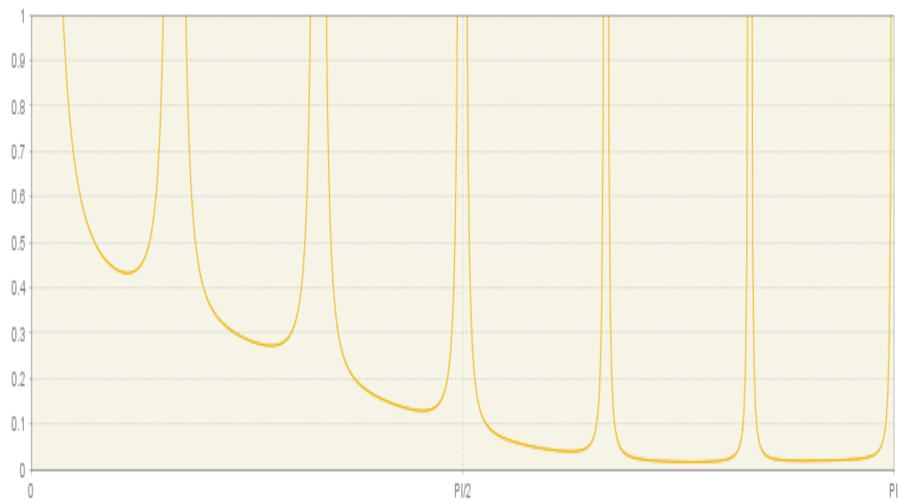

Figure 4.3: Theoretical spectrum of the ARIMA model.

As for decomposition, SEATS receives the linearised series from TRAMO. It is worth noting that the decomposition made by SEATS may include four components. In particular, the N_t in (3.3), is here broken down into two further additive components, a White noise (called "Irregular") and a further component , here called "Transitory". Transitory, if present, it will typically capture short-lived, fairly erratic behavior that is not white noise, sometimes associated with ackward frequencies. It is also assumed that all components in time series - trend, seasonal, transitory and irregular - are orthogonal and can be expressed by ARIMA models. SEATS performs the canonical decomposition of the components. In our example the overall ARIMA and the components models are below

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```
Model
D: 1.00000 - B - B^12 + B^13
MA: 1.00000 - 0.0195308 B - 0.383777 B^2 - 0.327771 B^3 - 0.497166 B^12 + 0.00971004 B^13 + 0.190801 B^14 + 0.162956 B^15

sa
D: 1.00000 - 2.00000 B + B^2
MA: 1.00000 - 0.989617 B - 0.330560 B^2 + 0.0414253 B^3 + 0.293885 B^4
Innovation variance: 0.55449
trend
D: 1.00000 - 2.00000 B + B^2
MA: 1.00000 + 0.0516560 B - 0.948344 B^2
Innovation variance: 0.01190
seasonal
D: 1.00000 + B + B^2 + B^3 + B^4 + B^5 + B^6 + B^7 + B^8 + B^9 + B^10 + B^11
MA: 1.00000 + 1.93213 B + 2.10979 B^2 + 1.62162 B^3 + 1.25081 B^4 + 0.917644 B^5 + 0.613070 B^6 + 0.431016 B^7 + 0.229508 B^8 + 0.0975139 B^9 + 0.0694222 B^10 - 0.0727428 B^11
Innovation variance: 0.12297
transitory
MA: 1.00000 + 1.57567 B + B^2
Innovation variance: 0.16296
irregular
Innovation variance: 0.09729
```

Figure 4.4: The ARIMA models for original series, seasonally adjusted series and components.

That is for example, the model for the trend component is an ARIMA(0,2,2)(0,0,0) with innovation variance 0.012 and the model for the seasonal component is an ARIMA(0,11,11)(0,0,0) with innovation variance 0.123.

WK analysis – Components section presents the pseudo-spectra of the components and seasonally adjusted series calculated from the ARIMA models presented in the main panel of the *Decomposition* section. The sum of the spectra of the components should be equal to the spectrum of the linearised time series, which is presented in the figure 4.3, (note that when the TRAMO model has not been accepted by SEATS, the spectra of the components derive from the ARIMA model changed by SEATS). In the figure 4.5 is presented the spectrum of the seasonally adjusted series (yellow), that is the sum of the spectra of the trend component (green), the transitory component (black), and the irregular component (orange).

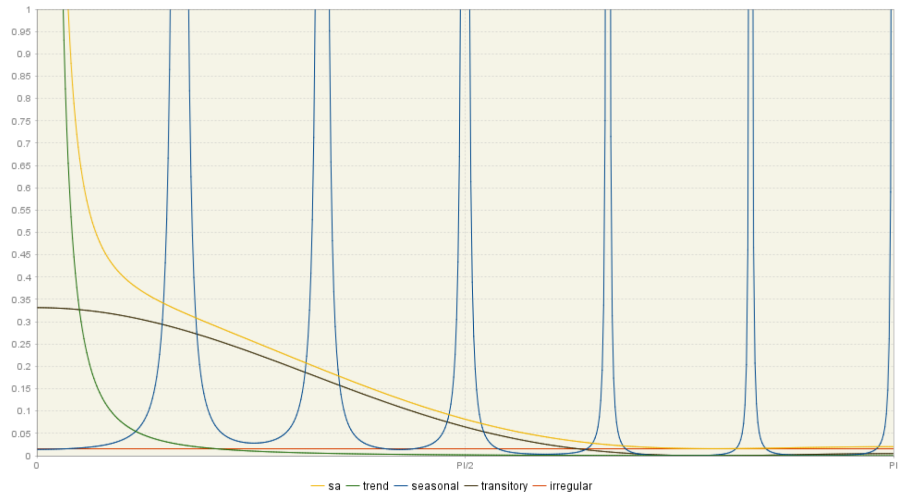

Figure 4.5: The theoretical spectra of the components and the seasonally adjusted series.

It seems that the spectrum of trend component presents a peak at the frequency $w = 0$. The spectrum of seasonal component presents peaks at seasonal frequencies, i.e $\pi/6, \pi/3, \pi/2, 2\pi/3, 5\pi/6$ and π as well as the the peak in the transitory component is evidence of a trading day effect. The irregular spectrum is flat (see section 2.5). Further, stable trend and seasonal components are those with thin spectral peaks, while unstable ones are characterised by wide spectral peaks.

The ACGF (stationary) window displays the pseudo-autocovariance generating functions of the stationary components. They are theoretical values (i.e. they are not computed on the linearised series, but on the ARIMA model).

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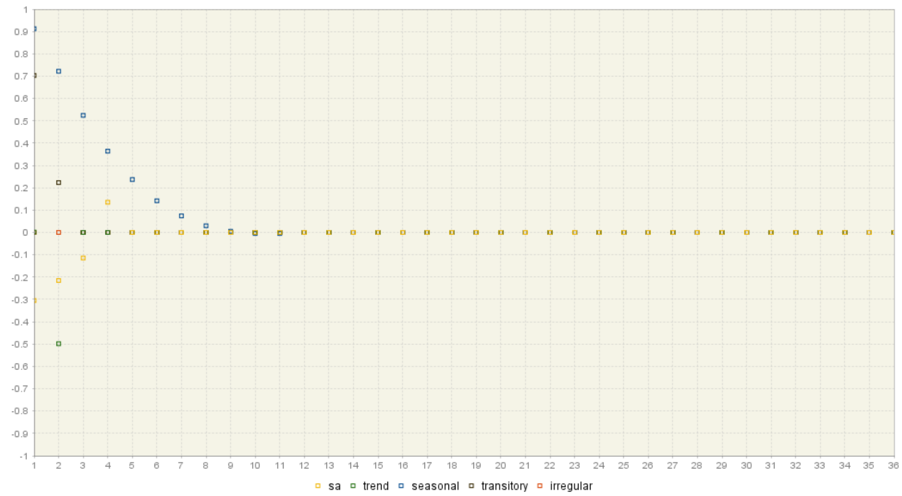

Figure 4.6: ACGF function of the stationary components.

Further, in *WK analysis – Final estimators* node, is presented various graphs that can be shown the results of the estimation of the components performed with Wiener-Kolmogorov (WK) filters. As it is explained in 3.4.3, WK filters are not available at the beginning and end of the finite time series. In order to apply a filter to all observations from X_t, the original, linearised time series is extended with forecasts and backcasts using an ARIMA model, which has been chosen in the TRAMO phase of seasonal adjustment. Then, SEATS applies the filter to extended series.

Regarding the importance of final estimators (see 3.4.1) derived applying the WK filters, *JDemetra+* presents several graphs showing their properties. The corresponding graphs for the components and for the final estimators of the components vary, as components and final estimators follow different models (see 3.4.2). The spectra of the final estimators (see 3.21) are shown in the first graph.

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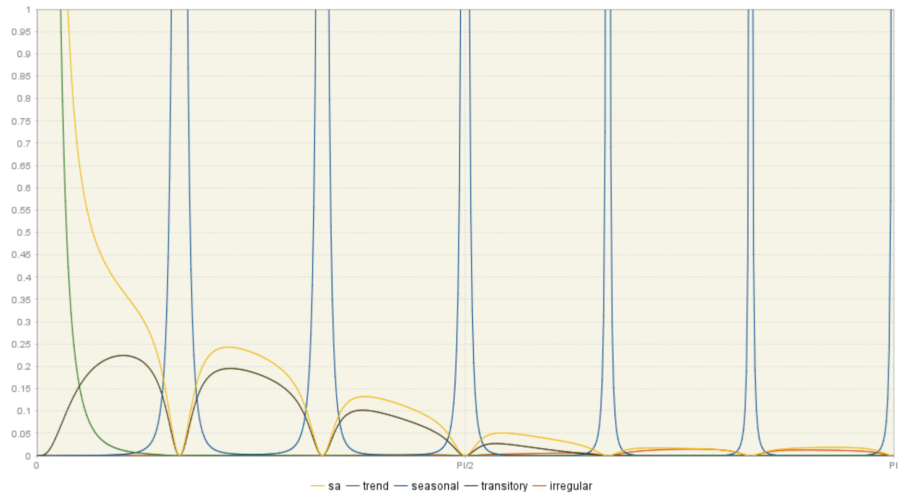

Figure 4.7: Spectra of the final estimators.

It is clear that these spectra are similar to those of the components, although estimator spectra show spectral zeros at the frequencies where the component spectra are close but not exactly zero. The estimator adapts to the structure of the analysed series, i.e. the width of the spectral holes in the seasonally adjusted series (yellow line) depends on the width of the seasonal peaks in the seasonal component estimator spectrum (blue lines). The second graph is for squared gain.

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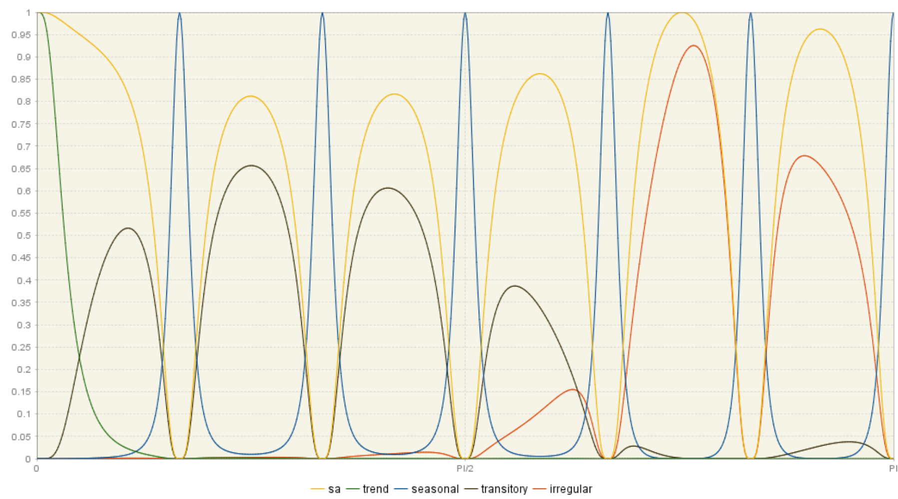

Figure 4.8: Squared gains for filters of component series – the case of a stochastic seasonal component.

Figure 4.8 shows that the seasonal frequencies are assigned to the seasonal component while the seasonally adjusted series captures the non-seasonal frequencies. As a consequence, it is expected that the seasonal component estimator captures only the seasonal frequencies, so its peaks assume unitary values at seasonal frequencies. By contrast, the estimator of the non-seasonal part of the time series is expected to eliminate seasonal frequencies, leaving unmodified non-seasonal frequencies. Therefore, the squared gain of seasonally adjusted data should be nearly zero for seasonal frequencies (see 3.4.2). Large troughs suppress the highly stochastic seasonal component.

Since WK filters are symmetric, centered and convergent, they are valid for computing the estimators in the central periods of the sample. The following graph demonstrates the weights that are applied to each observation for each component to calculate the estimate of each component.

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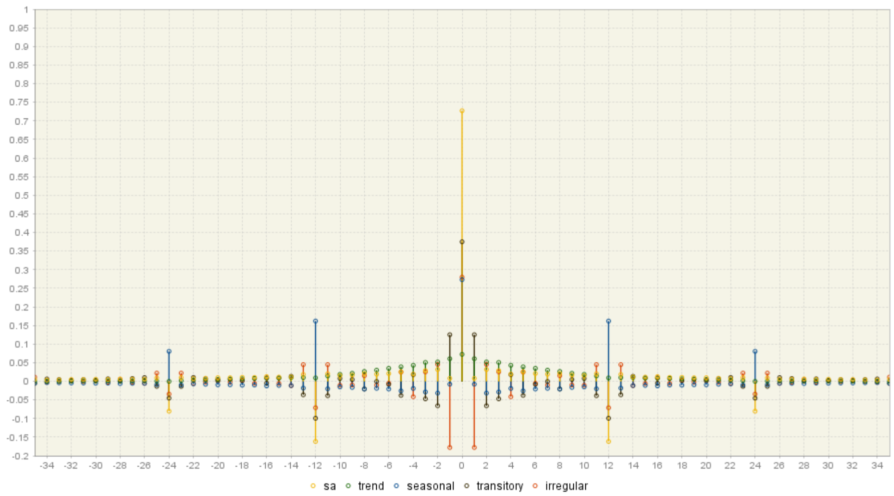

Figure 4.9: WK filter weights for the seasonally adjusted series and for the components.

The highest weights are applied to the central observations and the weights decrease for observations further away in time. Therefore the estimated value of each component is highly influenced by the linearised series value. The weighting pattern depends on the component. For example, in the case of the seasonal component the greatest weights are applied to the current value and the past and future values from this same period. On the contrary, the estimate of the trend at a given point in time is mostly influenced by the current value and few preceding and following values of the linearised series.

Last figure in this node is for ψ-weights, a different representation of the final estimator. This representation shows the estimator as a filter applied to the innovations, rather than the series X_t (see 3.19).

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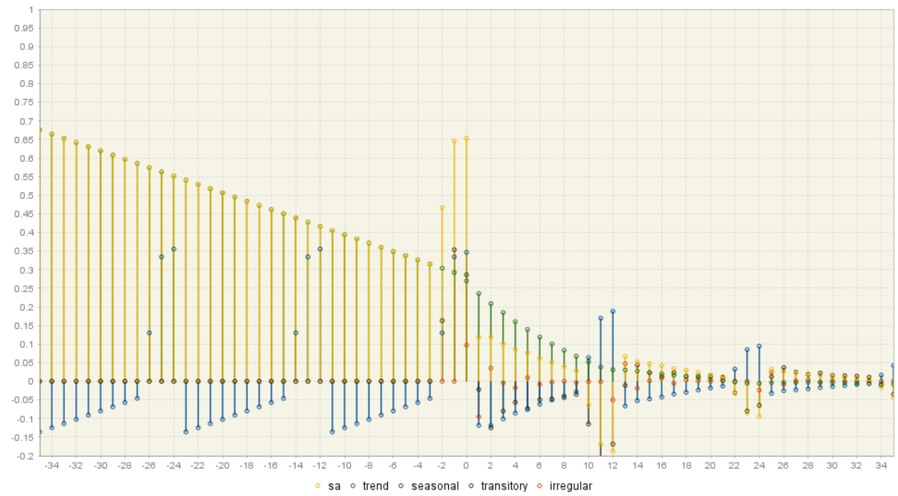

Figure 4.10: ψ-weights for the components.

Figure (4.10) shows for each component how the contribution of the total innovation to the component estimator varies in time (the size of this contribution is shown on the Y-axis). For non-negative values on the X-axis, ψ-weights show the effect of starting conditions, present and past innovations in series, while for negative observations they present the effect of future innovations. It can be seen that they are non-convergent in the past (they are convergent when series X_t stationary). On the contrary, the effect of future innovations is a zero-mean and convergent process. ψ-weights are important to analyse the convergence of estimators and revision errors.

Finally, the *Decompotisision* $\rightarrow$ *Stochastic series* node presents the results of the decomposition of the stochastic series resulting from linearization procedure performed by TRAMO. The two subsections allow for some insights into the development of the two components (trend and seasonal) in the last 7 years and a forecast for the next year. For each component its values are displayed with the associated 95% confidence intervals, highlighting the fact that these values result from the estimation procedure. The width of the confidence intervals shows the size of uncertainty of the estimation results, which in general is greater at the end of the time series. The

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prediction intervals, which are available for the forecasts, are even wider than the confidence intervals. The graph is available for the trend (figure 4.11) and for the seasonal component (figure 4.12).

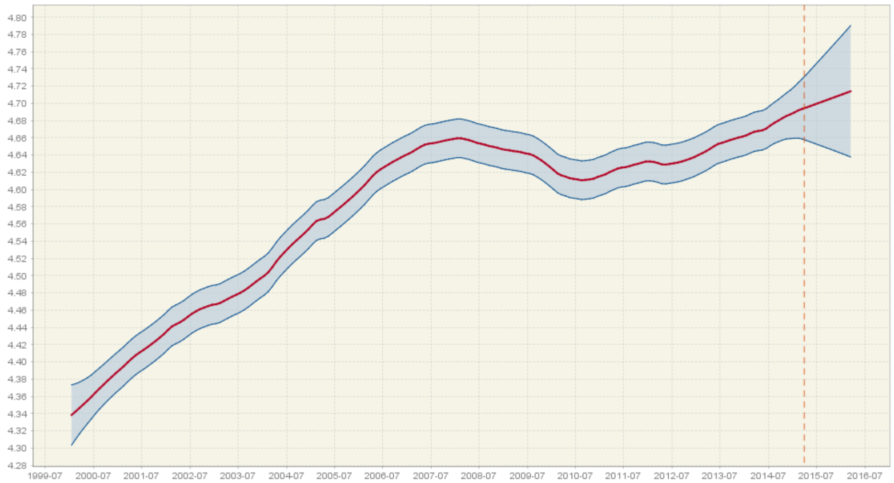

Figure 4.11: The trend estimate with the confidence interval and the prediction interval.

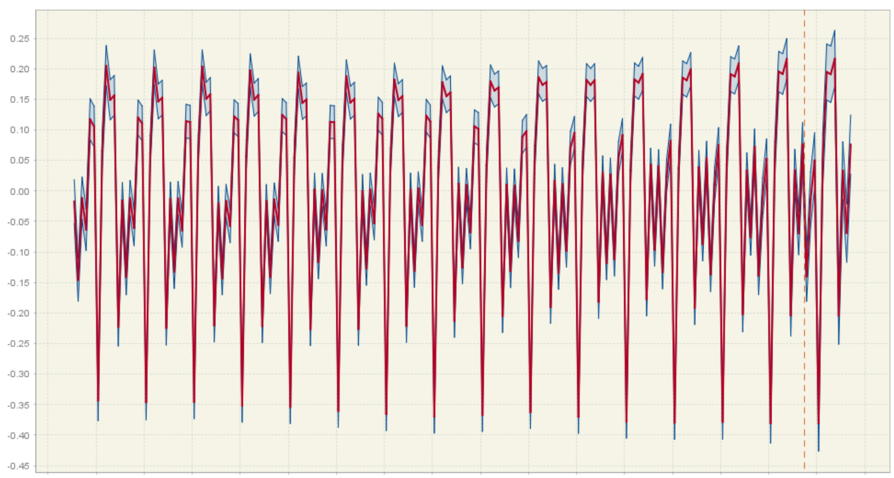

Figure 4.12: The seasonal component estimate with the confidence interval and the prediction interval.

To conclude, *JDemetra+* provides many more tools that make seasonal adjust-

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ment perfectly clear. For readers who are dealing with seasonal adjustment is recommended *JDemetra+* software, detailed description of which is provided in *JDemetra+ Reference Manual (2017)*.

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Appendix A

Autocovariance Generating Function

For (2.14) , let X_t be a stationary process with autocovariance function $\gamma(.)$, its **autocovariance generating function** is defined by

$$G(z) = \sum_{k=-\infty}^{\infty} \gamma(k) z^k \tag{A.1}$$

provided the series converges for all z in some annulus $r^{-1} < |z| < r$ with $r > 1$. Frequently the generating function is easy to calculate, in which case the autocovariance at lag-k may be determined by identifying the coefficients of either z_k or z_{-k}. Clearly X_t is white noise if and only if the autocovariance generating function $G(z)$ is constant for all z. If

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \quad Z_t \sim N(0, V_\sigma^2) \tag{A.2}$$

and there exists $r > 0$ such that,

$$\sum_{j=-\infty}^{\infty} |\psi_j| < \infty, \quad r^{-1} < |z| < r \tag{A.3}$$

, the generating function $G(.)$ takes a very simple form. It is easy to see that

$$\gamma(k) = Cov(X_{t+k}, X_t) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+|k|}$$

and hence that

$$\begin{aligned} G(z) &= \sigma^2 \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+|k|} z^k \\ &= \sigma^2 \Big[\sum_{j=-\infty}^{\infty} \psi_j^2 + \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+k} (z^k + z^{-k}) \Big] \\ &= \sigma^2 \Big(\sum_{j=-\infty}^{\infty} \psi_j z^j \Big) \Big(\sum_{k=-\infty}^{\infty} \psi_k z^k \Big) \end{aligned}$$

Defining

$$\psi(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j, \qquad r^{-1} < |z| < r$$

we can write this result more neatly in the form

$$G(z) = \sigma^2 \psi(z) \psi(z^{-1}), \qquad r^{-1} < |z| < r.$$

(Brockwell et al., 1991)

References

Bell, W. R., & Hillmer, S. C. (1984). Issues involved with the seasonal adjustment of economic time series. *Journal of Business & Economic Statistics*, *2*(4), 291–320.

Box, G. E., Hillmer, S., & Tiao, G. C. (1979). Analysis and modeling of seasonal time series. In *Seasonal analysis of economic time series* (pp. 309–346). NBER.

Box, G. E., Jenkins, G. M., Reinsel, G. C., & Ljung, G. M. (2015). *Time series analysis: forecasting and control.* John Wiley & Sons.

Brockwell, P. J., Davis, R. A., & Calder, M. V. (2002). *Introduction to time series and forecasting* (Vol. 2). Springer.

https://doi.org/10.26219/heal.aueb.5912

Brockwell, P. J., Davis, R. A., & Fienberg, S. E. (1991). *Time series: Theory and methods: Theory and methods.* Springer Science & Business Media.

Burman, J. P. (1980). Seasonal adjustment by signal extraction. *Journal of the Royal Statistical Society: Series A (General)*, *143*(3), 321–337.

Cleveland, W. P., & Tiao, G. C. (1976). Decomposition of seasonal time series: A model for the census x-11 program. *Journal of the American statistical Association*, *71*(355), 581–587.

Gomez, V., & Maravall, A. (1996). Programs tramo (time series regression with arima noise, missing observations, and outliers) and seats (signal extraction in arima time series). instructions for the user. *Documento de Trabajo*, *9628*.

Gómez, V., & Maravall, A. (2001). Seasonal adjustment and signal extraction in economic time series. *A course in time series analysis*, 202–247.

Hatanaka, M., & Suzuki, M. (1963). *A theory of the pseudospectrum and its application to non-stationary dynamic econometric models.* Princeton University.

Hillmer, S. C. (1985). Measures of variability for model-based seasonal adjustment procedures. *Journal of Business & Economic Statistics*, *3*(1), 60–68.

Hillmer, S. C., & Tiao, G. C. (1982). An arima-model-based approach to seasonal adjustment. *Journal of the American Statistical Association*, *77*(377), 63–70.

Kendall, M. G. (1976). *Time-series* (Tech. Rep.).

Maravall, A. (n.d.). In economic time series.

Shiskin, J. (1965). *The x-11 variant of the census method ii seasonal adjustment program* (No. 15). US Government Printing Office.