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ATHENS UNIVERSITY
OF ECONOMICS AND BUSINESS

DEPARTMENT OF STATISTICS

POSTGRADUATE PROGRAM

REPRESENTATION OF STOCHASTIC INTEGRALS WITH RESPECT TO LÉVY RANDOM MEASURES

By

Anastasios A. Haralampopoulos

A THESIS
Submitted to the Department of Statistics
of the Athens University of Economics and Business
in partial fulfilment of the requirements for
the degree of Master of Science in Statistics

Athens, Greece
2003

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ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ

ΤΜΗΜΑ ΣΤΑΤΙΣΤΙΚΗΣ

Αναπαράσταση Στοχαστικών Ολοκληρωμάτων Θετικών Συναρτήσεων ως προς Lévy τυχαία μέτρα

Αναστάσιος Α. Χαραλαμπόπουλος

ΔΙΑΤΡΙΒΗ

Που υποβλήθηκε στο Τμήμα Στατιστικής
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Anastasios A. Haralampopoulos

Approved by the Graduate Committee

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VITA

I was born in Athens in 1976. I finished High School in 1993. I became a student in the University of Athens, department of Mathematics, in September 1993. I graduated in January 1999. In June 1999 I was accepted as a post-graduate student in the International Master Program in Statistics of the Athens University of Economics and Business.

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ABSTRACT

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Representation of Stochastic Integrals with respect to Lévy random measures

November 2002

The aim of the present thesis is to provide a comprehensive review of the existing recent work on the stochastic integrals and differential equations with respect to Lévy processes. Furthermore, we also introduce also a new representation of stochastic integrals of possitive functions with respect to any Lévy process.

We provide in detail the Gamma Ornstein-Uhlenbeck process, which is the solution of the corresponding Ornstein-Uhlenbeck differential equation with respect to Lévy process when the latter is a Compound Poisson. The most important difference with the usual case when the corresponding Lévy process of the stochastic differential equation is the Brownian motion, is that now the process is not continuous and may have jumps over time.

We also provide some general theorems which charecterize the marginals distributions of such a process and show the strong relation between the resulted Ornstein-Uhlenbeck process and the corresponding Lévy process . Furthermore we utilize two completely different methods in order to observe the characteristic function of the integrated OU process in the Gamma case.

In addition, based on the fact that the Ornstein –Uhlenbeck processes give as result a stochastic integral of a possitive function with respect to a Lévy process we classify these integrals into a more general class of possitive functions with respect to any Lévy processes such Compound Poisson, α-Stable process, Brownian-motion etc. and we introduce a new representation for the marginals distributions of these integrals as a subordinated process.

Finally we compare our results theoreticaly in order to demostrate the aggreement of our representation with some well known results. We also use sampling methods in order to compare the distribution of our representation of the stochastic integral in Gamma – OU case.

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ΠΕΡΙΛΗΨΗ

Χαραλαμπόπουλος Αναστάσιος

Αναπαράσταση Στοχαστικών Ολοκληρωμάτων ως προς Lévy τυχαία μέτρα

Νοέμβριος 2002

Σκοπός της παρούσας διατριβής είναι να δώσει μια παρουσίαση του πρόσφατα σχετικά έργου πάνω στα στοχαστικά ολοκληρώματα και στοχαστιστικές διαφορικές εξισώσεις ως προς Lévy διαδικασίες καθώς και να εισάγει μια νέα αναπαράσταση στοχαστικών ολοκληρωμάτων θετικών συναρτήσεων ως προς Lévy διαδικασίες.

Παρουσιάζουμε αναλυτικά τη Gamma Ornstein-Uhlenbeck διαδικασία που είναι η λύση της αντίστοιχης Ornstein-Uhlenbeck στοχαστικής διαφορικής εξίσωσης όταν η Lévy διαδικασία σε αυτή είναι Compound Poisson. Η βασική διαφορά του μοντέλου που εξετάζουμε με το σύνηθες Ornstein – Uhlenbeck μοντέλο όταν δηλαδή ή Lévy διαδικασία από την οποία προέρχεται το μοντέλο είναι κίνηση Brown, είναι ότι η Lévy διαδικασία δεν είναι απαραίτητα συνεχής και περιέχει άλματα.

Παρουσίαζονται επίσης μερικά γενικά θεωρήματα που χαρακτηρίζουν τις κατανομές τέτοιων διαδικασιών και φανερώνουν τη σχέση μεταξύ Ornstein – Uhlenbeck και των αντίστοιχων Lévy διαδικασιών. Επιπλέον υπολογίζουμε με δύο διαφορετικούς τρόπους την χαρακτηριστική συνάρτηση για το ολοκλήρωμα της Ornstein-Uhlenbeck διαδικασίας στην περίπτωση που αυτή είναι Gamma OU.

Επιπλέον βασιζόμενοι στο γεγονός ότι το αποτέλεσμα τέτοιων διαδικασιών περιέχει ένα στοχαστικό ολοκλήρωμα μιας θετικής συνάρτησης ως προς μια Lévy διαδικασία, κατηγοριοποιούμε αυτά τα ολοκληρώματα σε μία γενικότερη ομάδα στοχαστικών ολοκληρωμάτων θετικών συναρτήσεων ως προς οποιαδήποτε Lévy διαδικασία όπως για παράδειγμα Compound Poisson, α-Stable, Brownian motion, και εισάγουμε μια νέα αναπαράσταση για τις κατανομές τους.

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Τέλος συγκρίνουμε θεωρητικά τα αποτελεσματα αυτής της αναπαράστασης με είδη γνωστά αποτελέσματα. Στην περίπτωση της Gamma OU διαδικασίας παρουσιάζουμε γραφικές και στατιστικές συγκριτικές μεθόδους.

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Chapter 1

Introduction

In this thesis we will present some basic results for Lévy processes that are solutions of Ornstein-Uhlenbeck (OU) stochastic differential equations with respect to a general Lévy process. In particular we will examine the so-called Gamma–OU process.

These models were introduced in Barndorff-Nielsen and Shephard (2001). The Ornstein-Uhlenbeck process as they have shown it can be a stationary process its one dimension marginals are completely determined by the background Lévy process. Barndorff-Nielsen and Shephard identify the corresponding background driven Lévy process and the marginals of the OU process in most cases like Gamma–OU, Inverse–Gaussian Ornstein–Uhlenbeck (IG-OU), Normal-Inverse-Gaussian (NIG-OU) and more. For these kinds of processes the existence is connected with the unique property of a subclass of infinitely divisible distributions the selfdecomposable distributions. More specific cf. Wolfe (1982) and Jurek and Vervaat (1983) (see also Barndorff-Nielsen, Jensen, and Sorensen (1998)) there exists a stationary process of Ornstein-Uhlenbeck type whose one dimensional marginal law is D iff D is selfdecomposable.

The OU-process is defined to be the solution of the stochastic differential equation

$$dX(t) = -\lambda X(t)dt + dZ(t) \tag{1.1}$$

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where the driving process, $\{Z(t); t \geq 0\}$ is a Lévy process. The solution of (1.1) can be written in the form

$$X(t) = e^{-\lambda t}X(0) + e^{-\lambda t}\int_0^{\lambda t} e^s dZ(s)$$

and using the fact that the stochastic integral and the random variable $X(0)$ are independent we can think of this model as a subclass of a more general family of processes which includes stochastic processes of the form

$$X(t) = \int_0^t f(s)dZ(s)$$

where f is a positive, real function.

In the next chapters we derive the characteristic function of the integrated OU-process in the Gamma case which as a process plays a crucial role in stochastic volatility models. Note that this process is of the form

$$\int_0^t X(s)ds = \frac{1}{\lambda}\int_0^{\lambda t}(1 - e^{-\lambda t}e^s)dZ(s)$$

which belongs to the more general class we described above.

For such integrals, Rosinski (1990, 2000) introduced a series representation which besides its theoretical interest, is particularly useful in practice since it provides a way of generating random variables with this distribution in Monte–Carlo simulation. In this thesis we introduce a new representation for such integrals. We will represent the above stochastic integral in the form

$$\int_0^t f(s)dZ(s) \overset{d}{=} Z(A(t))$$

where the new process $A(t)$ is completely determined for the Laplace exponent and the inverse Laplace exponent of the background driven Lévy process $Z(t)$. Furthermore we prove the existence of the inverse Laplace exponent (as a function) for the process $Z(t)$ in the Ornstein–Uhlenbeck case. The whole procedure is due to the fact that the one

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dimension marginals are selfdecomposable. We derive the process $A(t)$ for the Gamma-OU case and we examine the representation of some general stochastic integrals, for example when $Z(t)$ is stable process, Compound - Poisson etc. and for different functions. Finally we will use the above representation in order to sample from $Z(A(t)$ in Gamma -OU case and we compare the results with Rosinski series method.

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Chapter 2

Lévy process and random fields.

2.1 Lévy Process

In this chapter we include some basic theorems which will be used in the sequel. We will start with the definitions of a real valued Lévy process and we will proceed in some very famous theorems for such a process. Next we will define a random field from these processes .

Definition 2.1 : *A real valued stochastic process* $\{Z(t)\}_{t>0}$ *with* $Z(0) \equiv 0$ *defined on a probability space* (Ω, F, P), *is called a Lévy process, if it satisfies the following conditions:*

i) For any finite partition of the closed interval $[0, t]$ *with* $t_0, t_1, ..., t_n$ *their points, the increments,* $Z_{t_i} - Z_{t_{i-1}}$, $i = 1, \ldots, n$ *are independent random variables.*

ii) For any t, s *in the closed interval* $[0, \infty)$, *the distribution of the random variable* $Z_{t+s} - Z_t$ *do not depend on* s.

iii)The process Z_t *is continuous in probability, for every* $s \geq 0$ *and any* $\varepsilon > 0$ *we have that:*

$\lim_{t \to 0} P(|Z_{s-t} - Z_s| > \epsilon) = 0.$

iv) For almost every $\omega \in \Omega$, *the sample path :* $t \longrightarrow X_t(\omega)$ *is right continuous for every* $t \geq 0$ *and has left limits for positive* t.

If the conditions *i) – iii)* are satisfied we will say that Z_t is a Lévy process in law and if the conditions *i)–iv)* holds we will call Z an additive process. In the following we will study the Lévy processes which have only positive independent increments means that they have positive jumps, as an example we will refer the Compound Poisson process.

After the definition we can know represent some basic and very famous theorems which are well known. The first one in the bibliography is presented most times, as the definition of a Lévy process and the truth is, that is equivalent with the definition as describes the unique property of the family of Lévy processes which has to do with the infinity divisibility. First of all we give the definition of an infinitely divisible random variable.

Definition 2.2: *We will say that the random variable Z has infinitely divisible distribution, if for every n, there are exist independent and identically distributed random variables, $X_1^{(n)}, \ldots, X_n^{(n)}$ such that $Z \stackrel{d}{=} X_1^{(n)} + \cdots + X_n^{(n)}$ in distribution.*

The next theorem describes one unique property of such random variable, the reader can find the proof and more results in (Breiman, 1967)

Theorem 2.1 : *A random variable Z is infinitely divisible if and only if has the characteristic function of the form (Lévy–Khintchine representation)*

$$E\left[e^{isZ}\right] = \exp\left\{ is\alpha - \frac{1}{2}s^2 b + \int_R \left\{e^{isz} - 1 - ish(x)\right\} v(dx) \right\} \tag{2.1}$$

where v is a finite measure that assigns zero mass to the origin. The function $h(x)$ is a truncated function, means measurable bounded with bounded support that satisfies

$$h(x) = \begin{cases} \frac{x}{|x|} \text{ for } |x| > 1 \\ x \text{ for } |x| < 1 \end{cases}$$

We notice here that the proof for this theorem can be done with any truncated function which is equal to x in an neighborhood of $x = 0$. Actually the above representation of characteristic function indicates that the random variable Z is the sum of two independent random variables, one normal $N(\alpha, b^2)$ (the Gaussian part of Z), and the other is again an infinitely divisible random variable without Gaussian part. From know on we take

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the logarithm of the characteristic function as it is easier to work with this in most cases.

Theorem 2.2 : *A stochastic process* $\{Z(t)\}_{t>0}$ *with* $Z(0) \equiv 0$ *is a Lévy process if and only if the characteristic function of the random variable* $Z(1)$ *has the Lévy-Khintchine representation and furthermore we have that :*

$$\phi(s) = E[e^{isZ(t)}] = \left(E[e^{isZ(1)}]\right)^t \tag{2.2}$$

According to the previous results $Z(t)$ is a sum of two independent random process $Z(t) = \alpha t + \sqrt{b}W(t) + z'(t)$ where the process $W(t)$ is a Brownian motion and $z'(t)$ is a Lévy process without a Gaussian part. This well known result is obvious from the representation of the characteristic function of $Z(t)$

$$\phi(s) = E[e^{isZ(t)}] = e^{\alpha t - \frac{1}{2}s^2 bt} \exp\left\{\int_R \{e^{isx} - 1 - ish(x)\}v(dx)\right\}$$

where $h(x)$ and $v(x)$ as above.

If $b = 0$ then the process $Z(t)$ has jumps and we say that Z is a Lévy jump process. If $\alpha = 0$ it is called pure jump process and finally if Z has positive increments then we say that Z is a subordinator. (for more details see Bertoin, 1996,1999).

2.2 Random Fields

In order to define the stochastic integral with respect to a Lévy process we will start with the definitions of a Lévy random measure and field.

Definition 2.3: *A random measure on a measurable space* $(J, \Im)$ *(*J *the state space of a process) over the probability space* (Ω, F, P) *taking its values in* $\bar{R}_+$ *is a function* $Z : \Omega \times \Im \longrightarrow \bar{R}_+$ *such that*

a) $\forall$ *fixed* $A \in \Im$ *the function* $Z[\bullet, A] : \omega \longrightarrow Z(A, \omega)$ *is a random variable*

b) *for any fixed* ω *the function* $Z[\omega, \bullet] : A \longrightarrow Z(A, \omega)$ *is almost surely a measure on* $\Im$.

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Based on this definition of a random measure we define the random field.

Definition 2.4: *A Lévy Field, is a random measure on a region W such that for every countable number of disjoint subsets of W, the values of Z corresponding to these subsets are independent random variables and such that for every subset $A \subset W$ the random variable $Z(A)$ is infinitely divisible (Barndorff–Nielsen)*

From previous paragraph and the above definition it is clear that for such a field Z the characteristic function of the random variable $Z(d\omega)$ has the Lévy–Khintching representation

$$E[e^{isZ(d\omega)}] = is\alpha(d\omega) - \frac{1}{2}b^2(d\omega) + \int_R \{e^{isx} - 1 - ish(x)\}v(dx, d\omega) \tag{2.3}$$

The measure v assigns zero mass: $\forall\ A \subset W$, $v(\{0\}, A) = 0$ and such that if $A = \{\omega\}$ for every ω. Furthermore v has the property

$$\int_R \min(1, \|x\|^2)v(dx, A) < \infty.$$

The triplet (α, b, v) is called the characteristic triplet of the random field. If $\alpha = 0$, $b = 0$, Z called the Lévy jump field. In addition it can be seen that a Lévy random measure determines a Lévy process starting from the property that $Z((\alpha, \beta]) = Z(\beta) - Z(\alpha)$. The inverse is also true.

2.3 Selfdecomposability and Subordination

In this section we will give the basic ideas for the selfdecomposable distributions and subordinator processes. As we will see recently issues provide a strong relation between the class of selfdecomposable distributions and the class of Ornstein-Uhlenbeck type processes. Furthermore we will see that if we subordinate a Brownian motion with a selfdecomposable subordinator the subordinated process arising from them is again selfdecomposable. We may observe also useful results for the quadratic variation for the

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subordinated process. We refer to [10],[11],[20],[22] for the proofs and more issues.

2.3.1 Selfdecomposability

As we will see, there is one to one correspondence between selfdecomposability and the Ornstein-Uhlenbeck type processes. More specific a process is of an Ornstein-Uhlenbeck type if and only if its marginals are selfdecomposable.. We start with the definition of the selfdecomposable distribution.

Definition 2.5: *A distribution F on $\Re$ is called selfdecomposable if, for every $b > 1$, there exists a distribution F_b on $\Re$ such that their characteristic functions $\phi(z), \phi_b(z)$ satisfy*

$$\phi(z) = \phi(b^{-1}z)\phi_b(z) \ , \ z \in \Re \tag{2.4}$$

The class of selfdecomposable distributions is a subclass of the class of infinitely divisible distributions i.e. every selfdecomposable distribution is infinitely divisible. After the definition of selfdecomposable distribution we give the definition for the selfdecomposable Lévy process.

Definition 2.6:*A Lévy process is selfdecomposable if at each time t, its marginal distribution is selfdecomposable.*

Selfdecomposability can be preserved under various transformations, for example the transformation of background driven Lévy process to process of an Ornstein-Uhlenbeck type (Sato and Yamazato 1983).Furthermore there exists a relation between selfdecomposability and subordination. More specific let $\{X_t : t \geq 0\}$ be the subordinated process arising from $\{B_t : t \geq 0\}, \{A_t : t \geq 0\}$,where B_t is a Brownian motion and A_t is a selfdecomposable process independent from B_t, then $X_t = B(A_t)$ is selfdecomposable. Furthermore Halgreen (1979) and Ismail and Kelker (1979) noticed essentially the above result but for Brownian motion without drift. Sato had show that the same exist and with a Brownian motion with drift as the subordinand. Also now it is well known that

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we can use any strictly stable subordinand in the above procedure in order to take again a selfdecomposable subordinated process. Under some regularity conditions, if we replace the brownian motion with a selfdecomposable process, the subordinated process is selfdecomposable.

Another important property of a selfdecomposable processes refers to their Lévy measures.

Lemma 2.1 : *The process $\{X_t : t \geq 0\}$ is selfdecomposable if and only if the Lévy measure v of X_t has the representation*

$$v(du) = \frac{1}{|u|}k(u)du \tag{2.5}$$

where $k(u)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, +\infty)$ in the weak sense.

If the Lévy measure of the process is concentrated on $(0, +\infty)$, then it is enough to show that $k(u)$ is decreasing on $(0, +\infty)$. As an example we refer to gamma process with parameters a, c and Lévy measure $v(du) = \frac{1}{u}ae^{-cu}du$, $u > 0$.

2.3.2 Subordination

In this section we will refer only to Bohner's subordination procedure. Suppose that we have a Levy process let say $\{Z_t : t \geq 0\}$ and we evaluate this process at a random time change points which defined by an independent from Z_t, Lévy process $\{A_t : t \geq 0\}$. The process $\{A_t : t \geq 0\}$ is called the subordinator and the process Z_{A_t} is called the subordinated process. We can think the process A_t as a random clock. Most readers separate the subordinators in two categories, the continuous and the purely discontinuous processes.

We can defined the Subordinators as processes with independent and homogenous increments which are right continuous. Then the following theorem holds and characterize the probability measure of these processes.

Theorem 2.3 : *Φ is the Laplace exponent of a subordinator iff there exists a unique*

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pair (d, k) with $d, k \geq 0$ and a unique measure v on $\Re^+$ such that Φ has the representation :

$$\Phi(\theta) = d + k\theta + \int_0^{+\infty} (1 - e^{-\theta x}) v(dx).$$

As an example consider the Gamma process with parameters a, b, with Laplace exponent

$$\Phi(\theta) = a \log(1 + \frac{\theta}{b}) = \int_0^{+\infty} (1 - e^{-\theta x}) a x^{-1} e^{-bx} dx$$

where $v(dx) = ax^{-1}e^{-bx}dx$.

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Chapter 3

Stochastic Integral.

3.1 Introduction.

In this chapter we define the stochastic integral with respect to a Lévy process of a function f which may depend or not from ω. The stochastic integral in general is a short of operator which take a stochastic process let say f as an input and produces as an output a stochastic variable. In other words let f be a process in a space of measurable functions $L(\Omega \mathrm{x}[0,t], R^n)$ and X in a space $L(\Omega, R^n)$,then the stochastic integral with respect to $X(t)$ is an operator

$$\int_0^t [\bullet] dX(s) : L(\Omega \times \ [0,t]) \rightarrow L(\Omega, R^n) \tag{3.1}$$

Furthermore this operator should be linear with respect to Lebesgue – Stieltjes integral since it represents some kind of integration. We will see that under regularity conditions the stochastic integral with respect to Lévy process coincide with the usual stochastic integral (one has to keep in mind that we expecting this result when the Brownian motion is a Lévy process).We notice here that in general there are exist no random functions that they are not integrable with respect to Lévy process. There is a key measure which

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characterizes the integrable functions, for more details see Rajput and Rosinski (1989).

3.2 Stochastic integral with respect to Lévy process.

First of all we define the integral for a real simple function and then for the general case. Suppose then a probability space $\Omega = \mathbf{R}^n$ and the σ–field $\sigma(\Omega)$ on this space. A real simple function can be defined as a linear combination of characteristic functions: $f = \sum_{n=1}^{k} x_n \chi_{A_n}$ where A_n are subset of $\sigma(\Omega)$ such that if $A \subset B$ then $B \backslash A \in \sigma(\Omega)$.

Definition 3.1 *For every subset B, A_j of $\sigma(\Omega)$ such that $B \cap A_j \in \sigma(\Omega)$ and a real function f on Ω we define:*

$$\int_B f dz \stackrel{d}{=} \sum_{n=1}^{k} x_n z(B \cap A_n)$$

we proceed forward with the definition for the general case, in other words with functions which depends on ω.

Definition 3.2 : *Let Z be a Lévy process. A function $f : (\Omega, \sigma(\Omega)) \longrightarrow (R, B(R))$ is said to be Z–integrable if there exist a sequence of simple functions $\{f_n\}_{n \in N}$ such that:*

i)$f_n \longrightarrow f$ *a.e.*

ii)$\forall\ B \subseteq \sigma(\Omega)$ *the sequence of random variables $\left\{\int_B f_n dZ\right\}_n$ converges in probability as n goes to infinity.*

iii) *We define :* $\int_B f dZ \stackrel{d}{=} p - \lim_{n \to \infty} \int_B f_n dZ$

Based on this definition we present a well known property of the characteristic function of the above integrals that is very usual for calculations in the case where Z is a random measure on $\mathbf{R}_+$ with characteristic triplet $(0, 0, v(dx, d\omega)$ and v factorizes in two measures (often v is said to be product measure) $M(dx, d\omega)$and $\mu(d\omega)$ and μ is a measure on $\mathbf{R}_+$. Then Z determines a Lévy process and furthermore $Z((a, b]) = Z(b) - Z(a)$. The most important thing with the following theorem is that if one knows the distribution of the random variable $Z(1)$ then he can calculate the characteristic function of the stochastic

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integral and furthermore by using the inverse Fourier transform if this is possible to find out the distribution of the integral for every t.

Theorem 3.1 : *Let $Z(t)$ be a Levy process and Z is a subordinator, f a real Z-integrable negative function. The cumulant transform $\phi(s)$ of the random variable $\int_0^t f(y)dZ(y)$ satisfies the following :*

$$E[e^{s\int_0^t f(y)dZ(y)}] = \exp\left\{\int_0^t \Phi_z(f(y)s)dy\right\} \tag{3.2}$$

therefore for a real positive Z-integrable function the Laplace transform of the random variable $\int_0^t f(y)dZ(y)$ may be written

$$E[e^{-s\int_0^t f(y)dZ(y)}] = \exp\{\int_0^t Log[e^{-sf(y)Z(1)}]dy = \exp\left\{-\int_0^t \Phi_z(f(y)s)dy\right\} \tag{3.3}$$

where $\Phi(s)$ is the Laplace exponent of the random variable $Z(1)$.

For a proof of the above theorem we refer the reader to Barndorff-Nielsen (2001).

3.2.1 Series representation of Stochastic Integral

In this section we present the series representation of a stochastic integral with respect to Lévy process. One has to keep in mind that the following relation refers only to the distribution of the integral. The series representation is one very powerful tool which we need in order to sample from the stochastic integral. More details and the proof one can find in[4], [6], [18], [19].

Theorem 3.2 : *Let $Z(t)$ be a Lévy process and let Z be the Lévy random field which this process determines. Furthermore suppose that the Lévy measure is a product measure and factorizes to $v(dx, d\omega) = M(dx, d\omega)\mu(d\omega)$. We define $M^+(x,\omega) = M([x,\infty),\omega)$ and the inverse mapping $M^{-1}(y,\omega) = \inf\{x \geq 0 : M^+(x,\omega) \leq y\}$. Suppose that $\{\alpha_i\}_{i\in N}$ is the arrival times of a Poisson process with intensity 1 and $\{\omega_n\}_{n\in N}$ an i.i.d. sequence from the $\mu-$ law and the two sequence are independent. Then for nonnegative functions*

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f which are Z−integrable we have

$$\int_0^t f(s)dZ(s) \stackrel{d}{=} \sum_{j=1}^{\infty} f(\omega_j)M^{-1}(t_j,\omega_j) \tag{3.4}$$

The above equality holds in law. Furthermore according to the above representation one can easily derive the characteristic function of the stochastic integral if he calculate the characteristic function of the sum.

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Chapter 4

Ornstein-Uhlenbeck process.

4.1 Stochastic differential equations of the Ornstein-Uhlenbeck type.

In this chapter we examine one special stochastic differential equation of the Ornstein–Uhlenbeck type :

$$dX(t) = -\lambda X(t)dt + dZ(t). \tag{4.1}$$

The term λ in the above equations is used in order keep the marginal distributions of the process $X(t)$ unchanged in time. The general definition of the Ornstein–Uhlenbeck process is the solution of the stochastic differential equation of the above form. We mention here that Z is in general a homogeneous Lévy process with $Z(0) = 0$ and it is called the background driven Lévy process. From now on we will refer to the process $X(t)$ as the OU–process.

In this very common model, in the usual case, the process $Z(t)$ is a standard Brownian motion. In our case we can think the process $Z(t)$ as a general Lévy process, for example $Z(t)$ can be a process with Gamma or Inverse Gaussian, etc. marginals. The most important difficulty of this models comparing to the usual case where $Z(t)$ is a Brownian

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motion is that we allow $Z(t)$ to have jumps over time, if $Z(t)$ is for example a compound Poisson process. We will examine this case and we suppose that the jumps follow the Gamma distribution. The solution of the above equation is

$$X(t) = e^{-\lambda t} X(0) + e^{-\lambda t} \int_0^{\lambda t} e^s dZ(s). \tag{4.2}$$

In he above equation the random variable $X(0)$ and the stochastic integral are independent random variables and as we will see from the theorem below $X(0) \stackrel{d}{=} X(t)$.

The following theorem gives the existence and furthermore a very useful property of the marginal distributions of such a process. First of all suppose that we have a random variable x such that x is selfdecomposable (the characteristic function of x satisfies $\phi(\zeta) = \phi(k\zeta)\phi_k(\zeta)$ for all $\zeta \in R$ and every $k \in (0,1)$) then :

Theorem 4.1 *(Wolfe (1982) and Jurek and Vervaat (1983)): If x is a selfdecomposable random variable, there exists a stationary stochastic process $X(t)$ and a Lévy process $Z(t)$ such that $X(t) \stackrel{d}{=} x$ and $X(t)$ satisfies the equation (4.2). The converse is also true.*

From the above it is clear that the marginals of the OU process $X(t)$ is determined completely from the background driven process $Z(t)$. Furthermore the marginals of $X(t)$ remain unchanged over time. We mention here that $X(t)$ may have jumps and it is not necessary continuous process. On the other hand the integrated OU process (the integral of $X(t)$ with respect to time) is a continuous time process even though $X(t)$ and $Z(t)$ have jumps. Having now the distribution of $Z(1)$ and combine the results from the previous chapters we can completely determine the marginal distributions of the OU process. The above model introduced first by Barndorff–Nielsen and Shephard and give as an advantage a simple structure for the OU–process $X(t)$ and for the integrated process $\int X(t)dt$. Some times mostly in finance the process $X(t)$ referred to us as the stochastic volatility and the integrated process as the integrated volatility. A lot of these models have been completely solved by Barndorff-Nielsen and Shephard for different types of the background driven Lévy process such as the Inverse Gaussian, Compound Poisson, and Gamma case.

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4.2 Integrated Ornstein-Uhlenbeck process.

The integrated OU–process is simply the integral with respect to time of the OU–process $X(t)$ in the equation (4.2). Taking this integral we conclude that

$$\begin{aligned}\int_0^t X(s)ds &= X(0)\int_0^t e^{-\lambda s}ds + \int_0^t e^{-\lambda s}\int_0^{\lambda s} e^u dZ(u)ds \\ \int_0^t X(s)ds &= \frac{1}{\lambda}(Z(\lambda t) - X(t) + X(0)) \\ \int_0^t X(s)ds &= \frac{1}{\lambda}(1-e^{-\lambda t})X(0) + \frac{1}{\lambda}\int_0^{\lambda t}(1-e^{-\lambda t}e^s)dZ(s). \end{aligned} \tag{4.3}$$

From the above form it is clear that in order to find out the one dimension marginals and the other properties of the integrated OU process it is enough to examine the stochastic integral. Furthermore comparing these forms for OU and integrated OU process it is clear that both of them are of the more general form:

$$Y(t) = \int_B f(s)dZ(s) \tag{4.4}$$

where $Z(t)$ is a Lévy process and the function f is a positive function. In addition if f is bounded then all process which belong to above family are stationary positive process. Furthermore it is clear that all the properties of the stochastic integrals which we have described in the previous chapter holds.

4.3 The Laplace exponent of the BDLP of an OU process.

Let $Z(t)$ be the background driven Lévy process of an OU-type process $\sigma^2(t)$. This means that $\sigma^2(t)$ is the solution of the following stochastic differential equation

$$d\sigma^2(t) = -\lambda\sigma^2(t)dt + dZ(t) \tag{4.5}$$

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and straight forward calculus give us the solution for $\sigma^2(t)$

$$\sigma^2(t) = e^{-kt}\sigma^2(0) + e^{-\lambda t}\int_0^t e^{\lambda s} dZ(\lambda s).$$

Since the process $\sigma^2(t)$ is of an OU type then is selfdecomposable and furthermore its Lévy density will have the representation

$$U(dx) = \frac{1}{|x|}k(x)dx$$

where U is the Lévy measure of σ^2 and $k(x)$ is an increasing function on $(-\infty, 0)$ and decreasing on $(0, +\infty)$ (in the weak sense). In this section we will assume that the Lévy measure $U(x)$ concentrated only on $(0, +\infty)$ and therefore it is enough to define the $k(x) = 0$ in $(-\infty, 0)$. Barndorff -Nielsen and Shephard was found out that there exists the following relation between the Lévy measure of the process $Z(t)$ and the Lévy density of the corresponding OU process $\sigma^2(t)$

$$W^+(x) = xu(x) \tag{4.6}$$

where

$$W^+(x) = \int_x^{+\infty} w(u)du$$

and $w(x)$ is the Lévy density of the random variable $Z(1)$ and $u(x)$ the Lévy density of the OU process. Assume now that $u(x)$ is differentiable then $w(x)$ is related with the Lévy density $u(x)$ according to the following equation

$$w(x) = -u(x) - xu'(x) \tag{4.7}$$

Using now the fact that the process $\sigma^2(t)$ is selfdecomposable then

$$U(dx) \;=\; \frac{1}{|x|}k(x)dx$$

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$$u(x) = \frac{1}{|x|}k(x) \tag{4.8}$$

Furthermore if we assume that $\sigma^2(t) \geqslant 0$ then

$$u(x) = \frac{1}{x}k(x) \tag{4.9}$$

with $k(x)$ decreasing function. Under the above assumptions the Lévy density of the BDLP $Z(t)$ become

$$\begin{aligned} w(x) &= -u(x) - xu^{'}(x) \\ w(x) &= -k^{'}(x) \end{aligned} \tag{4.10}$$

Theorem 4.2 : *If the subordinator $Z(t)$ is the BDLP of an OU type stochastic differential equation and the solution*

$$\sigma^2(t) = e^{-kt}\sigma^2(0) + e^{-kt}\int_0^t e^{ks}dZ(\lambda s)$$

satisfies the conditions :

i) $\sigma^2(t) \geqslant 0$ and

ii) $\sigma^2(t)$ has differentiable Lévy density,

then the Laplace exponent of $Z(t)$,

$$\Phi(\theta) = d + k\theta + \int_0^{+\infty}(1 - e^{-\theta x})W(dx),$$

is an increasing function in the strong sense and therefore it can be inverted.

Proof: First of all note that $W(dx) = w(x)dx$ and using equation (4.10) $W(dx) = -k^{'}(x)dx$. Where $k(x)$ is decreasing and therefore $k^{'}(x) < 0$. Using this result we can

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rewrite the Laplace exponent

$$\begin{aligned}\Phi(\theta) &= d + k\theta + \int_0^{+\infty}(1-e^{-\theta x})W(dx) \\ \Phi(\theta) &= d + k\theta - \int_0^{+\infty}(1-e^{-\theta x})k^{'}(x)dx\end{aligned}$$

Taking now the first derivative with respect to θ we observe

$$\Phi^{'}(\theta) = k - \int_0^{+\infty} xe^{-\theta x}k^{'}(x)dx \tag{4.11}$$

Note now that

$$-\int_0^{+\infty} xe^{-\theta x}k^{'}(x)dx \geqslant 0$$

since $k(x)$ is decreasing and therefore $\Phi^{'}(\theta) \geqslant 0$. If $k > 0$ then $\Phi^{'}(\theta) > 0$ and therefore is an increasing function. $\Phi^{'}(\theta)$ vanishes when $k = 0$ and

$$\int_0^{+\infty} xe^{-\theta x}k^{'}(x)dx = 0.$$

The last integral is the Laplace transform of the function $f(x) = xk^{'}(x)$ and by the uniqueness of Laplace transform we get that $xk^{'}(x) = 0\ \forall\ x$ and therefore $k^{'}(x) = 0$ so $\Phi(\theta) = d$ and therefore $Z(t) = t$ so in this case the process $Z(t)$ does not depend on ω.

We mention here that the family of the Laplace exponent that can be inverted includes many interesting processes such as Poisson, Compound Poisson, stable process with index $\alpha \in (0,2)$ Gamma Process etc.

4.4 Gamma OU process and Integrated Gamma OU process.

In this paragraph we introduced the Gamma OU process a process with Gamma (one dimension) marginals which satisfies the equation (4.2).We will first derive the background

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driven Lévy process and then the characteristic and the moment generating function of the integrated Gamma OU process.

Let suppose that we have the process $X(t)$ which satisfies the equation (4.2) with the $\Gamma(\nu, \alpha)$ marginals. The probability density is then

$$f(x) = \frac{\alpha^{\nu}}{\Gamma(\nu)} x^{\nu-1} e^{-\alpha x}, x > 0. \tag{4.12}$$

The corresponding Lévy density of this distribution is well known that is

$$f_L(x) = \nu x^{-1} e^{-\alpha x}. \tag{4.13}$$

Let as assume that f_L and f_z are the Lévy densities of $X(t)$ and $Z(1)$ respectively, then from (4.7) we have:

$$f_z(x) = -f_L(x) - x f_L'(x)$$

Applying this equation to our case we find that

$$f_z(x) = \alpha \nu e^{-\alpha x}.$$

Using know the equation (2.2) we conclude that

$$\begin{aligned} t^{-1} \log E[e^{isZ(t)}] &= \log E[e^{isZ(1)}] \\ \log E[e^{isZ(t)}] &= e^{\nu t \{\frac{\alpha}{\alpha - is} - 1\}} \end{aligned}$$

which means that the background driven Lévy process $Z(t)$, is a compound Poisson process and the jumps follows the $\Gamma(1, \alpha)$ distribution, more specific $Z(t)$ is of the form

$$Z(t) = \sum_{n=1}^{N(t)} g_n \tag{4.14}$$

where $g_n \sim \Gamma(1, \alpha)$ and $N(t)$ is a Poisson process with $E[N(t)] = \nu t$.

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The reader can find a figure from this process in Appendix section figures 6-1 for the BDLP and 6-2 for the Gamma-OU process

4.4.1 Characteristic function of the Integrated Gamma OU process.

Having known now the background driven process $Z(t)$ and using the fact that $X(t)$ one dimension marginals follows the Gamma distribution we will find the characteristic function of the integrated process. We will derive our result using two completely different methods which we have described in previous paragraph. One is the series representation of the stochastic integral and the other one is given by key formula (3.2). Furthermore we will assume that $X(0)$ is known and $X(0) = 0$. The integrated process then have the form

$$\int_0^t X(s)ds = \frac{1}{\lambda}\int_0^{\lambda t}(1 - e^{-\lambda t}e^s)dZ(s).$$

Using now (3.2) for the characteristic function of the Integrated - OU process we observe

$$\begin{aligned}
\phi(s) &= E\left[\exp\left(i\frac{s}{\lambda}\int_0^{\lambda t}(1 - e^{-\lambda t}e^s)dZ(s)\right)\right] \\
\phi(s) &= \exp\left(\int_0^{\lambda t} Log\left(Ee^{\frac{s}{\lambda}(1-e^{-\lambda t}e^u)Z(1)}\right) ds\right)
\end{aligned}$$

Since $Z(t)$ is a compound Poisson process with Gamma, $\Gamma(1, a)$ jumps then

$$E[e^{isZ(1)}] = e^{\nu\{\frac{\alpha}{\alpha - is} - 1\}}$$

and therefore the integrated term in the above equation is

$$Log\left(E\left[\exp\left(\tfrac{s}{\lambda}(1 - e^{-\lambda t}e^u)Z(1)\right)\right]\right) = \nu\left(\frac{\alpha}{a - i\frac{s}{\lambda}(1 - e^{-\lambda t}e^u)} - 1\right)$$

Comparing the two equations above

$$\phi(s) = \exp\left(\int_0^{\lambda t}\nu(\frac{\alpha}{a - i\frac{s}{\lambda}(1 - e^{-\lambda t}e^u)} - 1)du\right)$$

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$$= \exp\left(\nu \int_0^{\lambda t} \frac{is(1-e^{-\lambda t}e^u)}{\lambda\alpha - is(1-e^{-\lambda t}e^u)} du\right).$$

Since $\phi(is) = M(s)$ where M is the moment generating function we get that

$$M(s) = \exp\left(\nu \int_0^{\lambda t} \frac{s(1-e^{-\lambda t}e^u)}{\lambda\alpha - s(1-e^{-\lambda t}e^u)} du\right).$$

If we make the substitution $y = (1-e^{-\lambda t}e^u)$ then

$$\begin{aligned} M(s) &= \exp\left(\nu \int_{1-e^{-\lambda t}}^{0} \frac{sy}{\lambda\alpha - sy} \frac{1}{1-y} du\right) \\ &= \exp\left(\nu \int_{1-e^{-\lambda t}}^{0} \left(\frac{A}{\lambda\alpha - sy} + \frac{B}{1-y}\right) dy\right) \end{aligned}$$

where $A = \frac{\alpha\lambda s}{\alpha\lambda - s}$, $B = \frac{s}{\alpha\lambda - s}$ and finally

$$M(s) = \exp\left(\nu\lambda t\left(\frac{\lambda\alpha}{\lambda\alpha - s} - 1\right)\right)\left(1 - \frac{s}{\lambda\alpha}(1-e^{-\lambda t})\right)^{\frac{\nu}{1-\frac{s}{\lambda\alpha}}}. \tag{4.15}$$

From this we conclude that the integrated Gamma OU process is a sum of two independent processes

$$\int_0^t X(s)ds = Z_0(t) + Y(t)$$

where $Z_0(t)$ is a compound Poisson process

$$Z_0(t) = \sum_{n=1}^{N(t)} g_n$$

where g_n have the $\Gamma(1, \lambda\alpha)$ distribution and $N(t)$ is a Poisson process with $E[N(t)] = \nu\lambda t$.The unknown process $Y(t)$ has the moment generating function

$$M(s) = \left(1 - \frac{s}{\lambda\alpha}(1-e^{-\lambda t})\right)^{\frac{\nu}{1-\frac{s}{\lambda\alpha}}}.$$

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We will now derive the same result in a different way using the series representation (3.4) of stochastic integral .

$$\int_0^t f(s)dZ(s) \stackrel{d}{=} \sum_{j=1}^{\infty} f(\omega_j)M^{-1}(t_j,\omega_j)$$

Where $M^{-1}(t,\omega)$ in our case is the inverse function of the upper tail Lévy measure. Barndorff-Nielsen (1998b) shows the relation which exists between the Lévy density of the OU–process and the Lévy measure of the background driven process $Z(t)$. More specifically, if we define

$$M^{+}(x) = \int_x^{+\infty} \mu(u)du$$

where $\mu(u)$ is the Lévy density of the Lévy measure M then

$$M^{+}(x) = xl(x)$$

where $l(x)$ is the Lévy density of the OU-process. In our case we have Gamma marginals for $X(t)$ so the density $l(x)$ will be

$$l(x) = \nu x^{-1} e^{-\alpha x}$$

and therefore,

$$M^{+}(x) = xl(x) = \nu e^{-\alpha x}$$

since $M^{-1}(t,\omega)$ is the inverse function defined:

$$\begin{aligned} M^{-1}(t) &= \inf\{\vartheta : \vartheta \text{ positive}, M^{+}(\vartheta) < t\} \\ M^{-1}(t) &= \max\{0, -\alpha^{-1}\log(\frac{x}{\nu})\} \end{aligned} \tag{4.16}$$

(see also Barndorff–Nielsen, 2000 , Bertoin, 1999).

Having known the above results and define t_j, $j \in \mathrm{N}$ as the arrivals times of a

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Poisson process $N(t)$ with $E[N(t)] = \nu\lambda t$, and τ_j i.i.d. sequence (and independent of t_j) of random variables from the uniform distribution $U(0,1)$ then we have

$$\int_0^t f(s)dZ(s) \stackrel{d}{=} \sum_{j=1}^{\infty} f(\omega_j) M^{-1}(t_j, \omega_j)$$

$$\int_0^{\lambda t} X(s)dZ(s) = \int_0^{\lambda t} (1 - e^{-\lambda t} e^s)dZ(s) \stackrel{d}{=} \frac{1}{\lambda\alpha} \sum_{j=1}^{\infty} M^{-1}(\frac{t_j}{\nu\lambda t}) f(\lambda t \tau_j)$$

$$\int_0^{\lambda t} (1 - e^{-\lambda t} e^s)dZ(s) \stackrel{d}{=} -\frac{1}{\lambda\alpha} \sum_{j=1}^{\infty} 1_{[0,\nu]}(\frac{t_j}{\lambda t}) \log(\frac{t_j}{\nu\lambda t})(1 - e^{-\lambda t} e^{\lambda t \tau_j})$$

$$\int_0^{\lambda t} (1 - e^{-\lambda t} e^s)dZ(s) \stackrel{d, u = \frac{t}{\nu\lambda t}}{=} -\frac{1}{\lambda\alpha} \sum_{j=1}^{\infty} 1_{[0,1]}(u_j) \log(u_j)(1 - e^{-\lambda t} e^{\lambda t \tau_j})$$

$$\int_0^{\lambda t} (1 - e^{-\lambda t} e^s)dZ(s) \stackrel{d}{=} \sum_{j=1}^{\infty} 1_{[0,1]}(u_j) \log(u_j^{-\frac{1}{\lambda\alpha}(1 - e^{-\lambda t} e^{\lambda t \tau_j})})$$

and finally

$$\int_0^{\lambda t} (1 - e^{-\lambda t} e^s)dZ(s) \stackrel{d}{=} \sum_{j=1}^{N(1)} \log(u_j^{-\frac{1}{\lambda\alpha}(1 - e^{-\lambda t} e^{\lambda t \tau_j})}) \tag{4.17}$$

Where u_j are the arrivals times of the Poisson process $N(1)$ with intensity 1. Using the above representation for the integrated OU process we are ready to calculate now the characteristic and moment generating function. But before this step we have to mention some important things of the above representation. First it is clear that one can sample from the integrated OU process and furthermore to simulate the process using equation (4.17). Another important thing is that the random variables u_j are independent from the uniform variables τ_j for each j. Furthermore since u_j denotes the arrivals of a Poisson process up to time 1, if we order them and conditioning with respect to random variable $N(1) = n$ then u_j become independent copies from the uniform $U(0,1)$ distribution. Furthermore, the sequence of τ_j remains independent from our new uniform random variables because r_j are independent from the Poisson process $N(t)$. So for the

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characteristic function we have

$$\begin{aligned} E[e^{is\int_0^{\lambda t} X(s)ds}] &= E[e^{is\sum_{j=1}^{N(1)} \log u_j^{-\frac{1}{\lambda\alpha}(1-e^{-\lambda t}e^{\lambda t\tau_j})}}] \\ &= \sum_{n=1}^{\infty} E[\exp[is\sum_{j=1}^{n} \log(u_j^{-\frac{1}{\lambda\alpha}(1-e^{-\lambda t}e^{\lambda t\tau_j})})|N(1)=n]P(N(1)=n). \end{aligned}$$

Since now u_j become independent the above expression can be written as

$$\sum_{n=1}^{\infty}\{\prod_{j=1}^{n} E[\exp[is\log(u_j^{-\frac{1}{\lambda\alpha}(1-e^{-\lambda t}e^{\lambda t\tau_j})})|N(1)=n]\}P(N(1)=n)$$

or

$$\sum_{n=1}^{\infty}\{\prod_{j=1}^{n} E[\exp[is\log(u_j^{-\frac{1}{\lambda\alpha}(1-e^{-\lambda t}e^{\lambda t\tau_j})})|N(1)=n]\}P(N(1)=n)$$

or

$$\sum_{n=1}^{\infty}\left(E[u^{-\frac{is}{\lambda\alpha}(1-e^{-\lambda t}e^{\lambda t\tau})}|N(1)=n]\right)^n P(N(1)=n). \tag{4.18}$$

We will calculate first the expected value and then, the series so we have

$$E[u^{-\frac{is}{\lambda\alpha}(1-e^{-\lambda t}e^{\lambda t\tau})}|N(1)=n] = \int_0^1 E[u^{-\frac{is}{\lambda\alpha}(1-e^{-\lambda t}e^{\lambda t\tau})}|\tau, N(1)=n]d\tau.$$

(Recall here that u, τ, are independent, $U(0,1)$ random variables.) By making the substitutions $y = 1-\tau$, (then y is again uniform distribution on $U(0,1)$), $r(y) = \frac{-s}{\lambda\alpha}(1-e^{-\lambda t y})$ and using again the fact that $M(s) = \phi(-is)$ we have

$$E[u^{-\frac{s}{\lambda\alpha}(1-e^{-\lambda t y})}|N(1)=n] = \int_0^1 E[u^{r(y)}|y, N(1)=n]dy$$

and

$$\int_0^1\int_0^1 u^{r(y)}dudy = \int_0^1 \frac{1}{r(y)+1}dy = \int_0^1 \frac{1}{1-\frac{s}{\lambda\alpha}(1-e^{-\lambda t y})}dy$$

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using the substitution $x = e^{-\lambda t y}$ the above becomes

$$-\frac{1}{\lambda t}\int_1^{e^{-\lambda t}} \frac{1}{1+\frac{sx}{\lambda\alpha}-\frac{s}{\lambda\alpha}}\frac{1}{x}dx = -\frac{1}{\lambda t}\int_1^{e^{-\lambda t}} \frac{A}{1+\frac{sx}{\lambda\alpha}-\frac{s}{\lambda\alpha}} + \frac{B}{x}dx$$

where $A = \frac{s}{s-\lambda\alpha}$, $B = \frac{\lambda\alpha}{\lambda\alpha - s}$. Then simple calculations give

$$E[u^{-\frac{s}{\lambda\alpha}(1-e^{-\lambda t y})}|N(1)=n] = \frac{\lambda\alpha}{\lambda\alpha - s}\frac{1}{\lambda t}(\lambda t + \log(1-\frac{s}{\lambda\alpha}(1-e^{-\lambda t}))$$

We will return now to the equation (4.18) in order to complete the proof

$$E[e^{s\int_0^{\lambda t} X(s)ds}] = \sum_{n=1}^{\infty}\{E[u^{-\frac{is}{\lambda\alpha}(1-e^{-\lambda t}e^{\lambda t\tau})}|N(1)=n]\}^n P(N(1)=n)$$

$$E[e^{s\int_0^{\lambda t} X(s)ds}] = \sum_{n=0}^{\infty}\left\{\frac{\lambda\alpha}{\lambda\alpha - s}\frac{1}{\lambda t}(\lambda t + \log(1-\frac{s}{\lambda\alpha}(1-e^{-\lambda t}))\right\}^n e^{-\nu\lambda t}\frac{(\nu\lambda t)^n}{n!}$$

$$E[e^{s\int_0^{\lambda t} X(s)ds}] = e^{-\nu\lambda t}\exp(\frac{\lambda\alpha}{\lambda\alpha - s}(\lambda t + \log(1-\frac{s}{\lambda\alpha}(1-e^{-\lambda t})))$$

$$E[e^{s\int_0^{\lambda t} X(s)ds}] = e^{-\nu\lambda t}\exp(\frac{\nu\lambda\alpha}{\lambda\alpha - s}\lambda t + \frac{\nu\lambda\alpha}{\lambda\alpha - s}\log(1-\frac{s}{\lambda\alpha}(1-e^{-\lambda t}))$$

$$E[e^{s\int_0^{\lambda t} X(s)ds}] = \exp(\nu\lambda t\{\frac{\lambda\alpha}{\lambda\alpha - s} - 1\}\{1-\frac{s}{\lambda\alpha}(1-e^{-\lambda t})\}^{\frac{\nu}{1-\frac{s}{\lambda\alpha}}}.$$

We note here that in order to define some of the above integrals we must use the fact that in our case $s < 1$. Furthermore as far as concerned the unknown process $Y(t)$ with characteristic function

$$M(is) = \left(1-\frac{is}{\lambda\alpha}(1-e^{-\lambda t})\right)^{\frac{\nu}{1-\frac{is}{\lambda\alpha}}}.$$

one can use the inverse Fourier transform of the characteristic function in order to have the one dimension marginals of this process, but it is very hard to get a close form solution. In the Appendix section we present a sample path from the integrated OU process in figure 6-3

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Chapter 5

Representation of the stochastic integral as a subordinated process.

5.1 Representation of the stochastic Integral

Let $X(t) = \int_0^t f(s)dZ(s)$, where f is a positive function and does not depend on ω. Furthermore assume that $Z(t)$ is a subordinator with a strictly monotone Laplace exponent so that can be inverted. Define for example the process

$$X(t) = \int_0^t e^{ks} dZ(ks).$$

Our aim is to find a process $A(t)$ where $A(t)$ is a process with independent increments independent of $Z(t)$ such that

$$X(t) = \int_0^t f(s)dZ(s) \stackrel{d}{=} Z(A(t)) \tag{5.1}$$

Note here that the process $Z(t)$ can be the BDLP of an OU process as theorem 4.2 indicates. This has to do with the fact that we want the process $Z(t)$ to have strictly monotone Laplace exponent and theorem 4.2 proves that all BDLP of an OU process have this property. We will need the following preposition

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Theorem 5.1 : *Let $Z(t), A(t)$ two independent processes and $Z(t)$ be a subordinator and $A(t)$ be a process with independent increments, then for the process $Z(A(t))$ we have*

$$E[e^{-\lambda Z(A(t))}] = e^{M_A(\Phi_z(\lambda))}, \tag{5.2}$$

where M_A, is the logarithm of the Laplace transform of the marginals at time t of the process $A(t)$ and $\Phi_z(\lambda)$ is the Laplace exponent of the process $Z(t)$.

Proof: We have that :

$$E[e^{-\lambda Z(A(t))}] = E[E[e^{-\lambda Z(A(t))}]|A(t)] = E[e^{-A(t)\Phi_Z(\lambda)}] = e^{M_A(\Phi_z(\lambda))}$$

We are ready now to identify with the following theorem the process $A(t)$

Theorem 5.2 : *Let $X(t) = \int_0^t f(s)dZ(s)$, where f is a positive function and does not depend on ω. Furthermore assume that $Z(t)$ is a subordinator with a strictly monotone Laplace exponent so that can be inverted then there exists a process $A(t)$* such that

$$X(t) = \int_0^t f(s)dZ(s) \stackrel{d}{=} Z(A(t))$$

with

$$(-\frac{\partial M_A(\lambda, t)}{\partial t})^{-1} = \Phi_z(\frac{\Phi_z^{-1}(\lambda)}{f(t)})$$

where $M_A(\lambda, t)$ is the Laplace transform of the process $A(t)$ and $\Phi_Z(\lambda)$ is the Laplace exponent of the process $Z(t)$.

Proof:

As we mention before (3.3) for the stochastic integral holds the following relation:

$$E\left[e^{-s\int_0^t f(s)dZ(s)}\right] = \exp\{\int_0^t Log\{E[e^{-sf(u)Z(1)}]\}du\}.$$

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Using now equations (5.2),(3.3) we want

$$M_A(\Phi_z(\lambda), t) = \int_0^t -\Phi_z(\lambda f(u))du\}$$

and taking the derivative with respect to t, the above equation becomes

$$\frac{\partial M_A(\Phi_z(\lambda), t)}{\partial t} = -\Phi_z(\lambda f(t)).$$

this determines the Laplace transform of the process A. Using now our assumption that $\Phi_z(s)$ is strictly monotone function so $M_A(\lambda)$ is we get that

$$\Phi_z(\lambda) = (-\frac{\partial M_A}{\partial t})^{-1}(\Phi_z(\lambda f(t)))$$

from which we get the solution

$$(-\frac{\partial M_A(\lambda, t)}{\partial t})^{-1} = \Phi_z(\frac{\Phi_z^{-1}(\lambda)}{f(t)}) \tag{5.3}$$

Note here that $(\frac{\partial M_A(\lambda,t)}{\partial t})^{-1}$ is well defined since $f(t) > 0, \Phi_z^{-1}(s) \geqslant 0$ because the function $\Phi_z : [0, +\infty) \to [0. + \infty)$ and therefore $(\frac{\partial M_A(\lambda,t)}{\partial t})^{-1}$is decreasing as we expected. Furthermore we have assumed that Φ_z it can be inverted so the function $\Phi_z(\frac{\Phi_z^{-1}(\lambda)}{f(t)})$ is well defined. In addition in 5.3 the inverse functions it have been taken with respect to λ.

5.1.1 Examples

We will use the previous result in order to derive the above representation for various Lévy processes $Z(t)$ and functions f in very common cases. Note that from now on for simplicity we will use the form $(-M_A'(\lambda))^{-1}$ instead of $(-\frac{\partial M_A(\lambda,t)}{\partial t})^{-1}$.

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Gamma-OU

We will try to find out the process $A(t)$ in the Gamma-OU case. Note here that the BDLP is a Compound Poisson Process with $\Gamma(1,a)$ jumps and the OU process

$$\sigma^2(t) = e^{-kt}\sigma^2(0) + e^{-kt}\int_0^t e^{ks}dZ(ks)$$

is a stationary process with $\Gamma(v,\alpha)$ marginals for each t. Furthermore the stochastic variable $\sigma^2(0) \stackrel{d}{=} \Gamma(v,\alpha)$ and is independent from the process $Z(t)$. The above stationary solution exists for each $k > 0$ so for simplicity we will take $k = 1$. Since $Z(t)$ is a Compound Poisson Process with $\Gamma(1,a)$ jumps we have for the Laplace exponent

$$\begin{aligned}
\Phi_z(\lambda) &= v\left(1 - \frac{a}{a+\lambda}\right)\\
\Phi_z^{-1}(\lambda) &= \frac{a\lambda}{v-\lambda}
\end{aligned}$$

Our aim is to represent the stochastic integral $\int_0^t e^s dZ(s)$ in the form $Z(A(t))$. So our function f will be the e^s. Therefore we have (5.1)that

$$\begin{aligned}
(-M_A'(\lambda))^{-1} &= \Phi_z\left(\frac{\Phi_z^{-1}(\lambda)}{f(t)}\right)\\
(-M_A'(\lambda))^{-1} &= v\left(1 - \frac{a}{a + \frac{a\lambda}{(v-\lambda)f(t)}}\right)\\
(-M_A'(\lambda))^{-1} &= v\left(1 - \frac{1}{1 + \frac{\lambda}{(v-\lambda)f(t)}}\right)\\
M_A'(\lambda) &= -\frac{v\lambda f(t)}{\lambda f(t) + v - \lambda}
\end{aligned}$$

and in our case

$$M_A'(\lambda) = -\frac{v\lambda e^t}{\lambda e^t + v - \lambda}$$

and therefore

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$$\begin{aligned} M_A(\lambda) &= -v \int_0^t \frac{\lambda e^s}{\lambda e^s + v - \lambda} ds \\ M_A(\lambda) &= -v \log[1 + \lambda \frac{(e^t - 1)}{v}] \end{aligned}$$

which means that $A(t)$ is a process with $\Gamma(v, \frac{v}{e^t-1})$ marginals.

The only think which left to do is to verify that

$$e^{-t}\sigma^2(0) + e^{-t}Z(A(t)) \stackrel{d}{=} X$$

where X is a random variable distributed as $\Gamma(v, a)$. Note that $\sigma^2(0), Z(A(t))$ are independent and such that

$$\begin{aligned} E[e^{-\lambda e^{-t}\sigma^2(0) - \lambda e^{-t} Z(A(t))}] &= E[e^{-\lambda e^{-t}\sigma^2(0)}] E[e^{-\lambda e^{-t} Z(A(t))}] \\ &= \exp\{-v \log[1 + \frac{\lambda e^{-t}}{a}] - vLog[1 + \frac{\lambda e^{-t}}{\lambda e^{-t} + a}(e^t - 1)] \\ &= \exp\{-v \log[\frac{a + \lambda e^{-t}}{a}(1 + \frac{\lambda}{\lambda e^{-t} + a}(1 - e^{-t}))] \\ &= \exp\{-v \log[\frac{a + \lambda e^{-t} + \lambda - \lambda e^{-t}}{a}] \\ &= \exp\{-v \log[\frac{a + \lambda}{a}]\} \\ &= \exp\{-v \log[1 + \frac{\lambda}{a}]\} \end{aligned}$$

In this special case we can verify the above representation in a more simple way. We can compute directly the Laplace transform of the stochastic integral in the integrated equation

$$\sigma^2(t) = e^{-kt}\sigma^2(0) + e^{-kt} \int_0^t e^{ks} dZ(ks)$$

We will use (we will take again $k = 1$) the following:

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i)

$$e^{-t}\int_0^t e^s dZ(s) \stackrel{d}{=} \int_0^t e^{-s} dZ(s)$$

ii)

$$E[e^{-\lambda \int_0^t e^{-s} dZ(s)}] = \exp\left(\int_0^t \Phi_Z(\lambda e^{-s}) ds\right)$$

iii)

$$\Phi_Z(\lambda) = v\left(1 - \frac{a}{a+\lambda}\right) = v\{1 - \ell(Y)\}$$

where $\ell(Y)$ is the Laplace transform of the exponential jump size of the Compound Poisson Process $Z(t)$. Straightforward calculus in the second equation above give us the result

$$E[e^{-\lambda \int_0^t e^{-s} dZ(s)}] = \exp\left(vt\left(\frac{1}{t}\int_0^t \ell(Ye^{-s}) ds - 1\right)\right) \tag{5.4}$$

If we look more carefully the above equation we will see that it is nothing more than a Compound Poisson random variable with rate vt and jumps from the Ye^{-S} distribution where Y is exponential and S is a uniform random variable on $[0, t]$ independent of Y. Note here that one can take a more general result if we replace the function e^{-s} with any positive integrable function f_s with respect to random field $Z(B)$ defined from the process $Z(t)$. Then the random variable $\int_0^t e^{-s} dZ(s)$ is distributed as a Compound Poisson random variable with rate vt and jumps size from $Yf(S)$where Y, S as before. We will verify now that $e^{-t}Z(A(t)) \stackrel{d}{=} CPO(vt, Ye^{-S})$. Taking the Laplace transforms in both sides we have that

a)

$$\begin{aligned} \ell(e^{-t}Z(A(t))) &= -v\log[1 + \frac{\lambda e^{-t}}{\lambda e^{-t} + a}(e^t - 1)] \\ &= -v\log[\frac{a+\lambda}{\lambda e^{-t} + a}] \end{aligned}$$

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b)

$$\begin{aligned}\ell(Ye^{-S}) &= \exp\left\{vt\left(\frac{1}{t}\int_0^t \ell(Ye^{-s})ds-1\right)\right\}=\exp\left\{vt\left(\int_0^t \frac{a}{a+\lambda e^{-s}}\frac{1}{t}ds-1\right)\right\}\\ &= -v\log[\frac{a+\lambda}{\lambda e^{-t}+a}].\end{aligned}$$

As a conclusion we write all the results for the Gamma-OU case in the following table

Table 5.1

Random variable	Laplace transform	Distribution
$\sigma^2(t)$	$\exp\{-v\log[1+\frac{\lambda}{a}]\}$	$\Gamma(v,a)$
$A(t)$	$-v\log[1+\lambda\frac{(e^t-1)}{v}]$	$\Gamma(v,\frac{v}{e^t-1})$
$e^{-t}\int_0^t e^s dZ(s)\overset{d}{=}Z(A(t))$	$-v\log[\frac{a+\lambda}{\lambda e^{-t}+a}]$	$CPO(vt,Ye^{-S})$,Y~Exp(a),S~$U(0,t)$

The stochastic integral with respect to α stable process with index $\alpha\,\epsilon\,(0,2)$.

Let $Z(t)$ be a stable process with index $\alpha\epsilon(0,2)$ then the Laplace exponent of Z it is well know that it is

$$\Phi_Z(\lambda)=\lambda^a$$

Define now the process

$$X(t)=\int_0^t f(s)dZ(s)$$

with f as before. Then using the above representation(5.1) we can write the integral in the following form

$$\int_0^t f(s)dZ(s)\overset{d}{=}Z(A(t))$$

where $A(t)$ is a process with Laplace transform defined from

$$\begin{aligned}(-M_A'(\lambda))^{-1} &= \Phi_z(\frac{\Phi_z^{-1}(\lambda)}{f(t)})\\ (-M_A'(\lambda))^{-1} &= \frac{\lambda}{f^a(t)}\\ M_A'(\lambda) &= -\lambda f^a(t)\end{aligned}$$

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$$M_A(\lambda) = -\lambda \int_0^t f^a(s)ds$$

Assume now that the function f is stable $f(t) = c > 0$ then the process $A(t)$ has Laplace transform

$$\begin{aligned} M_A(\lambda) &= -\lambda \int_0^t c^a ds \\ M_A(\lambda) &= -c^a t\lambda \end{aligned}$$

and therefore the process $A(t) = c^a t$. This means that the stochastic integral

$$\int_0^t cdZ(s) = cZ(t)$$

is distributed as

$$\int_0^t cdZ(s) = cZ(t) \stackrel{d}{=} Z(c^a t). \tag{5.5}$$

If we take $\alpha = 2$ then Z is a Brownian motion, we observe from (5.5) the well known scaling property for Brownian motion :

$$c^{-\frac{1}{2}} Z(t) \stackrel{d}{=} Z(c^{-1}t)$$
$$Z(t) \stackrel{d}{=} \sqrt{c} Z(\frac{t}{c})$$

Another simple structure we can observe for the function $f(t) = t$ then the process $A(t)$ is defined from

$$\begin{aligned} M_A(\lambda) &= -\lambda \int_0^t s^a ds \\ M_A(\lambda) &= -\frac{1}{\alpha+1} t^{\alpha+1} \lambda \end{aligned}$$

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and therefore $A(t) = \frac{1}{\alpha+1}t^{a+1}$ which means that :

$$\int_0^t sdZ(s) \stackrel{d}{=} Z(\frac{1}{\alpha+1}t^{a+1}).$$

Compound Poisson

Let now $Z(t)$ be a Compound Poisson Process with $\Gamma(v, a)$ jumps. Then we can define the process $\{cZ(t)\}$ in terms of the following stochastic integral

$$cZ(t) = \int_0^t cdZ(s).$$

Using now some results of first example the Gamma - OU process we have that

$$\begin{aligned}
\Phi_z(\lambda) &= v\left(1 - \frac{a}{a+\lambda}\right) \\
M_A^{'}(\lambda) &= -\frac{v\lambda f(t)}{\lambda f(t) + v - \lambda} = -\frac{v\lambda c}{\lambda c + v - \lambda}
\end{aligned}$$

and therefore

$$\begin{aligned}
M_A(\lambda) &= -t\frac{v\lambda c}{\lambda c + v - \lambda} \\
M_A(\lambda) &= -t\frac{vc}{c-1}\,\frac{\lambda(c-1)}{\lambda(c-1)+v} \\
M_A(\lambda) &= -t\frac{vc}{c-1}\left(1 - \frac{v/(c-1)}{v/(c-1)+\lambda}\right)
\end{aligned}$$

which means that the process $cZ(t)$ has marginals laws of the form

$$cZ(t) \stackrel{d}{=} Z(A(t))$$

where $A(t)$ is a compound Poisson process with intensity $\{t\frac{vc}{c-1}\}$ and jumps from the

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$\Gamma(1, v/(c-1))$ distribution. Note that straight forward calculus gives the expected result

$$M_A(\Phi_Z(\lambda)) = -t\Phi_{cZ}(\lambda).$$

Proof: We have that

$$\begin{aligned}
M_A(\Phi_Z(\lambda)) &= -t\frac{vc}{c-1}\{1 - \frac{v/(c-1)}{v/(c-1) + v\{1 - \frac{a}{a+\lambda}\}}\} \\
&= -t\frac{vc}{c-1}\{1 - \frac{1}{1 + \{1 - \frac{a}{a+\lambda}\}(c-1)}\} \\
&= -t\frac{vc}{c-1}\{1 - \frac{1}{1 + \frac{\lambda}{a+\lambda}(c-1)}\} \\
&= -tv\{\frac{\frac{\lambda c}{a+\lambda}}{\frac{a+\lambda c}{a+\lambda}}\} \\
&= -tv\{1 - \frac{a}{a+\lambda c}\}
\end{aligned}$$

The above result tell us that the simple process $cZ(t)$ is distributed as

$$cZ(t) \stackrel{d}{=} \sum_{j=1}^{N(\sum_{i=1}^{M(t)} X_i)} Y_j$$

where $N(t)$ is a Poisson point process with intensity vt, Y_j are independent and distributed as $\Gamma(1, a)$, $M(t)$ is a Poisson point process with intensity $t\frac{vc}{c-1}$ and finally X_i are independent $\Gamma(1, v/(c-1))$ distributions.

Constant function f

Examine more carefully the above example we can derive a more general result for the above representation of the stochastic integral. More specifically the subordinated process $A(t)$ it can be a subordinator if the function f is constant. To see this we will start from the beginning and we will conclude that the function f is constant.

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Let assume that there exists a process $A(t)$ which is subordinator such that

$$\int_0^t f(s)dZ(s) \stackrel{d}{=} Z(A(t))$$

therefore using again (5.2),(3.3), we have

$$M_A(\Phi_z(\lambda)) = \int_0^t -\Phi_z(\lambda f(u))du\}$$

but from our assumption $A(t)$ is subordinator and therefore we may rewrite the above equation

$$\log[E[e^{-\lambda Z(A(t))}]] = -t\Phi_A \Phi_Z(\lambda) = \int_0^t -\Phi_z(\lambda f(u))du\}$$

taking now the second derivative with respect to t we get that

$$0 = -\Phi_Z'(\lambda f(t))f'(t)$$

which obviously means that $f' = 0$ and therefore f is constant. Even in the case where $\Phi_Z'(\lambda f(t)) = 0$ that means $\Phi_Z(\lambda f(t)) = c$ and therefore $f(t)$ does not depend on t so again f is a constant function.

Brownian motion

Another simple and well known example is the usual case of the stochastic integral

$$\int_0^t f(s)dB(s)$$

where f is a real function and $B(t)$ is a Brownian motion. We will use the (5.1) in order to identify the process $A(t)$. Using now the fact that

$$\Phi_{B(t)}(\lambda) = \frac{t}{2}\lambda^2$$

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we get that

$$E[e^{-B(A(t))}] = E[e^{-A(t)\Phi_{B(1)}(\lambda)}] = e^{M_A(\Phi_{B(1)}(\lambda))}$$

and therefore we can use again the 5.1. So we have

i) $\Phi_{B(1)}(\lambda) = \frac{1}{2}\lambda^2$

ii) $\Phi^{-1}_{B(1)}(\lambda) = \left|\sqrt{2\lambda}\right|$

and therefore

$$\begin{aligned}
(-M_A'(\lambda))^{-1} &= \Phi_{B(1)}(\frac{\Phi^{-1}_{B(1)}(\lambda)}{f(t)}) \\
(-M_A'(\lambda))^{-1} &= \frac{\lambda}{f^2(t)} \\
M_A(\lambda) &= -\int_0^t f^2(s)ds\lambda
\end{aligned}$$

which means that the process $A(t)$ is of the form

$$A(t) = \int_0^t f^2(s)ds$$

and therefore we can write

$$\int_0^t f(s)dB(s) \stackrel{d}{=} B(\int_0^t f^2(s)ds). \tag{5.6}$$

We mention here that this result is equivalent to the Itô-isometry since

$$E[(\int_0^t f(s)dB(s))^2] = E[\int_0^t f^2(s)ds] = V(B(\int_0^t f^2(s)ds)).$$

As an example we will show that in the very simple and well known case $\int_0^t sdB(s)$ the distribution is $N(0, \frac{t^3}{3})$.Using (5.6) for this stochastic integral we take

$$\int_0^t sdB(s) \stackrel{d}{=} B(\int_0^t s^2ds) \stackrel{d}{=} B(\frac{t^3}{3})$$

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which obviously is $N(0, \frac{t^3}{3})$.

Write now (5.6)

$$\int_0^t f(s)dB(s) \stackrel{d}{=} B(\int_0^t f^2(s)ds) = f(t)B(t) - \int_0^t B(s)df(s).$$

We have that

$$\int_0^t B(s)df(s) = f(t)B(t) - B(\int_0^t f^2(s)ds)$$

or

$$\int_0^t B(s)df(s) \stackrel{d}{=} (f(t)-1)B(t) - (B(\int_0^t f^2(s)ds) - B(t))$$

and therefore

$$\int_0^t B(s)df(s) \backsim N(0, (f(t)-1)^2 t + \int_0^t f^2(s)ds - t), iff > 1.$$

5.2 Sampling from Stochastic integral.

In this section we will use two different methods in order to sample from the stochastic integral in Gamma -OU case. The first method is due to Rosinski series representation (3.4) and the second via the representation above (5.1). In Barndorff -Nielsen and Shepard (2001) the equations (4.2) and (3.4) for the $\Gamma(v,a)-$OU case takes the form

$$\int_0^t f(s)dZ(s) \stackrel{d}{=} \sum_{j=1}^{\infty} f(\omega_j) M^{-1}(t_j, \omega_j)$$

$$e^{-t}\int_0^t e^s dZ(s) \stackrel{d}{=} \frac{1}{a} e^{-t} \sum_{j=1}^{N(1)} \log(c_j^{-1}) e^{tr_j}$$

where $c_1 < c_2 < ... < c_{N(1)}$ the arrival times from Poisson with intensity $v\lambda t$ and $N(1)$ the corresponding events up until time 1, and finally r_j independent copies from uniform random variable in $[0, 1]$.

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On the other hand using the representation above one has that

$$e^{-t}\int_0^t e^s dZ(s) \stackrel{d}{=} e^{-t}Z(A(t))$$

where $A(t)$ has $\Gamma(v, \frac{v}{e^t-1})$ marginals and Z is a Compound Poisson with jumps $\Gamma(1, a)$ independent from $A(t)$. Note that we can write

$$Z(A(t)) \stackrel{d}{=} \sum_{j=1}^{N(A(t))} X_j$$

where X_j's are i.i.d. from $\Gamma(1, a)$. One can sample first from $A(t)$ and then given this value, to sample from the sum, when $A(t)$ is independent from $Z(t), N(t)$ and finally to repeat the above procedure in order to take the sample from $Z(A(t))$. In the figures bellow we will take two samples of size 10.000 values and we compare the empirical densities also we present some basic statistics for these densities.

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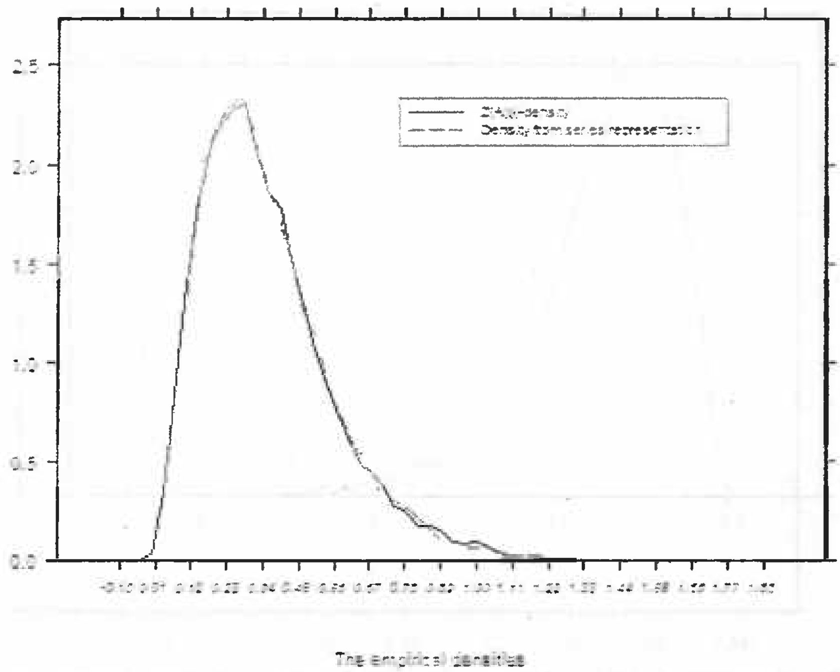

Figure 5-1. Empirical densities together using a sample of 10.000 for each one.

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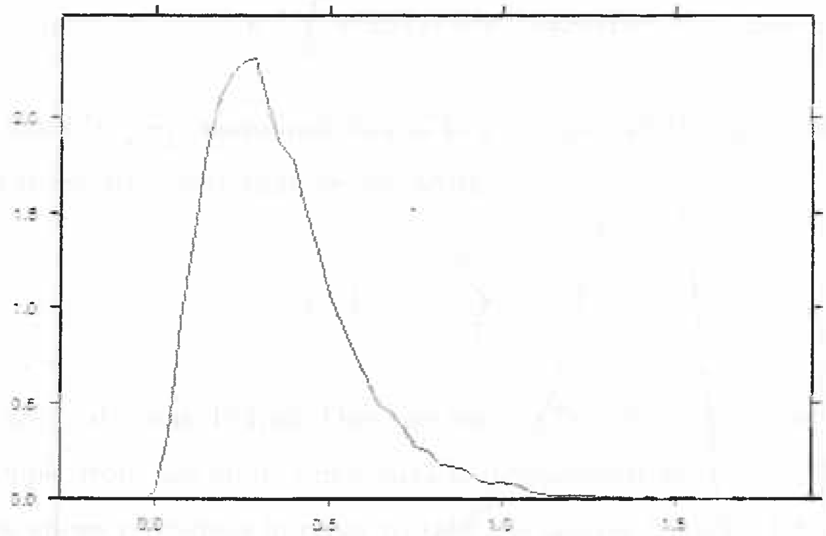

Figure 5-2.The density of the stochastic integral using the representation Z(A(t))

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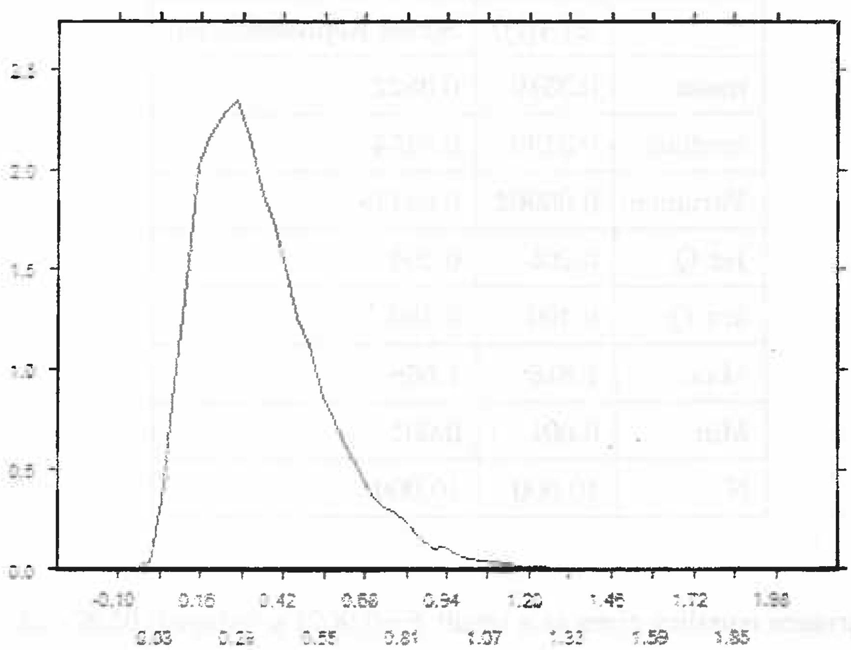

Figure 5-3.The density of series representation method

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In the following table we present some basic statistics for these samples

*Table*5.2

	$Z(A(t))$	Series Representation
mean	0.3519	0.3522
median	0.3139	0.3154
Variance	0.03962	0.04118
1st Q	0.205	0.204
3rd Q	0.460	0.459
Max	1.816	1.668
Min	0.001	0.001
N	10.000	10.000

An a F-test for variance equality gives as a result F=0.9621 p-value=0.0536 and the t-test for means t=0.0789,df=19998,p-value=0.9371.

Appendix

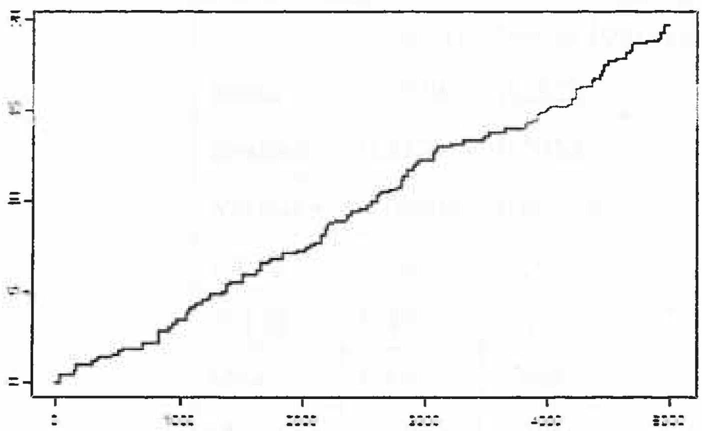

Figure 6.1 The BDLP a Compound Poisson process

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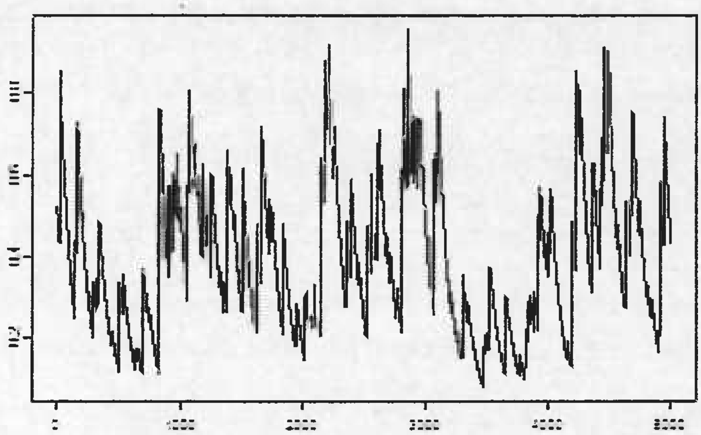

Figure 6-2. The Gamma OU process jumps when Z(t) jumps

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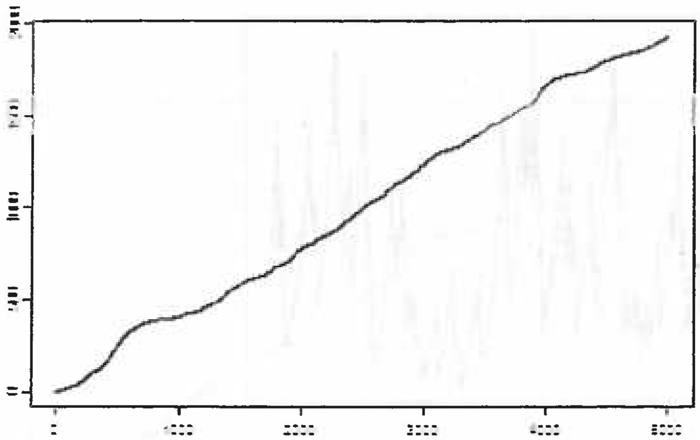

Figure 6-3 The Integrated OU process

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S-Plus Programmes options(object.size=5e8)

```
#Parameters
lamda<-0.01
a<-8.5
v<-3
N<-seq(1,5000,1)
#Simulate the BDLP and the OU-Process (Gamma-OU)
Process<-seq(1,5000,1)
Z<-seq(1,5000,1)
Z1<-seq(1,5000,1)
Process1<-seq(1,5000,1)
#Calculate the series representation
for (i in 1:5000) {
N[i]<-rpois(1,lamda*v)
if (N[i]<1) {
Process1[i]<-0
Z1[i]<-0
}
else{
c<-runif(N[i], 0, 1)
r<-runif(N[i], 0, 1)
c<-sort(c)
sum<-0
sum1<-0
for (k in 1:length(c)) {
sum<-(1/(a*exp(lamda)))*(sum+log(1/c[k],exp(1))*(exp(lamda*r[k])))
sum1<-(1/a)*(sum1+log(1/c[k],exp(1)))
}
```

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```
Z1[i]<-sum1
Process1[i]<-sum
}
}
#Construct the path
Process[1]<-rgamma(1,v,a)
Z[1]<-0
for (i in 1:4999) {
Z[i+1]<-Z[i]+Z1[i+1]
Process[i+1]<-(exp(-lamda))*Process[i]+Process1[i+1]
}
tsplot(Z)
tsplot(Process)
#Simulate the Integrated OU process-Construction of Path base on the above results
INTOU<-seq(1,5000,1)
for (i in 1:5000) {
INTOU[i]<-(1/lamda)*(Z[i]-Process[i]+Process[1])
}
tsplot(INTOU)
#S-Plus-Programme
#Parameters
lamda<-1
a1<-8.5
a<-1/a1
v<-7
t<-8
#dist =sample from Series Representation
dist<-seq(1,10000,1)
```

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```
for (i in 1:10000) {
N<-rpois(1,v*lamda*t)
c<-runif(N,0,1)
r<-runif(N,0,1)
sum<-0
c<-sort(c)
for (j in 1:N) {
sum<-sum+(exp(-t)*a*exp(lamda*t*r[j])*log(1/c[j],exp(1)))
}
dist[i]<-sum
}
Dist1<-seq(1,10000,1)
for (i in 1:10000){
A<-rgamma(1,v,v/(exp(t)-1))
N<-rpois(1,v*A)
q<-rgamma(N,1,a1)
s<-sum(q)
Dist1[i]<-exp(-t)*s
}
mean(dist)
mean(Dist1)
median(dist)
median(Dist1)
var(dist)
var(Dist1)
var.test(dist, Dist1, conf.level=.99)
```

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