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DEPARTMENT OF STATISTICS

# Itô Stochastic Integral and Applications

Master Thesis for the Graduate Programme

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# Abstract

A stochastic process is, in some way, a collection of paths evolving to time, chosen within some law (ie, probability measure). These paths can be either discrete –eg sequences, or functions defined on intervals. In the latter case we get what we call continuous-time stochastic processes. And this is the object of study in this text.

Analysis as a branch of mathematics, offers roughly, procedures by which we can respond in limit-finding problems: That is, where a function is about to converge to? Or if we are given a collection of functions, can we extract results about the convergence of this collection? Compactness plays a fundamental role in this direction.

With this in mind, we look for compactness arguments so that we are lead to existence of limits. In probabilities, the concept of Martingale is more or less a compactness argument about a stochastic process: If we get the instance of a stochastic process at a given moment in time and look at it through the information of a previous moment, we'll fall back on the instance of the process at that previous moment. Not by chance, this argument of compactness, often comes as a natural law in a wide range of applications: from game theory, physics experiments, to stocks on the stock market.

Despite the nice properties and naturality of Martingales, it is impossible to put into operation one of the most fundamental tools of analysis: the integral. Even continuous Martingales (i.e. processes whose trajectories are continuous functions) are inappopriate integrators in Lebesgue-Stieltjes integrals. This is where the Itô integral comes in.

The Itô integral is therefore a generalization of the Lebesgue-Stieltjes integral over a whole stochastic process. It gives us the tools on which we can build differential equations for stochastic processes and consequently, tools with which we can predict the evolution of a stochastic process, a random natural system, etc.

We here present the Itô integral for continuous Martingales and see some fundamental applications of it. We preferred to define the integral using the quadratic variation of a martingale.

In the first chapter we present the general theory we will need as well as define the idea of quadratic variation.

In the second chapter, we give very briefly the definition of Brownian motion. This is the most fundamental continuous Martingale pattern and perhaps the principal integrator in the Itô

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integral. Brownian motion is nothing other than the continuous version of the random walk.

In the third chapter, we are ready to define the stochastic integral for continuous Martingales and to give a formula through which we can evaluate stochastic integrals in a similar way that we seek for antiderivatives in the classical integration theory. This is the famous Itô formula.

In the fourth chapter we will see the first applications of this formula. A key notion that will concern us is that of exponential Martingales, the laws they induce and how through these laws we can look at other Martingales. So we'll finally get –and stop, to the Girsanov Theorem, presented in its most modern version.

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# Περίληψη

Μία στοχαστική ανέλιξη, είναι κατά κάποιον τρόπο μία συλλογή από μονοπάτια στον χρόνο, που επιλέγονται στα πλαίσια κάποιου νόμου (δηλαδή, μέτρου πιθανότητας). Τα μονοπάτια αυτά μπορούν να είναι είτε διακριτά –πχ ακολουθίες, είτε συναρτήσεις ορισμένες σε διαστήματα. Στην τελευταία περίπτωση έχουμε αυτό που λέμε στοχαστικές ανελίξεις συνεχούς χρόνου. Και αυτές μας απασχολούν σε αυτό το κείμενο.

Η ανάλυση –ως κλάδος των μαθηματικών, προσφέρει χοντρικά, διαδικασίες με τις οποίες μπορούμε να απαντήσουμε σε προβλήματα *εύρεσης ορίων*: Δηλαδή, *προς τα πού τείνει να πάει μία συνάρτηση*; Ή αν μας δόσουν μια συλλογή συναρτήσεων, μπορούμε να εξάγουμε αποτελέσματα για το αν αυτή η συλλογή συγκλίνει κάπου; Η συμπάγεια παίζει θεμελιώδη ρόλο προς αυτή την κατεύθυνση.

Έχοντας αυτό υπ' όψιν, αναζητούμε επιχειρήματα συμπάγειας, ώστε να πάρουμε αποτελέσματα ορίων. Στις πιθανότητες, η έννοια του Martingale είναι κατά κάποιο τρόπο ένα επιχείρημα συμπάγειας σε μια στοχαστική ανέλιξη: Αν πάρουμε την εικόνα μιας στοχαστικής ανέλιξης σε μια δεδομένη χρονική στιγμή και την κοιτάξουμε μέσα από την πληροφορία μιας πρότερης κατάστασης, θα ξαναπέσουμε πάνω στην πρότερη αυτή στιγμή. Καθόλου τυχαία, αυτό το επιχείρημα συμπάγειας, έρχεται συχνά ως φυσικός νόμος σε πληθώρα εφαρμογών: από τυχερά παιχνίδια, πειράματα φυσικής, μέχρι τις μετοχές στο χρηματιστήριο.

Παρά τις καλές ιδιότητες και τη φυσικότητα των Martingales, είναι *αδύνατο* να θέσουμε σε λειτουργία ένα από τα βασικότερα εργαλεία της ανάλυσης: το ολοκλήρωμα. Ακόμα και τα συνεχή Martingales (δηλαδή ανελίξεις των οποίων οι τροχιές οι συνεχείς συναρτήσεις) είναι απρόσφοροι ολοκληρωτές σε ολοκληρώματα Lebesgue-Stieltjes. Εδώ είναι που έρχεται το ολοκλήρωμα Itô.

Το ολοκλήρωμα Itô, είναι μία γενίκευση λοιπόν της ολοκλήρωσης Lebesgue-Stieltjes πάνω σε μία ολόκληρη στοχαστική ανέλιξη. Μας δίνει το εργαλείο πάνω στο οποίο μπορούμε να χτίσουμε διαφορικές εξισώσεις για στοχαστικές ανελίξεις και κατά συνέπεια, εργαλεία με τα οποία μπορούμε να προβλέψουμε την πορεία μιας ανέλιξης, ενός τυχαίου φυσικού συστήματος κλπ.

Εμείς εδώ θα παρουσιάσουμε το ολοκλήρωμα Itô για συνεχή Martingales και θα δούμε μερικές βασικές εφαρμογές του. Προτιμήσαμε να ορίσουμε το ολοκλήρωμα μέσω της τετραγωνικής κύμανσης ενός martingale.

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Στο πρώτο κεφάλαιο παρουσιάζουμε τη γενική θεωρία που θα μας χρειαστεί καθώς και ορίζουμε την έννοια της τετραγωνικής κύμανσης.

Στο δεύτερο κεφάλαιο, δίνουμε πολύ συνοπτικά, την έννοια της κίνησης Brown. Πρόκειται για το βασικότερο πρότυπο συνεχούς Martingale και ίσως τον κυριότερο ολοκληρωτή στο ολοκλήρωμα Itô. Η κίνηση Brown δεν είναι τίποτα άλλο από τη συνεχή έκδοση του τυχαίου περιπάτου.

Στο τρίτο κεφάλαιο, είμαστε έτοιμοι να ορίσουμε το στοχαστικό ολοκλήρωμα για τα συνεχή Martingales και να δώσουμε ένα τύπο μέσω του οποίου μπορούμε να υπολογίζουμε στοχαστικά ολοκληρώματα, με έναν τρόπο παρόμοιο με τον οποίο αναζητούμε αρχικές συναρτήσεις στην κλασσική ολοκλήρωση. Πρόκειται για τον περίφημο τύπο του Itô.

Στο τέταρτο κεφάλαιο θα δούμε τις πρώτες βασικές εφαρμογές αυτού του τύπου. Μία βασική έννοια που θα μας απασχολήσει είναι τα εκθετικά Martingales, οι νόμοι που αυτά επάγουν και πώς μέσα από αυτούς τους νόμους μπορούμε να δούμε ξανά άλλα Martingales. Θα φτάσουμε λοιπόν τελικά –και θα σταματήσουμε, στο Θεώρημα Girsanov, παρουσιασμένο στην πιο σύγχρονή του εκδοχή.

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# 1. Martingales

## 1.1 General theory

We make a quick introduction to the general theory of stochastic processes. Most of the material presented in this section can be found at [KS14, ch. 1].

**1.1.1 Definition** Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(S, \mathcal{S})$ a measurable space (usually $\left(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\right)$) and $T \subseteq \mathbb{R}$ (usually $T = [0, +\infty)$). An indexed family $(X_t : \Omega \longmapsto S)_{t \in T}$ of random variables is called **stochastic process**. The probability space $(\Omega, \mathcal{F}, P)$ is called **sample space**, the measurable space $(S, \mathcal{S})$ is called **state space**.

For every fixed $\omega \in \Omega$ the mapping $T \ni t \longmapsto X_t(\omega)$ is called path of the process at $\omega$.

In this text, $T$ will be $T = [0, +\infty)$ by default. Any other choices of $T$ will be explicitly mentioned. In case $T = [0, +\infty]$ we can say that the process has a *last element* $\mathrm{X}_\infty$.

Two stochastic processes $(X_t)_{t \in T}, (Y_t)_{t \in T}$ defined on the same sample space with common state space are called **indistinguishable** whenever almost all of their paths coincide, that is whene $\exists N \in \mathcal{F}$ with $\mathrm{P}(N) = 0$ and

$$\omega \in N^{\complement} \longrightarrow \forall t \in T (X_t(\omega) = Y_t(\omega))$$

They are called **modification** of each other when

$$\forall t \in T \quad \mathrm{P}([X_t = Y_t]) = 1.$$

Two stochastic processes $(X_t)_{t \in T}, (Y_t)_{t \in T}$ with common state space are called **equivalent** if for every $t_1 < t_2 < \ldots < t_n \in T$ the random vectors $\left(X_{t_1}, X_{t_2}, \ldots, X_{t_n}\right)$ and $\left(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}\right)$ have identical distributions [†].

It is evident that

$$\text{indistinguishable} \longrightarrow \text{modification} \longrightarrow \text{equivalent}$$

and the converse general fails. But we have the following proposition.

[†] that is, they induce identical measures on $S^n$

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**1.1.2 Proposition** Let $(X_t)_{t\in T}, (Y_t)_{t\in T}$ be two processes such that they are modification of each other. Then they are indistinguishable.

***Proof***: Indeed, a right continuous function can be completely determined by its values on $\mathbb{Q}$. If $(q_n)_{n\in\mathbb{N}}$ is an enumeration of the rationals, and for every $n$ we set $N_n = \left[X_{q_n} \neq Y_{q_n}\right]$, then $N = \bigcup_{n\in\mathbb{N}} N_n \in \mathcal{F}$ is negligible. Let $\omega \in \Omega \setminus N$ be fixed and $t \in T$. Then there is a decreasing sequence of rationals $t_n \downarrow t$ and $X_t(\omega) = \lim X_{t_n}(\omega) = \lim Y_{t_n}(\omega) = Y_t(\omega)$. □

**1.1.3 Definition** Let $(X_t)_{t\in T}$ be a stochastic process. The process is called **measurable** if the mapping

$$\Omega \times T \ni (\omega, t) \longmapsto X_t(\omega) \in S$$

is $\mathcal{F} \otimes \mathcal{B}(T) - \mathcal{S}$ measurable.

A nondecreasing family $(\mathcal{F}_t)_{t\in T}$ of σ-algebras, subalgebras of $\mathcal{F}$ is called **filtration** and whenever for every $t \in T$, $X_t$ is $\mathcal{F}_t$-measurable, then the process is called **adapted** to the filtration. The process is called **progressively measurable** relative to the filtration $(\mathcal{F}_t)_{t\in T}$ if for every $t \in T$ the mapping

$$\Omega \times [0, t] \ni (\omega, s) \longmapsto X_s(\omega) \in S$$

is $\mathcal{F}_t \otimes \mathcal{B}([0, t]) - \mathcal{S}$ measurable.

When $T = \{0, 1, 2, \dots\}$, a stochastic process $(\xi_n)_n$ such that $\xi_n$ is $\mathcal{F}_{n-1}$-measurable for all $n \geq 1$ is called **predictable**.

A filtration $\mathfrak{F} = (\mathcal{F}_t)_{t\in T}$ is said to **satisfy the usual conditions** if $\mathcal{F}_0$ is $P$-complete and for every $t \in T$, $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ (ie, the filtration is right continuous).

A progressively measurable stochastic process is by definition measurable and adapted (to its filtration). Generally the converse does not hold true. But it is possible to find a modification of a measurable and adapted process which is also progressively measurable. The proof of this fact is far from being trivial and is omitted. But we can prove the following somehow weaker result, which suffices for us.

**1.1.4 Lemma** Let $(X_t)_{t\in T}$ be a process with all of its paths right (or left) continuous. Then it is measurable.

***Proof***: For every $n \geq 1$ define

$$\mathrm{X}_t^{(n)}(\omega) = \mathrm{X}_{\lceil nt\rceil/n}(\omega) \text{ if } T = [0, +\infty) \text{ or } \mathrm{X}_t^{(n)}(\omega) = \mathrm{X}_{\lceil nt/b\rceil b/n}(\omega) \text{ if } \mathrm{T} = [0, b]$$

Each process $\left(X_t^{(n)}\right)_t$ is measurable and the whole sequence converges pointwise to $(X_t)_t$:

$$\lim_n X_t^{(n)}(\omega) = X_t(\omega), \qquad t \in T, \omega \in \Omega$$

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We conclude that $(X_t)_t$ measurable as well, being the limit of measurable processes. □

**1.1.5 Corollary** If the process $(X_t)_{t\in T}$ is adapted to the filtration $(\mathcal{F}_t)_{t\in T}$ and has all of its paths right (or left) continuous, then it is also progressively measurable.

## 1.2 Stopping times

In this section we present the notion of stopping time, crucial to the further development of our theory. We have chosen not to interfere with optional or predictable times and the related σ-algebras on the product space. The interested reader may be referred to [Del72, ch. III]. We follow the presentation of [KS14, ch. 1].

**1.2.1 Definition** Let $(\Omega, \mathcal{F})$ be a measure space, furnished with a filtration $(\mathcal{F}_t)_{t\in T}$. A random variable $\tau\colon \Omega \longmapsto [0,+\infty]$ is called **random time**. A random time also satisfying $[\tau \le t] \in \mathcal{F}_t$ for every $t \in T$ is called **stopping time**. The finest sub σ-algebra $\mathcal{F}_\tau \subseteq \mathcal{F}$ in $\Omega$, making the injections

$$\psi_t = \mathrm{Id}_{[\tau\le t]}\colon \left([\tau \le t], \mathcal{F}_t|_{[\tau\le t]}\right) \longmapsto \Omega, \quad t \in T$$

measurable is called the **σ-algebra of events prior to** $\tau$.

**1.2.2 Comment** a) One can easily deduce that

$$\mathcal{F}_\tau = \bigcap_{t\in T} \{A \in \mathcal{F} : A \cap [\tau \le t] \in \mathcal{F}_t\}$$

which is indeed a σ-algebra, for it is closed under countable unions and it contains complements: $[\tau \le t] \setminus A = [\tau \le t] \setminus ([\tau \le t] \cap A)$.

The $\mathcal{F}_\tau$ has also the following universal property: A random variable $Z\colon (\Omega, \mathcal{F}) \longmapsto (Q, \mathcal{J})$ is $\mathcal{F}_\tau - \mathcal{J}$ measurable if and only if for all $t \in T$ the compositions $Z \circ \psi_t = Z|_{[\tau\le t]}$ are $\mathcal{F}_t - \mathcal{J}$ measurable.

b) A stopping time $\tau$ is trivially $\mathcal{F}_\tau$ -measurable.

c) A stopping time $\tau$ also satisfies $[\tau < t] \in \mathcal{F}_t$, for $[\tau < t] = \bigcup_{n\ge 1} \left[\tau \le t - \frac{1}{n}\right]$. Random times satisfying this property are called *optional times*.

**1.2.3 Example** If $c \ge 0$, $C \in \mathcal{F}$, then the random time

$$\tau\colon \Omega \longmapsto [0,+\infty] : \omega \longmapsto \tau(\omega) = \begin{cases} c & \text{if } \omega \in C \\ +\infty & \text{if } \omega \notin C \end{cases}$$

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is a stopping time if and only if $C \in \mathcal{F}_c$ and in that case one can easily verify

$$\begin{aligned}\mathcal{F}_\tau &= \bigcap_{t\geq 0} \{A \in \mathcal{F} : A \cap [\tau \leq t] \in \mathcal{F}_t\} \\ &= \bigcap_{t\geq c} \{A \in \mathcal{F} : A \cap C \in \mathcal{F}_t\} \\ &= \{A \in \mathcal{F} : A \cap C \in \mathcal{F}_c\} \\ &= \{(A \cap C) \cup (B \setminus C) \mid A \in \mathcal{F}_c, B \in \mathcal{F}\}\end{aligned}$$

So that $\mathcal{F}_\tau$ is a direct sum of the trace σ-algebra $\mathcal{F}_t|_C$ (the events prior to $t$) and $\mathcal{F}|_{C^\complement}$ (the events after t).

In the special case that $C = \Omega$ we also obtain that a constant $\tau = t$ is trivially a stopping time and $\mathcal{F}_\tau = \mathcal{F}_t$.

**1.2.4 Example** Let $\tau \leq \sigma$ be two random times. We suppose that $\tau$ is a stopping time. If $\sigma$ is $\mathcal{F}_\tau$-measurable then it is also a stopping time. Indeed, since $\sigma$ is $\mathcal{F}_\tau$-measurable, then for all $t \in T$ the restrictions $\sigma|_{[\tau\leq t]}$ are $\mathcal{F}_t$-measurable. So $[\sigma \leq t] = [\sigma \leq t] \cap [\tau \leq t] = (\sigma|_{[t\leq t]})^{-1}((-\infty, t]) \in \mathcal{F}_t$.

**1.2.5 Proposition** Let $\tau, \sigma$ be stopping times. Then so are $\tau \wedge \sigma, \tau \vee \sigma, \tau + \sigma$.

Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of stopping times. Then $\sup_{n\in\mathbb{N}} \tau_n$ is a stopping time too.

***Proof***: For the minimum and maximum, just observe that $[\tau \wedge \sigma \leq t] = [\tau \leq t] \cap [\sigma \leq t]$ and $[\tau \vee \sigma \leq t] = [\tau \leq t] \cup [\sigma \leq t]$. For the sum, we decompose the second event in a standard way:

$$\begin{aligned}[\tau + \sigma > t] = ([\tau = 0] \cap [\sigma > t]) \cup ([0 < \tau < t] \cap [\sigma + \tau > t]) \\ \cup ([\tau > t] \cap [\sigma = 0]) \cup ([\tau \geq t] \cap [\sigma > 0])\end{aligned}$$

and then use the also standard argument about the sum of random variables:

$$[0 < \tau < t] \cap [\sigma + \tau > t] = \bigcup_{q\in\mathbb{Q}\cap(0,t)} [q < \tau < t] \cap [\sigma > t - q]$$

to obtain that all the events are in $\mathcal{F}_t$.

Finally, observe that $\left[\sup_{n\in\mathbb{N}} \tau_n \leq t\right] = \bigcap_{n\in\mathbb{N}} [\tau_n \leq t]$. □

**1.2.6 Proposition** Let $\tau, \sigma$ be stopping times. Then $\mathcal{F}_{\tau\wedge\sigma} = \mathcal{F}_\tau \cap \mathcal{F}_\sigma$ and

$$[\tau < \sigma], [\sigma < \tau], [\tau \leq \sigma], [\sigma \leq \tau], [\sigma = \tau] \in \mathcal{F}_\tau \cap \mathcal{F}_\sigma$$

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***Proof***: Let $\rho$ be a stopping time and $t \in T$ be fixed. First observe that the random times $\rho \wedge t, \sigma \wedge t$ are $\mathscr{F}_t$ measurable. Then for any $A \in \mathscr{F}_\rho$,

$$\begin{aligned} A \cap [\rho \leq \sigma] \cap [\sigma \leq t] &= A \cap [\rho \leq t] \cap [\rho \leq \sigma] \cap [\sigma \leq t] \\ &= A \cap [\rho \leq t] \cap [\rho \wedge t \leq \sigma \wedge t] \cap [\sigma \leq t] \in \mathscr{F}_t \end{aligned}$$

But $t$ was arbitrary, so $A \cap [\rho \leq \sigma] \in \mathscr{F}_\sigma$. Similarly, $A \cap [\rho \leq \tau] \in \mathscr{F}_\tau$. Replace $\rho = \tau \wedge \sigma$, so $[\rho \leq \sigma] = [\rho \leq \tau] = \Omega$, whence $\mathscr{F}_{\tau\wedge\sigma} \subseteq \mathscr{F}_\tau \cap \mathscr{F}_\sigma$.

For the reverse inclusion, if $A \in \mathscr{F}_\tau \cap \mathscr{F}_\sigma$ then for all $t \in T$,

$$A \cap [\sigma \wedge \tau \leq t] = (A \cap [\sigma \leq t]) \cup (A \cap [\tau \leq t]) \in \mathscr{F}_t$$

From the first part of the proof (for $A = \Omega$) we get that $[\sigma \leq \tau] \in \mathscr{F}_\tau$. The stopping time $\rho = \tau \wedge \sigma$ is $\mathscr{F}_{\tau\wedge\sigma} \subseteq \mathscr{F}_\tau$-measurable, so $[\rho < \tau] = [\sigma < \tau] \in \mathscr{F}_\tau$. Interchanging the roles of $\sigma$ and $\tau$, we get also that $[\tau < \sigma], [\tau \leq \sigma] \in \mathscr{F}_\sigma$. This finishes the proof. □

**1.2.7 Corollary** Let $\tau, \sigma$ be stopping times and $Z$ be an integrable random variable. Then

a. $1_{[\tau\leq\sigma]}\mathbb{E}\left[Z \mid \mathscr{F}_\tau\right] = 1_{[\tau\leq\sigma]}\mathbb{E}\left[Z \mid \mathscr{F}_{\tau\wedge\sigma}\right]$, $P$-a.e.
b. $\mathbb{E}\left[\mathbb{E}\left[Z \mid \mathscr{F}_\tau\right] \mid \mathscr{F}_\sigma\right] = \mathbb{E}\left[Z \mid \mathscr{F}_{\tau\wedge\sigma}\right]$, $P$-a.e.

***Proof***: a). Observe that for any $A \in \mathscr{F}_\tau$, $A \cap [\tau \leq \sigma] \in \mathscr{F}_\sigma$ and as $[\tau \leq \sigma] \in \mathscr{F}_{\tau\wedge\sigma} \subseteq \mathscr{F}_\tau$, $A \cap [\tau \leq \sigma] \in \mathscr{F}_\tau$ as well. So using the definition of conditional expectation, $A \cap [\tau \leq \sigma] \in \mathscr{F}_\tau \cap \mathscr{F}_\sigma = \mathscr{F}_{\tau\wedge\sigma}$. So

$$\begin{aligned} \int_A 1_{[\tau\leq\sigma]}\mathbb{E}\left[Z \mid \mathscr{F}_\tau\right] \, \mathrm{d}P &= \int_A 1_{[\tau\leq\sigma]} Z \, \mathrm{d}P = \int_{A\cap[\tau\leq\sigma]} Z \, \mathrm{d}P \\ &= \int_{A\cap[\tau\leq\sigma]} \mathbb{E}\left[Z \mid \mathscr{F}_{\tau\wedge\sigma}\right] \, \mathrm{d}P = \int_A 1_{[\tau\leq\sigma]}\mathbb{E}\left[Z \mid \mathscr{F}_{\tau\wedge\sigma}\right] \, \mathrm{d}P \end{aligned}$$

b). The first part of the proof also implies that $1_{[\tau<\sigma]}\mathbb{E}\left[Z \mid \mathscr{F}_\tau\right] = 1_{[\tau<\sigma]}\mathbb{E}\left[Z \mid \mathscr{F}_{\tau\wedge\sigma}\right]$ $P$-a.e. Now swapping $\tau$ and $\sigma$, and replacing $Z \leftarrow \mathbb{E}\left[Z \mid \mathscr{F}_\tau\right]$ we can extract the requested equality. □

**1.2.8 Definition** Let $X = (X_t)_{t\in T}$ be a stochastic process and $\tau$ be a stopping time. We define the stopped process $X^\tau = (X_t^\tau)_{t\in T}$ as follows:

$$X_t^\tau(\omega) = X_{\tau(\omega)\wedge t}(\omega) = \begin{cases} X_t(\omega) & \text{if } t \leq \tau(\omega) \\ X_{\tau(\omega)}(\omega) & \text{if } t > \tau(\omega) \end{cases}$$

We also put $X_\tau$ for the random variable $\Omega \ni \omega \longmapsto X_{\tau(\omega)}(\omega)$.

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**1.2.9 Proposition** Let $(X_t)_{t\in T}$ be an adapted process to the filtration $(\mathscr{F}_t)_{t\in T}$, progressively measurable and $\tau$ be a stopping time of this filtration. Then $X_\tau\colon \Omega \longmapsto S : \omega \longmapsto X_\tau(\omega) = X_{\tau(\omega)}(\omega)$ is $\mathscr{F}_\tau$-measurable and the stopped process $\left(X_{\tau\wedge t}\right)_{t\in T}$ is progressively measurable.

***Proof***: Just observe that for fixed $t \in T$, the mapping

$$[0,t]\times\Omega \ni (s,\omega) \longmapsto (\tau(\omega)\wedge s, \omega) \in [0,t]\times\Omega$$

is measurable when both product spaces are endowed with $\mathscr{F}_t \otimes \mathscr{B}([0,t])$. Progressive measurability of the stopped process is immediate consequence of the composition of the former mapping and $(\omega,t) \longmapsto X_{t(\omega)}(\omega)$.

Now to this end, if $\mathrm{B} \in \mathcal{S}$, $t \in T$ are given,

$$[\mathrm{X}_\tau \in B] \cap [\tau \le t] = [\mathrm{X}_{\tau\wedge t} \in B] \cap [\tau \le t] \in \mathscr{F}_t$$

which proves the $\mathscr{F}_\tau$-measurability of $X_\tau$. □

**1.2.10 Example** An important class of stopping times, is what we call *hitting times*. Let $(X_t)_{t\in T}$ be an adapted process to the filtration $(\mathscr{F}_t)_{t\in T}$. Let $\Gamma \subseteq S$ be a closed subspace of the state space $S$. We further suppose that the process has continuous paths. Then the function

$$\Omega \ni \omega \longmapsto \tau(\omega) = \inf\{t \in T : X_t(\omega) \in \Gamma\} \in [0,+\infty]$$

is indeed a stopping time; this time describes the first time that the process reaches the set $\Gamma$. Let $t \in T$. Observe that

$$[\tau \le t] = \bigcap_{n=1}^{\infty} \bigcup_{r\in[0,t)\cap\mathbb{Q}} \left[\operatorname{dist}(X_r,\Gamma) < \frac{1}{n}\right]$$

If $\omega \in \Omega$ such that $\tau(\omega) \le t$. Let $n \ge 1$. Since $u \longmapsto X_u(\omega)$ is continuous at $\tau(\omega)$, we can find some $\delta > 0$ such that whenever $|s - \tau(\omega)| < \delta$ we get $d(X_s(\omega), X_{\tau(\omega)}(\omega)) < 1/2n$. Take any rational $r > \tau(\omega) - \delta$ so that $d(X_r(\omega), X_{\tau(\omega)}(\omega)) < 1/2n$. By the characterization of the infimum, we can find some $\tau(\omega) \le s < \tau(\omega) + \delta$ such that $X_s(\omega) \in \Gamma$. $d(X_s(\omega), X_{\tau(\omega)}(\omega)) < 1/2n$. Hence

$$\operatorname{dist}(X_r(\omega),\Gamma) \le d(X_r(\omega), \mathrm{X}_s(\omega)) \le d(X_r(\omega), \mathrm{X}_{\tau(\omega)}(\omega)) + d(X_s(\omega), \mathrm{X}_{\tau(\omega)}(\omega)) < \frac{1}{n}$$

If $\omega$ lies in the right-side set, then we can find a sequence of rationals $(q_n)_{n=1}^{\infty}$, $0 \le q_n < t$ such that $\operatorname{dist}(X_{q_n}(\omega),\Gamma) \longrightarrow 0$. Since $(q_n)_n$ is bounded, passing to a subsequence, $q_n \longrightarrow q \le t$ and $X_q(\omega) \in \Gamma$. So $\tau(\omega) \le q \le t$ and $\omega \in [\tau \le t]$.

Now $\left[\operatorname{dist}(X_r,\Gamma) < \frac{1}{n}\right] \in \mathscr{F}_t$, and so does $[\tau \le t]$. This proves our claim: hitting times on closed sets are stopping times.

Note that, if $\tau(\omega) > 0$, then $X_{\tau(\omega)}(\omega) \in \partial E$. For if $X_{\tau(\omega)}(\omega) \in \operatorname{int} E$, this means that a whole open interval $J$ around $\tau(\omega)$ would make the path $X(\omega)$ stay inside $E$ which is impossible.

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## 1.3 Martingales, inequalities, optional stopping

In this section we introduce the notion of Martingale. We follow the development of [KS14, ch 1.3] and [RY13, ch II].

**1.3.1 Definition** Let $(X_t)_{t\in T}$ be an integrable adapted process to the filtration $(\mathscr{F}_t)_{t\in T}$. The process is called **submartingale** (resp. **supermartingale**) if for all $s < t \in T$

$$\mathbb{E}\left[X_t \mid \mathscr{F}_s\right] \geq X_s \quad (\text{resp. } \mathbb{E}\left[X_t \mid \mathscr{F}_s\right] \leq X_s) \text{ a.s.}$$

A process wich is both submartingale and supermartingale is called **martingale**.

**1.3.2 Remark** If $T = [0, +\infty]$ then we talk about a **(sub)martingale with last element** $X_\infty$. The σ-algebra $\mathscr{F}_\infty$ in this case, should contain at least $\sigma\left(\bigcup_{t\geq 0}\mathscr{F}_t\right)$. This as a supplement to the last definition.

The following proposition is a direct consequence of the Jensen's inequality for conditional expectation:

**1.3.3 Proposition** Let $(X_t)_{t\in T}$ be a martingale (resp. submartingale) and $\varphi\colon \mathbb{R} \longmapsto \mathbb{R}$ be a convex (resp. convex nondecreasing) function such that $\varphi(X_t)$ is integrable for all $t \in T$. Then $(\varphi(X_t))_{t\in T}$ is a submartingale.

Many of the properties of continuous time martingales come from arguments used in discrete time processes

**1.3.4 Proposition** Let $(X_n)_{n\in T}$, $T = \{0, 1, 2, \dots\}$ be a (sub)martingale adapted to the filtration $(\mathscr{F}_n)_{n\in T}$ and $(H_n)_{n\in T}$ be a non-negative bounded predictable process. Then the process $(Y_n)_{n\in T}$ defined by

$$Y_0 = X_0, \quad Y_n = Y_{n-1} + H_n \cdot (X_n - X_{n-1}) \ , \quad n \geq 1$$

is also a (sub)martingale. If $\tau$ is a stopping time then the stopped process $\left(X_{n\wedge\tau}\right)_{n\in T}$ is a (sub)martingale.

***Proof***: The first result is direct consequence of the properties of conditional expectation. If $\tau$ is a stopping time, then $H_n = 1_{[\tau\geq n]} = 1 - 1_{[\tau\leq n-1]}$ satisfies the measurability condition and further yields $Y_n = X_{n\wedge\tau}$. □

**1.3.5 Remark** a) In the literature, the operation $(H, X) \longmapsto Y$ of the last proposition is written as $(H \bullet X)_{n\in T}$. It is a discrete analogue of continuous time stochastic integral.

b) If $X$ is a bounded martingale and $H$ is a predictable such that $\mathbb{E}\left[|H_n|\right] < +\infty$ for all $n$, then the transform $(H \bullet X)$ is again a martingale.

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**1.3.6 Proposition** Let $(X_n)_{n\in T}$, $T = \{0, 1, 2, \dots\}$ be a (sub)martingale adapted to the filtration $(\mathscr{F}_n)_{n\in T}$ and $\sigma \leq \tau$ be two bounded stopping times. Then

$$X_\sigma \,(\leq) = \mathbb{E}\left[X_\tau \,|\, \mathscr{F}_\sigma\right] \quad P - \text{a.s.}$$

***Proof***: We suppose that $(X_n)_n$ is a submartingale. The key point is to note that $H_n := 1_{[\tau\geq n]} - 1_{[\sigma\geq n]}$, $n \geq 1$, yields

$$(H \bullet X)_n - X_0 = X_{n\wedge\tau} - X_{n\wedge\sigma}\,, \quad n \in T$$

and the submartingale property implies $\mathbb{E}\left[(H \bullet X)_n\right] \geq \mathbb{E}\left[X_0\right]$, for all $n \in T$. So $\mathbb{E}\left[X_{n\wedge\tau}\right] \geq \mathbb{E}\left[X_{n\wedge\sigma}\right]$, for all $n$. But $\sigma, \tau$ are bounded and taking any $n > \mathrm{M} := \|\tau\|_\infty$ leads to $\mathbb{E}\left[X_\tau\right] \geq \mathbb{E}\left[X_\sigma\right]$.

We need to show $\mathbb{E}\left[X_\tau \,|\, \mathscr{F}_\sigma\right] \geq \mathrm{X}_\sigma$, $P$-a.e. Let $A \in \mathscr{F}_\sigma$ and define $\rho_A = \begin{cases} M & \text{on } A \\ +\infty & \text{on } A^{\complement} \end{cases}$. $\rho_A$ is indeed a stopping time and apply the previous equality το $\tau \leftarrow \tau \wedge \rho_A$ and $\sigma \leftarrow \sigma \wedge \rho_A$ to obtain $\mathbb{E}\left[X_{\tau\wedge\rho_A}\right] = \mathbb{E}\left[X_{\sigma\wedge\rho_A}\right]$, that is,

$$\mathbb{E}\left[1_A \cdot X_\tau + 1_{A^{\complement}} \cdot X_M\right] \geq \mathbb{E}\left[1_A \cdot X_\sigma + 1_{A^{\complement}} \cdot X_M\right]$$

But $A$ was arbitrarily chosen. This leads us to the desired result.

If $(X_n)_n$ is a martingale, then we apply all the previous to both $-X_n$ and $X_n$ to obtain equalities. □

A moment's reflection to the second part of the previous proof leads us to the

**1.3.7 Corollary** Let $(X_n)_{n\in T}$, $T = \{0, 1, 2, \dots\}$ be a submartingale adapted to the filtration $(\mathscr{F}_n)_{n\in T}$. Then, $(X_n)_n$ is a martingale if and only if for any two bounded stopping times $\sigma \leq \tau$,

$$\mathbb{E}\left[X_\tau\right] = \mathbb{E}\left[X_\sigma\right]$$

holds.

The following are generalizations to maximal inequalities.

**1.3.8 Proposition** Let $(X_1, X_2, \dots, X_N)$ be a submartingale ($T$ is finite). Then for any $\lambda > 0$.

$$\lambda \cdot \mathrm{P}\left(\left[\max_{1\leq n\leq N} X_n \geq \lambda\right]\right) \leq \mathbb{E}\left[1_{\left[\max_{1\leq n\leq N} X_n \geq \lambda\right]} \cdot X_N\right]$$

$$\lambda \cdot \mathrm{P}\left(\left[\min_{1\leq n\leq N} X_n \leq -\lambda\right]\right) \leq \mathbb{E}\left[X_N - X_1\right] - \int_{\left[\min_{1\leq n\leq N} X_n \leq -\lambda\right]} X_N \,\mathrm{d}P \leq \mathbb{E}\left[X_N^+\right] - \mathbb{E}\left[X_1\right]$$

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Further, for any $p \geq 1$,

$$\lambda^p \, \mathrm{P}\left( \max_{1 \leq n \leq N} X_n \geq \lambda \right) \leq \mathbb{E}\left[ |X_N|^p \right]$$

and for any $p > 1$,

$$\mathbb{E}\left[ |X_N|^p \right] \leq \mathbb{E}\left[ \max_{1 \leq n \leq N} |X_n|^p \right] \leq \left( \frac{p}{p-1} \right)^p \mathbb{E}\left[ |X_N|^p \right]$$

***Proof***: Put $Z = \max\limits_{1 \leq n \leq N} X_n$. Let $\tau = \min\{n \in T : X_n \geq \lambda\} \wedge N$. This is a bounded stopping time: $\tau \leq N$. Now observe that $Z < \lambda$ implies $\tau = N$. Hence

$$\begin{aligned} \mathbb{E}[X_N] \geq \mathbb{E}\left[X_\tau\right] &= \mathbb{E}\left[1_{[Z \geq \lambda]} \cdot X_\tau + 1_{[Z < \lambda]} \cdot X_\tau\right] \\ &= \mathbb{E}\left[1_{[Z \geq \lambda]} \cdot X_\tau\right] + \mathbb{E}\left[1_{[Z < \lambda]} \cdot X_N\right] \\ &= \lambda \cdot P([Z \geq \lambda]) + \mathbb{E}\left[1_{[Z < \lambda]} \cdot X_N\right] \end{aligned}$$

Subtract $\mathbb{E}\left[1_{[Z<\lambda]} \cdot X_N\right]$ from both sides and get the desired inequality.

Put $Y = \min\limits_{1 \leq n \leq N} X_n$. Let $\sigma = \min\{n \in T : X_n \geq \lambda\} \wedge N$. This is a bounded stopping time: $\sigma \leq N$. Let $M_n = \left[\min\limits_{1 \leq k \leq n} X_k \leq -\lambda\right]$, $n = 1, 2, \ldots, N$. Then $\Omega$ is the disjoint union of $[\sigma \leq \mathrm{N} - 1] = M_{N-1}$, $M_N \setminus M_{N-1}$ and $M_N^{\complement}$, further $X_\sigma \leq -\lambda$ on $M_N$. So we get

$$\begin{aligned} \mathbb{E}[X_1] \leq \mathbb{E}\left[X_\sigma\right] &= \mathbb{E}\left[1_{[\sigma \leq N-1]} X_\sigma + 1_{M_N \setminus M_{N-1}} X_\sigma + 1_{M_N^{\complement}} X_\sigma\right] \\ &= \mathbb{E}\left[1_{[\sigma \leq N-1]} X_\sigma + 1_{M_N \setminus M_{N-1}} X_N + 1_{M_N^{\complement}} X_N\right] \\ &\leq -\lambda P(M_N) + \mathbb{E}[X_N] - \mathbb{E}\left[1_{M_N} X_N\right] \end{aligned}$$

which leads us to the second inequality.

For $p \geq 1$ the result is immediate consequence of the Jensen's inequality, so that $|X_n|$, $1 \leq n \leq N$ becomes a submartingale.

For the last part, let $W = \max\limits_{1 \leq n \leq N} |X_n|$, and from $\lambda \cdot P([W \geq \lambda]) \leq E[1_{[W \geq \lambda]} \cdot |X_N|]$, we get,

$$\begin{aligned} \mathbb{E}[W^p] &= \int_0^{+\infty} p\lambda^{p-1} \, \mathrm{P}([W \geq \lambda]) \, \mathrm{d}\lambda = \int_0^{+\infty} p\lambda^{p-1} \, \mathrm{P}([W \geq \lambda]) \, \mathrm{d}\lambda \\ &\leq \int_0^{+\infty} p\lambda^{p-2} \mathbb{E}\left[1_{[W \geq \lambda]} \cdot |X_N|\right] \, \mathrm{d}\lambda = \mathbb{E}\left[ |X_N| \int_0^{+\infty} p\lambda^{p-2} \cdot 1_{[W \geq \lambda]} \, \mathrm{d}\lambda \right] \\ &= \mathbb{E}\left[ |X_N| \int_0^{W} p\lambda^{p-2} \, \mathrm{d}\lambda \right] = \frac{p}{p-1} \cdot \mathbb{E}\left[ |X_N| \cdot W^{p-1} \right] \\ &\leq \frac{p}{p-1} \mathbb{E}[W^p]^{\frac{p-1}{p}} \cdot \mathbb{E}\left[ |X_N|^p \right]^{\frac{1}{p}} \qquad \text{(Hölder inequality)} \end{aligned}$$

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and dividing by $\mathbb{E}\left[(W \wedge \eta)^p\right]^{\frac{p-1}{p}}$ (whenever nonzero) to get

$$\mathbb{E}\left[W^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|X_N|^p\right]$$

□

The last proposition can be easily generalized in right-continuous martingales and in any finite interval $[u, v] \subseteq T$. The technique is more or less standard: take an increasing sequence of finite sets of rationals to fill $[u, v] \cap \mathbb{Q}$. Then pass to limits, exploiting continuity and convergence.

**1.3.9 Proposition** Let $(X_t)_{t \in T}$ be a submartingale ($T = [0, +\infty)$) with right continuous paths. Let $I = [u, v] \subseteq T$ be an interval and $\lambda > 0$. Then

$$\lambda \cdot \mathrm{P}\left(\left[\sup_{u \leq t \leq v} X_t \geq \lambda\right]\right) \leq \mathbb{E}\left[1_{\left[\sup_{u \leq t \leq v} X_t \geq \lambda\right]} \cdot X_v\right]$$

$$\lambda \cdot \mathrm{P}\left(\left[\inf_{u \leq t \leq v} X_t \leq -\lambda\right]\right) \leq \mathbb{E}\left[X_v - X_u\right] - \int_{\left[\inf_{u \leq t \leq v} X_t \leq -\lambda\right]} X_v \, \mathrm{d}P \leq \mathbb{E}\left[X_v^+\right] - \mathbb{E}\left[X_u\right]$$

Further, for any $p \geq 1$,

$$\lambda^p \, \mathrm{P}\left(\sup_{u \leq t \leq v} X_t \geq \lambda\right) \leq \mathbb{E}\left[|X_v|^p\right]$$

and for any $p > 1$,

$$\mathbb{E}\left[|X_v|^p\right] \leq \mathbb{E}\left[\max_{u \leq t \leq v} |X_t|^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|X_v|^p\right]$$

For the following proposition, recall that given a function $f : T \longmapsto \mathbb{R}$, a finite set $F \subseteq T$ and $a < b \in \mathbb{R}$ we define an nondecreasing sequence of numbers ($+\infty$ is allowed) as follows: Let $\eta = \max F$,

$$\tau_1 = \inf\{t \in F : f(t) \geq b\} \wedge \eta$$

and for all $n \geq 1$

$$\sigma_n = \inf\{t \in F : t > \tau_n \text{ and } f(t) \leq a\} \wedge \eta$$
$$\tau_{n+1} = \inf\{t \in F : t > \sigma_n \text{ and } f(t) \geq b\} \wedge \eta$$

Then we define the number of *downcrossings* over $F$, and over $T$:

$$D_F(f; a, b) = \sup\{n \geq 1 : \sigma_n < \eta\} \vee 0$$
$$D(f; a, b) = \sup\{D_F(f; a, b) \mid \subseteq T \text{ finite}\}$$

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The convention is $\sup \varnothing = -\infty, \inf \varnothing = +\infty$.

We can have analogous definitions in case of a process $(X_t)_{t\in T}$. In this case all $\tau_i, \sigma_i$ are bounded stopping times, $D_F, D$ are random variables. Further, for $N$ big enough (say $> |F|/2$),

$$\tau_1 \leq \sigma_1 \leq \tau_2 \leq \sigma_2 \leq \ldots \leq \tau_N = \sigma_N \ldots = \eta$$

**1.3.10 Proposition** Let $(X_t)_{t\in T}$ be a submartingale. Then for any $a < b \in \mathbb{R}$,

$$\mathbb{E}\left[D(X; a < b)\right] \leq \frac{\sup_{t\in T} \mathbb{E}\left[(X_t - b)^+\right]}{b - a}$$

***Proof***: First suppose that $T$ is finite. Using the above notation, let $A_n = \left[\tau_n < \eta\right] \in \mathscr{F}_{\tau_n}$, $B_n = \left[\sigma_n < \eta\right] \in \mathscr{F}_{\sigma_n}$, for $n = 1, 2, \ldots$. Then $A_1 \supseteq B_1 \supseteq A_2 \supseteq B_2 \supseteq \ldots \supseteq A_N = B_N = \varnothing$. On $A_n$, $X_{\tau_n} \geq b$ and on $B_n$, $X_{\sigma_n} \leq a$. Applying the optional stopping on random times $\tau_n \leq \sigma_n$, we obtain $\mathbb{E}\left[X_{\tau_n} \,\middle|\, \mathscr{F}_{\sigma_n}\right] \geq X_{\sigma_n}$. Whence

$$0 \leq \int_{A_n} X_{\tau_n} - b \, \mathrm{d}P \leq \int_{A_n} X_{\sigma_n} - b \, \mathrm{d}P \leq (a - b)\,\mathrm{P}(B_n) + \int_{A_n \setminus B_n} X_{\sigma_n} - b \, \mathrm{d}P$$

Now $\sigma_n = \eta$ on $B_n^{\complement}$, and this leads to

$$(b - a)\,\mathrm{P}(B_n) \leq \int_{A_n \setminus B_n} X_\eta - b \, \mathrm{d}P \leq \int_{A_n \setminus B_n} \left(X_\eta - b\right)^+ \mathrm{d}P$$

Observe that $\mathrm{P}(B_n) = \mathrm{P}([D_F(X; a < b) \geq n])$, and that $(A_n \setminus B_n)_{n\geq 1}$ is a sequence of pairwise disjoint sets, All theses lead to

$$\begin{aligned}\mathbb{E}\left[D_F(X; a < b)\right] &= \sum_{n=1}^{\infty} \mathrm{P}(B_n) \leq \frac{1}{b-a} \sum_{n=1}^{\infty} \int_{A_n \setminus B_n} \left(X_\eta - b\right)^+ \mathrm{d}P \\ &\leq \frac{1}{b-a} \mathbb{E}\left[\left(X_\eta - b\right)^+\right] \leq \frac{1}{b-a} \sup_{\eta\in T} \mathbb{E}\left[\left(X_\eta - b\right)^+\right]\end{aligned}$$

In the case $T$ is not finite, just use the monotone convergence theorem with an a appropriate increasing sequence of finite sets $F_n \subseteq T$, such that $D_{F_n}(X; a < b) \uparrow D(X; a < b)$ pointwise. □

The number of downcrossings is important for the existence of discontinuities of second kind. When $D(f; a < b) < +\infty$ for all $a < b$ ($a, b$ may be even chosen rational), then $f$ can only have jump discontinuities. In the case of a process $(X_t)_{t\in T}$, the integrability of $D\left(X|_{[0,\eta]}; a < b\right)$ implies that it is finite almost surely, for all $\eta \in \mathbb{N}$, and all rationals $a < b \in \mathbb{Q}$. Taking intersections, we obtain the following

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**1.3.11 Proposition** Let $(X_t)_{t\in T}$ be a submartingale ($T=[0,+\infty)$). Then almost surely, it has left and right limits.

***Proof***: Let $\eta\in\mathbb{N}$. Then for any $a<b\in\mathbb{Q}$

$$\begin{aligned} U\left(X|_{[0,\eta]};a,b\right) &\leq (b-a)^{-1}\sup_{t\in[0,\eta]}\mathbb{E}\left[(X_t-b)^+\right] \\ &\leq (b-a)^{-1}\left(\sup_{t\in[0,\eta]}\mathbb{E}\left[X_t^+\right]+|b|\right) \\ &\leq (b-a)^{-1}\left(\mathbb{E}\left[X_\eta^+\right]+|b|\right)<+\infty \qquad \text{(submartingale)} \end{aligned}$$

So there is a $P$-null set $N_{\eta;a,b}$ such that $D\left(X|_{[0,\eta]};a<b\right)(\omega)<+\infty$, for all $\omega\in N_{\eta;a,b}$. Setting $N=\bigcup_{\substack{\eta\in\mathbb{N}\\ a<b\in\mathbb{Q}}} N_{\eta,a,b}$ gives us the result in $N^{\complement}$. Indeed, let $t\in T$ and $\omega\in N^{\complement}$. Let $\eta>t$. If the left limit, $\lim_{s\longrightarrow t^-}X_s(\omega)$ did not exist, then we could find two sequences $(t_n)_n,(s_n)_n$ and $a<b\in\mathbb{Q}$ such that $t_1<s_1<t_2<s_2<\ldots$, both $\lim_n t_n, s_n=t$ and $X_{t_n}(\omega)<a<b<X_{s_n}(\omega)$. This would imply that the path $[0,\eta]\ni t\longmapsto X(\omega)$ has infinitely many downcrossings between $[a,b]$ on $[0,\eta]$, which is impossible. □

**1.3.12 Proposition** Let $(X_n)_{n\in\mathbb{N}}$ be a discrete time submartingale, such that $\sup_n\mathbb{E}\left[X_n^+\right]<+\infty$. Then $X_n\longrightarrow X$ pointwise a.s. and $X$ is finite.

***Proof***: Again, its the same argument: the finiteness of downcrossings implied by the boundedness of the martingale. First note that by the submartingale property $\mathbb{E}\left[X_n\right]=\mathbb{E}\left[X_n^+\right]-\mathbb{E}\left[X_n^-\right]\geq\mathbb{E}\left[X_0\right]$ whence $\mathbb{E}\left[X_n^-\right]\leq\mathbb{E}\left[X_n^+\right]-\mathbb{E}\left[X_0\right]$, which implies that $(X_n)_n$ is $L^1$-bounded. Thus, Fatou's lemma asserts that $\mathbb{E}\left[\liminf_n X_n\right]\leq\liminf_n\mathbb{E}\left[X_n\right]<+\infty$. So $\liminf_n X_n$ is a.s. finite.

Further, whenever $D(X(\omega);a<b)<+\infty$ for all $a<b\in\mathbb{Q}$, we should have $\liminf_n X_n(\omega)=\limsup_n X_n(\omega)$ else we would count infinite downcrossings between some interval $[a,b], a<b\in\mathbb{Q}$ contained in $\liminf_n X_n(\omega)<a<b<\limsup_n X_n(\omega)$. But $D(X;a<b)<+\infty$ a.s. for any $a<b\in\mathbb{Q}$. So $\lim_n X_n$ exists a.s. and it is finite. □

In a similar fashion, we can prove

**1.3.13 Proposition** Let $(X_t)_{t\in T}$ be a submartingale ($T=[0,+\infty)$), with right continuous paths, such that $\sup_{t\in T}\mathbb{E}\left[X_t^+\right]<+\infty$. Then the limit $X=\lim_{t\longrightarrow+\infty}X_t$ exists a.s. and is finite a.s.

**1.3.14 Comment** Keep in mind that the limit random variables $X$ of the previous proposition is $\sigma\left(\bigcup_{t\in T}\mathscr{F}_t\right)$-measurable

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**1.3.15 Proposition** Let $(X_t)_{t\in T}$ be a martingale ($T=[0,+\infty)$), with right continuous paths. The following are equivalent

a) $X_\infty = \lim_{t\longrightarrow+\infty} X_t$ in the $L^1$ sense.

b) There is some $X_\infty \in L^1(\Omega;P)$ such that $X_t = \mathbb{E}\left[X_\infty \mid \mathscr{F}_t\right]$, $t\in T$.

c) $\{X_t \mid t\in T\}$ is uniformly integrable.

***Proof***: **a** $\rightarrow$ **b**): By the martingale property, for any $s>t$ $X_t = \mathbb{E}\left[X_s \mid \mathscr{F}_t\right]$. The operator $L^1(\Omega)\ni f \longmapsto E_{\mathscr{F}_t} f = \mathbb{E}\left[f \mid \mathscr{F}_t\right]$ is bounded, whence letting $s\longrightarrow+\infty$ justifies our claim.
**b** $\rightarrow$ **c**): Let $\xi>0$. We compute

$$\begin{aligned}\int_{[|X_t|\geq\xi]} |X_t|\ \mathrm{d}P &= \int_{[|X_t|\geq\xi]} \left|\mathbb{E}\left[X_\infty \mid \mathscr{F}_t\right]\right|\ \mathrm{d}P \leq \int_{[|X_t|\geq\xi]} \mathbb{E}\left[|X_\infty|\ \mid \mathscr{F}_t\right]\mathrm{d}P\\ &= \int_{[|X_t|\geq\xi]} |X_\infty|\ \mathrm{d}P\end{aligned}$$

The event $\left[|X_t|\geq\xi\right]$ can have arbitrarily small probability, for

$$\mathrm{P}\left(\left[|X_t|\geq\xi\right]\right) \leq \frac{1}{\xi}\mathbb{E}\left[|X_t|\right] \leq \frac{1}{\xi}\mathbb{E}\left[|X_\infty|\right]$$

Integrability of $X_\infty$ can make $\int_A |X_\infty|\ \mathrm{d}P$ arbitrarily close to 0, whenever $P(A)$ is small enough.
**c** $\rightarrow$ **a**): Uniform integrability implies boundedness in $L^1$, so we get that $X_\infty = \lim_{t\longrightarrow+\infty} X_t$ exists and is finite a.s. Pointwise limits of uniformly integrable families are limits in $L^1$. □

Now we can prove the continuous version of optional stopping theorem.

**1.3.16 Proposition** Let $(X_t)_{t\in T}$ be a martingale ($T=[0,+\infty)$), with right continuous paths and uniformly integrable. Then the family $\{X_\sigma \mid \sigma \text{ is a stopping time}\}$ is uniformly integrable too, and whenever $\sigma\leq\tau$ are two stopping times,

$$X_\sigma = \mathbb{E}\left[X_\tau \mid \mathscr{F}_\sigma\right] = \mathbb{E}\left[X_\infty \mid \mathscr{F}_\sigma\right]$$

$X_\infty$ is the limit of the previous proposition.

***Proof***: If $\tau$ is a stopping time, we put $X_\tau = X_\infty$ on $[\tau=+\infty]$. Let $U = \{\mathbb{E}[X_\infty \mid \mathcal{S}] \mid \mathcal{S}\subseteq\mathscr{F} \text{ is subalgebra}\}$. Then, repeating the arguments of the previous proof, we obtain the uniform integrability of $U$.

The discrete optional stopping theorem ensures that whenever $\sigma$ is a stopping time taking only finite number of values, then $X_\sigma = \mathbb{E}\left[X_\infty \mid \mathscr{F}_\sigma\right]$ ($\tau=\infty\geq\sigma$). Whence $X_\sigma\in U$.

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Now every stopping time $\sigma$ can be written as the pointwise limit of a nonincreasing sequence $(\sigma_n)_n$ of finite-valued stopping times:

$$\sigma_n = \begin{cases} +\infty & \text{whenever } \sigma \geq n \\ \dfrac{\lceil 2^n \sigma \rceil + 1}{2^n} & \text{whenever } \sigma < n \end{cases}$$

So $\sigma_n \downarrow \sigma$ pointwise and due to right continuity of the process $X_{\sigma_n} \longrightarrow X_\sigma$ pointwise, and also in $L^1(\Omega)$ (due to uniform integrability of $\left\{\mathrm{X}_{\sigma_n}\right\}_{n=1}^{\infty} \subseteq U$). We need to show $X_\sigma = \mathbb{E}\left[X_\infty \mid \mathscr{F}_\sigma\right]$. Indeed, let $A \in \mathscr{F}_\sigma$. Then $A \in \mathscr{F}_{\sigma_n}$ for every $n$, so

$$\int_A X_\sigma \, \mathrm{d}P = \lim_n \int_A \mathrm{X}_{\sigma_n} \, \mathrm{d}P = \int_A X_\infty \, \mathrm{d}P$$

which proves the desired equality and also $X_\sigma \in U$. As the stopping time $\sigma$ is arbitrary, we obtain that $\{X_\sigma \mid \sigma \text{ is a stopping time}\} \subseteq U$ is also uniformly integrable.

Finally, let $\tau \geq \sigma$ be two stopping times. Then $\mathrm{X}_\tau = \mathbb{E}\left[\mathrm{X}_\infty \mid \mathscr{F}_\tau\right]$, $\mathrm{X}_\sigma = \mathbb{E}\left[\mathrm{X}_\infty \mid \mathscr{F}_\sigma\right]$, $\mathscr{F}_\sigma \subseteq \mathscr{F}_\tau$ and we obtain

$$X_\sigma = \mathbb{E}\left[\mathrm{X}_\infty \mid \mathscr{F}_\sigma\right] = \mathbb{E}\left[\mathrm{X}_\infty \mid \mathscr{F}_{\sigma\wedge\tau}\right] = \mathbb{E}\left[\mathbb{E}\left[\mathrm{X}_\infty \mid \mathscr{F}_\tau\right] \,\middle|\, \mathscr{F}_\sigma\right] = \mathbb{E}\left[X_\tau \mid \mathscr{F}_\sigma\right]$$

which finishes the proof. □

We now pass to quite an important result which is known as Doob's decomposition of a sumbartingale. Again we start with discrete-time processes.

**1.3.17 Proposition** Let $(X_n)_{n\in T}$, $T = \{0, 1, 2, \ldots\}$ be a submartingale adapted to the filtration $\mathfrak{F} = (\mathscr{F}_n)_{n\in T}$. Then there exist unique $\mathfrak{F}$-adapted martingale $(M_n)_n$, and increasing predictable process $(A_n)_n$ such that $A_0 = 0$ and

$$X_n = M_n + A_n \,, \quad P - \text{a.s., for all } n$$

If $X$ is $L^1$-bounded or uniformly integrable, then so are $M$ and $A$.

***Proof***: Put

$$\begin{aligned} &\xi_0 = X_0 \,, \quad \xi_{n+1} = X_{n+1} - X_n \\ &m_0 = \xi_0 \,, \quad m_{n+1} = \xi_{n+1} - \mathbb{E}\left[\xi_{n+1} \mid \mathscr{F}_n\right] \,, \quad M_n = \sum_{k=0}^{n} m_k \,, \quad n \in T \\ &a_0 = 0 \,, \quad a_{n+1} = \mathbb{E}\left[\xi_{n+1} \mid \mathscr{F}_n\right] \,, \quad A_n = \sum_{k=0}^{n} a_k \,, \quad n \in T \end{aligned}$$

and observe that $\xi, m$ are $\mathfrak{F}$-adapted and so do $M$, $a$ and $A$ are predictable. Further $M$ is a martingale and $a_n \geq 0$ for all $n$ (due to the sub`martingale property of $X$) which implies that $A$ is increasing.

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If $X_n = M_n + A_n = M'_n + A'_n$ for all $n$ then $A_n - A'_n$ is a predictable martingale. But a predictable martingale must be constant.

If $X$ is $L^1$-bounded, then $\mathbb{E}\left[|A_n|\right] = \mathbb{E}\left[A_n\right] = \mathbb{E}\left[X_n - M_n\right] \leq \mathbb{E}\left[|X_n|\right] - \mathbb{E}\left[M_n\right] = \mathbb{E}\left[|X_n|\right] - \mathbb{E}\left[M_0\right] = \mathbb{E}\left[|X_n|\right] - \mathbb{E}\left[X_0\right]$, whence $A$ is bounded. And so is $M$.

If $X$ is uniformly integrable, then it is bounded and so do $A, M$. Whence the limit $A_\infty = \sup_n A_n$ exists a.s. and it is integrable. This implies that since $0 \leq A_n \leq A_\infty$ the dominated family $\{A_n\}_n$ is uniformly integrable. □

**1.3.18 Definition** Let $(\Omega, \mathscr{F}, P)$ be a probability space furnished with a filtration $\mathfrak{F} = (\mathscr{F}_t)_{t\in T}$. For $a > 0$ put

$$\mathscr{S}_a(\mathfrak{F}) = \{\tau \mid \tau \text{ is an } \mathfrak{F}\text{-stopping time such that } \mathrm{P}([\tau \leq a]) = 1\}$$
$$\mathscr{S}(\mathfrak{F}) = \{\tau \mid \tau \text{ is an } \mathfrak{F}\text{-stopping time such that } \mathrm{P}([\tau < +\infty]) = 1\}$$

An $\mathfrak{F}$-adapted, right continuous process $X = (X_t)_{t\in T}$ is said to be **of class D** if the set $\{X_\tau \mid \tau \in \mathscr{S}(\mathfrak{F})\}$ is uniformly integralbe. It is said to be **of class DL** if, for every $a > 0$ the set $\{X_\tau \mid \tau \in \mathscr{S}_a(\mathfrak{F})\}$ is uniformly integralbe.

**1.3.19 Remark** a) Clearly uniform integrable martingale $\longrightarrow$ class D $\longrightarrow$ class DL.

b) If $(X_t)_{t\in T}$ is a right continuous, non-negative submartingale then it is of class DL, for if $a > 0$ then $\|X_t\|_1 = \mathbb{E}\left[X_t\right] \leq \mathbb{E}\left[X_a\right]$ so the submartingale is uniformly integrable and so is the family

$$\{X_\tau \mid \tau \in \mathscr{S}_a(\mathfrak{F})\} \subseteq \left\{f : \Omega \longmapsto \mathbb{R} \;\middle|\; f \text{ measurable}, 0 \leq f \leq \mathbb{E}\left[X_a \mid \mathscr{F}_\tau\right],\ \tau \in \mathscr{S}_a(\mathfrak{F})\right\}$$

c) If $X$ admits a decomposition $X_t = M_t + A_t$ where $(M_t)_t$ is a Martingale and $(A_t)_t$ is a nondecreasing process, then we immediately conclude that it is of class DL.

## 1.4 Quadratic variation and brackets

In order to define the stochastic integral for continuous martingales, we shall pass through the quadratic variation. Given a continuous martingale, its square is merely a submartingale. It is well known, that a submartingale can be uniquely decomposed to a sum of a martingale and an increasing process (see [KS14, theorem 1.4.10]). This is the Doob-Meyer decomposition. The increasing process, in the case of continuous martingales is nothing more than the quadratic variation of the martingale and plays important role to the definition of the stochastic integral. Here we follow the approach of [RY13, ch IV.1].

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In the following we suppose that $T = [0, +\infty)$, that the filtration $\mathfrak{F} = (\mathscr{F}_t)_{t\in T}$ has its $\mathscr{F}_0$ $P$-complete.

**1.4.1 Definition** A process $A = (A_t)_{t\in T}$ is said to be **nondecreasing**, if it is $\mathfrak{F}$-adapted, almost all of its paths are right continuous and nondecreasing. We write $A \in \mathscr{A}(\Omega, \mathscr{F}; \mathfrak{F})$ or if it is clear from the context, $A \in \mathscr{A}$. Similarly we define the space of processes of **bounded variation** as $\mathscr{BV} = \mathscr{A} - \mathscr{A}$.

Let $X = (X_t)_{t\in T}$, $T = [0, +\infty)$, be a progressively measurable, $\mathfrak{F}$-adapted process, and $A = (A_t)_t \in \mathscr{BV}$. The following Lebesgue-Stieltjes integral defines a new process

$$(X \bullet A)_t(\omega) = \begin{cases} \int_0^t X_s(\omega)\, \mathrm{d}A_s(\omega) & \text{whenever the path } A_{(\cdot)}(\omega) \text{ is of bounded variation} \\ 0 & \text{else} \end{cases}$$

**1.4.2 Remark** a) An increasing process $(A_t)_{t\in T}$ defines a kernel: For every fixed $\omega \in \Omega$, there is a unique locally finite measure $\mu_\omega$ defined on $\mathscr{B}([0,t])$ such that $\mu_\omega((c,d]) = A_d(\omega) - A_c(\omega)$, for all $0 \le c < d \le t$.

Since for any $0 \le c < d \le t \in T$ the function $\Omega \ni \omega \longmapsto \mu_\omega((c,d])$ is $\mathscr{F}_t$-measurable and the family of all sets $(c,d]$ generates $\mathscr{B}([0,t])$, we get that the mapping $\Omega \times \mathscr{B}([0,t]) \ni (\omega, A) \longmapsto \mu(\omega, A) = \mu_\omega(A)$ is indeed a kernel from $(\Omega, \mathscr{F}_t)$ to $([0,t], \mathscr{B}([0,t]))$. Now the progressive measurability of the process $X$ implies that the mapping $\Omega \ni \omega \longmapsto \int_{[0,t]} X_s(\omega)\, \mathrm{d}\mu_\omega(s) = \int_0^t X_s(\omega)\, \mathrm{d}A_s(\omega)$ is $\mathscr{F}_t$-measurable.

b) Note that $(X \bullet A)$ is again of bounded variation.

c) The integral $(X \bullet A)$ is defined pathwise

Unfortunately, the class $\mathscr{BV}$ is way too small to contain useful integrators as we see in the following proposition.

**1.4.3 Proposition** Let $M = (M_t)_{t\in T}$ be a continuous $\mathfrak{F}$-martingale. If $M \in \mathscr{BV}$ then the martingale is constant, that is $M_t = M_0$ for all $t \in T$.

***Proof***: We can suppose that, without loss of generality, $M_0 = 0$. We first treat the case that $M$ is of bounded variation uniformly on $\Omega$, say by a constant $K > 0$. Let $0 \le t \in T$ be fixed and $\Pi = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ be any partition. Then

$$\begin{aligned} \mathbb{E}\left[M_t^2\right] &= \mathbb{E}\left[\sum_{k=1}^n M_{t_n}^2 - M_{t_{n-1}}^2\right] \\ &= \mathbb{E}\left[\sum_{k=1}^n (M_{t_n} - M_{t_{n-1}})^2\right] \qquad (M \text{ is a martingale}) \end{aligned}$$

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$$\leq \mathbb{E}\left[\sup_{1\leq k\leq n}\left|M_{t_n}-M_{t_{n-1}}\right|\cdot\sum_{k=1}^{n}\left|M_{t_n}-M_{t_{n-1}}\right|\right]$$
$$\leq K\cdot\mathbb{E}\left[\sup_{1\leq k\leq n}\left|M_{t_n}-M_{t_{n-1}}\right|\right]$$

But, due to continuity of the paths, for (almost) every single $\omega\in\Omega$, $\sup_{1\leq k\leq n}\left|M_{t_n}-M_{t_{n-1}}\right|\longrightarrow 0$ as $|\Pi|\longrightarrow 0$, while the supremum is bounded by $K$. So we apply the dominated convergence theorem for some sequence of partitions $(\Pi_n)_n$ such that $|\Pi_n|\longrightarrow 0$ to get that $\mathbb{E}\left[M_t^2\right]=0$.

In the general case let $V_t$ be the (almost surely) finite variation of $M$, which is again an $\mathfrak{F}$-adapted process and let $\tau_n=\inf\{s\in T\mid V_s\geq n\}$, for every $n$. Clearly $(\tau_n)_n$ is a nondecreasing sequence of stopping times such that $\tau_n\longrightarrow+\infty$. Apply the previous argument for the martingale $\left(M_{t\wedge\tau_n}\right)_t$. So $\mathbb{E}\left[M_{t\wedge\tau_n}^2\right]=0$ for all $n$. Use the monotone convergence to finish the proof. □

**1.4.4 Remark** In the previous proposition, the process $M$ is *modification* of the constant $M_0$. But since it is continuous, it turns out that $M$ is indistinguishable from $M_0$. In what follows, whenever we mention uniqueness of quadratic variation, it will be about indistinguishable processes.

So nontrivial continuous martingales cant be integrators in the sense of pathwise integration. We need to construct another kind of integration.

Let $\Pi=\{0=t_0<t_1<\ldots\}$ be an increasing sequence such that for every $t\geq 0$, the set $\Pi_t=(\Pi\cup\{t\})\cap[0,t]$ is a partition of $[0,t]$. We call such $\Pi$ a partition of $[0,+\infty)$. Let also $t(\Pi)=\max\{i\mid t_i\leq t\}$. Given two stochastic processes $X=(X_t)_t$, $Y=(Y_t)_t$ we define a) The $p$-variation of $X$ over $\Pi$ to be a new stochastic process

$$V_t^p(X,\Pi)=\sum_{k=1}^{t(\Pi)}\left|X_{t_k}-X_{t_{k-1}}\right|^p+\left|X_t-X_{t_{t(\Pi)}}\right|^p$$
$$=\sum_{k=1}^{\infty}\left|X_{t\wedge t_k}-X_{t\wedge t_{k-1}}\right|^p \qquad \text{(equivalent definition)}$$

and b) The covariation of $X,Y$ over $\Pi$ to be the new stochastic process

$$C_t(X,Y;\Pi)=\sum_{k=1}^{t(\Pi)}(X_{t_k}-X_{t_{k-1}})\cdot(Y_{t_k}-Y_{t_{k-1}})+(X_t-X_{t_{t(\Pi)}})\cdot(Y_t-Y_{t_{t(\Pi)}})$$
$$=\sum_{k=1}^{\infty}(X_{t\wedge t_k}-X_{t\wedge t_{k-1}})\cdot(Y_{t\wedge t_k}-Y_{t\wedge t_{k-1}}) \qquad \text{(equivalent definition)}$$

Keep in mind that we have some kind of triangle inequality for $p$-variaton:

$$V^p(X+Y,\Pi)\leq 2^{p-1}(V^p(X,\Pi)+V^p(Y,\Pi)),\quad \text{on } \Omega$$

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and thats due to $|x+y|^p \le 2^{p-1}(|x|^p + |y|^p)$ which holds for any real numbers $x, y$ and $p \ge 1$. Further $V^2(X;\Pi) = C(X,X;\Pi)$. Covariation is bilinear, symmetric and $C(X,Y;\Pi)^2 \le V^2(X;\Pi)\cdot V^2(Y;\Pi)$.

**1.4.5 Proposition** Let $M = (M_t)_{t\in T}$, $T = [0,+\infty)$, be a bounded continuous, $\mathfrak{F}$-adapted martingale. Then there is a unique continuous, nondecreasing process $\langle\langle M, M\rangle\rangle \in \mathscr{A}(\mathfrak{F})$ vanishing at 0, such that $M^2 - \langle\langle M, M\rangle\rangle$ is a martingale. The process $\langle\langle M, M\rangle\rangle$ is the uniform limit in probability of $V^2(M,\Pi)$ on every bounded interval as $|\Pi| \longrightarrow 0$:

$$\forall \varepsilon > 0, \forall a > 0 \left( \lim_{\substack{|\Pi|\longrightarrow 0 \\ \Pi \text{ partition of } [0,a]}} \mathrm{P}\left( \left[ \sup_{0\le t\le a} \left| V_t^2(\Pi) - \langle\langle M, M\rangle\rangle_t \right| \ge \varepsilon \right] \right) = 0 \right)$$

The process $\langle\langle M, M\rangle\rangle$ is called **quadratic variation** of $M$.

***Proof***: First, if $\langle\langle M, M\rangle\rangle$ exists, it should be unique. Due to the previous proposition, if $A_1, A_2$ are stochastic processes, continuous, nondecreasing such that $M^2 - A_1, M^2 - A_2$ are martingales, then so are $A_1 - A_2 \in \mathscr{BV}(\mathfrak{F})$. So $A_1 = A_2$.

Let $\Pi = \{0 = t_0 < t_1 < \ldots\}$ be a partition of $[0,+\infty)$. and let $s < t$ and integers $i, j$ sucht that $t_i \le s < t_{i+1} < \ldots < t_j \le t < t_{j+1}$. From now on we will write $V_t^2(\Pi)$ instead of $V_t^2(M,\Pi)$. So,

$$\begin{aligned}
\mathbb{E}\left[V_t^2(\Pi) - V_s^2(\Pi) \,\middle|\, \mathscr{F}_s\right] &= \mathbb{E}\left[ \sum_{k=i+1}^{j} (M_{t_k} - M_{t_{k-1}})^2 + (M_t - M_{t_j})^2 - (M_s - M_{t_i})^2 \,\middle|\, \mathscr{F}_s \right] \\
&= \mathbb{E}\left[ (M_{t_{i+1}} - M_{t_i})^2 + \sum_{k=i+2}^{j} (M_{t_k}^2 - M_{t_{k-1}}^2) + M_t - M_{t_j}^2 - (M_s - M_{t_i})^2 \,\middle|\, \mathscr{F}_s \right] \\
&= \mathbb{E}\left[ (M_{t_{i+1}} - M_{t_i})^2 + M_t^2 - M_{t_{i+1}}^2 - (M_s - M_{t_i})^2 \,\middle|\, \mathscr{F}_s \right] \\
&= \mathbb{E}\left[ (M_{t_{i+1}}^2 - 2M_s M_{t_i} + M_{t_i}^2) + M_t^2 - M_{t_{i+1}}^2 - (M_s^2 - 2M_s M_{t_i} + M_{t_i}^2) \,\middle|\, \mathscr{F}_s \right] \\
&= \mathbb{E}\left[ M_t^2 - M_s^2 \,\middle|\, \mathscr{F}_s \right]
\end{aligned}$$

that is, $\left(M_t^2 - V_t^2(M,\Pi)\right)_{t\in T}$ is an $\mathfrak{F}$-martingale.

Now let two partitions $\Pi_1, \Pi_2$ of $[0,+\infty)$ and fix some $0 < a \in \Pi_1 \cap \Pi_2$. Define $X = V^2(\Pi_2) - V^2(\Pi_1) = \left(M^2 - V^2(\Pi_1)\right) - \left(M^2 - V^2(\Pi_2)\right)$, which is a martingale and apply the last argument to $X$, but this time with the partition $\Sigma = \Pi_2 \cup \Pi_1$. We obtain

$$\begin{aligned}
\mathrm{Var}\left[X_a\right] = \mathbb{E}\left[X_a^2\right] &= \mathbb{E}\left[(V_a^2(\Pi_2) - V_a^2(\Pi_1))^2\right] = \mathbb{E}\left[V_a^2(X,\Sigma)\right] \\
&\le 2\left(\mathbb{E}\left[V_a^2(V^2(\Pi_1),\Sigma)\right] + \mathbb{E}\left[V_a^2(V^2(\Pi_2),\Sigma)\right]\right)
\end{aligned}$$

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Let $\Pi_1 = \{0 = t_0 \leq t_1 < t_2 < \ldots\}$, $\Sigma = \Pi_1 \cup \Pi_2 = \{0 = s_0 \leq s_1 < s_2 < \ldots\}$, fix any $s_i \leq a \in \Pi_1 \cap \Pi_2$ and $t_{j_i} \in \Pi_1$ be the rightmost point in $\Pi_1$ such that $t_{j_i} \leq s_i < s_{i+1} \leq t_{j_i+1} \leq a$. Then

$$\begin{aligned} V^2_{s_{i+1}}(\Pi_1) - V^2_{s_i}(\Pi_1) &= (M_{s_{i+1}} - M_{t_{j_i}})^2 - (M_{s_i} - M_{t_{j_i}})^2 \\ &= (M_{s_{i+1}} + M_{s_i} - 2M_{t_{j_i}}) \cdot (M_{s_{i+1}} - M_{s_i}) \end{aligned}$$

so since $a \in \Sigma$,

$$\begin{aligned} V^2_a(V^2(\Pi_1), \Sigma) &= \sum_{k=1}^{a(\Sigma)} (V^2_{s_k}(\Pi_1) - V^2_{s_{k-1}}(\Pi_1))^2 \\ &\leq \sum_{k=1}^{a(\Sigma)} (M_{s_{i+1}} + M_{s_i} - 2M_{t_{j_i}})^2 (M_{s_{i+1}} - M_{s_i})^2 \\ &\leq \max_{1 \leq i \leq a(\Sigma)} \left| M_{s_{i+1}} + M_{s_i} - 2M_{t_{j_i}} \right|^2 \cdot \sum_{k=1}^{a(\Sigma)} (M_{s_{i+1}} - M_{s_i})^2 \\ &= \max_{1 \leq i \leq a(\Sigma)} \left| M_{s_{i+1}} + M_{s_i} - 2M_{t_{j_i}} \right|^2 \cdot V^2_a(\Sigma) \end{aligned}$$

and by Cauchy-Schwarz's inequality,

$$\mathbb{E}\left[V^2_a(V^2(\Pi_1), \Sigma)\right] \leq \mathbb{E}\left[\max_{1 \leq i \leq a(\Sigma)} \left| M_{s_{i+1}} + M_{s_i} - 2M_{t_{j_i}} \right|^4\right]^{\frac{1}{2}} \cdot \mathbb{E}\left[(V^2_a(\Sigma))^2\right]^{\frac{1}{2}}$$

The first factor on the right converges to 0 as $|\Pi_1| \longrightarrow 0$, since $M$ has continuous (hence uniformly continuous on $[0, a]$) paths and is uniformly bounded on $\Omega$ (if $|M| \leq C$ then the factor is bounded by $16C^2$). The rightmost factor is also bounded, but even more, its bound does not depend on the chosen partition. Indeed,

$$\begin{aligned} (V^2_a(\Pi_1))^2 &= \left( \sum_{k=1}^{n} (M_{t_k} - M_{t_{k-1}})^2 \right) \\ &= \sum_{k=1}^{n} (M_{t_k} - M_{t_{k-1}})^4 + 2 \sum_{k=1}^{n} \sum_{k'=1}^{k-1} (M_{t_k} - M_{t_{k-1}})^2 (M_{t_{k'}} - M_{t_{k'-1}})^2 \\ &= \sum_{k=1}^{n} (M_{t_k} - M_{t_{k-1}})^4 + 2 \sum_{k=1}^{n} (M_{t_k} - M_{t_{k-1}})^2 (V^2_a(\Pi_1) - V^2_{t_k}(\Pi_1)) \\ &= \sum_{k=1}^{n} (M_{t_k} - M_{t_{k-1}})^4 + 2 \sum_{k=1}^{n} (V^2_{t_k}(\Pi_1) - V^2_{t_{k-1}}(\Pi_1))(V^2_a(\Pi_1) - V^2_{t_k}(\Pi_1)) \end{aligned}$$

We can exploit the martingale property of $M^2 - V^2(\Pi_1)$ to obtain

$$\mathbb{E}\left[V^2_a(\Pi_1) - V^2_{t_k}(\Pi_1) \,\middle|\, \mathcal{F}_{t_k}\right] = \mathbb{E}\left[M^2_a - M^2_{t_k} \,\middle|\, \mathcal{F}_{t_k}\right] = \mathbb{E}\left[(M_a - M_{t_k})^2 \,\middle|\, \mathcal{F}_{t_k}\right]$$

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and passing to the expectation,

$$\begin{aligned}
\mathbb{E}\left[(V_a^2(\Pi_1))^2\right] &= \mathbb{E}\left[\sum_{k=1}^{n}(M_{t_k}-M_{t_{k-1}})^4 + 2\sum_{k=1}^{n}(V_{t_k}^2(\Pi_1)-V_{t_{k-1}}^2(\Pi_1))(V_a^2(\Pi_1)-V_{t_k}^2(\Pi_1))\right] \\
&= \mathbb{E}\left[\sum_{k=1}^{n}(M_{t_k}-M_{t_{k-1}})^4 + 2\sum_{k=1}^{n}(V_{t_k}^2(\Pi_1)-V_{t_{k-1}}^2(\Pi_1))(M_a-M_{t_k})^2\right] \\
&\leq \mathbb{E}\left[\left(2\max_{k\leq a(\Pi_1)}\left|M_a-M_{t_k}\right|^2 + \max_{k\leq a(\Pi_1)}\left|M_{t_k}-M_{t_{k-1}}\right|^2\right)\cdot V_a^2(\Pi_1)\right] \\
&\leq \mathbb{E}\left[\left(2\cdot(2C)^2+(2C)^2\right)\cdot V_a^2(\Pi_1)\right] \\
&= 12C^2\mathbb{E}\left[V_a^2(\Pi_1)\right] = 12C^2\mathbb{E}\left[M_a^2-M_0^2\right] \leq 12C^4
\end{aligned}$$

Thus we have proven, that for every $\varepsilon > 0$, there is some $\delta > 0$ that whenever $\Pi_1, \Pi_2$ are partitions of $[0, +\infty)$, both containing $a$,

$$|\Pi_1|, |\Pi_2| < \delta \longrightarrow \operatorname{Var}\left[V_a^2(\Pi_1) - V_a^2(\Pi_2)\right] < \varepsilon$$

so there is a unique random variable $\langle\langle M, M\rangle\rangle_a$ that is the $L^2$ limit of $V_a^2(\Pi)$ as $|\Pi| \longrightarrow 0$ (for partitions of $[0, +\infty)$ containing $a$, or equivalently, for partitions of $[0, a]$). Existence of the limit $\langle\langle M, M\rangle\rangle_a$ implies existence of a whole process in $[0, a]$: Using Doob's maximal inequality, applied on the martingale $X = V^2(\Pi_1) - V^2(\Pi_2)$

$$\mathbb{E}\left[\sup_{0\leq t\leq a}(V_t^2(\Pi_1)-V_t^2(\Pi_2))^2\right] \leq 4\operatorname{Var}\left[V_a^2(\Pi_1)-V_a^2(\Pi_2)\right]$$

This has two implications: First, for every $t$, $V_t^2(\Pi)$ converges to a limit in $L^2$ (and thus, in probability), say $\langle\langle M, M\rangle\rangle_t$, –which must be appropriately chosen from its equivalence class. Second, can find a sequence of such partitions $(\Pi_n)_{n\in\mathbb{N}}$ such that $\Pi_n \subseteq \Pi_{n+1}$ for all $n$, $\lim_n |\Pi_n| = 0$ (which in turn implies that $\bigcup_{n\in\mathbb{N}}\Pi_n$ is dense in $[0, a]$) and for all $k \in \mathbb{N}$, $\operatorname{Var}\left[V_a^2(\Pi_{n+k}) - V_a^2(\Pi_n)\right] < 2^{-n}$, thus, by monotone convergence and Markov inequality

$$\varepsilon^2\cdot \mathrm{P}\left(\left[\sup_{k\in\mathbb{N}}\sup_{0\leq t\leq a}\left|V_t^2(\Pi_{n+k})-V_t^2(\Pi_n)\right|\geq\varepsilon\right]\right) \leq \mathbb{E}\left[\sup_{k\in\mathbb{N}}\sup_{0\leq t\leq a}(V_t^2(\Pi_{n+k})-V_t^2(\Pi_n))^2\right] < 2^{-n}$$

Whence we obtain that $V_t^2(\Pi_n) \longrightarrow \langle\langle M, M\rangle\rangle_t$ a.s. (and hence in probability) on $\Omega$ and uniformly on $[0, a]$. So $\langle\langle M, M\rangle\rangle$ has a.s. continuous paths.

Now since

$$\sup_{0\leq t\leq a}\left|V_t^2(\Pi_1) - \langle\langle M, M\rangle\rangle_t\right| \leq \sup_{0\leq t\leq a}\left|V_t^2(\Pi_n) - \langle\langle M, M\rangle\rangle_t\right| + \sup_{0\leq t\leq a}\left|V_t^2(\Pi_1) - V_t^2(\Pi_n)\right|$$

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we obtain the desired limit in probability.

We need to show $\langle\langle M, M\rangle\rangle$ is a.s. increasing, and it suffices to show it is increasing on $\bigcup_{n\in\mathbb{N}} \Pi_n$. If $s < t \in \bigcup_{n\in\mathbb{N}} \Pi_n$ then $\exists n_0 \in \mathbb{N} : s, t \in \Pi_{n_0}$ and so $s, t \in \Pi_n$ for all $n \geq n_0$. But then $V_s^2(\Pi_n) \leq V_t^2(\Pi_n)$, for all $n \geq n_0$, and passing to the limit as $n \longrightarrow +\infty$, $\langle\langle M, M\rangle\rangle_s \leq \langle\langle M, M\rangle\rangle_t$. Density of the partitions leads us to the desired monotonicity of $\langle\langle M, M\rangle\rangle$.

To this end, we need to observe that $M^2 - \langle\langle M, M\rangle\rangle$ is a martingale. But this is the case, since $M_t^2 - V_t^2(\Pi_n) \xrightarrow{L^2} M_t^2 - \langle\langle M, M\rangle\rangle_t$, and thus in $L^1$ and conditional expectation is continuous on $L^1$. □

Using the uniqueness of the quadratic variation and that the stopped process is still a (bounded) martingale, we obtain:

**1.4.6 Corollary** With the assumptions of the previous proposition, and for every stopping time $\tau$,

$$\langle\langle M^\tau, M^\tau\rangle\rangle = \langle\langle M, M\rangle\rangle^\tau$$

The class of martingales having quadratic variation as in the previous propositions are quite limited. We need to extend this class. We introduce the following definitions

**1.4.7 Definition** Let $(\Omega, \mathcal{F}, P)$ be a probability space furnished with a filtration $\mathfrak{F} = (\mathcal{F}_t)_{t\in T}$. Let $X = (X_t)_{t\in T}$ be an $\mathfrak{F}$-adapted process.

a) A stopping time $\tau$ **reduces** $X$ if $\left(1_{[\tau>0]} \cdot X_{\tau\wedge t}\right)_{t\in T}$ is a uniformly integrable martingale.

b) The process $X$ is called **local martingale** if it is right continuous and there exists a sequence of stopping times $(\tau_n)_{n=1}^{\infty}$ such that

LM1. The sequence $(\tau_n)_n$ is a.s. nondecreasing and $\lim_n \tau_n = +\infty$ a.s.

LM2. For every $n$, $\tau_n$ reduces $X$

**1.4.8 Remark** a) As one sees, $1_{[\tau=0]} \cdot X_0$ is irrelevant when talking about local martingales; on the other hand $1_{[\tau>0]}X_0$ should be integrable. Generally this is always the case when, e.g. $X_0$ is a.s. constant. Reduction hence, consists of uniform integrability of $\left(X_{\tau\wedge\tau} - X_0\right)_{t\in T}$ and –whenever $X_0$ is nonconstant, integrability of $1_{[\tau>0]}X_0$.

b) If we take the sequence of stopping times in the definition of local martingale, to be bounded for each $n$ (that is, $\operatorname{ess\,sup}_\Omega \tau_n < +\infty$ a.s. for every $n$), by replacing $\tau_n \leftarrow n \wedge \tau_n$ or –in the case of continuous processes, $\tau_n \leftarrow \tau_n \wedge \sigma_n$, where $\sigma_n = \inf\{t \in T : |X_t| \geq n\}$, then the uniform integrability can be taken for granted.

c) A right continuous martingale is always a martingale: take $\tau_n = n$, $n \geq 1$.

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**1.4.9 Proposition** A local martingale $X$ is a martingale if and only if is of class DL

***Proof***: First, a right continuous martingale is always of class DL as $\mathrm{X}_\tau = \mathbb{E}\left[X_a \,|\, \mathscr{F}_\tau\right]$, for every $a$-bounded stopping time, and this family is uniformly integrable.

For the converse, let $X$ be a local martingale which is of class DL. Fix some $s < t \in T$. Then by the martingale property of $X_{\tau_n\wedge(\cdot)}$, $X_{\tau_n\wedge s} = \mathbb{E}\left[\mathrm{X}_{\tau_n\wedge t} \,\middle|\, \mathscr{F}_s\right]$, for all $n \geq 1$. But the family $\left\{X_{\tau_n\wedge t} \,\middle|\, n \geq 1\right\} \subseteq \{X_\sigma \,|\, \sigma \text{ stopping time } \leq t\}$ is uniformly integrable by the DL property. Further, for every single $u \leq t$, $X_{\tau_n\wedge u} \longrightarrow X_u$ a.s. and we can use the extended dominated convergence theorem to obtain that

$$X_s = \lim_n X_{\tau_n\wedge s} = \lim_n \mathbb{E}\left[X_{\tau_n\wedge t} \,\middle|\, \mathscr{F}_s\right] = \mathbb{E}\left[\lim_n X_{\tau_n\wedge t} \,\middle|\, \mathscr{F}_s\right] = \mathbb{E}\left[X_t \,|\, \mathscr{F}_s\right]$$

which proves our claim. □

Now we can prove the existence of quadratic variation for a larger class of martingales

**1.4.10 Proposition** Let $M = (M_t)_{t\in T}$, $T = [0, +\infty)$, be continuous, $\mathfrak{F}$-adapted local martingale. Then there is a unique continuous, nondecreasing process $\langle\!\langle M, M\rangle\!\rangle \in \mathscr{A}(\mathfrak{F})$ vanishing at 0, such that $M^2 - \langle\!\langle M, M\rangle\!\rangle$ is a continuous local martingale. The process $\langle\!\langle M, M\rangle\!\rangle$ is the uniform limit in probability of $V^2(M, \Pi)$ on every bounded interval as $|\Pi| \longrightarrow 0$:

$$\forall \varepsilon > 0, \forall a > 0 \left( \lim_{\substack{|\Pi| \longrightarrow 0 \\ \Pi \text{ partition of } [0,a]}} \mathrm{P}\left( \left[ \sup_{0\leq t\leq a} \left| V_t^2(M, \Pi) - \langle\!\langle M, M\rangle\!\rangle_t \right| \geq \varepsilon \right] \right) = 0 \right)$$

***Proof***: Let $(\tau_n)_{n\geq 1}$ be a nondecreasing sequence of finite stopping times such that $\lim_n \tau_n = +\infty$ a.s. and for every $n$, the process $X_n = 1_{[\tau_n>0]} \cdot M_{\tau_n\wedge(\cdot)}$ is a bounded martingale. So for every $n$, there is a unique continuous process $A_n \in \mathscr{A}(\mathfrak{F})$ such that $X_n^2 - A_n$ is a martingale. Observe now that $(X_{n+1} - A_{n+1})_{\tau_n\wedge t} \cdot 1_{[\tau_n>0]} = 1_{[\tau_n>0]} \cdot M^2_{\tau_n\wedge t} - 1_{[\tau_n>0]} \cdot (A_{n+1})_{\tau_n\wedge t}$ is again a martingale, and uniqueness of quadratic variation leads to $(A_n)_t = 1_{[\tau_n>0]}(A_{n+1})_{\tau_n\wedge t}$ a.s. on $\Omega$ for all $t \in T$. So we can define unambiguously (the following sums are finite for every single pair $(t, \omega) \in T \times \Omega$)

$$\langle\!\langle M, M\rangle\!\rangle = \sum_{n=1}^{\infty} 1_{(\tau_{n-1},\tau_n]} \cdot A_n$$

where we put $\tau_0 = 0$ identically and $(\tau_{n-1}, \tau_n] = \{(t, \omega) \in T \times \Omega \mid \tau_{n-1}(\omega) < t \leq \tau_n(\omega)\}$, for all $n \geq 1$. We can easily verify that $\langle\!\langle M, M\rangle\!\rangle_{\tau_n\wedge(\cdot)} = A_n$, for all $n \geq 1$.

To prove the limit, observe that, since $\tau_n \uparrow +\infty$ a.s., $0 = \mathrm{P}\left([\sup_n \tau_n \leq a]\right) = \mathrm{P}\left(\bigcap_n [\tau_n \leq a]\right) = \inf_n \mathrm{P}\left([\tau_n \leq a]\right)$, so, given $\delta > 0$ find some $n$ such that $\mathrm{P}\left([\tau_n \leq a]\right) < \frac{\delta}{2}$. Now,

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since the stopped process $M_{\tau_n \wedge t}$ is a martingale, we can apply Doob's inequality to obtain

$$C \cdot \mathrm{P}\left(\left[\sup_{0\le s\le a} \mid M_{\tau_n\wedge s} \mid \ge C\right]\right) \le \mathbb{E}\left[\mid M_{\tau_n\wedge a} \mid\right]$$

With any $C > \frac{2}{\delta} \cdot \mathbb{E}\left[\mid M_{\tau_n\wedge a} \mid\right]$, put $\sigma = \inf\left\{t \ge 0 \,\middle|\, \left|M_{\tau_n\wedge t}\right| \ge C\right\}$, so for the stopping time $\tau = \tau_n \wedge \sigma$,

$$\mathrm{P}([\tau \le a]) \le \mathrm{P}([\sigma \le a]) + \mathrm{P}([\tau_n \le a]) \le \mathrm{P}\left(\left[\sup_{0\le s\le a} \mid M_{\tau_n\wedge s} \mid \ge C\right]\right) + \mathrm{P}([\tau_n \le a]) < \delta$$

while $|M_{\tau\wedge t}| \le C$ a.s. in $\Omega$, for all $t \in T$. Thus

$$\begin{aligned}
\mathrm{P}\left(\left[\sup_{0\le s\le a} \left|V_s^2(M,\Pi) - \langle\!\langle M, M\rangle\!\rangle_s\right| \ge \varepsilon\right]\right) \le & \\
&\le \mathrm{P}([\tau \le a]) + \mathrm{P}\left(\left[\sup_{0\le s\le a} \left|V_s^2(M,\Pi) - \langle\!\langle M, M\rangle\!\rangle_s\right| \ge \varepsilon\right] \cap [\tau > a]\right) \\
&\le \frac{\delta}{2} + \mathrm{P}\left(\left[\sup_{0\le s\le a} \left|V_s^2(M^\sigma,\Pi) - \langle\!\langle M^\sigma, M^\sigma\rangle\!\rangle_s\right| \ge \varepsilon\right]\right)
\end{aligned}$$

with the rightmost term converging to 0, due to our main result about quadratic variation of continuous bounded martingales. □

**1.4.11 Proposition** Let $M, N$ be continuous local $\mathfrak{F}$-martingales. Then there exists a unique continuous process $\langle\!\langle M, N\rangle\!\rangle \in \mathscr{BV}(\mathfrak{F})$, vanishing at 0, such that $MN - \langle\!\langle M, N\rangle\!\rangle$ is a local $\mathfrak{F}$-martingale. The process $\langle\!\langle M, N\rangle\!\rangle$ is the uniform limit in probability of $C(M, N; \Pi)$ on every bounded interval as $|\Pi| \longrightarrow 0$:

$$\forall \varepsilon > 0, \forall a > 0 \left(\lim_{\substack{|\Pi|\longrightarrow 0 \\ \Pi \text{ partition of } [0,a]}} \mathrm{P}\left(\left[\sup_{0\le t\le a} |C_t(M,N;\Pi) - \langle\!\langle M, N\rangle\!\rangle_t| \ge \varepsilon\right]\right) = 0\right)$$

The process $\langle\!\langle M, N\rangle\!\rangle$ is called **bracket** of $M$ and $N$.

***Proof***: Just use the parallelogram law to define

$$\langle\!\langle M, N\rangle\!\rangle = \frac{1}{4}\left[\langle\!\langle M+N, M+N\rangle\!\rangle - \langle\!\langle M-N, M-N\rangle\!\rangle\right]$$

and and the parallelogram law to substitute

$$C(M, N; \Pi) = \frac{1}{4}\left[C(M+N, M+N; \Pi) - C(M-N, M-N; \Pi)\right]$$

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Then taking limits in probability yields the result. □

**1.4.12 Proposition** With the notation and the hypotheses of the previous proposition and for any stopping time $\tau$

$$\langle\langle M, N\rangle\rangle^{\tau} = \langle\langle \mathrm{M}^{\tau}, N^{\tau}\rangle\rangle = \langle\langle M^{\tau}, N\rangle\rangle = \langle\langle M, N^{\tau}\rangle\rangle$$

***Proof***: Let $\Pi$ be a partition of $[0, +\infty)$. Observe that, since the difference $M_{\tau\wedge t\wedge t_k} - M_{\tau\wedge t\wedge t_{k-1}} = 0$ whenever $\tau \wedge t \leq t_{k-1}$,

$$\begin{aligned} C_t(M^{\tau}, N; \Pi) &= \sum_{k=1}^{\infty} (M_{\tau\wedge t\wedge t_k} - M_{\tau\wedge t\wedge t_{k-1}}) \cdot (Y_{t\wedge t_k} - N_{t\wedge t_{k-1}}) \\ &= \sum_{k=1}^{\infty} (M_{\tau\wedge t\wedge t_k} - M_{\tau\wedge t\wedge t_{k-1}}) \cdot (Y_{\tau\wedge t\wedge t_k} - N_{\tau\wedge t\wedge t_{k-1}}) \\ &= C_t(M^{\tau}, N^{\tau}; \Pi) = C_{\tau\wedge t}(M, N; \Pi) \end{aligned}$$

This yields the desired equalities. □

**1.4.13 Proposition** If $M$ is a continuous local martingale then

$$\langle\langle M, M\rangle\rangle = 0 \leftrightarrow M_t = M_0 \quad P\text{-a.s. for all } t \in T$$

***Proof***: If $M$ is bounded, then (it has second order moments) for every $t \in T$, $\mathbb{E}\left[(M_t - M_0)^2\right] = \mathbb{E}\left[M_t^2 - M_0^2\right] = \mathbb{E}\left[\langle\langle M, M\rangle\rangle_t - 0\right]$. If $M$ is a local martingale and $\tau$ is a stopping time such that $M^{\tau}$ is bounded martingale, then we obtain that $M^{\tau} = M_0$ since $\langle\langle M^{\tau}, M^{\tau}\rangle\rangle = \langle\langle M, M\rangle\rangle^{\tau}$. We can find an nondecreasing sequence of stopping times $(\tau_n)_n$ such that $\tau_n \uparrow +\infty$ $P$-a.s., $M^{\tau_n}$ is bounded local martingale. Then $M^{\tau_n} = M_0$. Passing to the limit finishes the proof. □

**1.4.14 Proposition** Let $M, N$ be two continuous local martingales and $H, K$ be two measurable processes then, for every $t \in [0, +\infty]$

$$\int_0^t |H_s|\,|K_s|\;|\mathrm{d}\langle\langle M, N\rangle\rangle|_s \leq \left(\int_0^t H_s^2\,\mathrm{d}\langle\langle M, M\rangle\rangle_s\right)^{\frac{1}{2}} \cdot \left(\int_0^t K_s^2\,\mathrm{d}\langle\langle N, N\rangle\rangle_s\right)^{\frac{1}{2}},\ P\text{-a.s.}$$

*Notation: The symbol* $|\mathrm{d}\langle\langle M, N\rangle\rangle|$ *abusively stands for the measure kernel implied by the (increasing) variation* $t \longmapsto V_0^t\langle\langle M, N\rangle\rangle$ *of the process* $\langle\langle M, N\rangle\rangle$.

***Proof***: Let $\langle\langle M, N\rangle\rangle_u^v = \langle\langle M, N\rangle\rangle_v - \langle\langle M, N\rangle\rangle_u$ for any $u \leq v$. Then, for every rational $r$, there exists a $P$-null set $N_r \in \mathscr{F}$ such that $\langle\langle M \pm rN, M \pm rN\rangle\rangle_u^v \geq 0$. Expanding the bracket we obtain that everywhere on $\Omega$ except the neglible set $\bigcup_{r\in Q} N_r$,

$$0 \leq r^2\langle\langle N, N\rangle\rangle_u^v \pm 2r\langle\langle M, N\rangle\rangle_u^v + \langle\langle M, M\rangle\rangle_u^v$$

thus,

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$$0 \le r^2 \langle\!\langle N, N \rangle\!\rangle_u^v - 2r \left| \langle\!\langle M, N \rangle\!\rangle \right|_u^v + \langle\!\langle M, M \rangle\!\rangle_u^v$$

and this inequality holds also for every real $r$ due to continuity. Whence,

$$\left| \langle\!\langle M, N \rangle\!\rangle_u^v \right| \le \left( \langle\!\langle M, M \rangle\!\rangle_u^v \right)^{\frac{1}{2}} \left( \langle\!\langle N, N \rangle\!\rangle_u^v \right)^{\frac{1}{2}} \ P\text{-a.s.}$$

We can even take $u, v$ rationals and exploit continuity of the bracket processes to obtain that the last inequality holds a.s. for every $u \le v$. If $H, K$ are simple processes, defined on a common partition $\Pi = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ of $[0, t]$, that is, for $K_0, H_0, K_1, H_1, \ldots, K_n, H_n$ $\mathcal{F}$-measurable random variables,

$$H_t = 1_{\{0\}} H_0 + \sum_{i=1}^{n} 1_{(t_{i-1}, t_i]} H_i, \ K_t = 1_{\{0\}} K_0 + \sum_{i=1}^{n} 1_{(t_{i-1}, t_i]} K_i$$

then

$$\begin{aligned}
\int_0^t H_s K_s \, \mathrm{d} \langle\!\langle M, N \rangle\!\rangle_s &= \sum_{i=1}^{n} H_i K_i \langle\!\langle M, N \rangle\!\rangle_{t_{i-1}}^{t_i} \le \sum_{i=1}^{n} |H_i| \, |K_i| \left| \langle\!\langle M, N \rangle\!\rangle_{t_{i-1}}^{t_i} \right| \\
&\le \sum_{i=1}^{n} |H_i| \, |K_i| \left( \langle\!\langle M, M \rangle\!\rangle_{t_{i-1}}^{t_i} \right)^{\frac{1}{2}} \left( \langle\!\langle N, N \rangle\!\rangle_{t_{i-1}}^{t_i} \right)^{\frac{1}{2}} \\
&\le \left( \sum_{i=1}^{n} |H_i|^2 \langle\!\langle M, M \rangle\!\rangle_{t_{i-1}}^{t_i} \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} |K_i|^2 \langle\!\langle N, N \rangle\!\rangle_{t_{i-1}}^{t_i} \right)^{\frac{1}{2}} \\
&= \left( \int_0^t |H_s|^2 \ \mathrm{d} \langle\!\langle M, M \rangle\!\rangle_s \right)^{\frac{1}{2}} \left( \int_0^t |K_s|^2 \ \mathrm{d} \langle\!\langle N, N \rangle\!\rangle_s \right)^{\frac{1}{2}}
\end{aligned}$$

It's a standard density argument that the latter inequality is also valid for continuous bounded measurable processes, and again by molification, valid for measurable processes such that the right integrals are finite. Eventually the inequality is valid even for $H, K$ such that the right side is infinite.

Now take $G(\cdot, \omega) = \dfrac{\mathrm{d} \langle\!\langle M, N \rangle\!\rangle (\omega)}{\left| \mathrm{d} \langle\!\langle M, N \rangle\!\rangle \right| (\omega)}$ by the Radon-Nikodym theorem, so that for allmost every $\omega$, $|G(\cdot, \omega)| = 1$ holds almost everywhere in $[0, t]$ and $\int_{[0,t]} HKG \, \mathrm{d} \langle\!\langle M, N \rangle\!\rangle = \int_{[0,t]} HK \ \left| \mathrm{d} \langle\!\langle M, N \rangle\!\rangle \right|$; then replace $H \leftarrow HG \cdot \operatorname{sgn} H$, $K \leftarrow K \operatorname{sgn} K$ to obtain the desired inequality. □

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**1.4.15 Corollary** Let $M, N$ be two continuous local martingales and $H, K$ be two measurable processes then, for every $t \in [0, +\infty]$, and $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\mathbb{E}\left[\int_0^t |H_s K_s|\ |\mathrm{d}\langle\langle M, N\rangle\rangle|_s\right] \leq \mathbb{E}\left[\left(\int_0^t H_s^2\ \mathrm{d}\langle\langle M, M\rangle\rangle_s\right)^{\frac{p}{2}}\right]^{\frac{1}{p}} \cdot \mathbb{E}\left[\left(\int_0^t K_s^2\ \mathrm{d}\langle\langle N, N\rangle\rangle_s\right)^{\frac{p'}{2}}\right]^{\frac{1}{p'}}$$

To finish this section, we proceed now to define the most generic integrators in the class of continuous martingales:

**1.4.16 Definition** Let $(\Omega, \mathscr{F}, P)$ be a probability space, equipped with filtration $\mathfrak{F} = (\mathscr{F}_t)_{t\in T}$, $T = [0, +\infty)$. A **continuous semimartingale** is an $\mathfrak{F}$-adapted stochastic process $X$ that is the sum of a continuous local $\mathfrak{F}$-martingale $M$ and a bounded variation process $A \in \mathscr{BV}(\mathfrak{F})$: $X = M + A$.

**1.4.17 Remark** Clearly, the decomposition $X = M + A$ is unique. We want to stress that, the decomposition depends on the filtration.

We shall see many reasons for the introduction of this class of processes, but we may already observe that their definition recalls the decomposition of many physical systems into a signal (the finite variation process) and a noise (the local martingale).

## 1.5 Time changes

In this section we make a quick introduction to time change of a martingale. We shall use these results to provide a representation of a continuous martingale as a time-changed Brownian motion. We follow the approach of [RY13, ch. V.1].

For the following let us summarize a few facts about the generalized inverse of an nondecreasing function $f : \overline{\mathbb{R}} \longmapsto \overline{\mathbb{R}}$: First define,

$$f^{\vee} : \overline{\mathbb{R}} \longmapsto \overline{\mathbb{R}} : y \longmapsto f^{\vee}(y) = \inf\left\{x \in \overline{\mathbb{R}} : f(x) > y\right\}$$
$$f^{\wedge} : \overline{\mathbb{R}} \longmapsto \overline{\mathbb{R}} : y \longmapsto f^{\wedge}(y) = \inf\left\{x \in \overline{\mathbb{R}} : f(x) \geq y\right\}$$

Then $f^{\wedge} \leq f^{\vee}$, both $f^{\vee}, f^{\wedge}$ are nondecreasing functions, $f^{\vee}$ is right continuous with left limits, $f^{\wedge}$ is left continuous with right limits and $\lim_{t\uparrow x^-} f^{\vee}(t) = f^{\wedge}(x)$. $f^{\vee}$ is of interest for us: For

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every $x, y \in \overline{\mathbb{R}}$,

$$f(x^-) > y \longleftrightarrow x > f^\vee(y)$$
$$f(x^+) < y \longleftrightarrow x < f^\vee(y^-)$$
$$f^\vee(f(x)) \geq x$$
$$f(f^\vee(y)^+) \geq y$$

and

$$f \text{ is maximal right continuous} \longleftrightarrow (f^\vee)^\vee = f$$
$$f \text{ strictly increasing} \longleftrightarrow f^\vee \text{ continuous}$$
$$f \text{ continuous} \longleftrightarrow f^\vee \text{ strictly increasing}$$

Finally, $f^\vee$ is constant and equal to $x$ on an open interval $J \subseteq \overline{\mathbb{R}}$, if and only if $J \subseteq (f(x^-), f(x^+))$. And reversely, $f$ is constant and equal to $y$ on an open interval $I \subseteq \overline{\mathbb{R}}$, if and only if $I \subseteq (f^\vee(y^-), f^\vee(y^+))$.

If $f : [0, +\infty] \longmapsto [0, +\infty]$, let $\overline{f} = 1_{[-\infty,0)} f(0) + 1_{[0,+\infty]} f$ and put $f^\vee(y) = 0 \vee \overline{f}^\vee(y)$, $y \geq 0$. Then, $f^\vee$ satisfies all the above properties restricted on $[0, +\infty]$.

**1.5.1 Proposition** Let $g : \mathbb{R} \longmapsto \mathbb{R}$ be a right continuous function of bounded variation, $I \subseteq \mathbb{R}$ be an interval, $\varphi : I \longmapsto \mathbb{R}$ be non-decreasing, right continuous function. We suppose that for every $s \in \mathbb{R}$, $g$ is constant on $[\varphi(s^-), \varphi(s)]$. Let $f : \mathbb{R} \longmapsto \mathbb{R}$ be a borel function. Then for any $a < b \in I$,

$$\int_{\varphi(a)}^{\varphi(b)} f(u)\, \mathrm{d}g(u) = \int_a^b f(\varphi(s))\, \mathrm{d}g(\varphi(s))$$

***Proof***: First suppose that $g$ is nondecreasing. Extent $\varphi$ and $g$ in a right continuous fashion on $\overline{\mathbb{R}}$.

If $y \in \mathbb{R}$ then using the first two properties of $\varphi^\vee$, we obtain

$$\varphi(\varphi^\vee(y^-)^-) = \varphi(\varphi^\vee(y)^-) \leq y \leq \varphi(\varphi^\vee(y^-)) \leq \varphi(\varphi^\vee(y))$$

But $g$ is constant on $[\varphi(\varphi^\vee(y)^-), \varphi(\varphi^\vee(y))]$, whence

$$g(\varphi(\varphi^\vee(y^-)^-)) = g(\varphi(\varphi^\vee(y)^-)) = g(y) = g(\varphi(\varphi^\vee(y^-))) = g(\varphi(\varphi^\vee(y)))$$

Put $\mu, \nu$ for the measures defined by $\mu((c, d]) = g(c) - g(d)$ and $\nu((a, b]) = g(\varphi(b)) - g(\varphi(a))$, $a < b \in I, c < d \in \mathbb{R}$, respectively. Let $c < d \in \mathbb{R}$. From the properties of $\varphi^\vee$, for any $x \in \mathbb{R}$, $c \leq \varphi(x) < d \longleftrightarrow \varphi^\vee(c^-) \leq x < \varphi^\vee(d^-)$. Thus, if $c, d$ are continuity points of $g$, we obtain

$$(\varphi^* \nu)([c, d)) = \nu\left(\varphi^{-1}([c, d))\right) = \nu([\varphi^\vee(c^-), \varphi^\vee(d^-)))$$

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$$
\begin{aligned}
&= (g \circ \varphi)(\varphi^{\vee}(d^{-})^{-}) - (g \circ \varphi)(\varphi^{\vee}(c^{-})^{-}) \\
&= \sup_{x<\varphi^{\vee}(d^{-})} g(\varphi(x)) - \sup_{x<\varphi^{\vee}(c^{-})} g(\varphi(x)) \\
&= g\left(\varphi(\varphi^{\vee}(d^{-})^{-})\right) - g\left(\varphi(\varphi^{\vee}(c^{-})^{-})\right) \\
&= g(d) - g(c) = \mu((c,d])
\end{aligned}
$$

Hence $\varphi^{*}\nu = \mu$ on $\mathscr{K} = \{(c,d] \mid c < d : g \text{ is continuous at } c, d\}$ which is a $\pi$-system. And as $g$ is continuous everywhere but a countable set, a fortiori, continuous in a dense set, $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{K}) \subseteq [\varphi^{*}\nu = \mu] \subseteq \mathscr{B}(\mathbb{R})$ proving that $\varphi^{*}\nu = \mu$.

Now, by change of measures for the Lebesgue integral, we obtain that for any borel function $f$, $f$ is $\varphi^{*}\nu$ integrable if and only if $f \circ \varphi$ is $\nu$-integrable, and in that case for any borel set $A \subseteq I$,

$$\int_{\varphi(A)} f(y)\, \mathrm{d}\mu(y) = \int_{\varphi(I)} f(y)\, \mathrm{d}\varphi^{*}\nu(y) = \int_{A} f(\varphi(x))\, \mathrm{d}\nu(x)$$

Choose $A = [a,b]$ and observe that $\mu = \mathrm{d}g$ and $\nu = \mathrm{d}g \circ \varphi$.

In the case $g$ is of bounded variation, just pick a decomposition: $g = g_1 - g_2$ with $g_1, g_2$ increasing, right continuous and constant on the jumps of $\varphi$. □

**1.5.2 Proposition** Let $g\colon \mathbb{R} \longmapsto \mathbb{R}$ be a continuous function of bounded variation, $I \subseteq \mathbb{R}$ be an interval, $\varphi\colon I \longmapsto \mathbb{R}$ be non-decreasing, right continuous function. Let $f\colon \mathrm{I} \longmapsto \mathbb{R}$ be a borel function. Then for any $a < b \in I$,

$$\int_{\varphi(a)}^{\varphi(b)} f(\varphi^{\vee}(y))\, \mathrm{d}g(y) = \int_{a}^{b} f(s)\, \mathrm{d}g(\varphi(s))$$

***Proof***: Put $\mu, \nu$ for the measures defined by $\mu((c,d]) = g(c) - g(d)$ and $\nu((a,b]) = g(\varphi(b)) - g(\varphi(a))$, $a < b \in I, c < d \in \mathbb{R}$. Then by the properties of $\varphi^{\vee}$,

$$a \le \varphi^{\vee}(x) < b \longleftrightarrow \varphi(a^{-}) \le x < \varphi(b^{-}), a < b \in \mathbb{R}$$

and so

$$(\varphi^{\vee})^{*}\mu([a,b)) = \mu\left((\varphi^{\vee})^{-1}([a,b))\right) = \mu\left([\varphi(a^{-}), \varphi(b^{-}))\right) = g(\varphi(b^{-})) - g(\varphi(a^{-})) = \nu([a,b)) \quad \square$$

□

**1.5.3 Proposition** Let $\mathfrak{F} = (\mathscr{F}_t)_{t\in T}$ be a right continuous filtration, and $A \in \mathscr{A}(\mathfrak{F})$ which is right continuous and $A_0 \ge 0$ $P$-a.s. Define for any $s \ge 0$,

$$C_s = \inf\{t \ge 0 : A_t > s\}$$
$$C_{s^{-}} = \inf\{t \ge 0 : A_t \ge s\}$$

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Then $(C_s)_{s\geq 0}$ is a nondecreasing, right-continuous family of $\mathfrak{F}$-stopping times. Further for any $t \in T$, $A_t = \inf\{s \geq 0 : C_s > t\}$

The filtration $\mathfrak{F}_C = \left(\mathscr{F}_{C_s}\right)_{s\in T}$ is a right continuous filtration and for every $t \in T$, $A_t$ is a $\mathfrak{F}_C$-stopping time and for any stopping time $\tau$ of $\mathfrak{F}$, the random time $C_\tau^\vee := A_\tau$ is an $\mathfrak{F}_C$-stopping time.

***Proof***: Every $C_s$ is a stopping time, for $\left[C_s < t\right] = \bigcup_{q<t,q\in\mathbb{Q}} \left[A_q > s\right] \in \mathscr{F}_t$ and $\mathfrak{F}$ is right-continuous.

We show that $\mathfrak{F}_C$ is a right continuous filtration. First observe that the monotonicity of $C$ transfers to the monotonicity of $\mathfrak{F}_C$. Now let $s \in T$ and $u_n = s + {}^1\!/_n$, $n \geq 1$. By right continuity of $C_s$, we obtain $C_s = \inf C_{u_n}$ and $\mathscr{F}_{C_s} = \bigcap_{n=1}^{\infty} \mathscr{F}_{C_{u_n}} = \bigcap_{u<s} \mathscr{F}_{C_u}$.

Let now $t, s \in T$. We observe that $[A_t < s] = \bigcup_{q<s,q\in\mathbb{Q}} [C_q < t] \in \mathscr{F}_{C_s}$. Right continuity of $\mathfrak{F}_C$ allows us to conclude that $A_t$ is a stopping time of $\mathfrak{F}_C$.

Let $s \geq 0$, then $\left[A_\tau < s\right] = \left[\tau < C_{s^-}\right] = \bigcup_{n=1}^{\infty} \left[\tau < C_{s-{}^1\!/_n}\right] \in \mathscr{F}_{C_s}$. Right continuity of $\mathfrak{F}_C$ concludes the claim that $A_\tau$ is a $\mathfrak{F}_C$ stopping time. □

**1.5.4 Definition** A stochastic process $C = (C_s)_{s\geq 0}$ is called **time-change** if $P$-a.s. $s \longmapsto C_s$ is nondecreasing right continuous and for every $s \geq 0$, $C_s$ is a $\mathfrak{F}$-stopping time.

If $C$ is a time-change, then the process $X = (X_t)_{t\in T}$ is called $C$**-continuous**, if, $P$-a.s. $X$ is constant on $\left[C_{s^-}, C_s\right]$, for every $s \geq 0$. We put $X_C = \left(X_{C_s}\right)_{s\geq 0}$ for the time changed process $X$.

**1.5.5 Proposition** If $K$ is progressively measurable, $C$ is a time change, then $K_C$ is $\mathfrak{F}_C$-progressively measurable. If $X \in BV(\mathfrak{F})$ is a nonnegative $C$-continuous process, then $X_C \in \mathscr{BV}(\mathfrak{F}_C)$ and

$$\int_{C_0}^{C_s} K_t \, \mathrm{d}X_t = \int_{C_0}^{C_s} 1_{[0,+\infty)}(t) \cdot K_t \, \mathrm{d}X_t = \int_0^s 1_{[C_u<+\infty]} \cdot K_{C_u} \, \mathrm{d}X_{C_u}$$

***Proof***: This is an application to the variable change formula for the Lebesgue-Stieltjes integral. We just keep track of the infinities, whence we put the indicator. The reader can verify the details. □

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# 2. Brownian Motion

In this chapter we make a quick introduction to the Brownian motion. We shall not present any constructions of the Brownian motion; we refer to [KS14, ch. 2] for this. Our development follows [KS14, ch. 2], [RY13, ch. I] and [Σπη04, ch. 2].

## 2.1 Definition and basic properties

**2.1.1 Definition** Let $(\Omega, \mathcal{F}, P)$ be a probability space, $T = [0, +\infty)$ and $\mathfrak{F} = (\mathcal{F}_t)_{t\in T}$ be a filtration in $\Omega$. An a.s. continuous, $\mathbb{R}$-valued ($\mathbb{R}^d$-valued, $d$ positive integer), $\mathfrak{F}$-adapted process $B = (B_t)_{t\in T}$ such that

i. $B_0 = 0$ a.s.
ii. For every $s < t \in T$ the increment $B_t - B_s$ is independent of $\mathcal{F}_s$
iii. For every $s < t \in T$ the increment $B_t - B_s \sim \mathrm{N}(0, t-s)$, ie normally distributed with zero mean and variance equal to $t - s$ (or in the case $d > 1$, $B_t - B_s$ follows the multivariate normal law $\sim \mathrm{N}(0, (t-s)\mathbb{I}_d)$) is called **standard one-dimensional** (**standard $d$-dimensional**) Brownian motion.

**2.1.2 Proposition** Let $(B_t)_{t\in T}$ be a standard one dimensional Brownian motion along with its filtration $(\mathcal{F}_t)_{t\in T}$. Then the following hold:

i. For any $0 < t_1 < \ldots < t_n < +\infty$, the random variables $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent
ii. $\mathrm{Cov}(B_s, B_t) = s \wedge t$
iii. For any $0 < t_1 < \ldots < t_n < +\infty$, the random vector $\left(B_{t_1}, B_{t_2}, \ldots, B_{t_n}\right)$ is normally distributed $\mathrm{N}(0, \Sigma)$ with covariance matrix $\Sigma = \left(t_i \wedge t_j\right)_{i,j=1}^{n}$.
iv. All $(B_t)_t$, $\left(B_t^2 - t\right)_t$, $\left(\exp(\lambda B_t - \lambda^2/2t)\right)_t$ are Martingales adapted to $\mathfrak{F}$.
v. If $a \geq 0$, then the process $U_t = B_{t+a} - B_a$ is again a Brownian motion relative to its natural filtration, and independent to $\mathcal{F}_a$.
vi. If $\tau$ is a stopping time with $\mathrm{P}([\tau < +\infty]) = 1$, then the process $V_t = B_{t+\tau} - B_\tau$ is again a Brownian motion relative to its natural filtration, and independent to $\mathcal{F}_\tau$.

***Proof***: i. For any $1 \leq k \leq n$ the r.v. $B_{t_k} - B_{t_{k-1}}$ is independent of $\mathcal{F}_{t_{k-1}}$, a fortiory independent of $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_{k-1}} - B_{t_{k-2}}$.

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ii. Let $s < t$. Then $\mathbb{E}\left[B_s B_t\right] = \mathbb{E}\left[B_s(B_t - B_s) + B_s^2\right] = \mathbb{E}\left[B_s\right]\mathbb{E}\left[B_t - B_s\right] + \mathbb{E}\left[B_s^2\right] = s$ and $\operatorname{Cov}(B_s, B_t) = \mathbb{E}\left[B_s B_t\right] - \mathbb{E}\left[B_s\right]\mathbb{E}\left[B_t\right] = s$.

iii. Put $L_n \in \mathsf{GL}_n(\mathbb{R})$ the linear transformation acting on the standard basis $(\mathbf{e}_i)_{i=1}^n$ of $\mathbb{R}^n$ as follows: $L_n(\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_k) = \mathbf{e}_k$, $1 \leq k \leq n$, and observe that

$$\left(B_{t_1}, B_{t_2}, \ldots, B_{t_n}\right) = L_n\left(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}\right)$$

iv. Every $B_t$, $t \in T$ has moments of any order being normally distributed ($\mathbb{E}\left[|B_t|^n\right] = t\mathbb{E}\left[|Z|^n\right]$, where $Z \sim \mathrm{N}(0,1)$ and the integral $\int_0^\infty x^n \exp(-x^2/2)\,\mathrm{d}x$ converges, for the exponential dominates over the polynomial $x^n$). Further for any $s < t$,

$$\mathbb{E}\left[B_t \mid \mathscr{F}_s\right] = \mathbb{E}\left[B_t - B_s \mid \mathscr{F}_s\right] + \mathbb{E}\left[B_s \mid \mathscr{F}_s\right] = \mathbb{E}\left[B_t - B_s\right] + B_s$$

and

$$\begin{aligned}\mathbb{E}\left[B_t^2 - t \,\middle|\, \mathscr{F}_s\right] &= \mathbb{E}\left[(B_t - B_s)^2 - B_s^2 + 2B_s B_t \,\middle|\, \mathscr{F}_s\right] - t \\ &= \mathbb{E}\left[(B_t - B_s)^2\right] - B_s^2 + 2B_s^2 - t = t - s + B_s^2 - t = B_s^2 - s\end{aligned}$$

As for the last, it is a part of theory of exponential martingales; see remark 4.3.3. One can rigorously prove that it is indeed a martingale only using moment generating functions of $B_t$; see [Σπη04, ex. II.4.4]

v. Every $U_t$ is integrable, and every path is continuous. Further $U_0 = B_a - B_a = 0$ a.s. Let $s < t$. We get $U_t - U_s = B_{t+a} - B_{s+a} \sim \mathrm{N}(0, t-s)$ and furhter for any $0 \leq s_1 < s_2 < \ldots < s_n = s$ the random variable $B_{t+a} - B_{s+a}$ is independent to the algebra $\mathscr{F}_{s+a} \supseteq \mathscr{F}_s \cup \sigma\left(B_{s_1+a}, B_{s_2+a}, \ldots, B_{s_n+a}\right)$.

vi. First we suppose that the stopping time $\tau$ takes countable values, that is, there is an at most countable set $F \subseteq T$, and a function $A\colon F \longmapsto \mathscr{F} : t \longmapsto A(t) = A_t$ such that $A_t \cap A_s = \varnothing$ whenever $t \neq s \in F$, $\forall t \in T\ \forall s \in F(s \leq t \longrightarrow A_s \in \mathscr{F}_t)$ and $\tau = \sum_{s \in F} s \cdot 1_{A_s}$. Let now $t_1 < t_2 < \ldots < t_n \in T$, $C \in \mathscr{F}_\tau$ and $H_1, H_2, \ldots, H_n \in \mathscr{B}(\mathbb{R})$. We have

$$\begin{aligned}\mathrm{P}\left(C \cap \bigcap_{i=1}^n [V_{t_i} \in H_i]\right) &= \mathrm{P}\left(\bigcup_{s \in F}\left[C \cap [\tau = s] \cap \bigcap_{i=1}^n [V_{t_i} \in H_i]\right]\right) \\ &= \sum_{s \in F} \mathrm{P}\left(C \cap [\tau = s] \cap \bigcap_{i=1}^n [B_{t_i+s} - B_s \in H_i]\right) \\ &\qquad (C \cap [\tau = s] \in \mathscr{F}_s \text{ and } B_{t_i+s} - B_s \text{ independent of } \mathscr{F}_s) \\ &= \sum_{s \in F} \mathrm{P}\left(C \cap [\tau = s]\right) \cdot \mathrm{P}\left(\bigcap_{i=1}^n [B_{t_i+s} - B_s \in H_i]\right)\end{aligned}$$

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$$= \mathrm{P}\left(\bigcap_{i=1}^{n}[B_{t_i} \in H_i]\right) \cdot \sum_{s\in F} \mathrm{P}\left(C \cap [\tau = s]\right)$$
$$= \mathrm{P}\left(\bigcap_{i=1}^{n}[B_{t_i} \in H_i]\right) \cdot \mathrm{P}\left(C\right)$$

When $C = \Omega$ we obtain $\mathrm{P}\left(\bigcap_{i=1}^{n}[V_{t_i} \in H_i]\right) = \mathrm{P}\left(\bigcap_{i=1}^{n}[B_{t_i} \in H_i]\right)$ and substituting,

$$\mathrm{P}\left(C \cap \bigcap_{i=1}^{n}[V_{t_i} \in H_i]\right) = \mathrm{P}\left(\bigcap_{i=1}^{n}[V_{t_i} \in H_i]\right) \cdot \mathrm{P}\left(C\right) = \mathrm{P}\left(\bigcap_{i=1}^{n}[B_{t_i} \in H_i]\right) \cdot \mathrm{P}\left(C\right) \qquad (*)$$

The latter equality holds for all $C \in \mathcal{F}_\tau$ and every tuple $t_1 < t_2 < \ldots < t_n \in T$ whenever $\tau$ takes at most countable values. For the general case, for every $n \in \mathbb{N}$ we define $\tau_n = 2^{-n} \cdot (\lceil 2^n \sigma \rceil + 1)$ and observe that it is a decreasing sequence of stopping times such that $\tau_n \downarrow \tau$ pointwise. Further we define the sequence of processes: $V_t^{(n)} := B_{t+\tau_n} - B_{\tau_n}$, $n \in \mathbb{N}, t \in T$, and exploiting the continuity of the paths of $(B_t)_t$ we obtain $\lim_n V_t^{(n)} = V_t$ pointwise. Whence $(*)$ holds for any stopping time, for all $C \in \mathcal{F}_\tau$ and every tuple $t_1 < t_2 < \ldots < t_n \in T$, leads us to a) $(V_t)_t$ is independent of $\mathcal{F}_\tau$ and b) it is equivalent to $(B_t)_t$. As a result $(V_t)_{t\in T}$ is a Brownian motion relative to its own filtration and independent of $\mathcal{F}_\tau$. □

**2.1.3 Comment** a) Note that the definition of the standard Brownian motion comes along with its filtration $\mathfrak{F}$. Many times the filtration will be the default one: $\mathcal{F}_t = \sigma(X_s \,|\, 0 \le s \le t)$. But there will be times this filtration will not work and we 'll need strictly finer ones.

b) A continuous process $(W_t)_t \ge T$ having independent normally distributed increments (in the sense of (iii) of definition 2.1.1) is a Brownian motion relative to its natural filtration. Similarly, a continuous process $(W_t)_t$ which is equivalent to a Brownian motion, is again a Brownian motion. Hence, all Brownian motions are equivalent.

c) If $(B_t)_t$ is a Brownian motion relative to $\mathfrak{F}$, then so is $(cB_{t/c^2})_t$ for any $c \in \mathbb{R} \setminus \{0\}$.

d) We have proven that $\left(B_t^2 - t\right)_t$ is a martingale. This leads us to conclude that $\langle\langle B, B\rangle\rangle_t = t$, for all $t$.

e) The multidimensional Brownian motion is essentially $d$ independent standard one-dimensional Brownian motions. We will see later when a local continuous $d$-dimensional martingale can be a $d$-dimensional Brownian motion.

In order to prove the Markov property of the Brownian motion, we need a special case of a disintegration tool, which is interesting on its own:

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**2.1.4 Lemma** Let $(\Omega, \mathscr{F}, P)$ be a probability space, $\mathscr{A} \subseteq \mathscr{F}$ be a σ-algebra, $g: \mathbb{R} \times \mathbb{R} \longmapsto \mathbb{R}$ be a borel function, and $X, Y$ be random varialbes such that $Y$ is $\mathscr{A}$-measurable and $X$ is independent of $\mathscr{A}$. We suppose that $g(X, Y)$ is integrable and define $\mathbb{R} \ni y \longmapsto \varphi_g(y) = \mathbb{E}[g(X, y)]$. Then

$$\mathbb{E}[g(X, Y) \mid \mathscr{A}] = \varphi_g(Y)\,, \quad P\text{-a.s.}$$

So $\mathbb{E}[g(X, Y) \mid \mathscr{A}]$ is $Y$-measurable and $\mathbb{E}[g(X, Y) \mid \mathscr{A}] = \mathbb{E}[g(X, Y) \mid Y]$ $P$-a.s.

***Proof***: It is more or less standard: Let

$$\mathscr{C} = \left\{C \in \mathscr{B}(\mathbb{R}^2) : \mathbb{E}[g(X, Y) \mid \mathscr{A}] = \varphi_g(Y) \text{ with } g = 1_C\right\}$$

$\mathscr{C}$ is a Dynkin system: $\mathbb{R} \times \mathbb{R} \in \mathscr{C}$ trivially, whenever $C \in \mathscr{C}$ then $C^{\complement} \in \mathscr{C}$ for putting $g = 1_{C^{\complement}} = 1 - 1_C$ makes the equality valid too. Now let $(C_n)_{n=1}^{\infty}$ be an increasing sequence of sets in $\mathscr{C}$, and write $C = \bigcup_{n\geq 1} C_n$, $g_n = 1_{C_n}$, $\varphi_n = \varphi_{g_n}$ for every $n$ and $g = 1_C$, $\varphi_g(y) = \mathbb{E}\left[1_C(X, y)\right] = \mathrm{P}([X \in C^y])$. Then $\mathbb{E}\left[g_n(X, Y) \mid \mathscr{A}\right] = \varphi_n(Y)$ a.s. for all $n$ and $g_n \uparrow g$, $\varphi_n \uparrow \varphi_g$ pointwise. Using the monotone convergence theorem we obtain

$$\mathbb{E}[g(X, Y) \mid \mathscr{A}] = \sup_{n\geq 1} \mathbb{E}\left[g_n(X, Y) \mid \mathscr{A}\right] = \sup_{n\geq 1} \varphi_n(Y) = \varphi(Y)\,, \quad P\text{-a.s.}$$

Observe now that the family $\mathscr{C}_0 = \{U \times V \mid U, V \in \mathscr{B}(\mathbb{R})\}$, which is closed to finite intersections, is contained in $\mathscr{C}$. Indeed, if $C = U \times V \in \mathscr{C}_0$, then $\varphi_{1_{U\times V}}(y) = 1_V(y) \cdot \mathbb{E}\left[1_{[X\in U]}\right]$ so

$$\begin{aligned}
\varphi_{1_{U\times V}}(Y) &= 1_{[Y\in V]} \cdot \mathbb{E}\left[1_{[X\in U]}\right] \\
&= 1_{[Y\in V]} \cdot \mathbb{E}\left[1_{[X\in U]} \,\middle|\, \mathscr{A}\right] && (X \text{ is independent of } \mathscr{A}) \\
&= \mathbb{E}\left[1_{[Y\in V]} \cdot 1_{[X\in U]} \,\middle|\, \mathscr{A}\right] && (Y \text{ is } \mathscr{A}\text{-measurable}) \\
&= \mathbb{E}\left[1_{U\times V}(X, Y) \mid \mathscr{A}\right]
\end{aligned}$$

Now we can apply the Dynkin π-λ theorem and conclude that $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{C}_0) \subseteq \mathscr{C} \subseteq \mathscr{B}(\mathbb{R})$. So the desired equality holds for any indicator function of a borel set, hence for any simple function and hence for their monotone limits, ie, for every borel measurable function.

To this end, observe that $\varphi_g(Y)$ is $Y$-measurable, so $\mathbb{E}[g(X, Y) \mid Y] = \mathbb{E}[\mathbb{E}[g(X, Y) \mid \mathscr{A}] \mid Y] = \mathbb{E}\left[\varphi_g(Y) \,\middle|\, Y\right] = \varphi_g(Y)$. □

**2.1.5 Corollary** Let $(B_t)_t$ be a standard Brownian motion relative το the filtration $\mathfrak{F}$. Then for any finite stopping time $\tau$ (that is, $\mathrm{P}([\tau = +\infty]) = 0$) and any $C \in \mathscr{B}(\mathbb{R})$,

$$\mathrm{P}\left([B_{\tau+t} \in C] \mid \mathscr{F}_\tau\right) = \mathrm{P}\left([B_{\tau+t} \in C] \mid B_\tau\right) = \varphi(B_\tau)\,, \quad P - \text{a.s.}$$

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where $\varphi(y) = \mathrm{P}([B_t + y \in C])\,, y \in \mathbb{R}$. This is known as the strong Markov Property of the Brownian Motion.

***Proof***: Use the preceeding lemma with $g(x,y) = 1_C(x+y)$ , $X = B_{\tau+t} - B_\tau$ and $Y = B_\tau$. As $B_t$ and $B_{t+\tau} - B_\tau$ have common distributions and with the notation used before,

$$\begin{aligned}\varphi(y) = \varphi_g(y) &= \mathbb{E}\left[1_C(B_{\tau+t} - B_\tau + y)\right] = \mathbb{E}\left[1_{C-y}(B_{\tau+t} - B_\tau)\right] \\ &= E[1_{C-y}(B_t)] = \mathrm{P}([B_t + y \in C])\end{aligned}$$

which finishes the proof. □

The following proposition is not needed, for we have already seen that $\langle\!\langle B, B \rangle\!\rangle_t = t$. But this proof is too direct to omit; further it introduces a bound in $L^2$ for the quadratic variation.

**2.1.6 Proposition** Let $(B_t)_{t\in T}$ be a standard Brownian motion. Then for any $t \in T$ and partition $\Pi = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ of $[0,t]$.

$$\mathbb{E}\left[V_t^2(B,\Pi)\right] = t\,, \quad \mathrm{Var}\left[V_t^2(B,\Pi)\right] \le \mathrm{Var}\left[Z^2\right] \cdot t \cdot |\Pi|$$

where $Z \sim \mathrm{N}(0,1)$ is a normally distributed random variable.

***Proof***: The first equality is an immediate consequence of the properties of the Brownian motion. As all summands in $V^2_{[0,t]}(B,\Pi)$ are independent, we obtain

$$\begin{aligned}\mathrm{Var}\left[V_t^2(X,\Pi)\right] &= \sum_{i=1}^n \mathrm{Var}\left[\left(B_{t_j} - B_{t_{j-1}}\right)^2\right] \\ &= \sum_{i=1}^n \mathrm{Var}\left[\left(t_j - t_{j-1}\right) Z^2\right] \\ &\qquad\qquad \left(B_{t_j} - B_{t_{j-1}} \sim \left(t_j - t_{j-1}\right)^{\frac{1}{2}} Z\right) \\ &= \mathrm{Var}\left[Z^2\right] \cdot \sum_{i=1}^n \left(t_j - t_{j-1}\right)^2 \\ &\le \mathrm{Var}\left[Z^2\right] \cdot t \cdot |\Pi|\end{aligned}$$

which completes our proof. □

**2.1.7 Example** Let now $B$ be a standard Brownian motion and an interval $J = (a,b) \subseteq \mathbb{R}$ containing 0. Put $\lambda = \lambda(a) = \inf\{t \in T : B_t \le a\}$, $\rho = \rho(b) = \inf\{t \in T : B_t \ge b\}$ and $\tau = \lambda \wedge \rho$. All $\rho, \lambda, \tau$ are stopping times, $\lambda \ne \rho$ a.s. and we can apply the optional stopping theorem for the martingale $B_t^2 - t, t \in T$ and the bounded stopping time $\tau \wedge n$. The sequence of random variables $B_{\tau\wedge n}$ is bounded by $|a| \vee |b|$ and so $B_{\tau\wedge n} \to B_\tau$ pointwise implies convergence also in $L^1$ and $L^2$. So,

$$\mathbb{E}[\tau] = \sup_n \mathbb{E}[\tau \wedge n] = \lim_n \mathbb{E}[\tau \wedge n] = \lim_n \mathbb{E}\left[B^2_{\tau\wedge n}\right] = \mathbb{E}\left[B^2_\tau\right]$$

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and similarly $\mathbb{E}\left[B_\tau\right] = 0$. From these two equalities we obtain that $\mathrm{P}([\rho < \lambda]) = \frac{-a}{b-a}$, $\mathrm{P}([\lambda < \rho]) = \frac{b}{b-a}$ and $\mathbb{E}[\tau] = -ab$.

Further $a' \leq a$ implies $\lambda(a') \geq \lambda(a)$ and for every $\omega \in \Omega$,

$$\lim_{a\to-\infty}(\lambda(a))(\omega) = \sup_{a<0}(\lambda(a))(\omega) = \inf\bigcap_{a<0}[B(\omega) \leq a] = \inf\varnothing = +\infty$$

proving that $\rho$ is a finite stopping time:

$$\mathrm{P}([\rho = +\infty]) = \mathrm{P}\left(\bigcap_{\kappa=1}^{\infty}[\rho > \lambda(-k)]\right) = \inf_{\substack{k\geq 1\\ k\in\mathbb{N}}}\mathrm{P}([\rho > \lambda(-k)]) = \inf_{\substack{k\geq 1\\ k\in\mathbb{N}}}\frac{b}{b+k} = 0$$

## 2.2 Convergence and path properties

**2.2.1 Proposition** Let $B$ be a Brownian motion. Then

$$\lim_{t\to+\infty}\frac{B_t}{t} = 0, P\text{-a.s.}$$

***Proof***: For any $s < t$, application of the Doob's maximal inequality to $B$ on $[s,t]$, leads to the following estimation

$$\mathbb{E}\left[\sup_{s\leq u\leq t}\frac{B_u^2}{u^2}\right] \leq \frac{1}{s^2}\mathbb{E}\left[\sup_{s\leq u\leq t}B_u^2\right] \leq \frac{4}{s^2}\mathbb{E}\left[B_t^2\right] = \frac{4t}{s^2}$$

Now let $t_k = 2^k, k = 1, 2, \ldots$ and apply Chebyshev's inequality with $\varepsilon > 0$ and sum over $k$:

$$\sum_{k=1}^{\infty}\mathrm{P}\left(\left[\sup_{t_{k-1}\leq u\leq t_k}\left|\frac{B_u}{u}\right| \geq \varepsilon\right]\right) \leq \sum_{k=1}^{\infty}\frac{4t_k}{\varepsilon^2\cdot t_{k-1}^2} = \sum_{k=1}^{\infty}\frac{2^{-k}}{\varepsilon^2} = \frac{1}{\varepsilon^2}$$

Then apply the Borel - Cantelli lemma to obtain

$$\lim_{k\to\infty}\sup_{t_{k-1}\leq u\leq t_k}\left|\frac{B_u}{u}\right| = 0, \quad P\text{-a.s.}$$

which leads to the desired result. □

**2.2.2 Example** As en example, let $(B_t)_t$ be a standard Brownian motion and consider $X_t = tB_{1/t}$, whenver $t > 0$ and $X_0 = 0$. It is clear that $(X_t)_t$ has its paths continuous on $(0, +\infty)$

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and all of its finite dimensional distributions coincide with the finite dimensional distributions of $(B_t)_t$ on $(0,+\infty)$. Continuity at 0, is a consequence of the last proposition.

**2.2.3 Proposition** Let $B$ be a Brownian motion. Then

$$\mathrm{P}\left(\left[\limsup_{t\downarrow 0^+} \frac{B_t}{\sqrt{2t\ln(\ln(1/t))}} = 1\right]\right) = 1$$

***Proof***: Apply the Doob's inequality on the martingale $M_t = \exp(\lambda B_t - \lambda^2 t/2)$, $t \in T$ to obtain

$$\mathrm{P}\left(\left[\max_{0\le s\le t}(B_s - \lambda s/2) \ge \beta\right]\right) = \mathrm{P}\left(\left[\max_{0\le s\le t} M_s \ge e^{\lambda\beta}\right]\right) \le e^{-\lambda\beta}$$

Put $h(t) = \sqrt{2t\ln(\ln(1/t))}$, $t \in (0, 1/e)$, let $\theta, \delta \in (0,1)$ and apply the latter inequality with $\lambda = (1+\delta)\theta^{-n}h(\theta^n)$, $\beta = \frac{1}{2}h(\theta^n)$, $t = \theta^n$ and $n$ large enough so that $h(\theta^n)$ is defined (say $n > \frac{1}{|\ln\theta|}$). The bound obtained thus is $e^{-\lambda\beta} = (n\ln(1/\theta))^{-(1+\delta)}$ which is summable up to $n$. We apply the Borel-Cantelli lemma to the sequence of events $\left[\max_{0\le s\le\theta^n}(W_s - \lambda s/2) \ge \beta\right]$ to obtain an event $\Omega_{\theta,\delta}$ with $\mathrm{P}(\Omega_{\theta,\delta}) = 1$ and an integer-valued random variable $N_{\theta,\delta}$ such that

$$\forall\omega\in\Omega_{\theta,\delta}\ \forall n \ge N_{\theta,\delta}(\omega)\ \forall s\in[0,\theta^n]\ \left(W_s - \frac{(1+\delta)s}{2\theta^n}h(\theta^n) < \frac{1}{2}h(\theta^n)\right)$$

If we only let $s \in [\theta^{n+1}, \theta^n]$, so that $s/\theta^n \le 1$,

$$B_s(\omega) < \left(1+\frac{\delta}{2}\right)h(\theta^n) \le \left(1+\frac{\delta}{2}\right)\frac{1}{\sqrt{\theta}}h(s)$$

leading to $\sup_{\theta^{n+1}\le s\le\theta^n}\frac{B_s(\omega)}{h(s)} \le \left(1+\frac{\delta}{2}\right)\frac{1}{\sqrt{\theta}}$, for every $n \ge N_{\theta,\delta}(\omega)$. Taking the unions over $n \ge N_{\theta,\delta}(\omega)$, we obtain that the same inequality holds for $s \in (0, \theta^{N_{\theta,\delta}(\omega)}]$, hence

$$\limsup_{t\downarrow 0^+}\frac{B_t(\omega)}{h(t)} \le \left(1+\frac{\delta}{2}\right)\frac{1}{\sqrt{\theta}}$$

But $\omega \in \Omega_{\theta,\delta}$ was arbitrarily chosen, so the inequality is valid on $\Omega_{\theta,\delta}$. Take now two sequences $\theta_n \uparrow 1, \delta_n \downarrow 0$ and obtain that on the intersection $\bigcap_n \Omega_{\theta_n,\delta_n}$, that is, $P$-almost surely,

$$\limsup_{t\downarrow 0^+}\frac{B_t}{h(t)} \le \inf_n\left(1+\frac{\delta_n}{2}\right)\frac{1}{\sqrt{\theta_n}} = 1$$

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Observe now, that as $-B$ is also a Brownian motion we obtain that $\limsup_{t\downarrow 0^+} \frac{B_t}{h(t)} \leq 1$. So there is an event $\Omega^*$ with $\mathrm{P}(\Omega^*) = 1$ such that

$$\forall \omega \in \Omega^* \ \exists \delta = \delta(\omega) > 0 \ \forall s \in (0, \delta) \ (-B_s \leq 2h(s))$$

We need to prove the opposite direction of the latter inequality. To this end, we are considering the second Borel-Cantelli lemma. Put for any integer $n > \frac{1}{|\ln \theta|}$,

$$A_n = \left[ B_{\theta^n} - B_{\theta^{n+1}} \geq \sqrt{1-\theta} h(\theta^n) \right] = \left[ \frac{B_{\theta^n} - B_{\theta^{n+1}}}{\sqrt{\theta^n - \theta^{n+1}}} \geq \frac{\sqrt{1-\theta} h(\theta^n)}{\sqrt{\theta^n - \theta^{n+1}}} \right]$$

$$= \left[ \underbrace{\frac{B_{\theta^n} - B_{\theta^{n+1}}}{\sqrt{\theta^n - \theta^{n+1}}}}_{\sim \mathrm{N}(0,1)} \geq \underbrace{\sqrt{2 \ln n + 2 \ln(|\ln \theta|)}}_{z} \right]$$

so that

$$\mathrm{P}(A_n) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-u^2/2} \, \mathrm{d}u \geq \frac{e^{-z^2/2}}{\sqrt{2\pi}(z + z^{-1})} \geq \frac{\text{const.}}{n}$$

thus $\sum_{n > |\ln \theta|^{-1}} \mathrm{P}(A_n) = +\infty$ hence there exists an event $\Omega_\theta$ with $\mathrm{P}(\Omega_\theta) = 1$ such that

$$\forall \omega \in \Omega_\theta \ \forall k \in \mathbb{N} \ \exists m = m(k, \omega) \geq k \left( B_{\theta^m}(\omega) - \mathrm{B}_{\theta^{m+1}}(\omega) \geq \sqrt{1-\theta} h(\theta^m) \right)$$

And for every $\omega \in \Omega^*$ and any integer $n > \mathrm{N}^*(\theta, \omega) \stackrel{\text{def}}{=} \left\lceil \left| \frac{\ln \delta(\omega)}{\ln \theta} \right| \right\rceil$,

$$-B_{\theta^{n+1}}(\omega) \leq 2h(\theta^{n+1}) \leq 4\sqrt{\theta} h(\theta^n)$$

So if $\omega \in \Omega_\theta \cap \Omega^*$ and $k \in \mathbb{N}$ we can find an integer $m = m(k \vee N^*(\theta, \omega), \omega) \geq k \vee N^*(\theta, \omega)$ such that

$$-B_{\theta^{m+1}}(\omega) \leq 4\sqrt{\theta} h(\theta^m) \text{ and } B_{\theta^m}(\omega) - \mathrm{B}_{\theta^{m+1}}(\omega) \geq \sqrt{1-\theta} h(\theta^m)$$

and this leads to

$$\frac{B_{\theta^m}(\omega)}{h(\theta^m)} \geq \sqrt{1-\theta} - 4\sqrt{\theta}$$

That is,

$$\limsup_{t\downarrow 0^+} \frac{B_t(\omega)}{h(t)} \geq \limsup_n \frac{B_{\theta^n}(\omega)}{h(\theta^n)} \geq \sqrt{1-\theta} - 4\sqrt{\theta}$$

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Now take any sequence $\theta_n \downarrow 0$ and on $\Omega^* \cap \bigcap_n \Omega_{\theta_n}$,

$$\limsup_{t\downarrow 0^+} \frac{B_t(\omega)}{h(t)} \geq \sup_n \sqrt{1-\theta_n} - 4\sqrt{\theta_n} = 1$$

But $\mathrm{P}\left(\Omega^* \cap \bigcap_n \Omega_{\theta_n}\right) = 1$. Proof is complete. □

**2.2.4 Corollary** Let $B$ be a Brownian motion and $\tau$ any a.s. finite stopping time. Then $P$-a.s.

$$\limsup_{t\downarrow 0^+} \frac{B_{t+\tau} - B_\tau}{\sqrt{2t\ln(\ln(1/t))}} = 1 \text{ and } \liminf_{t\downarrow 0^+} \frac{B_{t+\tau} - B_\tau}{\sqrt{2t\ln(\ln(1/t))}} = -1$$

$$\limsup_{t\to+\infty} \frac{B_{t+\tau} - B_\tau}{\sqrt{2t\ln(\ln t)}} = 1 \text{ and } \liminf_{t\to+\infty} \frac{B_{t+\tau} - B_\tau}{\sqrt{2t\ln(\ln t)}} = -1$$

**2.2.5 Remark** The last corollary shows that immediately after 0, the Brownian motion oscilates infinitely many times between -1 and 1, so its zeros accumulate at 0. Further, at $\infty$, the Brownian motion oscilates between large negative and large positive values; whence the level set $L(\omega) = [B(\omega) = x], \omega \in \Omega$ is a.s. unbounded (and of zero measure in $\mathbb{R}$). This is a recurrence property.

We deduce also a weak result about non-differentiability of the Brownian motion.

**2.2.6 Corollary** If $B$ is a Brownian motion, then for any $t \in T$, and $\gamma > 1/2$

$$\limsup_{h\downarrow 0^+} \frac{B_{t+h} - B_t}{h^\gamma} = +\infty \text{ and } \liminf_{h\downarrow 0^+} \frac{B_{t+h} - B_t}{h^\gamma} = -\infty, \quad P\text{-a.s.}$$

Actually the following result is valid, but we omit the proof.

**2.2.7 Proposition** Let $B$ be a Brownian motion and $\gamma > 1/2$. Then there exists some $P$-neglible event $N$ such that whenever $\omega \in N^{\complement}$, the path $B(\omega)$ is nowhere Hölder continuous with exponent $\gamma$.

**2.2.8 Remark** We have proved that $\frac{B_t}{t} \longrightarrow 0$, $P$-a.s. On the other end $\frac{B_t}{\sqrt{t}} \sim \mathrm{N}(0,1)$, thus $\frac{B_t}{\sqrt{t}} \xrightarrow{\mathrm{d}} \mathrm{N}(0,1)$. But this convergence is not even in probability: The following difference can be broken down into a sum of independent normal distributions

$$\frac{B_{2t}}{\sqrt{2t}} - \frac{B_t}{\sqrt{t}} = \underbrace{\frac{\sqrt{2}}{2}\frac{B_{2t} - B_t}{\sqrt{t}}}_{\sim \mathrm{N}(0,1/2)} - \underbrace{\left(1 - \frac{\sqrt{2}}{2}\right)\frac{B_t}{\sqrt{t}}}_{\sim \mathrm{N}(0,(3-2\sqrt{2})/2)} \sim \mathrm{N}(0, 2-\sqrt{2})$$

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which is impossible to converge in probability to 0. Somewhere in the middle between $t$ and $\sqrt{t}$, we obtain than if $h\colon (0,+\infty) \longmapsto (0,+\infty)$ is a function such that $h(t) \in \omega(\sqrt{t})$, then $\frac{\boldsymbol{B}_t}{h(t)} \xrightarrow{L^2} 0$. But how about a.s. convergence of $\frac{\boldsymbol{B}_t}{h(t)}$? Well, the last propositions show that when we choose $h(t) = \sqrt{t \ln(\ln(t))}$, a.s. convergence fails, while still $\frac{\boldsymbol{B}_t}{h(t)} \xrightarrow{\text{P}} 0$.

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# 3. Stochastic integration

This chapter is devoted to the presentation of the stochastic integral for continuous martingales, based on the characterization through quadratic variation. We follow [RY13, ch. IV.2, IV.3].

## 3.1 Integrator spaces, definition and basic properties

We need to introduce some martingale and process spaces. In the whole chapter, unless stated otherwise, $(\Omega, \mathscr{F}, P)$ will be a probability space equipped with a filtration $\mathfrak{F} = (\mathscr{F}_t)_{t\in T}$, $T = [0, +\infty)$, that satisfies the usual conditions (that is, it contains all the $P$-neglible sets of $\mathscr{F}$ and it is right continuous). Further we define $\mathscr{F}_\infty = \sigma\left(\bigcup_{t\in T}\mathscr{F}_t\right)$.

We begin with the definition of the classes of integrators:

**3.1.1 Definition** Let $\mathscr{M}$ be the space of $\mathfrak{F}$-martingales defined on $T\times\Omega$. The space of $L^2$-bounded martingales:

$$\mathscr{M}^2 = \left\{ M = (M_t)_{t\in T} \;\middle|\; \begin{array}{l} M \in \mathscr{M} \text{ with almost all its paths right continuous} \\ \text{with left limits and } \sup\limits_{t\in T} \mathbb{E}\left[M_t^2\right] < +\infty \end{array} \right\}$$

equipped with the norm $\|M\|_{\mathscr{M}} = \sup\limits_{t\in T} \mathbb{E}\left[M_t^2\right]^{\frac{1}{2}}$, $M \in \mathscr{M}^2$. When we say $M_\alpha \xrightarrow{\mathscr{M}} M$, we mean $\|M_\alpha - M\|_{\mathscr{M}} \longrightarrow 0$, that is, $M_\alpha$ converges to $M$ in the norm of $\mathscr{M}^2$.

We also define the spaces

$$\begin{aligned} \mathscr{M}_0^2 &= \{M \in \mathscr{M} : M_0 = 0\} \\ \mathscr{M}_c^2 &= \{M \in \mathscr{M} : M \text{ has a.s. its paths continuous}\} \\ \mathscr{M}_{0,c}^2 &= \mathscr{M}_0^2 \cap \mathscr{M}_c^2 \end{aligned}$$

**3.1.2 Proposition** $\mathscr{M}^2$ is a Hilbert space isometric to $L^2(\Omega, \mathscr{F}_\infty, P)$ under the isometry

$$J\colon L^2(\Omega, \mathscr{F}_\infty, P) \longmapsto \mathscr{M} : M_\infty \longmapsto JM_\infty = \left(\mathbb{E}\left[M_\infty \mid \mathscr{F}_t\right]\right)_{t\in T}$$

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Further, $\mathscr{M}_c^2$, $\mathscr{M}_0^2$ are closed subspaces of $\mathscr{M}^2$.

***Proof***: First observe that if $M \in \mathscr{M}^2$, its $L^2$-boundedness implies that it is uniformly integrable, thus there is some $M_\infty \in L^2(\Omega, \mathscr{F}_\infty, P)$ such that $\lim_{t \longrightarrow +\infty} = M_\infty$, a.s. and also in $L^1$, but even more in $L^2$ and that's due to Doob's inequality for $M_t^* = \sup_{s \le t} |M_s|$:

$$\mathbb{E}\left[(M_\infty^*)^2\right] \le 4 \sup_{t \in T} \mathbb{E}\left[M_t^2\right]$$

so that $M^2$ is uniformly integrable as well. Whence $\sup_{t \in T} \mathbb{E}\left[M_t^2\right] = \mathbb{E}\left[M_\infty^2\right]$.

By its definition, $JM_\infty$ is an $\mathfrak{F}$-martingale and because the filtration is right continuous, then so is $JM_\infty$. This martingale can be further modified to have a.s. left limits (cadlag). Due to continuity, $JM_\infty$ is indistinguishable from one with this property. So indeed, $JM_\infty \in \mathscr{M}$ as $M_\infty$ is $L^2$-bounded. Then the first argument of the proof applies and we conclude that $\mathbb{E}\left[M_\infty \mid \mathscr{F}_t\right] \longrightarrow M_\infty$ a.s. and in $L^2$. Thus we have an isometry.

Again by the first argument of the proof, given a martingale $M \in \mathscr{M}^2$, and its last element $M_\infty$, we obtain that $JM_\infty = M$ up to indistinguishability.

To prove that $\mathscr{M}_c$ is closed, take a sequence $(M_n)_n$ in $\mathscr{M}_c$ and suppose that $M_n \xrightarrow{\mathscr{M}} M \in \mathscr{M}$. Then, again, by Doob's inequality

$$\mathbb{E}\left[\sup_{t \in T} |(M_n)_t - M_t|^2\right] \le 4 \|M_n - M\|_{\mathscr{M}} \longrightarrow 0$$

or, equivalently $\sup_{t \in T} |(M_n)_t - M_t| \xrightarrow{L^2} 0$, thus for some subsequence, $\sup_{t \in T} \left|(M_{n_k})_t - M_t\right| \longrightarrow 0$ a.s. So for allmost all $\omega$ the path $M_{n_k}(\omega)$ converges uniformly on $T$ to the path $M(\omega)$. So $M(\omega)$ is continuous.

To show that $\mathscr{M}_0$ is closed, take

$$P_0 \colon L^2 \longmapsto L^2 : f \longmapsto P_0 f = \mathbb{E}\left[f \mid \mathscr{F}_0\right]$$

This is an orthogonal projection on $L^2$ and observe that $\ker P_0 = \left\{f - P_0 f \;\middle|\; f \in L^2\right\} = J^{-1}\mathscr{M}_0$. □

**3.1.3 Remark** We identify all the martingales in $\mathscr{M}^2$ that are indistinguishable from each other. Also note that $\|M_\infty^*\|_2 \le 2\|M\|_{\mathscr{M}}$ introduces an equivalent norm in $\mathscr{M}$ (though not a Hilbert one).

The topology of convergence in $\mathscr{M}^2$ is the one suggested by the Doob's inequality: $M^{(n)} \xrightarrow{\mathscr{M}} 0$ if and only if $\mathbb{E}\left[\sup_{t \in T} \left|M_t^{(n)}\right|^2\right] \longrightarrow 0$.

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**3.1.4 Proposition** A continuous local $\mathfrak{F}$-martingale $M = (M_t)_{t\in T}$ is in $\mathscr{M}^2$ if and only if $M_0 \in L^2$ and $\langle\langle M, M\rangle\rangle$ is integrable. In that case, the martingale $M^2 - \langle\langle M, M\rangle\rangle$ is uniformly integrable and for all $s < t \in T$,

$$\mathbb{E}\left[M_t^2 - M_s^2 \,\middle|\, \mathscr{F}_s\right] = \mathbb{E}\left[(M_t - M_s)^2 \,\middle|\, \mathscr{F}_s\right] = \mathbb{E}\left[\langle\langle M, M\rangle\rangle_s^t \,\middle|\, \mathscr{F}_s\right]$$

***Proof***: ($\Rightarrow$): We suppose that $M$ is a local martingale and $M \in \mathscr{M}^2$. Obviously $M_0 \in L^2$. Now let $(\tau_n)_n$ be a sequence of stopping times as in the definition of local martingale. We can further suppose that $1_{[\tau_n>0]} \cdot M^{\tau_n}$ is bounded. In a standard fashion,

$$\begin{aligned}
\mathbb{E}\left[\langle\langle M, M\rangle\rangle_{\tau_n\wedge t}\right] &= \mathbb{E}\left[\langle\langle M, M\rangle\rangle_{\tau_n\wedge t} \cdot 1_{[\tau_n>0]}\right] && (\langle\langle M, M\rangle\rangle_0 = 0)\\
&= \mathbb{E}\left[1_{[\tau_n>0]} \cdot M^2_{\tau_n\wedge t}\right] - \mathbb{E}\left[M_0^2 \cdot 1_{[\tau_n>0]}\right] && (M_0 \in L^2)\\
&= \mathbb{E}\left[M^2_{\tau_n\wedge t}\right] - \mathbb{E}\left[M_0^2\right]
\end{aligned}$$

Now let $n \longrightarrow +\infty$ and exploit the $L^2$-boundedness of $M$ to obtain

$$\mathbb{E}\left[\langle\langle M, M\rangle\rangle_t\right] = \mathbb{E}\left[M_t^2\right] - \mathbb{E}\left[M_0^2\right] \leq \mathbb{E}\left[M_\infty^2\right] - \mathbb{E}\left[M_0^2\right]$$

Now, by the monotone convergence, let $t \longrightarrow +\infty$ to obtain that $\langle\langle M, M\rangle\rangle_\infty = \sup_{n\in\mathbb{N}} \langle\langle M, M\rangle\rangle_n = \sup_{t\in T} \langle\langle M, M\rangle\rangle_t$ is $L^2$-integrable, for it is bounded:

$$\mathbb{E}\left[\langle\langle M, M\rangle\rangle_\infty^2\right] = \sup_{n\in\mathbb{N}} \mathbb{E}\left[\langle\langle M, M\rangle\rangle_n\right] \leq \mathbb{E}\left[M_\infty^2\right] - \mathbb{E}\left[M_0^2\right]$$

($\Leftarrow$): We suppose that $M$ is a local martingale such that $M_0 \in L^2(\Omega)$ and $\langle\langle M, M\rangle\rangle$ is integrable, thus the limit $\langle\langle M, M\rangle\rangle_\infty = \sup_{n\in\mathbb{N}} \langle\langle M, M\rangle\rangle_n = \sup_{t\in T} \langle\langle M, M\rangle\rangle_t$ exists pointwise and is integrable. Let again the $(\tau_n)_n$ sequence of stopping times as above. Then, integrability of $M_0^2$ and the martingale property of $1_{[\tau_n>0]} \cdot M^{\tau_n}$ lead to

$$\mathbb{E}\left[M^2_{\tau_n\wedge t}\right] = \mathbb{E}\left[M_0^2\right] + \mathbb{E}\left[\langle\langle M, M\rangle\rangle_{\tau_n\wedge t}\right] \leq \mathbb{E}\left[M_0^2\right] + \mathbb{E}\left[\langle\langle M, M\rangle\rangle_\infty\right], \quad \text{for all } n \in \mathbb{N}, t \in T$$

For fixed $t$ let $n \longrightarrow +\infty$ to obtain, by Fatou's lemma: $\mathbb{E}\left[M_t^2\right] \leq \liminf_n \mathbb{E}\left[M^2_{\tau_n\wedge t}\right] \leq \mathbb{E}\left[M_0^2\right] + \mathbb{E}\left[\langle\langle M, M\rangle\rangle_\infty^2\right]$. This prove the $L^2$ boundedness of $M$. Further, the whole family $\left\{M_t, M^2_{\tau_n\wedge t} \,\middle|\, t \in T, n \in \mathbb{N}\right\}$ is uniformly integrable (as $L^2$ bounded). For fixed $t$, let $n \longrightarrow +\infty$ to obtain

$$M_s = \lim_n M_{\tau_n\wedge s} = \lim_n \mathbb{E}\left[M_{\tau_n\wedge t} \,\middle|\, \mathscr{F}_s\right] = \mathbb{E}\left[M_t \,|\, \mathscr{F}_s\right], \quad \text{for all } s < t \in T$$

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To show that $M^2 - \langle\langle M, M \rangle\rangle$ is integrable, just observe

$$\sup_{t \in T} \left| M_t^2 - \langle\langle M, M \rangle\rangle_t \right| \leq (M_\infty^*)^2 + \langle\langle M, M \rangle\rangle_\infty$$

and the right side is integralbe. Eventually, use the optional stopping theorem to obtain the desired equality. □

**3.1.5 Remark** i) Whenever a continuous local martingale $M$ is also in $\mathscr{M}^2$, then we get also that

$$\| M \|_{\mathscr{M}}^2 = \| M_0 \|_2^2 + \| \langle\langle M, M \rangle\rangle_\infty \|_1$$

ii) A moment's reflection on the previous proof shows that the results may hold also for any stopping time $\tau$: $M^\tau$ is again a continuous local martingale and thus $M^\tau \in \mathscr{M}$ if and only if $\mathbb{E}\left[M_0^2\right]$ and $\mathbb{E}\left[\langle\langle M, M \rangle\rangle_\tau^2\right] < +\infty$.

We now proceed with the classes of integrands. For the next definitions, recall that if $M \in BV(\mathfrak{F})$ and $K$ is a progressively measurable process we defined a new process $(K \bullet M)$, pathwise as the Lebesgue-Stieltjes integral:

$$(K \bullet M)_t = \int_0^t K_s \, \mathrm{d}M_s \,, \quad t \in T$$

Further, if $M$ is a continuous local martingale, then it has a continuous quadratic variation, and we can define a measure $\mu_M$ on $(T \times \Omega, \mathscr{B}(T) \times \mathscr{F})$:

$$\mathscr{B}(T) \times \mathscr{F} \ni A \longmapsto \mu_M(A) = \int_\Omega \int_T 1_A(t, \omega) \, \mathrm{d} \langle\langle M, M \rangle\rangle_t \, \mathrm{d}P(\omega) \in [0, +\infty]$$

(or $\mu_M(A) = \mathbb{E}\left[\int_T 1_A \, \mathrm{d}\langle\langle M, M \rangle\rangle\right]$ for short). So for a process $K$, $\int_{[0,t] \times \Omega} K \, \mathrm{d}\mu_M = \mathbb{E}\left[\int_0^t K_s \, \mathrm{d}\langle\langle M, M \rangle\rangle_s\right]$.

**3.1.6 Definition** Let $M$ be a continuous local martingale. We define

$$L^2(M) = \left\{ K = (K_t)_{t \in T} \;\middle|\; \begin{array}{l} \text{K is a progressively measurable process such that} \\ \int_{T \times \Omega} |K|^2 \, \mathrm{d}\mu_M < +\infty \end{array} \right\}$$

for a $K \in L^2(M)$ we define $\| K \|_M = \left( \int_{T \times \Omega} |K|^2 \, \mathrm{d}\mu_M \right)^{\frac{1}{2}}$.

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We define

$$L^2_{\text{loc}}(M) = \left\{ K = (K_t)_{t\in T} \;\middle|\; \begin{array}{l} \text{K is a progressively measurable process such that,} \\ \text{there exists a nondecreasing sequence of stopping} \\ \text{times } (\tau_n)_n \text{ such that } \tau_n \uparrow +\infty \text{ and } 1_{[0,\tau_n]}K \in L^2(M) \end{array} \right\}$$

We identify any two processes $K, H \in L^2(M)$ or $L^2_{\text{loc}}(M)$ if $K = H$ a.e. for the measure $\mu_M$.

A progressively measurable process $K$ is called **locally bounded** if there exists a nondecreasing sequence of stopping times $(\tau_n)_n$ such that $\tau_n \uparrow +\infty$ and $\sup_{T\times\Omega} |K^{\tau_n}| < +\infty$.

The next proposition is the heart of the definition of the stochastic integral:

**3.1.7 Proposition** Let $M \in \mathscr{M}^2_c$. For every $K \in L^2(M)$ there is a unique $I_M(K) \in \mathscr{M}^2_{c,0}$ such that for every $N \in \mathscr{M}^2_c$

$$\langle\langle I_M(K), N\rangle\rangle = K \bullet \langle\langle M, N\rangle\rangle$$

Further, the mapping $I_M\colon L^2(M) \longmapsto \mathscr{M}^2_{c,0}$ is an isometry between $L^2(M)$ and $\mathscr{M}^2_{c,0}$. The mapping $I_M$ is called **stochastic integral** of $K$ with respect to $M$ and we use the symbol:

$$(I_M(K))_t = \int_0^t K_s \, \mathrm{d}M_s, \quad t \in T$$

***Proof***: First we suppose that $M \in \mathscr{M}^2_{c,0}$. We define

$$B\colon L^2(N) \times \mathscr{M}^2_{c,0} \longmapsto \mathbb{R} : (K, N) \longmapsto B(K, N) = \mathbb{E}\left[\int_0^\infty K_s \, \mathrm{d}\langle\langle M, N\rangle\rangle_s\right]$$

For every $N \in \mathscr{M}^2_{c,0}$, $K \in L^2(M)$,

$$|B(K, N)| \le \mathbb{E}\left[\int_0^\infty K_s^2 \, \mathrm{d}\langle\langle M, M\rangle\rangle_s\right]^{1/2} \mathbb{E}\left[\langle\langle N, N\rangle\rangle_\infty\right]^{1/2} = \|K\|_M \cdot \|N\|_{\mathscr{M}}$$

Thus $B$ is a bounded bilinear functional on $L^2(N) \times \mathscr{M}^2_{c,0}$ and so it uniquely defines a bounded linear mapping $I_M\colon L^2(M) \longmapsto \mathscr{M}^2_{c,0}$ such that

$$B(K, N) = \mathbb{E}\left[I_M(K)_\infty \, N_\infty\right], \quad K \in L^2(M), N \in \mathscr{M}^2_{c,0}$$

We need to show that $\langle\langle I_M(K), N\rangle\rangle = K \bullet \langle\langle M, N\rangle\rangle$ and it suffices to show that $X = I_M(K)N - K \bullet \langle\langle M, N\rangle\rangle$ is a martingale. To this end, let $\tau$ be a bounded stopping time.

$$\begin{aligned} \mathbb{E}\left[I_M(K)_\tau \, N_\tau\right] &= \mathbb{E}\left[\mathbb{E}\left[I_M(K)_\infty \mid \mathscr{F}_\tau\right] N_\tau\right] = \mathbb{E}\left[I_M(K)_\infty \, N_\tau\right] \\ &= \mathbb{E}\left[I_M(K)_\infty \, (N^\tau)_\infty\right] = B(K, N^\tau) \\ &= \mathbb{E}\left[\int_0^\infty K_s \, \mathrm{d}\langle\langle M, N^\tau\rangle\rangle_s\right] = \mathbb{E}\left[\int_0^\infty K_s \, \mathrm{d}(\langle\langle M, N\rangle\rangle^\tau)_s\right] \\ &= \mathbb{E}\left[\int_0^\tau K_s \, \mathrm{d}\langle\langle M, N\rangle\rangle_s\right] = \mathbb{E}\left[(K \bullet \langle\langle M, N\rangle\rangle)_\tau\right] \end{aligned}$$



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We have shown that $\mathbb{E}[X^\tau] = \mathbb{E}\left[X_0\right]$ for an arbitrary bounded stopping time $\tau$. This asserts the martingale property of $X$.

Further, $\langle\langle I_M(K), I_M(K)\rangle\rangle = K \bullet \langle\langle M, I_M(K)\rangle\rangle = K^2 \bullet \langle\langle M, M\rangle\rangle$ and this leads to

$$\|I_M(K)\|_{\mathscr{M}} = \mathbb{E}[\langle\langle I_M(K), I_M(K)\rangle\rangle_\infty] = \mathbb{E}\left[(K^2 \bullet \langle\langle M, M\rangle\rangle)_\infty\right] = \|K\|_M$$

thus proving that $I_M$ is an isometry.

Observe that the identity $\langle\langle I_M(K), N\rangle\rangle = K \bullet \langle\langle M, N\rangle\rangle$ extends to the whole $\mathscr{M}_c^2$, for adding a constant to $N$ does not change the bracket.

Finally, we extend the $I_M$ to martingales not vanishing at 0. If $M = M_0$ is a constant martingale, we proceed to define $I_M(K) = 0$ □

**3.1.8 Proposition** Let $M \in \mathscr{M}_c^2$. For every $K \in L^2(M)$, $H \in L^2\left(\int_0 K_s \, \mathrm{d}M_s\right)$

$$I_{I_M(K)}(H) = I_M(KH)$$

If $\tau$ is a stopping time and $H \in L^2(M)$ then

$$\left(\int_0 H_s \, \mathrm{d}M_s\right)^\tau = \int_0 1_{[0,\tau]} H_s \, \mathrm{d}M_s = \int_0 H_s \, \mathrm{d}M_{\tau\wedge s}$$

where the process $1_{[0,\tau]}$ is defined by $1_{[0,\tau]}(t,\omega) = \begin{cases} 1 & \text{if } t \le \tau(\omega) \\ 0 & \text{if } t > \tau(\omega) \end{cases}$.

***Proof***: This is a consequence of the defining property of the stochastic integral and the associativity of the Lebesgue-Stieltjes integral: For any $N \in \mathscr{M}_c^2$,

$$\begin{aligned} \left\langle\left\langle I_{I_M(K)}(H), N\right\rangle\right\rangle &= H \bullet \langle\langle I_M(K), N\rangle\rangle = H \bullet (K \bullet \langle\langle M, N\rangle\rangle) \\ &= HK \bullet \langle\langle M, N\rangle\rangle = \langle\langle I_M(HK), N\rangle\rangle \end{aligned}$$

Now apply the previous argument to $K = 1_{[0,\tau]}$ and observe that $\int_0 1_{[0,\tau]} \, \mathrm{d}M = M^\tau$: For any $N \in \mathscr{M}_c^2$,

$$\left\langle\left\langle \int_0 1_{[0,\tau]} \, \mathrm{d}M, N \right\rangle\right\rangle = 1_{[0,\tau]} \bullet \langle\langle M, N\rangle\rangle = \langle\langle M, N\rangle\rangle^\tau = \langle\langle M^\tau, N\rangle\rangle$$

This shows the right pair of the equalities. For the left side pair, we obtain for every $N \in \mathscr{M}_c^2$,

$$\begin{aligned} \left\langle\left\langle \left(\int_0 H_s \, \mathrm{d}M_s\right)^\tau, N \right\rangle\right\rangle &= \left\langle\left\langle \int_0 H_s \, \mathrm{d}M_s, N \right\rangle\right\rangle^\tau = (H \bullet \langle\langle M, N\rangle\rangle)^\tau \\ &= 1_{[0,\tau]} \bullet (H \bullet \langle\langle M, N\rangle\rangle) = 1_{[0,\tau]} H \bullet \langle\langle M, N\rangle\rangle \\ &= \left\langle\left\langle \int_0 1_{[0,\tau]} H \, \mathrm{d}M, N \right\rangle\right\rangle \end{aligned}$$ □

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**3.1.9 Remark** We let the symbol $\int_0^\tau K_s \, \mathrm{d}M_s$ stand for the random variable $\left(\int_0 K_s \, \mathrm{d}M_s\right)_\tau$. So for the stopped process $\left(\int_0 K_s \, \mathrm{d}M_s\right)^\tau$ one has

$$\left(\int_0 K_s \, \mathrm{d}M_s\right)_{\tau\wedge t} = \int_0^t 1_{[0,\tau]} K_s \, \mathrm{d}M_s = \int_0^{\tau\wedge t} K_s \, \mathrm{d}M_s$$

Further for any stopping times $a, b$ we define

$$\int_a^b K_s \, \mathrm{d}M_s = \int_0^b K_s \, \mathrm{d}M_s - \int_0^a K_s \, \mathrm{d}M_s$$

In the same time we want to stress the fact that the stochastic integral is not defined pathwise. In the case of a $M \in \mathscr{M}_c^2$ and $K \in L^2(M)$, one has $\int_0^\tau H_s \, \mathrm{d}M_s = \int_0^\infty 1_{[0,\tau]} H_s \, \mathrm{d}M_s$.

Now we can extend the stochastic integral to continuous local martingales

**3.1.10 Proposition** Let $M$ be a continuous local martingale and $K \in L^2_{\mathrm{loc}}(M)$. Then there exists a unique continuous local martingale $\int_0 K_s \, \mathrm{d}M_s$ vanishing at 0, such that for any continuous local martingale $N$,

$$\left\langle\!\!\left\langle \int_0 K_s \, \mathrm{d}M_s, N \right\rangle\!\!\right\rangle = K \bullet \langle\!\langle M, N \rangle\!\rangle$$

***Proof***: First suppose that $M_0 = 0$. We proceed by locality: Let $(\tau_n)_n$ be a nondecreasing sequence of stopping times such that $\tau_n \uparrow +\infty$ such that $M^{\tau_n} \in \mathscr{M}_c^2$ and $1_{[0,\tau_n]} K \in L^2(M^{\tau_n})$. The stochastic integral $X_n = \int_0 1_{[0,\tau_n]} K \, \mathrm{d}M^{\tau_n}$ is well defined, and $X_{n+1}^{\tau_n} = \int_0 1_{[0,\tau_n]} K \, \mathrm{d}M^{\tau_{n+1}} = \int_0 1_{[0,\tau_n]} K \, \mathrm{d}M^{\tau_n} = X_n$, for every $n$. So we define

$$X = \int_0 K_s \, \mathrm{d}M_s = \sum_{n=1}^\infty \int_0 1_{(\tau_{n-1},\tau_n]} K \, \mathrm{d}M^{\tau_n}$$

observe that

$$\begin{aligned}
\left(\int_0 K_s \, \mathrm{d}M_s\right)^{\tau_m} &= \sum_{n=1}^\infty \left(\int_0 1_{(\tau_{n-1},\tau_n]} K \, \mathrm{d}M^{\tau_n}\right)^{\tau_m} \\
&= \sum_{n=1}^m \int_0 1_{[0,\tau_n]} K \, \mathrm{d}M^{\tau_n} - \int_0 1_{[0,\tau_{n-1}]} K \, \mathrm{d}M^{\tau_n} \\
&= \sum_{n=1}^m \int_0 1_{[0,\tau_n]} K \, \mathrm{d}M^{\tau_n} - \int_0 1_{[0,\tau_{n-1}]} K \, \mathrm{d}M^{\tau_{n-1}} \\
&= \int_0 1_{[0,\tau_m]} K \, \mathrm{d}M^{\tau_m} = X_m
\end{aligned}$$

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Thus, $\int_0 K_s \, \mathrm{d}M_s$ is a continuous local martingale vanishing at 0, such that for every $N \in \mathscr{M}_c^2$ and every $n$,

$$\langle\langle X, N\rangle\rangle^{\tau_n} = \langle\langle X_n, N\rangle\rangle = 1_{[0,\tau_n]} K \bullet \langle\langle M^{\tau_n}, N\rangle\rangle = 1_{[0,\tau_n]} K \bullet \langle\langle M, N\rangle\rangle$$

Now we suppose that $N$ is a continuous local martingale. Let $(\sigma_k)_n$ be a nondecreasing sequence of stopping times such that $s_k \uparrow +\infty$ and $N^{\sigma_k} \in \mathscr{M}_c^2$ (or even bounded). Now for every $n, k$, $\langle\langle X, N^{\sigma_k}\rangle\rangle^{\tau_n} = 1_{[0,\tau_n]} K \bullet \langle\langle M, N^{\sigma_k}\rangle\rangle$, or equivalently, $\langle\langle X, N\rangle\rangle^{\tau_n \wedge \sigma_k} = 1_{[0,\tau_n \wedge \sigma_k]} K \bullet \langle\langle M, N\rangle\rangle$. Fix any $t \in T$ to obtain

$$\langle\langle X, N\rangle\rangle_{\tau_n \wedge \sigma_k \wedge t} = \int_0^{\tau_n \wedge \sigma_k \wedge t} K_s \, \mathrm{d}\langle\langle M, N\rangle\rangle_s$$

take limits as $n, k \longrightarrow +\infty$, exploit continuity of the bracket and the Lebesgue-Stieltjes integral (with respect to continuous integrator), to obtain the desired equality.

In the general case, whenever $M$ is a local martingale (warning: $M_0$ may not even be integrable) we define $\int_0 K_s \, \mathrm{d}(M - M_0)_s = \int_0 K_s \, \mathrm{d}M_s$. □

We also have:

**3.1.11 Proposition** Let $M$ be a continuous local martingale. For every $K \in L^2_{\mathrm{loc}}(M)$, $H \in L^2_{\mathrm{loc}}\left(\int_0 K_s \, \mathrm{d}M_s\right)$

$$I_{I_M(K)}(H) = I_M(KH)$$

If $\tau$ is a stopping time and $H \in L^2_{\mathrm{loc}}(M)$ then

$$\left(\int_0 H_s \, \mathrm{d}M_s\right)^{\tau} = \int_0 1_{[0,\tau]} H_s \, \mathrm{d}M_s = \int_0 H_s \, \mathrm{d}M_{\tau \wedge s}$$

**3.1.12 Proposition** Let $M$ be a local martingale, $(K_n)_{n=1}^{\infty}$ a sequence of progressively measurable processes converging pointwise to 0. We further suppose that $H \in L^2_{\mathrm{loc}}(M)$ such that $|K_n| \leq H$ for every $n = 1, 2, \ldots$. Then for any $b \in T$,

$$\sup_{0 \leq t \leq b} \left| \int_0^t K_n \, \mathrm{d}M \right| \xrightarrow{\mathrm{P}} 0 \quad \text{as } n \to +\infty$$

***Proof***: We can suppose without loss of generality that $M_0 = 0$. Let $(\tau_m)_n$ be a nondecreasing sequence of stopping times such that $\tau_m \uparrow +\infty$ and for all $m$, $M^{\tau_m} \in \mathscr{M}_{c,0}^2$ and $H^{\tau_m} \in L^2(M)$. Let $b \in T$ and $\varepsilon > 0$ be fixed. We can find now some $m$ such that $\mathrm{P}([\tau_m \leq b]) < \delta / 2$ and some $n_0$ such that for all $n \geq n_0$, $\mathbb{E}\left[\int_0^{\tau_m} K_n^2 \, \mathrm{d}\langle\langle M, M\rangle\rangle\right] < \delta\varepsilon^2 / 2$. But the stochastic integral is an

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isometry; whence $\mathbb{E}\left[\left(\int_0^{\tau_m} K_s \,\mathrm{d}M_s\right)^2\right] = \left\|\left(\int_0 K \,\mathrm{d}M\right)^{\tau_m}\right\|_{\mathscr{M}} = \mathbb{E}\left[\int_0^{\tau_m} K_n^2 \,\mathrm{d}\langle\!\langle M, M\rangle\!\rangle\right] < \delta\varepsilon^2/2$. By the Doob inequality and for all $n \geq n_0$,

$$\begin{aligned}
\mathrm{P}\left(\left[\sup_{0\leq t\leq b}\left|\int_0^t K_n \,\mathrm{d}M\right| \geq \varepsilon\right]\right) &= \mathrm{P}\left([\tau_m \leq b]\right) + \mathrm{P}\left(\left[\sup_{0\leq t\leq b}\left|\int_0^t K_n \,\mathrm{d}M\right| \geq \varepsilon\right] \cap [\tau_m > b]\right) \\
&= \mathrm{P}\left([\tau_m \leq b]\right) + \mathrm{P}\left(\left[\sup_{0\leq t\leq b}\left|\int_0^{\tau_m\wedge t} K_n \,\mathrm{d}M\right| \geq \varepsilon\right] \cap [\tau_m > b]\right) \\
&\leq \mathrm{P}\left([\tau_m \leq b]\right) + \mathrm{P}\left(\left[\sup_{0\leq t\leq b}\left|\int_0^{\tau_m\wedge t} K_n \,\mathrm{d}M\right| \geq \varepsilon\right]\right) \\
&\leq \mathrm{P}\left([\tau_m \leq b]\right) + \frac{1}{\varepsilon^2}\cdot\mathbb{E}\left[\left(\int_0^{\tau_m} K_s \,\mathrm{d}M_s\right)^2\right] < \delta
\end{aligned}$$ □

**3.1.13 Example** Let $M$ be a continuous local martingale, $(\tau_n)_{n=0}^{\infty}$ be a nondecreasing sequence of stopping times (not necessarily finite) such that $\tau_0 = 0$ and $\tau_n \uparrow +\infty$, $(\xi_n)_{n=0}^{\infty}$ be a sequence of random variables, with each of $\xi_n$ being $\mathscr{F}_{\tau_n}$-measurable. Define now the process $K$:

$$K = \sum_{k=0}^{\infty} \xi_k \cdot 1_{(\tau_k, \tau_{k+1}]} \tag{E}$$

and observe that $K$ is adapted and with all its paths left continuous (and with right limits). Whence it is progressively measurable. If we further suppose that $\exists C > 0$ such that $|\xi_n| \leq C$ for all $n$, we can be easily verify that $K \in L^2(M)$. So the stochastic integral $\int_0 K_s \,\mathrm{d}M_s$ exists and since $\left(\int_0 K_s \,\mathrm{d}M_s\right)^{\tau_n} = \int_0 1_{[0,\tau_n]} K_s \,\mathrm{d}M_s = \sum_{k=0}^{n-1} \int_0 1_{(\tau_k,\tau_{k+1}]}\xi_k \,\mathrm{d}M_s = \sum_{k=0}^{n-1} \xi_k(M_{\tau_{k+1}} - \mathrm{M}_{t_k})$, hence $\int_0^{\tau_n\wedge t} K_s \,\mathrm{d}M_s = \sum_{k=0}^{n-1} \xi_k(M_{\tau_{k+1}\wedge t} - \mathrm{M}_{t_k\wedge t})$. We can take limits, and exploiting continuity of $M$ and the stochastic integral,

$$\int_0^t K_s \,\mathrm{d}M_s = \lim_n \int_0^{\tau_n\wedge t} K_s \,\mathrm{d}M_s = \sum_{n=0}^{\infty} \xi_n(M_{\tau_{n+1}\wedge t} - \mathrm{M}_{t_n\wedge t}) \tag{SI}$$

While the left side limit cannot be evaluated directly for every single $\omega$, the limit on the right side is explicitly defined pathwise (and it is a finite sum for every single $\omega$); we obtain thus, up to modification and hence up to indistinguishability, a closed form of the stochastic integral, whenever $K$ has the special form defined earlier. We call such processes **elementary**, and we write $K \in \mathscr{E}(\mathfrak{F})$ or simply $\mathscr{E}$, whenever the filtration is obvious. When the stopping times are all constant we say that $K \in \mathscr{E}_0$.

Even as $\mathscr{E}_0$ is smaller than $\mathscr{E}$, it seems it suffices: Every adapted, left-continuous with right limits process, is the pointwise limit of a sequence of elementary processes in $\mathscr{E}_0$. And one can



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begin to build the stochastic integration by defining integrals pathwise on $\mathscr{E}_0$ or $\mathscr{E}$. Observe, that when we take the equation (SI) as a beginning definition for our stochastic integral, we dont really need the full continuity assumption over $M$. Actually stochastic integrals may be defined for a much broader class of integrators: right continuous semimartingales. And for this class of integrators, the appropriate integrands are the left-continuous adapted processes with right limits. We shall not develop this concept here.

**3.1.14 Proposition** Let $M$ be a continuous local martingale, $K \in L^2_{\text{loc}}(M)$ with almost all its paths left continuous. Then for every $t \in T$, $\varepsilon > 0$,

$$\sum_{k=0}^{n-1} K_{t_k}(M_{t_{k+1}} - M_{t_k}) \xrightarrow{\text{P}} \int_0^t K_s \, \mathrm{d}M_s \quad \text{as } |\Pi| \longrightarrow 0$$

with the limit taken over the partitions $\Pi = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ of $[0, t]$.

**3.1.15 Corollary** Let $\mathfrak{G} = (\mathscr{G}_t)_{t \in T}$ be a filtration (other that $\mathfrak{F}$). Let $M$ be a continuous local martingale with respect to both $\mathfrak{F}, \mathfrak{G}$ and $K \in L^2_{\text{loc}}(M; \mathfrak{G}) \cap L^2_{\text{loc}}(M; \mathfrak{F})$ with allmost all its path left continuous. Then, the stochastic integral $\int_0 K_s \, \mathrm{d}M_s$ does not depend on the filtration

***Proof***: Use the previous corollary to obtain a limit-in-probability evaluation of the stochastic integral, which is not dependent on the filtration. □

**3.1.16 Corollary** Let $B$ be a standard $\mathfrak{F}$-Brownian motion and $f$ be a locally bounded borel function $f: \mathbb{R}_+ \longmapsto \mathbb{R}$. Then for any $s < t$, the stochastic integral $\int_s^t f(s) \, \mathrm{d}B_s$ is independent of $\mathscr{F}_s$.

***Proof***: Consider the process $M = (M_t)_{t \in T}$, defined by

$$M_t = \begin{cases} 0 & \text{if } t \leq s \\ B_t - B_s & \text{if } t \geq s \end{cases} = (B - B^s)_t$$

Then $M$ is a continuous martingale (and a Brownian motion starting at time $s$), adapted to its own natural fintration $\mathfrak{G}$ (it can be taken to contain the neglible sets of $\mathscr{F}$) which is independent of $\mathscr{F}_s$. Further $f$ as a process, being constant on $\omega$, is in $L^2_{\text{loc}}(M) \cap L^2_{\text{loc}}(B)$.

First suppose that $f$ is left continuous: First $\int_0^t f(u) \, \mathrm{d}M_u$ is $\mathscr{G}_t$-measurable and hence independent of $\mathscr{F}_s$; further it equals (now take the integral on the filtration $\mathfrak{F}$) $\int_0^t f(u) \, \mathrm{d}(B - B^s)_u = \int_0^t f(u) \, \mathrm{d}B_u - \int_0^t f(u) \, \mathrm{d}B^s_u = \int_0^t f(u) \, \mathrm{d}B_u - \int_0^{s \wedge t} f(u) \, \mathrm{d}B_u = \int_s^t f(u) \, \mathrm{d}B_u$. Whence $\int_s^t f(u) \, \mathrm{d}B_u$ is independent of $\mathscr{F}_s$.

If $f$ is only borel measurable, then on $[0, t]$ it a.s. the pointwise limit of a sequence of continuous functions $f_n$ such that $|f_n| \leq K$, where $K$ is a bound of $|f|$ on $[0, t]$. We can apply the convergence result for stochastic integrals to obtain that $\int_0^t f(u) \, \mathrm{d}M_u = \int_s^t f(u) \, \mathrm{d}B_u$, with the left expression being $\mathscr{G}_t$-measurable and independent of $\mathscr{F}_s$. □

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## 3.2 Itô's formula

**3.2.1 Proposition** If $M$ is a continuous local martingale, then for any $t \in T$,

$$M_t^2 - M_0^2 = 2\int_0^t M_s \, \mathrm{d}M_s + \langle\langle M, M \rangle\rangle_t$$

***Proof***: If $\Pi = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ is a partition of $[0, t]$, then

$$\sum_{k=0}^{n-1} (M_{t_{k+1}} - M_{t_k})^2 = M_t^2 - M_0^2 - 2\sum_{i=0}^{n-1} M_{t_i}(M_{t_{k+1}} - M_{t_k})$$

Taking limit as $|\Pi| \longrightarrow 0$ we are lead to $\langle\langle M, M \rangle\rangle_t$ on the left, and to $M_t^2 - M_0^2 - 2\int_0^t M_s \, \mathrm{d}M_s$ on the right. □

**3.2.2 Proposition** If $M, N$ are continuous local martingales, then for any $t \in T$,

$$M_t N_t - M_0 N_0 = \int_0^t M_s \, \mathrm{d}N_s + \int_0^t N_s \, \mathrm{d}M_s + \langle\langle M, N \rangle\rangle_t$$

If $A \in \mathscr{BV}$ is continuous, then for any $t \in T$,

$$M_t A_t - M_0 A_0 = \int_0^t M_s \, \mathrm{d}A_s + \int_0^t A_s \, \mathrm{d}M_s$$

***Proof***: For the first equation, just use twice the previous corollary replacing $M$ by $\frac{1}{4}(M + N)$ and $\frac{1}{4}(M - N)$; then subtract.

For the second equation, fix some $t \in T$ and take a partition $\Pi = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ of $[0, t]$. Then

$$\begin{aligned} M_t A_t - M_0 A_0 &= \sum_{k=0}^{n-1} M_{t_{k+1}} A_{t_{k+1}} - M_{t_k} A_{t_k} \\ &= \sum_{k=0}^{n-1} \left[ M_{t_{k+1}} A_{t_k} + M_{t_k} A_{t_{k+1}} - 2M_{t_k} A_{t_k} \right] + \sum_{k=0}^{n-1} (M_{t_{k+1}} - M_{t_k})(A_{t_{k+1}} - A_{t_k}) \\ &= \sum_{k=0}^{n-1} A_{t_k}(M_{t_{k+1}} - M_{t_k}) + \sum_{k=0}^{n-1} M_{t_k}(A_{t_{k+1}} - A_{t_k}) + \sum_{k=0}^{n-1} (M_{t_{k+1}} - M_{t_k})(A_{t_{k+1}} - A_{t_k}) \end{aligned}$$

The first factor on the right converges in probability to $\int_0^t A_s \, \mathrm{d}M_s$, the second factor converges pointwise to $\int_0^t M_s \, \mathrm{d}A_s$ and the third factor is the covariation $C_t(M, A; \Pi)$ and we estimate:

$$|C_t(M, A; \Pi)| \leq \max_{1 \leq k \leq n} \left| M_{t_{k+1}} - M_{t_k} \right| \cdot \sum_{i=1}^{n} \left| A_{t_{k+1}} - A_{t_k} \right| \leq \max_{1 \leq k \leq n} \left| M_{t_{k+1}} - M_{t_k} \right| \cdot V_t^1(A)$$

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But for almost every single $\omega \in \Omega$, the variation $V_t^1(A(\omega))$ is finite, and $M(\omega)$ is uniformly continuous on $[0,t]$, hence $C_t(M(\omega), A(\omega); \Pi) \longrightarrow 0$. This completes the proof. □

The last corollary suggests us to extend the definition of the bracket: As $C_t(A, B; \Pi)$ is well defined for any two processes, and since $C_t(A, B; \Pi) \longrightarrow 0$ a.s. for any $A \in \mathscr{BV}$ and continuous $B$, we define

$$\langle\langle A, B \rangle\rangle = 0$$

and still be consistent with $\sup_{s \le t} |C_s(A, B; \Pi)| \xrightarrow{\mathrm{P}} 0$. This last definition, further allows us to evaluate the bracket for continuous semimartingales, $X = M + A, Y = N + B$, with $M, N$ local martingales and $A, B \in \mathscr{BV}$: $\langle\langle X, Y \rangle\rangle = \langle\langle M, N \rangle\rangle$. This further enables us to extend the stochastic integral to continuous semimartingales and to locally bounded processes:

$$\int_0^t K_s \, \mathrm{d}X_s = \underbrace{\int_0^t K_s \, \mathrm{d}M_s}_{\text{stochastic part}} + \underbrace{\int_0^t K_s \, \mathrm{d}A_s}_{\text{pathwise def'd}}$$

All of the properties proven earlier, still hold and the last corollary now unifies all the cases:

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s \, \mathrm{d}Y_s + \int_0^t Y_s \, \mathrm{d}X_s + \langle\langle X, Y \rangle\rangle_t$$

This formula is essentially the precursor to the next proposition, known in the literature as Itô's formula. It is the stochastic analogue of the fundamental theorem of calculus:

**3.2.3 Proposition** Let $M^1, M^2, \ldots, M^n$ be continuous local martingales, $A^1, A^2, \ldots, A^m \in \mathscr{BV}$ be continuous processes of bounded variation and

$$\mathbb{R}^n \times \mathbb{R}^m \ni (\underbrace{\mu_1, \mu_2, \ldots, \mu_n}_{\boldsymbol{\mu}}; \underbrace{\alpha_1, \alpha_2, \ldots, \alpha_m}_{\boldsymbol{\alpha}}) \longmapsto F(\boldsymbol{\mu}; \boldsymbol{\alpha}) \in \mathbb{R}$$

be a $C^2$ function up to $\boldsymbol{\mu}$ and $C^1$ up to $\boldsymbol{\alpha}$. Put $\mathbf{X}_t = \left(M_t^1, \ldots, M_t^n; A_t^1, \ldots, A_t^m\right)$ for short. Then for every $t \in T$,

$$\begin{aligned} F(\mathbf{X}_t) - F(\mathbf{X}_0) = &\sum_{i=1}^n \int_0^t \frac{\partial F}{\partial \mu_i}(\mathbf{X}_s) \, \mathrm{d}M_s^i + \sum_{i=1}^n \int_0^t \frac{\partial F}{\partial \alpha_i}(\mathbf{X}_s) \, \mathrm{d}A_s^i \\ &+ \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2 F}{\partial \mu_i \partial \mu_j}(\mathbf{X}_s) \, \mathrm{d}\langle\langle M^i, M^j \rangle\rangle_s \end{aligned} \tag{$*$}$$

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***Proof***: Let $\mathscr{C}$ the space of functions $F = F(\boldsymbol{\mu}, \boldsymbol{\alpha}) \in C^1(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R})$, $C^2$ up to $\boldsymbol{\mu}$, equipped with the seminorms

$$p_K(F) = \max \left\{ \|F_n - F\|_{K,\infty}, \left\| \frac{\partial}{\partial \mu_i}(F_n - F) \right\|_{K,\infty}, \left\| \frac{\partial}{\partial \alpha_i}(F_n - F) \right\|_{K,\infty}, \left\| \frac{\partial^2}{\partial \mu_i \partial \mu_j}(F_n - F) \right\|_{K,\infty} \right\}$$

as $K$ ranges in the compact subsets of $\mathbb{R}^{n+m}$. Take $(K_n)_{n=1}^{\infty}$ a sequence of compact rectangles in $\mathbb{R}^n \times \mathbb{R}^m$, such that $K_n \subseteq \operatorname{int} K_{n+1}$ for all $n$ and $\bigcup_{n=1}^{\infty} K_n = \mathbb{R}^n \times \mathbb{R}^m$. Then the sequence of seminorms $(p_{K_n})_n$ forms a basis for the topology of uniform convergence on compacts on $\mathscr{C}$.

Let $\mathscr{H} \subseteq \mathscr{C}$ the set of functions satisfying $(*)$ for all $(m+n)$-tuples of processes $\mathbf{X}$ in hypothesis. Clearly $1 \in \mathscr{H}$ and $\mathscr{H}$ is a vector space due to linearity. Further, if $F \in \mathscr{H}$, then so do $(\boldsymbol{\mu}; \boldsymbol{\alpha}) \longmapsto \mu_k F(\boldsymbol{\mu}; \boldsymbol{\alpha})$ for any $1 \leq k \leq n$ and $(\boldsymbol{\mu}; \boldsymbol{\alpha}) \longmapsto \alpha_k F(\boldsymbol{\mu}; \boldsymbol{\alpha})$ for any $1 \leq k \leq m$: Observe that $F(\mathbf{X})$ is a semimartingale and apply the integration by parts formula for the pairs of continuous semimartingales $M^i, F(\mathbf{X})$ and $A^j, F(\mathbf{X})$. Hence $\mathscr{H}$ contains all the polynomials in $\mu_1, \mu_2, \ldots, \mu_n, \alpha_1, \alpha_2, \ldots, \alpha_m$.

Now let $(F_n)_n$ be a sequence in $\mathscr{H}$ converging to $F = F(\boldsymbol{\mu}, \boldsymbol{\alpha}) \in \mathscr{C}$. Fix $\mathbf{X}$ a $(m+n)$-tuple of processes as in hypothesis and let

$$\tau_n = n \wedge \inf \left\{ t \geq 0 \;\middle|\; X_t \in (\operatorname{int} K_n)^{\complement} \right\}, n \geq 1$$

Then $(\tau_n)_n$ is a nondecreasing sequence of stopping times, $\tau_n \uparrow +\infty$, and $\mathbf{X}^{\tau_n}$ has its paths lying in $K_n$; hence $\mathbf{X}$ is locally bounded and so are all $\frac{\partial F}{\partial \mu_i}(\mathbf{X})$, $\frac{\partial F}{\partial \alpha_i}(\mathbf{X})$, $\frac{\partial^2 F}{\partial \mu_i \partial \mu_j}(\mathbf{X})$ (so we obtain bounds for the sequence). So we can apply convergence theorems to stochastic and Lebesgue-Stieltjes integrals. To this end, observe that the polynomials are dense in $\mathscr{C}$; whence $\mathscr{H} = \mathscr{C}$. □

**3.2.4 Example** Let $w: \mathbb{R} \longmapsto \mathbb{R} : x \longmapsto w(x) = \exp(-x^2/2)$ and put $Q$ for the measure that $w$ induces in $\mathbb{R}$, that is, $Q(A) = \int_A w(x)\, \mathrm{d}\lambda(x)$, $A \in \mathscr{L}(\mathbb{R})$, where $\lambda$ is the Lebesgue measure on $\mathbb{R}$. The *Hermite's polynomials*, is an orthogonal sequence of polynomials $(h_n)_{n=1}^{\infty}$, on $L^2(\mathbb{R}; \mathscr{L}, Q)$ with $\deg h_n = n$, $n \in \mathbb{N}$. Their defining equation can be taken to be (among others),

$$h_0 = 1, \quad \int h_n \cdot h_m \cdot w \, \mathrm{d}\lambda = \delta_{nm} \sqrt{2\pi} n!, \quad n, m \in \mathbb{N}$$

We further know that,

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left( e^{-\frac{x^2}{2}} \right)$$

Putting $H_n(x, a) = a^{\frac{n}{2}} h_n \left( \frac{x}{\sqrt{a}} \right)$, $x \in \mathbb{R}$, $a > 0$, $n \in \mathbb{N}$ one obtains,

$$\left( \frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial a} \right) H_n(x, a) = 0, \quad n \in \mathbb{N}, x \in \mathbb{R}, a > 0$$

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and if $M$ is a continuous local martingale, then so is $L^{(n)} = H_n(M, \langle\langle M, M \rangle\rangle)$. This is an immediate consequence of the Itô's formula. We further can obtain a recursive equation: $H_n$ can be proven to satisfy also $\frac{\partial}{\partial x} H_n = n H_{n-1}$, $n \geq 1$, whence we obtain

$$L^{(n)} = n \int_0 L_s^{(n-1)} \, \mathrm{d}M_s$$

The first four martingales generated by $H_1, H_2, H_3, H_4$ are

$$\begin{aligned}
H_1(M, \langle\langle M, M \rangle\rangle) &= M \\
H_2(M, \langle\langle M, M \rangle\rangle) &= M^2 - \langle\langle M, M \rangle\rangle \\
H_3(M, \langle\langle M, M \rangle\rangle) &= M^3 - 3 \langle\langle M, M \rangle\rangle M \\
H_4(M, \langle\langle M, M \rangle\rangle) &= M^4 - 6 \langle\langle M, M \rangle\rangle M^2 + 3 \langle\langle M, M \rangle\rangle^2
\end{aligned}$$

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# 4. Applications

In this chapter again $(\Omega, \mathscr{F}, P)$ is a probability space, $\mathfrak{F} = (\mathscr{F}_t)_{t\in T}$, $T = [0, +\infty)$ is a filtration, such that $\mathscr{F}_0$ contains all the $P$-negligible sets of $\mathscr{F}$.

## 4.1 Bounds for $M^*_\infty$

We have already seen that for a martingale vanishing at 0, $\mathbb{E}\left[\langle\!\langle M, M\rangle\!\rangle_\infty\right] \le \mathbb{E}\left[(M^*_\infty)^2\right] \le 4\mathbb{E}\left[\langle\!\langle M, M\rangle\!\rangle_\infty\right]$. The right side inequality is a consequence of Doob's inequality for $p = 2$. Actually analogous equalities hold for any $p \ge 0$. We prove a partial result which is useful in its own.

**4.1.1 Lemma** [RY13, prop. IV.4.3] For every $p \ge 2$ there is a constant $C_p > 0$ such that for any continuous local martingale $M$ vanishing at 0

$$\mathbb{E}\left[|M^*_\infty|^p\right] \le C_p \cdot \mathbb{E}\left[\langle\!\langle M, M\rangle\!\rangle_\infty^{p/2}\right]$$

***Proof***: If $M$ is a continuous local martingale, then using the Itô's formula for $|M_t|^p$, we obtain

$$|M_\infty|^p = \int_0^\infty p\,|M_s|^{p-1} \operatorname{sgn} M_s \, \mathrm{d}M_s + \frac{1}{2}\int_0^\infty p(p-1)\,|M_s|^{p-2} \, \mathrm{d}\langle\!\langle M, M\rangle\!\rangle_s$$

Thus, using Doob's inequality and after taking expectations

$$\begin{aligned}
\mathbb{E}\left[|M^*_\infty|^p\right] &\le \left(\frac{p}{p-1}\right)^p \cdot \mathbb{E}\left[|M_\infty|^p\right] \\
&= \frac{1}{2}p(p-1)\left(\frac{p}{p-1}\right)^p \cdot \mathbb{E}\left[\int_0^\infty |M_s|^{p-2} \, \mathrm{d}\langle\!\langle M, M\rangle\!\rangle_s\right] \\
&\le \frac{p^{p+1}}{2(p-1)^{p-1}} \cdot \mathbb{E}\left[|M^*_\infty|^{p-2}\langle\!\langle M, M\rangle\!\rangle_\infty\right] \\
&\le \frac{p^{p+1}}{2(p-1)^{p-1}} \cdot \mathbb{E}\left[|M^*_\infty|^p\right]^{\frac{p-2}{p}} \cdot \mathbb{E}\left[(\langle\!\langle M, M\rangle\!\rangle_\infty)^{\frac{p}{2}}\right]^{\frac{2}{p}}
\end{aligned}$$

Thus,

$$\mathbb{E}\left[|M^*_\infty|^p\right] \le \left(\frac{p^{p+1}}{2(p-1)^{p-1}}\right)^{\frac{p}{2}} \cdot \mathbb{E}\left[(\langle\!\langle M, M\rangle\!\rangle_\infty)^{\frac{p}{2}}\right] \qquad \square$$

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## 4.2 Stochastic integral of deterministic function up to Brownian motion

**4.2.1 Proposition** Let $f: T \longmapsto \mathbb{R}$ be a locally bounded borel function and $B = (B_t)_{t\in T}$ be a standard Brownian motion. Then the process $X = \left(\int_0^t f(s)\,\mathrm{d}B_s\right)_{t\in T}$ is a centered Gaussian process with covariance $\Gamma(s,t) = \mathrm{Cov}(X_s, X_t) = \mathbb{E}\left[\int_0^{t\wedge s} f(s)^2\,\mathrm{d}B_s\right]$.

***Proof***: Fix $n \in T, \lambda \in \mathbb{R}$, and let $\varphi_\lambda(t) = \mathbb{E}\left[e^{i\lambda X_t}\right], t \le n$. This expectation always exists. We apply the Itô formula to the function $F(x) = e^{i\lambda x}$ and the stopped process $X^n \in \mathscr{M}^2_{c,0}$ and any $t \le n$, to obtain

$$\begin{aligned} e^{i\lambda X_t} - 1 &= i\lambda \int_0^t e^{i\lambda X_s}\,\mathrm{d}X_s - \frac{1}{2}\lambda^2 \int_0^s e^{i\lambda X_s}\,\mathrm{d}\langle\!\langle X, X\rangle\!\rangle_s \\ &= i\lambda \int_0^t e^{i\lambda X_s} f(s)\,\mathrm{d}B_s - \frac{1}{2}\lambda^2 \int_0^s e^{i\lambda X_s} f(s)^2\,\mathrm{d}s \end{aligned}$$

The first term on the right is a martingale (as $f \in L^2([0,n])$), and we can take expectations, use the fubini's theorem to obtain

$$\varphi_\lambda(t) = 1 - \frac{1}{2}\lambda^2 \int_0^t \varphi_\lambda(s) f(s)^2\,\mathrm{d}s, \quad t \le n$$

This integral equation has a unique solution

$$\varphi_\lambda(t) = \exp\left(-\frac{1}{2}\int_0^t f(s)^2\,\mathrm{d}s \cdot \lambda^2\right), t \le n$$

But $n$ was arbitrary, and as solution is unique, this identity holds for all $t \in T$.

Thus, for every $t \in T$, $X_t$ is a normally distributed random variable, with mean 0 and variance $\int_0^t f(s)^2\,\mathrm{d}s$. Quite similarly, for $s < t \in T$, $X_t - X_s = \int_s^t f(s)\,\mathrm{d}B_s$ is normally distributed, with mean 0 and variance $\int_s^t f(s)^2\,\mathrm{d}s$ and further, independent of $\mathscr{F}_s$.

Let now $s \le t \in T$,

$$\begin{aligned} \mathrm{Cov}(X_s, X_t) &= \mathbb{E}\left[\int_0^s f(u)\,\mathrm{d}B_u \cdot \int_0^t f(u)\,\mathrm{d}B_u\right] \\ &= \mathbb{E}\left[\left(\int_0^s f(u)\,\mathrm{d}B_u\right)^2\right] + \mathbb{E}\left[\int_0^s f(u)\,\mathrm{d}B_u \cdot \int_s^t f(u)\,\mathrm{d}B_u\right] \\ &= \mathbb{E}\left[\int_0^s f(u)^2\,\mathrm{d}B_u\right] + \mathbb{E}\left[\int_0^s f(u)\,\mathrm{d}B_u\right] \cdot \mathbb{E}\left[\int_s^t f(u)\,\mathrm{d}B_u\right] \\ &= \mathbb{E}\left[\int_0^s f(u)^2\,\mathrm{d}B_u\right] \end{aligned}$$

□

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## 4.3 Exponential martingales

**4.3.1 Proposition** [RY13, prop. IV.3.4] Let $f:\mathbb{R}\times T\longmapsto\mathbb{C}:(x,t)\longmapsto f(x,t)$ such that $=\frac{\partial^2 f}{\partial x^2},\frac{\partial f}{\partial t}\in C(\mathbb{R}\times T)$ and satisfy $\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}=0$. If $M$ is a continuous local martingale, then so does $f(M,\langle\!\langle M,M\rangle\!\rangle)=(f(M_t,\langle\!\langle M,M\rangle\!\rangle_t))_{t\in T}$.

***Proof***: Direct application to the Itô formula for the function $f=f(x,t)$ replacing $x$ by $M_t$ and $t$ by $\langle\!\langle M,M\rangle\!\rangle_t$. □

**4.3.2 Remark** When $f(x,t)=\exp(\lambda x-\frac{1}{2}\lambda^2 t)$, the local martingale

$$\mathsf{E}^\lambda(M)=\left(\exp\left(\lambda M_t-\frac{1}{2}\lambda^2\langle\!\langle M,M\rangle\!\rangle_t\right)\right)_{t\in T}$$

is called the **exponential of** $M$. When $\lambda=1$ we just write $\mathsf{E}(\mathrm{M})$.

Actually $\mathsf{E}^\lambda(M)$ is a supermartingale as a non-negative local martingale: $\mathsf{E}^\lambda(M)\geq 0$. If we know that the expectation $\mathbb{E}\left[\mathsf{E}^\lambda(M)\right]$ exists and is constant, then for any $s<t$, the inequality $\mathbb{E}\left[\mathsf{E}^\lambda(M)_t\,\middle|\,\mathscr{F}_s\right]\leq\mathsf{E}^\lambda(M)_s$ and the equality $0=\mathbb{E}\left[\mathsf{E}^\lambda(M)_t-\mathsf{E}^\lambda(M)_s\right]=\mathbb{E}\left[\mathbb{E}\left[\mathsf{E}^\lambda(M)_t\,\middle|\,\mathscr{F}_s\right]-\mathsf{E}^\lambda(M)_s\right]$ lead to $\mathbb{E}\left[\mathsf{E}^\lambda(M)_t\,\middle|\,\mathscr{F}_s\right]=\mathsf{E}^\lambda(M)_s$.

One can prove a reverse: If $M$ is a continuous $\mathfrak{F}$-adapted process, $A\in\mathscr{BV}(\mathfrak{F})$ is continuous, and for every $\lambda\in\mathbb{R}$ the process $E(\lambda)=\left(\exp\left(\lambda M_t-\frac{1}{2}\lambda^2 A_t\right)\right)_{t\in T}$ is a local martingale, then so is $M$ and $\langle\!\langle M,M\rangle\!\rangle=A$. Indeed, pick any nonzero $\lambda\in\mathbb{R}$, and exploit the identity $E(2\lambda)e^{\lambda^2 A}=E(\lambda)^2$. Whence, for any $t\in T$, we obtain

$$E(\lambda)_t^2-E(\lambda)_0^2=\int_0^t E(2\lambda)_s\,\mathrm{d}e^{\lambda^2 A_s}+\int_0^t e^{\lambda^2 A_s}\,\mathrm{d}E(2\lambda)_s$$

and we conclude that $\langle\!\langle E(\lambda),E(\lambda)\rangle\!\rangle_t=\int_0^t E(2\lambda)_s\,\mathrm{d}e^{\lambda^2 A_s}$. Now $E(\lambda)>0$ always, so we can apply the Itô's formula to $F(x)=\ln x, x>0$ and to the local martingale $E(\lambda)$:

$$\lambda(M_t-M_0)-\frac{1}{2}\lambda^2(A_t-A_0)=\int_0^t\frac{\mathrm{d}E(\lambda)_s}{E(\lambda)_s}-\frac{1}{2}\int_0^t\frac{\mathrm{d}\langle\!\langle E(\lambda),E(\lambda)\rangle\!\rangle_s}{E(\lambda)_s^2}$$

But $\int_0^t\frac{\mathrm{d}\langle\!\langle E(\lambda),E(\lambda)\rangle\!\rangle_s}{E(\lambda)_s^2}=\int_0^t\frac{E(2\lambda)_s\,\mathrm{d}e^{\lambda^2 A_s}}{E(\lambda)_s^2}=\int_0^t e^{-\lambda^2 A_s}\,\mathrm{d}e^{\lambda^2 A_s}=\lambda^2(A_t-A_0)$. So we conclude immediately that $M$ is a local martingale, since $\lambda M_t=\lambda M_0+\int_0^t\frac{\mathrm{d}E(\lambda)_s}{E(\lambda)_s}$, for all $t\in T$. Further

$$\lambda^2\langle\!\langle M,M\rangle\!\rangle_t=\left\langle\!\!\left\langle\int_0\frac{\mathrm{d}E(\lambda)_s}{E(\lambda)_s},\int_0\frac{\mathrm{d}E(\lambda)_s}{E(\lambda)_s}\right\rangle\!\!\right\rangle_t=\int_0^t\frac{\mathrm{d}\langle\!\langle E(\lambda),E(\lambda)\rangle\!\rangle_s}{E(\lambda)_s^2}=\lambda^2(A_t-A_0)$$

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and this completes our claim.

Observe, further that $E^\lambda(M)$, due to Itô's formula, satisfies the following equation

$$E^\lambda(M)_t - E^\lambda(M)_0 = \lambda \int_0^t E^\lambda(M)_s \, \mathrm{d}M_s, \quad t \in T$$

and actually, this equation characterizes $\mathrm{E}(\lambda)$: For if $Z$ would satisfy the same equation, namely $Z_t - E^\lambda(M)_0 = \lambda \int_0^t Z_s \, \mathrm{d}M_s$, for all $t$, then applying the Itô's formula to the function $F(z,x) = z/x, z \in \mathbb{R}, x > 0$ and to the semimartingale $Z$ and the local martingale $E(\lambda)$, for every $t \in T$,

$$F(Z_t, E(\lambda)_t)|_0^t = \int_0^t \frac{\mathrm{d}Z_s}{E(\lambda)_s} - \int_0^t \frac{Z_s \, \mathrm{d}E(\lambda)_s}{E(\lambda)_s^2} - \int_0^t \frac{\mathrm{d}\langle\!\langle Z, E(\lambda)\rangle\!\rangle_s}{E(\lambda)_s^2} + \int_0^t \frac{Z_s \, \mathrm{d}\langle\!\langle E(\lambda), E(\lambda)\rangle\!\rangle_s}{E(\lambda)_s^3}$$

but this sum vanishes, for the terms on the right cancel each other:

$$\int_0^t \frac{\mathrm{d}Z_s}{E(\lambda)_s} = \int_0^t \frac{Z_s \, \mathrm{d}M_s}{E(\lambda)_s}, = \int_0^t \frac{Z_s \, \mathrm{d}E(\lambda)_s}{E(\lambda)_s^2} \quad \text{for every } t \in T$$
$$\frac{\langle\!\langle Z, E(\lambda)\rangle\!\rangle_s}{E(\lambda)_s^2} = \lambda^2 \frac{Z_s \langle\!\langle M, M\rangle\!\rangle_s}{\mathrm{E}(\lambda)_s} = \frac{Z_s \, \mathrm{d}\langle\!\langle E(\lambda), E(\lambda)\rangle\!\rangle_s}{E(\lambda)_s^3} \quad \text{for every } s \in T$$

**4.3.3 Remark** Especially, when $M_t = \int_0^t f(s) \, \mathrm{d}B_s$, where $f : \mathbb{R}_+ \longmapsto \mathbb{R}$ is a locally bounded borel function, the exponential $\mathsf{E}(M)$ of $M$, which will be denoted by

$$\mathsf{E}^f(B)_t = \exp\left(\int_0^t f(s) \, \mathrm{d}B_s - \frac{1}{2}\int_0^t f(s)^2 \, \mathrm{d}s\right), \quad t \in T$$

is a martingale: First observe that $M_t \sim N\left(0, \int_0^t f(s)^2 \, \mathrm{d}s\right)$ and so $\mathbb{E}\left[e^{M_t}\right] = \exp\left(\frac{1}{2}\int_0^t f(s)^2 \, \mathrm{d}s\right)$, that is, $\mathbb{E}\left[\mathsf{E}^f(B)_t\right] = 1$ for all $t \in T$. Now apply the previous remark.

The following two propositions, known as "Kazamaki's criterion" and "Novikov's condition" can give sufficient conditions so that the local martingale $\mathsf{E}(L)$ is a true martingale.

**4.3.4 Proposition** [RY13, prop. VIII.1.14] If $L$ is a continuous local martingale vanishing at 0, such that $e^{1/2 L}$ is a uniformly integrable submartingale, then $\mathsf{E}(L)$ is a uniformly integrable martingale.

***Proof***: Let $a \in (0,1)$ and $Z(a)_t = \exp\left(\frac{a}{(1+a)} L_t\right)$. Then $\mathsf{E}(aL) = \mathsf{E}(L)^{a^2} Z(a)^{1-a^2}$. By the optional stopping theorem, $\{\mathsf{E}(1/2 L)_\tau \mid \tau \text{ is a stopping time}\}$ is uniformly integrable and as $a/(1+a) < 1/2$, so does $\{Z(a)_\tau \mid \tau \text{ is a stopping time}\}$. Now observe that $\mathsf{E}(L)$ is also a

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supermartingale, whence $\mathbb{E}\left[\mathsf{E}(L)_\tau\right] \leq 1$ for any stopping time and by the Hölder inequality, for any stopping time $\tau$ and $A \in \mathscr{F}$,

$$\mathbb{E}\left[1_A \mathsf{E}(aL)_\tau\right] \leq \mathbb{E}\left[\mathsf{E}(L)_\tau\right]^{a^2} \cdot \mathbb{E}\left[1_A Z(a)_\tau\right]^{1-a^2} \leq \mathbb{E}\left[1_A Z(a)_\tau\right]^{1-a^2}, \quad t \in T$$

Hence $\{\mathsf{E}(aL)_\tau \mid \tau \text{ is a stopping time}\}$ is also uniformly integrable, so $\mathsf{E}(aL)$ is a uniformly integrable martingale. So $\mathsf{E}(aL)_\infty$ exists (and is integrable), it is the a.s. pointwise and $L^1$ limit of $\mathsf{E}(aL)_t$ as $t \to +\infty$ and so $\mathbb{E}\left[\mathsf{E}(aL)_\infty\right] = 1$. Further $L_\infty$ exists $P$-a.s. and so does $\exp(L_\infty)$. Further, $\exp(L_\infty / 2)$ is integrable (hypothesis) and $Z(a)_\infty \leq 1_{[L_\infty \leq 0]} + 1_{[L_\infty > 0]} \exp(L_\infty / 2)$. Now $\lim_{a \to 1} Z(a)_\infty = \exp(L_\infty / 2)$ $P$-a.s. and $1 = \mathbb{E}\left[\exp(L_\infty / 2)\right] = \lim_{a \to 1^-} \mathbb{E}\left[Z(a)_\infty\right]$. So $\mathbb{E}\left[\mathsf{E}(L)_\infty\right] = 1$ and this completes the proof. □

**4.3.5 Proposition** [RY13, prop. VIII.1.15] If $L$ is a continuous local martingale vanishing at 0, such that $\mathbb{E}\left[\exp\left(\frac{1}{2}\langle\!\langle L, L\rangle\!\rangle_\infty\right)\right] < +\infty$, then $\mathsf{E}(L)$ is a uniformly integrable martingale, $\mathbb{E}\left[\exp\left(\frac{1}{2} L^*_\infty\right)\right] < +\infty$ and $L^*_\infty$ has moments of any order.

***Proof***: $\langle\!\langle L, L\rangle\!\rangle_\infty \geq 0$ and for $x > 0$, $x^n \leq \frac{n!}{\lambda^n} \cdot e^{\lambda x}$, so $\mathbb{E}\left[\langle\!\langle L, L\rangle\!\rangle^n_\infty\right] < +\infty$ for all $n \geq 1$, and this implies that $\mathbb{E}\left[(L^*_\infty)^{2n}\right] < C_{2n} \mathbb{E}\left[\langle\!\langle L, L\rangle\!\rangle^n_\infty\right] < +\infty$ so both $\langle\!\langle L, L\rangle\!\rangle_\infty$ and $L^*_\infty$ have moments of any order. From the identity $e^{1/2 L^*_\infty} = (\mathsf{E}(L)_\infty)^{1/2} \cdot e^{1/4 \langle\!\langle L, L\rangle\!\rangle_\infty}$ applied to the Cauchy-Schwarz inequality and the fact that $\mathsf{E}(L)$ is a positive supermartingale,

$$\mathbb{E}\left[\exp\left(\frac{1}{2} L^*_\infty\right)\right] \leq \mathbb{E}\left[\mathsf{E}(L)_\infty\right]^{\frac{1}{2}} \cdot \mathbb{E}\left[\exp\left(\frac{1}{2}\langle\!\langle L, L\rangle\!\rangle_\infty\right)\right]^{\frac{1}{2}} \leq \mathbb{E}\left[\exp\left(\frac{1}{2}\langle\!\langle L, L\rangle\!\rangle_\infty\right)\right]^{\frac{1}{2}}$$

Thus, by the previous proposition, $\mathsf{E}(L)$ is a uniformly integrable martingale.

We now prove that $\mathbb{E}\left[\exp\left(\frac{1}{2} L^*_\infty\right)\right] < +\infty$. In a similar fashion for any $c \in \mathbb{R}$ and $t \in T$,

$$X(c)_t \stackrel{\text{def}}{=} \exp\left(\frac{1}{2} c L_t\right) = (\mathsf{E}(L)_t)^{\frac{c}{2}} \cdot \exp\left(\frac{1}{4} c \langle\!\langle L, L\rangle\!\rangle_t\right)$$

Let now $c < 1$ and apply the Doob's inequality to the positive submartingale $X(c)$ with $p = 1/c > 1$,

$$\begin{aligned}
\mathbb{E}\left[\sup_{t \in T} e^{1/2 L_t}\right] &= \mathbb{E}\left[\sup_{t \in T} X(c)^p_t\right] \leq K_p \cdot \mathbb{E}\left[(X(c)_\infty)^p\right] \\
&= K_p \cdot \mathbb{E}\left[(\mathsf{E}(L)_\infty)^{\frac{1}{2}} \cdot \exp\left(\frac{1}{4}\langle\!\langle L, L\rangle\!\rangle_\infty\right)\right] \\
&\leq K_p \cdot \mathbb{E}\left[(\mathsf{E}(L)_\infty)\right]^{\frac{1}{2}} \cdot \mathbb{E}\left[\exp\left(\frac{1}{2}\langle\!\langle L, L\rangle\!\rangle_\infty\right)\right]^{\frac{1}{2}} \\
&\leq K_p \cdot \mathbb{E}\left[\exp\left(\frac{1}{2}\langle\!\langle L, L\rangle\!\rangle_\infty\right)\right]^{\frac{1}{2}} < +\infty
\end{aligned}$$

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Nothing prevents us to put replace $L$ by $-L$ in the previous arguments, whence we also get $\mathbb{E}\left[\sup_{t\in T} e^{-1/2 L_t}\right] < +\infty$. Thus,

$$\mathbb{E}\left[\exp\left(\frac{1}{2}L_\infty^*\right)\right] = \mathbb{E}\left[\sup_{t\in T}\exp\left(\frac{1}{2}|L_t|\right)\right] \leq \mathbb{E}\left[\sup_{t\in T}\exp\left(\frac{1}{2}L_t\right) + \exp\left(-\frac{1}{2}L_t\right)\right]$$
$$\leq \mathbb{E}\left[\sup_{t\in T}\exp\left(\frac{1}{2}L_t\right)\right] + \mathbb{E}\left[\sup_{t\in T}\exp\left(-\frac{1}{2}L_t\right)\right] < +\infty$$

and this completes the proof. □

These two results, have also finite interval counterpatrs:

**4.3.6 Proposition** [RY13, prop. IV.1.14] If $L$ is a continuous local martingale vanishing at 0, such that $\exp\left(\frac{1}{2}L\right)$ is a submartingale or $\mathbb{E}\left[\exp\left(\frac{1}{2}\langle\langle L, L\rangle\rangle_t\right)\right]$ is finite for all $t \in T$. then $\mathsf{E}(L)$ is a martingale.

**4.3.7 Example** Let $W = (W^1, W^2, \ldots, W^d)$ be a $d$-dimensional Brownian motion and $X = (X^1, X^2, \ldots, X^d)$ be a $d$-dimensional progressively measurable process such that

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t |X_s|^2 \, \mathrm{d}s\right)\right] < +\infty, \quad t \in T$$

then $\mathsf{E}\left(\int_0 X_s \, \mathrm{d}W_s\right) = \exp\left(\int_0 X_s \, \mathrm{d}W_s - \frac{1}{2}\int_0 |X_s|^2 \, \mathrm{d}s\right)$ is a martingale.

The following proposition is known as "Lévy's characterization" of Brownian motion:

**4.3.8 Proposition** [RY13, prop. IV.3.6] Let $\mathbf{X} = \left((X_t^1, X_t^2, \ldots, X_t^d)\right)_{t\in T}$ be a $d$-dimensional $\mathfrak{F}$-adapted continuous process such that $X_0 = 0$. Then the following are equivalent.

i) $\mathbf{X}$ is a $d$-dimensional Brownian motion
ii) $\mathbf{X}$ is a continuous local martingale and $\langle\langle X^i, X^j\rangle\rangle_t = \delta_{ij}t$, for all $t \in T$, $1 \leq i, j \leq d$.
iii) $\mathbf{X}$ is a continuous local martingale and for every $d$-tuple of functions $\mathbf{f} = (f_1, f_2, \ldots, f_d)$, with $f_i \in L^2(\mathbb{R}_+)$, $1 \leq i \leq d$ the process $\mathsf{E}^{if}$ defined by

$$\mathsf{E}_t^{i\mathbf{f}} = \exp\left(i\sum_{k=1}^d \int_0^t f_k(s)\,\mathrm{d}X_s^k + \frac{1}{2}\sum_{k=1}^d \int_0^t f_k(s)^2\,\mathrm{d}s\right)$$

is a martingale.

***Proof***: i)⟶ii) is a known property. ii)⟶iii) The exponential $\mathsf{E}^i(M)$ with $M$ defined by

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$M_t = \sum_{k=1}^{d} \int_0^t f_k(s)\, \mathrm{d}X_s^k$, $t \in T$, is a (positive) bounded continuous local martingale:

$$\left|\mathsf{E}_t^{i\mathbf{f}}\right| = \exp\left(\frac{1}{2}\sum_{k=1}^{d}\int_0^t f_k(s)^2\, \mathrm{d}s\right) \leq C := \exp\left(\frac{1}{2}\sum_{k=1}^{d}\|f_k\|_2^2\right)$$

hence a martingale. iii)⟶i) Let $\xi \in \mathbb{R}^d$ and $b \in T$. Put $\mathbf{f} = \xi \cdot 1_{[0,b]}$ and we obtain $\mathsf{E}_t^{i\mathbf{f}} = \exp\left(i\,\langle \xi, \mathbf{X}_{b\wedge t}\rangle + \frac{1}{2}\,|\xi|^2 \cdot (b \wedge t)\right)$, for all $t \in T$. If $s \leq t \leq b$, $\eta \in \mathbb{R}$ and $A \in \mathscr{F}_s$, then

$$\mathbb{E}\left[e^{i\eta 1_A}\right] = = \mathbb{E}\left[e^{i\eta 1_A} \cdot \frac{\mathbb{E}\left[\mathsf{E}_t^{i\mathbf{f}} \,\middle|\, \mathscr{F}_s\right]}{\mathsf{E}_s^{i\mathbf{f}}}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{i\eta 1_A} \cdot \frac{\mathsf{E}_t^{i\mathbf{f}}}{\mathsf{E}_s^{i\mathbf{f}}} \,\middle|\, \mathscr{F}_s\right]\right] = \mathbb{E}\left[e^{i\eta 1_A} \cdot \frac{\mathsf{E}_t^{i\mathbf{f}}}{\mathsf{E}_s^{i\mathbf{f}}}\right]$$
$$= \mathbb{E}\left[e^{i\eta 1_A} \cdot \exp\left(i\,\langle \xi, \mathbf{X}_t - \mathbf{X}_s\rangle + \frac{1}{2}\,|\xi|^2\,(t-s)\right)\right]$$

But $\eta, \xi$ and $b$ were arbitrarily chosen, whence $\mathbf{X}_t - \mathbf{X}_s \sim \mathrm{N}(0, t-s)$, for every $s \leq t \in T$ and even more $\mathbf{X}_t - \mathbf{X}_s$ is independent of $1_A$, for every $A \in \mathscr{F}_s$, so $\mathbf{X}_t - \mathbf{X}_s$ is independent of $\mathscr{F}_s$.□

## 4.4 The reflection principle

**4.4.1 Example** Let us now see an application of Lévy's characterization on the Brownian motion. This is known as the **reflection principle**: At the time of stopping the reflected Brownian motion is equivalent to the original Brownian motion. Namely, if $\boldsymbol{B}$ is a $\mathfrak{F}$-Brownian motion and $\tau$ any $P$-a.s. finite $\mathfrak{F}$-stopping time, the process $\boldsymbol{B}^{(r)} = 2\boldsymbol{B}^{\tau} - \boldsymbol{B}$ is again a Brownian motion. Indeed, by its definition, $1_{(0,\tau]} \in L^2(\boldsymbol{B})$, so the process

$$\int_0^t 1_{[0,\tau]} \cdot 2 - 1\, \mathrm{d}\boldsymbol{B}_s = 2\boldsymbol{B}_{\tau\wedge t} - \mathrm{B}_t = \boldsymbol{B}_t^{(r)}, \quad t \in T$$

is indeed a local martingale, and further

$$\left\langle\!\left\langle \boldsymbol{B}^{(r)}, \boldsymbol{B}^{(r)} \right\rangle\!\right\rangle_t = \int_0^t (1_{[0,\tau]} \cdot 2 - 1)^2\, \mathrm{d}s = t, \quad t \in T$$

thus, by Lévy's characterization, $\boldsymbol{B}^{(r)}$ is indeed a Brownian motion.

Thus, for $h, b \geq 0$ and the stopping time $\tau = \inf\{t \in T : \boldsymbol{B}_t \geq b\}$ we obtain

$$[\tau \leq t] \cap \left[\boldsymbol{B}_t < b - h\right] = [\boldsymbol{B}_t^{(r)} > b + h]$$

Indeed, if $\tau \leq t$ then $\boldsymbol{B}_t = 2\boldsymbol{B}_\tau - \boldsymbol{B}_t^{(r)} = 2b - \boldsymbol{B}_t^{(r)}$ and hence $\boldsymbol{B}_t < b + h$ is equivalent to $\boldsymbol{B}_t^{(r)} > b - h$. And conversely, $\boldsymbol{B}_t^{(r)} > b + h$ and $\tau > t$ cannot hold together, else $\boldsymbol{B}_t^{(r)} = \boldsymbol{B}_t > b + h$

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which implies that $\tau \leq t$, a contradiction. Hence $t \leq t$ and $b + h < B^{(r)} = 2B_\tau - B_t = 2b - B_t$. Now, for the running maximum $B_t^*$, observe that $[\tau \leq t] = [B_t^* \geq b]$ and we are lead to

$$[B_t^* \geq b] \cap [B_t < b - h] = [B_t^{(r)} > b + h]$$

Hence, we obtain the joint law of $B_t^*, B_t$:

$$\mathrm{P}\left([B_t^* \geq b] \cap [B_t < b - h]\right) = \mathrm{P}\left([B_t^{(r)} > b + h]\right) = \frac{1}{\sqrt{2\pi t}} \int_{b+h}^{\infty} e^{-1/2t\, x^2}\, \mathrm{d}x$$

and

$$\begin{aligned}\mathrm{P}\left([\tau \leq t]\right) &= \mathrm{P}\left([B_t^* \geq b]\right) = \mathrm{P}\left([B_t^* \geq b] \cap [B_t < b]\right) + \mathrm{P}\left([B_t^* \geq b] \cap [B_t > b]\right) \\ &= 2\,\mathrm{P}\left([B_t > b]\right) = \frac{2}{\sqrt{2\pi t}} \int_b^{\infty} e^{-1/2t\, x^2}\, \mathrm{d}x = \mathrm{P}\left(|B_t| \geq b\right)\end{aligned}$$

One can easily prove that, for every $a, t > 0$,

$$\mathrm{P}\left([B_t^* \geq at]\right) \leq \exp\left(-at^2 / 2\right)$$

## 4.5 Representation of martingales

In this and the following section we follow [RY13, ch. V]

**4.5.1 Proposition** Let $B = (B_t)_{t \in T}$ be a Brownian motion along with its natural filtration $\mathfrak{F}^B = \left(\mathscr{F}_t^B\right)_{t \in T}$ augmented to satisfy the usual conditions. If $F \in L^2(\Omega, \mathscr{F}_\infty^B, P)$, then there is a $P$-unique progressively measurable process $H \in L^2(B)$ such that

$$F = \mathbb{E}[F] + \int_0^{\infty} H_s\, \mathrm{d}B_s$$

***Proof***: Define the (nonlinear) operator

$$\begin{aligned}&e\colon L^2(T) \longmapsto L^2(\Omega, \mathscr{F}_\infty, P)\,: \\ &\qquad f \longmapsto e(f) = \mathsf{E}^f(B)_\infty = \exp\left(\int_0^{\infty} f(s)\, \mathrm{d}B_s - \frac{1}{2}\int_0^{\infty} |f(s)|^2\, \mathrm{d}s\right)\end{aligned}$$

We already know that $\mathsf{E}^f$ is a martingale, and we estimate

$$\mathbb{E}\left[(\mathsf{E}^f(B)_t)^2\right] = \mathbb{E}\left[\mathsf{E}^{2f}(B)_t \cdot \exp\left(\int_0^t f(s)^2\, \mathrm{d}s\right)\right] = \exp\left(\int_0^t f(s)^2\, \mathrm{d}s\right) \leq e^{\|f\|_2^2} < +\infty$$

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thus $\mathsf{E}^f(B) \in \mathscr{M}_c^2$ and the latter is isometric to $L^2(\Omega, \mathscr{F}_\infty^B, P)$ through the mapping $\mathsf{E}^f(B) \longmapsto \mathsf{E}^f(B)_\infty$. Hence $e$ is indeed well defined[†].

Now consider the vector space of the step functions with compact support:

$$E_{00} = \operatorname{span}\left\{1_{(0,b]} \mid b > 0\right\} \subset L^2(T)$$

We claim that $e(E_{00})$ separates the points of $L^2(\Omega, \mathscr{F}_\infty, P)$. Indeed: Let $X \in L^2(\Omega, \mathscr{F}_\infty, P)$ such that $\langle e(f), X\rangle = 0$ for all $f \in E_{00}$. That is, for any $t_1 < \ldots < t_n \in T$, $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$,

$$\mathbb{E}\left[\exp\left(\sum_{k=1}^{n} \lambda_k B_{t_k} - \frac{1}{2}\lambda_k^2 t_k\right) \cdot X\right] = 0$$

Define $\nu$ to be the measure: $\nu(A) = \mathbb{E}[1_A \cdot X]$, $A \in \mathscr{F}_\infty^B$. Now fix, $t_1 < \ldots < t_n \in T$ and put $\mathbf{Y} = \left(B_{t_1}, B_{t_2}, \ldots, B_{t_n}\right)$. Then

$$\varphi_Y \colon \mathbb{C}^n \longmapsto \mathbb{C} : \mathbf{z} = (z_1, z_2, \ldots, z_n) \longmapsto \varphi_Y(\mathbf{z}) = \mathbb{E}\left[e^{-2\pi i\langle \mathbf{z}, \mathbf{Y}\rangle} \cdot X\right]$$

is analytic and vanishes when $\mathbf{z}$ is real, that is $\varphi_Y = 0$ identically. Thus the fourier transform of the measure $\mathbf{Y}^*\nu$ vanishes, and so does the measure $\mathbf{Y}^*\nu$. Whence for every bounded borel function $f$ in $\mathbb{R}^n$, $\int_{\mathbb{R}^n} f \, \mathrm{d}\mathbf{Y}^*\nu = \int_\Omega f(\mathbf{Y}) \, \mathrm{d}\nu = \mathbb{E}[f(\mathbf{Y}) \cdot X] = 0$. Now it is a standard monotone argument to conclude that $\mathbb{E}[f \cdot X] = 0$ also for every $\mathscr{F}_\infty^B = \sigma(\{B_t \mid t \in T\})$-measurable function $f$, thus proving that $X = 0$, $P$-a.s.

Let

$$H = \left\{\mu + \int_0^\infty K_s \, \mathrm{d}B_s \;\middle|\; \mu \in \mathbb{R}, K \in L^2(B)\right\}$$

This is a closed linear subspace of $L^2(\Omega, \mathscr{F}_\infty, P)$ containing $e(E_{00})$ which separates points of $L^2(\Omega, \mathscr{F}_\infty^B, P)$, so its linear hull is weakly dense. Thus

$$L^2(\Omega, \mathscr{F}_\infty^B, P) = \overline{\operatorname{span} e(E_{00})}^{w} = \overline{\operatorname{span} e(E_{00})}^{\|\|} \subseteq H \subseteq L^2(\Omega, \mathscr{F}_\infty^B, P)$$

which finishes our proof. □

[†] We can further estimate $\|e(f)\|_2^2 = e^{\|f\|_2^2} \geq 1$. In a similar fashion $\|e(f) - e(g)\|_2^2 = e^{\|f\|_2^2} + e^{\|g\|_2^2} - 2e^{\langle f, g\rangle}$. Apart from being continuous, $e(f)$, has closed image: Indeed, if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L^2(T)$ such that $e(f_n) \xrightarrow{L^2} F \in L^2(\Omega, \mathscr{F}_\infty^B, P)$, then $\|F\|_2 \geq 1$ and $\|f_n\|_2 = \ln \|e(f_n)\|_2 \longrightarrow \ln \|F\|_2$. Further,

$$0 \leq e^{\|f_n\|_2 \cdot \|f_m\|_2} - e^{\langle f_m, f_n\rangle} \leq e^{\frac{\|f_n\|_2^2 + \|f_m\|_2^2}{2}} - e^{\langle f_m, f_n\rangle} \leq \frac{1}{2}\left(e^{\|f_n\|_2^2} + e^{\|f_m\|_2^2} - e^{\langle f_m, f_n\rangle}\right) \longrightarrow 0$$

Thus $(f_n)_{n\in\mathbb{N}}$ is Cauchy and converges to some $f \in L^2(T)$.

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**4.5.2 Proposition** With the assumptions and the symbols of the previous proposition, every local $\mathfrak{F}^B$-martingale $M = (M_t)_{t\in T}$ can be written as

$$M_t = \mu + \int_0^t K_s \, \mathrm{d}B_s, \quad t \in T$$

where $\mu \in \mathbb{R}$ and $K \in L^2_{\mathrm{loc}}(B)$.

***Proof***: If $M \in \mathscr{M}^2$ then just apply the previous proposition to $M_\infty$ and observe that $M_t = \mathbb{E}\left[M_\infty \,\middle|\, \mathscr{F}_t^B\right]$ a.s. and $\int_0^t K_s \, \mathrm{d}B_s = \mathbb{E}\left[\int_0^\infty K_s \, \mathrm{d}B_s \,\middle|\, \mathscr{F}_t^B\right]$ a.s. as well, for all $t \in T$.

If $M$ is merely a uniformly integrable martingale, then exploit density of $L^2(\Omega, \mathscr{F}_\infty^B)$ in $L^1(\Omega, \mathscr{F}_\infty^B)$ to obtain a sequence $(M_n)_n$ of r.v. in $L^2(\Omega, \mathscr{F}_\infty^B)$ and converging in $L^1$ to $M_\infty$. Put $M_t^n = \mathbb{E}\left[M_n \,\middle|\, \mathscr{F}_t^B\right]$ for every $t \in T$. Let $\lambda > 0$; by the Doob's maximal inequality

$$\mathrm{P}\left(\left[\sup_{t\in T} |M_t - M_t^n| \geq \lambda\right]\right) \leq \frac{1}{\lambda} \|M_\infty - M_n\|_1$$

leads us to a subsequence $M^{n_k} \longrightarrow M$ uniformly on $T$, $P$-a.s. Thus $M$ has a continuous modification.

If $M$ is a $\mathfrak{F}^B$-local martingale then, it has a continuous modification, denoted again by $M$. So we can find a nondecreasing sequence of stopping times $(\tau_n)_n$ –its convenient to put $\tau_0 = 0$, such that $\tau_n \uparrow +\infty$ a.s. and $1_{[\tau_n>0]} M^{\tau_n}$ is bounded, continuous (hence in $\mathscr{M}_c^2$), for every $n$. Thus for every $n$ we can find some $K^n \in L^2(B)$ and $\mu_n \in \mathbb{R}$ such that

$$1_{[\tau_n>0]} M_{\tau_n \wedge t} = \mu_n + \int_0^t K_s^n \, \mathrm{d}B_s, n \geq 1, t \in T$$

Note that $\int_0^t K_s^m \, \mathrm{d}B_s = \int_0^t 1_{(0,\tau_m]} K_s^n \, \mathrm{d}B_s$ for all $t \in T$ and $m < n$, whence

$$\begin{aligned}
\int_0^t 1_{(0,\tau_m]} K_s^n \, \mathrm{d}B_s + 1_{[\tau_m>0]} \cdot \mu_n &= 1_{[\tau_m>0]} \cdot \left(\int_0^t 1_{(0,\tau_n]} K_s^n \, \mathrm{d}B_s + \mu_n\right)_{\tau_m \wedge t} \\
&= 1_{[\tau_m>0]} 1_{[\tau_n>0]} M_{\tau_n \wedge \tau_m \wedge t} \\
&= 1_{[\tau_m>0]} M_{\tau_m \wedge t} \\
&= \int_0^t 1_{(0,\tau_m]} K_s^m \, \mathrm{d}B_s + \mu_m
\end{aligned}$$

So $(\mu_n)_n$ is a constant sequence (take $t = 0$) and the last equation leads us to

$$\int_0^t 1_{(0,\tau_m]} (K_s^n - K_s^m) \, \mathrm{d}B_s = 0, \quad t \in T$$

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Thus we can now define

$$K = \sum_{n=1}^{\infty} 1_{(\tau_{n-1},\tau_n]} K^n$$

and be unambiguous when integrating. Obviously, this process is in $L^2_{\text{loc}}(B)$ and satisfies the desired equation. □

**4.5.3 Proposition** Let $M$ be a continuous local martingale relative to its own natural filtration $\mathfrak{F}^M = \left(\mathcal{F}_t^M\right)_{t\in T}$ which is augmented to satisfy the usual conditions. We suppose that the measure $\mathrm{d}\langle\langle M, M\rangle\rangle$ is equivalent to the lebesgue measure $\lambda$, a.s. Then, there is a progressively measurable process $K$, strictly positive $\lambda \otimes P$-a.s. and an $\mathfrak{F}^M$-Brownian motion $B$ such that

$$\int_0^t K_s \,\mathrm{d}s = \langle\langle M, M\rangle\rangle_t \text{ and } M_t = M_0 + \int_0^t \sqrt{K_s}\,\mathrm{d}B_s, \quad t \in T$$

***Proof***: Take $K_t = \lim\limits_{n\to+\infty} n \cdot \left(\langle\langle M, M\rangle\rangle_t - \langle\langle M, M\rangle\rangle_{t-\frac{1}{n}}\right)$, that is, the Radon Nikodym derivative of $\mathrm{d}\langle\langle M, M\rangle\rangle$. Then $K$ is by its definition progressively measurable, satisfies the first equation, $K(\omega) > 0$ $\lambda$-a.s. for every $\omega \in \Omega$, and by Fubini's theorem $K > 0$ a.s. in $\lambda \otimes P$. Further $K^{-1/2} \in L^2_{\text{loc}}(M)$ so we can define the process $B = (B_t)_{t\in T}$ by

$$B_t = \int_0^t K_s^{-\frac{1}{2}} \,\mathrm{d}M_s, \quad t \in T.$$

This is a continuous local martingale having $\langle\langle B, B\rangle\rangle_t = \int_0^t K_s^{-1}\,\mathrm{d}\langle\langle M, M\rangle\rangle_s = t$, for all $t \in T$. By the Levy's characterization, $B$ is a Brownian motion and we can compute

$$\int_0^t K_s^{\frac{1}{2}}\,\mathrm{d}B_s = \int_0^t K_s^{\frac{1}{2}} \cdot K_s^{-\frac{1}{2}}\,\mathrm{d}M_s = M_t - M_0, \quad t \in T$$

which completes our proof. □

We can even generalize when $\langle\langle M, M\rangle\rangle$ is only absolutely continuous up to the Lebesgue measure $\lambda$:

**4.5.4 Proposition** Let $M$ be a continuous local martingale relative to its own natural filtration $\mathfrak{F}^M = \left(\mathcal{F}_t^M\right)_{t\in T}$ which is augmented to satisfy the usual conditions. We suppose that $\mathrm{d}\langle\langle M, M\rangle\rangle \ll \lambda$, $P$-a.s. Then there is an extension measure space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}\right)$ to $(\Omega, \mathcal{F}, P)$, a Brownian motion $B$, a non-negative progressively measurable process $K$, such that

$$\int_0^t K_s \,\mathrm{d}s = \langle\langle M, M\rangle\rangle_t \text{ and } M_t = M_0 + \int_0^t \sqrt{K_s}\,\mathrm{d}B_s, \quad t \in T$$

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***Proof***: Let $(\Psi, \mathcal{G}, Q)$ be a measure space and $W = (W_t)_{t\in T}$ be a Brownian motion on it along with its natural filtration $\mathfrak{G}^W = \left(\mathcal{G}_t^W\right)_{t\in T}$. We define $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}\right) = \left(\Omega\times\Psi, \overline{\mathcal{F}\otimes\mathcal{G}}, P\times Q\right)$ and $\tilde{\mathfrak{F}} = \left(\mathcal{F}_t^M \otimes \mathcal{G}_t^W\right)_{t\in T}$ augmented to satisfy the usual conditions. Put again –as in the last proposition, $K_s$ to be the Radon-Nikodym derivative of $\langle\!\langle M, M\rangle\!\rangle$. Then observe that $M, W$ are independent when seen as processes in the extended space; thus

$$B_t(\omega,\psi) = \underbrace{\left(\int_0^t 1_{[K_s>0]} K_s^{-\frac{1}{2}}\, \mathrm{d}M_s\right)(\omega)}_{B_t^1(\omega,\psi)} + \underbrace{\left(\int_0^t 1_{[K_s=0]}(\omega)\, \mathrm{d}W_s\right)(\psi)}_{B_t^0(\omega,\psi)}, \quad t\in T, (\omega,\psi)\in\Omega\times\Psi$$

and we estimate ($M, W$ are seen in the product space but keep the same notation for clarity)

$$\begin{aligned}
\langle\!\langle B, B\rangle\!\rangle_t &= \left\langle\!\left\langle B^1 + B^0, B^1 + B^0\right\rangle\!\right\rangle_t \\
&= \left\langle\!\left\langle B^1, B^1\right\rangle\!\right\rangle_t + 2\left\langle\!\left\langle B^1, B^0\right\rangle\!\right\rangle_t + \left\langle\!\left\langle B^1, B^1\right\rangle\!\right\rangle_t \\
&= \int_0^t 1_{[K_s>0]} K_s^{-1}\, \mathrm{d}\langle\!\langle M, M\rangle\!\rangle_s + 2\int_0^t 1_{[K_s>0]} K_s^{-\frac{1}{2}} 1_{[K_s=0]}\, \mathrm{d}\langle\!\langle M, W\rangle\!\rangle_s + \int_0^t 1_{[K_s=0]}\, \mathrm{d}s \\
&= \int_0^t 1_{[K_s>0]}\, \mathrm{d}s + 0 + \int_0^t 1_{[K_s=0]}\, \mathrm{d}s = t, \quad t\in T
\end{aligned}$$

So, by the Levy's characterization $B$ is a Brownian motion and we can proceed as in the previous proposition to finish the proof. □

**4.5.5 Proposition** Let $M = \left(M^1, M^2, \ldots, M^d\right)$ be a continuous $d$-dimensional local martingale relative to its own natural filtration $\mathfrak{F}^M = \left(\mathcal{F}_t^M\right)_{t\in T}$ which is augmented to satisfy the usual conditions. We suppose that, for every $1 \le i, j \le d$, $\mathrm{d}\langle\!\langle M^i, M^j\rangle\!\rangle \ll \lambda$, $P$-a.s. Then there is an extension measure space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}\right)$ to $(\Omega, \mathcal{F}, P)$

***Proof***: This proof lies on similar ideas as the single dimensional case. If we knew that the brackets $\langle\!\langle M^i, M^j\rangle\!\rangle$ vanish when $i \neq j$, then we could repeat the previous proposition coordinatewise. Now we need to find an orthogonal coordinate transform $U^\top$ (it preserves the Brownian motion) and then reduce to this case. And eventually we return again to our original coordinates using the inverse transform $U$.

Write by $\langle\!\langle M, M\rangle\!\rangle$ the symmetric matrix process $\left(\langle\!\langle M^i, M^j\rangle\!\rangle\right)_{i,j=1}^d$ and $\Gamma_t = \frac{\mathrm{d}\langle\!\langle M, M\rangle\!\rangle_t}{\mathrm{d}t}$ the derivative, which is progressively measurable. We can verify that for every $\mathbf{a}\in\mathbb{R}^d$, $\langle\!\langle M\cdot\mathbf{a}, M\cdot\mathbf{a}\rangle\!\rangle = \mathbf{a}^\top\cdot\langle\!\langle M, M\rangle\!\rangle\cdot\mathbf{a}$ and hence $\Gamma$ is symmetric positive semidefinite. Thus we can find an orthogonal matrix process $U$ and a diagonal matrix process $D$, progressively measurable, such that, for every $t$, $U_t^\top\cdot\Gamma_t\cdot U_t = D_t^2$, $D_t = \mathrm{diag}\left(\sqrt{\lambda_t^{(1)}}, \ldots, \sqrt{\lambda_t^{(r_t)}}, 0, \ldots, 0\right)$

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where $\lambda_t^{(i)} > 0$ are the nonzero eigevalues of $\Gamma$ and $r_t$ its rank –the latter being also progressively integer-valued measurable process. Thus we obtain the symmetric, positive semidefinite square root $K = U \cdot D \cdot U^\top$ of $\Gamma$. Put $D^{-1}$ to be the diagonal matrix process $\left(D_t^{-1}\right)_{t\in T}$ where the formal symbol $D_t^{-1}$ stands for the diagonal matrix process having entries $\left((\lambda_t^{(1)})^{-1/2}, \ldots, (\lambda_t^{(r_t)})^{-1/2}, 0, \ldots, 0\right)$ on its main diagonal. Put also $I_t^{(r)} := D_t^{-1} \cdot D_t = \begin{pmatrix} \mathbb{I}_{r(t)} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}$ and $J_t^{(r)} = \mathbb{I}_d - I_t^{(r)}$. All processes mentioned are progressively measurable.

Let $(\Psi, \mathcal{G}, Q)$ be a measure space and $W = (W_t)_{t\in T}$ be a $d$-dimensional Brownian motion on it along with its natural filtration $\mathfrak{G}^W = \left(\mathcal{G}_t^W\right)_{t\in T}$. We define $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}\right) = \left(\Omega\times\Psi, \overline{\mathcal{F}\otimes\mathcal{G}}, P\times Q\right)$ and $\tilde{\mathfrak{F}} = \left(\mathcal{F}_t^M \otimes \mathcal{G}_t^W\right)_{t\in T}$ augmented to satisfy the usual conditions.

Now define $B = (B_t)_{t\in T}$ the matrix process:

$$B_t = \underbrace{\int_0^t U_s D_s^{-1} U_s^\top \, \mathrm{d}M_s}_{B^\Omega} + \underbrace{\int_0^t U_s J_s^{(r)} \, \mathrm{d}W_s}_{B^\Psi} \quad t \in T$$

Observe that as $\mathcal{G}$ and $\mathcal{F}$ are independent σ-algebras, $MW$ is a local martingale and hence the bracket $\langle\!\langle M, W\rangle\!\rangle$ vanishes and so does $\left\langle\!\left\langle B^\Omega, B^\Psi\right\rangle\!\right\rangle$. We calculate thus,

$$\begin{aligned}
\langle\!\langle B, B\rangle\!\rangle_t &= \left\langle\!\left\langle B^\Omega, B^\Omega\right\rangle\!\right\rangle_t + 2\left\langle\!\left\langle B^\Omega, B^\Psi\right\rangle\!\right\rangle_t + \left\langle\!\left\langle B^\Psi, B^\Psi\right\rangle\!\right\rangle_t \\
&= \int_0^t U_s D_s^{-1} U_s^\top \, \mathrm{d}\langle\!\langle M, W\rangle\!\rangle_s \, U_s D_s^{-1} U_s^\top + 0 + \int_0^t U_s \, \mathrm{d}\langle\!\langle W, W\rangle\!\rangle_s \, J_s^{(r)} U_s^\top \\
&= \int_0^t U_s D_s^{-1} U_s^\top \Gamma_s \, U_s D_s^{-1} U_s^\top \, \mathrm{d}s + \int_0^t U_s J_s^{(r)} U_s^\top \, \mathrm{d}s \\
&= \int_0^t U_s I_s^{(r)} U_s^\top \, \mathrm{d}s + \int_0^t U_s J_s^{(r)} U_s^\top \, \mathrm{d}s = \int_0^t U_s U_s^\top \, \mathrm{d}s = t
\end{aligned}$$

and again by the Levy's characterization, $B$ is a $d$-dimensional Brownian motion. To this end, we just need to verify that since $D_s J_s^{(r)} = 0$ for all $s$,

$$\begin{aligned}
\int_0^t K_s \, \mathrm{d}B_s &= \int_0^t K_s \, \mathrm{d}B_s^\Omega + \int_0^t K_s \, \mathrm{d}B_s^\Psi \\
&= \int_0^t U_s D_s U_s^\top U_s D_s^{-1} U_s^\top \, \mathrm{d}M_s + \int_0^t U_s D_s U_s^\top U_s J_s^{(r)} \, \mathrm{d}W_s = M_t - M_0 + 0 \qquad \square
\end{aligned}$$

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## 4.6 Time changes and representations

**4.6.1 Proposition** Let $C$ be a time change which is a.s. finite, $M$ is a continuous local martingale for the filtration $\mathfrak{F}$. If $M$ is $C$-continuous, then $M_C$ is an $\mathfrak{F}_C$-local martingale and $\langle\langle M_C, M_C\rangle\rangle = \langle\langle M, M\rangle\rangle_C$. If $K \in L^2_{\text{loc}}(M;\mathfrak{F})$ then $K \in L^2_{\text{loc}}(M_C;\mathfrak{F}_C)$ and thus, for any $\mathfrak{F}_C$-stopping time $\sigma$,

$$\int_{C_0}^{C_\sigma} K_t \, \mathrm{d}M_t = \int_0^\sigma K_{C_u} \, \mathrm{d}M_{C_u}, t \in T$$

***Proof***: If $Z$ is a uniform integrable (sub)martingale for $\mathfrak{F}$, then for any $u < s$, by the optional stopping theorem $\mathbb{E}\left[Z_{C_s} \,\middle|\, \mathcal{F}_{C_u}\right] =(\geq)\ Z_{C_u}$. Thus $Z$ is a $\mathfrak{F}_C$-(sub)martingale. Further, as $Z$ is of class D, and $C$ is a.s. finite, $\left\{Z_{C_s} \,\middle|\, s \geq 0\right\}$ is uniformly integrable, so $Z_C$ is a uniformly integrable (sub)martingale.

Let $\tau$ a $\mathfrak{F}$-stopping time. Then both $\sigma = C^\vee_\tau$, $\sigma^- = C^\wedge_\tau$ are $\mathfrak{F}_C$-stopping times, $C_{\sigma^-} \leq \tau \leq C_\sigma$ (also $C_{\sigma^-} \wedge C_s \leq \tau \wedge C_s \leq C_\sigma \wedge C_s$) and by $C$-continuity of $M$, $M_{C_{\sigma^-}} = M_\tau = \mathrm{X}_{C_\sigma}$ (also $M_{C_{\sigma^-} \wedge C_s} = M_{\tau \wedge C_s} = M_{C_\sigma \wedge C_s}$). Thus, if $1_{[\tau>0]}M^\tau$ is a uniformly integrable martingale, then so is $\left(1_{[\tau>0]}M^\tau\right)_C = 1_{[\tau>0]}(M_C)^{\sigma^-}$, and since $\sigma^- > 0 \longleftrightarrow \tau > C_0 \longrightarrow \tau > 0$, so does $1_{[\sigma^->0]}(M^\tau)_C = 1_{[\sigma^->0]}(M_C)^{\sigma^-}$.

Now observe that if $\tau_n \uparrow +\infty$ a.s. for a sequence of stopping times, then defining $\sigma_n^- = C^\vee_{\tau_n}$, we observe that for any $s \geq 0$, $n \in \mathbb{N}$, $\sigma_n^- > s \longleftrightarrow \tau_n > C_s$. This fact ensures that $\sigma_n^- \uparrow +\infty$. Thus, if $M$ is a $C$-continuous local martingale, $M_C$ is a local martingale too.

Next we know that $\langle\langle M, M\rangle\rangle$ is also $C$-continuous, hence $(M^2 - \langle\langle M, M\rangle\rangle)_C = M_C^2 - \langle\langle M, M\rangle\rangle_C$ is a continuous local martingale. Uniqueness of this property implies $\langle\langle M, M\rangle\rangle_C = \langle\langle M_C, M_C\rangle\rangle$.

If $K \in L^2(M;\mathfrak{F})$ then, the previous proposition asserts that $K \in L^2(M_C;\mathfrak{F}_c)$. If $K \in L^2_{\text{loc}}(M;\mathfrak{F})$ then, using similar arguments as in the first part of this proof, we conclude $K \in L^2_{\text{loc}}(M_C;\mathfrak{F}_C)$.

So, if $K \in L^2_{\text{loc}}(M;\mathfrak{F})$, $\int_0 K_t \, \mathrm{d}M_t \in \mathscr{M}^2_{c,0}$ is also $C$-continuous, for the stopped processes $M^{C_{s^-}}$ and $M^{C_s}$ coincide. Hence, for any $t \in T$,

$$\int_{C_{s^-} \wedge t}^{C_s \wedge t} K_u \, \mathrm{d}M_u = \int_0^t K_u \, \mathrm{d}(M^{C_s} - M^{C_{s^-}})_u = 0$$

thus $\left(\int_0 K_t \, \mathrm{d}M_t\right)^{C_{s^-}} = \left(\int_0 K_t \, \mathrm{d}M_t\right)^{C_{s^-}}$ proving our claim.

If $K \in L^2_{\text{loc}}(M;\mathfrak{F})$ and $M, N \in \mathscr{M}^2_{c,\text{loc}}$ are $C$-continuous martingales, then,

$$\left\langle\!\!\left\langle \left(\int_0 K_t \, \mathrm{d}M_t\right)_C, N_C \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \int_0 K_t \, \mathrm{d}M_t, N \right\rangle\!\!\right\rangle_C = \left(\int_0 K_t \, \mathrm{d}\langle\langle M, N\rangle\rangle_t\right)_C$$

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$$= \int_0 K_{C_u} H_{C_u} \, \mathrm{d}\langle\!\langle M_C, N_C \rangle\!\rangle_u = \left\langle\!\!\left\langle \int_0 K_{C_u} \, \mathrm{d}M_{C_u}, N_C \right\rangle\!\!\right\rangle$$

Put $N = \int_0 K_t \, \mathrm{d}M_t$ and further obtain,

$$\left\langle\!\!\left\langle \int_0 K_t \, \mathrm{d}M_t, \int_0 K_t \, \mathrm{d}M_t \right\rangle\!\!\right\rangle_C = \left\langle\!\!\left\langle \int_0 K_{C_u} \, \mathrm{d}M_{C_u}, \left( \int_0 K_t \, \mathrm{d}M_t \right)_C \right\rangle\!\!\right\rangle$$

so that $\left\langle\!\!\left\langle \int_0 K_{C_u} \, \mathrm{d}M_{C_u} - \left( \int_0 K_t \, \mathrm{d}M_t \right)_C, \int_0 K_{C_u} \, \mathrm{d}M_{C_u} - \left( \int_0 K_t \, \mathrm{d}M_t \right)_C \right\rangle\!\!\right\rangle = 0.$ □

The next proposition is known as Dambis, Dubins-Schwarz theorem, and it is, one more application of the Levy's characterization theorem.

**4.6.2 Proposition** Let $M$ be a continuous $\mathfrak{F}$-local martingale such that $M_0 = 0$ and $\langle\!\langle M, M \rangle\!\rangle_\infty = +\infty$. We suppose that $\mathfrak{F}$ is right continuous. Define

$$C_s = \inf \{ t \in T \mid \langle\!\langle M, M \rangle\!\rangle_t > s \}, s \geq 0$$

Then $C = (C_s)_{s \geq 0}$ is a time change, and the time-changed process $B \overset{\text{def}}{=} M_C$ is an $\mathfrak{F}_C$-Brownian motion and $M_t = B_{\langle\!\langle M,M \rangle\!\rangle_t}$.

***Proof***: That $C$ is indeed a time change is evident. What we also observe is that as $\langle\!\langle M, M \rangle\!\rangle$ is continuous, we obtain that $\langle\!\langle M, M \rangle\!\rangle_{C_s} = s$ almost surely (we havent yet applied any time change formula, just a composition of the function $\langle\!\langle M, M \rangle\!\rangle(\omega)$ and its right continuous pseudoinverse $C$). By the definition of the time change, we also get that $M$ is $C$-continuous ($C$ has a jump only whenever $\langle\!\langle M, M \rangle\!\rangle$ is constant only whenever $M$ is constant) and that $C$ is a.s. finite. Thus, by the previous proposition, $B = M_C$ is a continuous local martingale and $\langle\!\langle B, B \rangle\!\rangle_s = \langle\!\langle M, M \rangle\!\rangle_{C_s} = s$ and by Levy's characterization, $B$ is an $\mathcal{F}_C$ Brownian motion. To this end, the equivalence: "$\langle\!\langle M, M \rangle\!\rangle$ is constant only whenever $M$ is constant," implies that $M_t = M_{C_{\langle\!\langle M,M \rangle\!\rangle_t}} = B_{\langle\!\langle M,M \rangle\!\rangle_t}$, for every $t \in T$. □

We have an analogue in multiple dimensions, which under the hypothesis that $\langle\!\langle M^i, M^j \rangle\!\rangle = 0$, concludes that under appropriate time change $M^i_{C^i}, M^j_{C^j}$, can become independent processes.

**4.6.3 Proposition** Let $M = \left( (M_t^1, M_t^2, \ldots, M_t^d) \right)_{t \in T}$ be a continuous $\mathfrak{F}$-martingale in $d$-dimensions, vanishing at 0. We suppose that $\langle\!\langle M^i, M^i \rangle\!\rangle_\infty = +\infty$ while $\langle\!\langle M^i, M^j \rangle\!\rangle = 0$ whenever $i \neq j$. Define for $i = 1, 2, \ldots, d$,

$$C_s^i = \inf \left\{ t \in T \,\middle|\, \langle\!\langle M^i, M^i \rangle\!\rangle_t > s \right\}, s \geq 0$$

Then every $C^i = \left( C_s^i \right)_{s \geq 0}$ is a time change, and the time-changed process $B \overset{\text{def}}{=} \left( (M^1_{C_s^1}, M^2_{C_s^2}, \ldots, M^2_{C_s^2}) \right)_{t \in T}$ is a $d$-dimensional $\mathfrak{F}_C$-Brownian motion.

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The last result can be improved a bit and be extended to the cases that $\langle\!\langle M, M\rangle\!\rangle_\infty < +\infty$. But in this case, the probability space needs to be extended. But first

**4.6.4 Lemma** If $M$ is a continuous local $\mathfrak{F}$-martingale with $M_0 \in L^2(\Omega)$, then on $[\langle\!\langle M, M\rangle\!\rangle_\infty < +\infty]$ the limit $\lim_{t\to+\infty} M_t$ exists, except for a $P$-neglible set.

***Proof***: Define for any $n \in \mathbb{N}$, $\sigma_n = \inf\{t \in T : \langle\!\langle M, M\rangle\!\rangle_t \geq n\}$. Then $\sigma_n$ is a stopping time and as $M^{\sigma_n}$ is again a continuous local martingale and $\langle\!\langle M, M\rangle\!\rangle^{\sigma_n} \leq n$, we conclude that $M^{\sigma_n}$ is bounded and the limit $\lim_{t\to+\infty} M_t^{\sigma_n}$ exists everywhere except a $P$-null set $N_n$. Put $N = \bigcup_{n=1}^{\infty} N_n$. Then Now observe that $[\langle\!\langle M, M\rangle\!\rangle_\infty < +\infty] \subseteq \bigcup_{n=1}^{\infty} \left[\sigma_n = +\infty\right]$. If $\omega \in \Omega \setminus N$ such that $\langle\!\langle M, M\rangle\!\rangle_\infty(\omega)$ is finite, then for some $n$, $M^{\sigma_n}(\omega) = M(\omega)$ and the limit $M_t^{\sigma_n}(\omega)$, hence the limit $M_t(\omega)$ exists. □

**4.6.5 Proposition** Let $M$ be a continuous local $\mathfrak{F}$-martingale such that $M_0 = 0$. Then, there is an extension to $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, an extension $\tilde{\mathfrak{F}}$ to the filtration $\mathfrak{F}$, a $\tilde{\mathfrak{F}}$-Brownian motion $W$, independent of $M$, such that the process $B$ defined by (definition is consistent due to the last lemma)

$$B_s = M_{C_s} + W_s - W_{s\wedge\langle\!\langle M,M\rangle\!\rangle_\infty} = M_{C_s} + \int_0^s 1_{(\langle\!\langle M,M\rangle\!\rangle_\infty,+\infty)}\, \mathrm{d}W_u, \quad s \geq 0$$

The process $M_C$ is a Brownian motion stopped at $\langle\!\langle M, M\rangle\!\rangle_\infty$.

***Proof***: Let $(\Omega_1, \mathcal{F}_\infty^W, P_1)$ be a probability space, $W$ a Brownian motion on $\Omega_1$ along with its filtration $\mathfrak{F}^W$, and now define $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}) = (\Omega\times\Omega_1, \mathcal{F}\otimes\mathcal{F}_\infty^W, P\otimes P_1)$ and $\tilde{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{F}_t^W$, $t \in T$. Let $\pi, \pi_1$ be the natural projections to $\Omega, \Omega_1$ respectively and let $(\pi^* M)(t;\omega,\omega_1) = M_t(\omega)$, $(\pi_1^* W)(t;\omega,\omega_1) = W_t(\omega_1)$ be the pullbacks of $M$ and $W$ respectively on the product space. $\pi_1^* W$ is independent of $M$. For convenience, write again $M$ instead of $\pi^* M$ and $W$ instead of $\pi_1^* W$, since we work on the extended space from now on.

$B$ is a continuous local martingale (arguments hold even on infinite time change) and we can compute for any $s \geq 0$,

$$\begin{aligned}\langle\!\langle B, B\rangle\!\rangle_s &= \langle\!\langle M_C, M_C\rangle\!\rangle_s + \int_0^s 1_{(\langle\!\langle M,M\rangle\!\rangle_\infty,+\infty)}\, \mathrm{d}u + 2\int_0^s 1_{(\langle\!\langle M,M\rangle\!\rangle_\infty,+\infty)}\, \mathrm{d}\langle\!\langle M, W\rangle\!\rangle_u \\ &= \langle\!\langle M, M\rangle\!\rangle_{C_s} + s - s\wedge\langle\!\langle M, M\rangle\!\rangle_\infty \\ &= s\wedge\langle\!\langle M, M\rangle\!\rangle_\infty + s - s\wedge\langle\!\langle M, M\rangle\!\rangle_\infty = s\end{aligned}$$

The proof is complete by the Levy's characterization of Brownian motion. □

Now we can even provide a converse to the last lemma,

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**4.6.6 Proposition** If $M$ is a continuous local martingale, then $[\langle\langle M, M\rangle\rangle_\infty < +\infty]$ and $\left[\lim_{t\to+\infty} M_t \text{ exists}\right]$ differ only by a $P$-negligible set. Even more, on $[\langle\langle M, M\rangle\rangle_\infty = +\infty]$ we have $\liminf_{t\to+\infty} M_t = -\infty$ and $\limsup_{t\to+\infty} M_t = +\infty$.

***Proof***: Apply the last proposition to $M - M_0$: So for some enlargment of $\Omega$, some Brownian motion $W$ independent of $M$, the process

$$B_s = M_{C_s} - M_0 + W_s - W_{s\wedge\langle\langle M,M\rangle\rangle_\infty}, \quad s \geq 0$$

is a Brownian motion, and on $A = [\langle\langle M, M\rangle\rangle_\infty = +\infty]$ we obtain that $B_s = M_{C_s} - M_0$ and $M_t = M_0 + B_{\langle\langle M,M\rangle\rangle_t}$ and $\langle\langle M, M\rangle\rangle_t \to +\infty$ as $t \to +\infty$. Hence $\limsup_{t\to+\infty} M_t = +\infty$ and $\liminf_{t\to+\infty} M_t = -\infty$. □

## 4.7 The Girsanov's theorem

In this section we state the theorem of Girsanov for continuous semimartingales, in its most contemporary form. We follow [RY13, prop. VIII.1].

**4.7.1 Definition** Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathfrak{F} = (\mathcal{F}_t)_{t\in T}$, $T = [0, +\infty)$, be a filtration with last element $\mathcal{F}_\infty$. Let $Q$ a finitely additive function defined on $\bigcup_{t\in T} \mathcal{F}_t$. If for every $t \in T$, $Q_t \stackrel{\text{def}}{=} Q\big|_{\mathcal{F}_t}$ is a probability measure and $Q_t \ll P\big|_{\mathcal{F}_t}$, we write $Q \lhd P$. If $Q$ is also a measure on $\mathcal{F}_\infty$ and $Q \ll P\big|_{\mathcal{F}_\infty}$ we shal write simply $Q \ll P$. We shal also write $\mathsf{D}Q$ for the process defined by

$$\mathsf{D}Q_t = \frac{\mathrm{d}Q_t}{\mathrm{d}P\big|_{\mathcal{F}_t}}, \quad t \in T$$

**4.7.2 Proposition** Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathfrak{F}$ is a filtration with last element $\mathcal{F}_\infty = \sigma\left(\bigcup_{t\in T} \mathcal{F}_t\right) \subseteq \mathcal{F}$.

If $Q \lhd P$, then $Q$ extends to a finitely additive probability measure $\tilde{Q}$ on $\overline{\mathcal{F}}^P_\infty$ vanishing on the $P$-null sets. Further $\mathsf{D}Q$ is an $\mathfrak{F}$-martingale and if $\mathfrak{F}$ is right-continuous, then $\mathsf{D}Q$ has a right continuous modification.

Conversely, if $M = (M_t)_{t\in T}$ is a nonnegative $\mathfrak{F}$-martingale with $\mathbb{E}\left[M_0\right] = 1$, then

$$Q\colon \bigcup_{t\in T} \mathcal{F}_t \longmapsto \mathbb{R}_+ : B \longmapsto Q(B) = \mathbb{E}\left[1_B \cdot M_t\right], \exists t \in T : B \in \mathcal{F}_t$$

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defines unambiguously a finitely additive probability measure $Q \lhd P$ and $\mathsf{D}Q$ is a modification of $M$.

***Proof***: Define for every $t \in [0, +\infty]$, the linear subspace

$$F_t = \{f \in L^\infty(\Omega, \mathscr{F}_\infty, P) \mid f \text{ is bounded } \mathscr{F}_t\text{-measurable}\}$$

and $A = \bigcup_{t \in T} F_t$. We know from hypothesis that $Q$ vanishes on $P$-null sets, thus we may extend it to $\mathscr{N} = \{A \mid \exists N \supseteq A \in \mathscr{F}_\cdot P(N) = 0\}$ in the obvious way, and all hypotheses hold true for this extension as well. So, for convenience, put again $Q$ for this extension. We can now define a linear functional $\mu$ on $A$ such that $\langle \mu, f \rangle = \int_\Omega f \, \mathrm{d}Q$, $f \in A$. If $\exists t \in T : f \in F_t$, then the integral reduces to the usual Lebesgue integral. For every $f \in A$, $|\mu(f)| \leq \|f\|_\infty$; $\|\mu\| = 1$. We extend $\mu$ by Hahn-Banach, to a functional $\tilde{\mu} \in (L^\infty(\Omega, \mathscr{F}_\infty, P))^*$ of the same norm. Then it is known that there is a finitely additive measure $\tilde{Q}$ on $(\Omega, \overline{\mathscr{F}}^P_\infty)$, vanishing on $\mathscr{N}$ and $\langle \tilde{\mu}, f \rangle = \int f \, \mathrm{d}\tilde{Q}$, for every $f \in (L^\infty(\Omega, \mathscr{F}_\infty, P))$. If $B \in \bigcup_{t \in T} \mathscr{F}_t$ then $\exists t \in T : B \in \mathscr{F}_t$ and $\tilde{Q}(B) = \langle \tilde{\mu}, 1_B \rangle = \langle \mu, 1_B \rangle = \int_B \mathrm{d}Q = Q(B)$, thus $\tilde{Q}$ extends $Q$.

Let $s < t \in T$ and $B \in \mathscr{F}_s$. Then $B \in \mathscr{F}_t$ too and $\mathbb{E}\left[1_B \mathsf{D}Q_t\right] = Q(B) = \mathbb{E}\left[1_B \mathsf{D}Q_s\right]$, which proves the martingale property of $\mathsf{D}Q$. Thus $\mathsf{D}Q$ has right and left limits $P$-a.s. If $\mathfrak{F}$ is right continuous, we know that $Z \overset{\text{def}}{=} (\mathsf{D}Q_{t^+})_{t \in T}$ is cadlag, it is a martingale and a modification of $\mathsf{D}Q$.

The last part of the proposition is obvious. □

We also mention the following result, which is notable on its own:

**4.7.3 Proposition** Let $E$ be a polish space (that is, a topological space complete and separable by some metric). If $\Omega = W = C(\mathbb{R}_+, E)$ is the canonical process, with its own filtration $\mathfrak{F}^W$. If $Q : \bigcup_{t>0} \mathscr{F}_t^W \longmapsto [0, 1]$ is a finitely additive probability measure, such that for every $t$, $Q_t \overset{\text{def}}{=} Q|_{\mathscr{F}_t^W}$ is a probability measure, then there is a unique probability measure $\overline{Q}$ on $\mathscr{F}_\infty^W$ that extends $Q$.

For the proof we refer the interested reader to [SV07, thorem. 1.1.10].

**4.7.4 Proposition** Let $(\Omega, \mathscr{F}, P)$ be a probability space, $\mathfrak{F}$ is a right continuous filtration. Let $Q \lhd P$. The following are equivalent:

i) $\mathsf{D}Q$ is uniformly integrable

ii) $Q$ uniquely extends on $\overline{\mathscr{F}}^P_\infty$ to a probability measure $\overline{Q}$ such that for the completion $\overline{P}$ of $P$ on $\overline{\mathscr{F}}^P_\infty$, $\overline{Q} \ll \overline{P}$

If any of i), ii), hold then

$$\mathsf{D}Q_t = \mathbb{E}\left[\left.\frac{\mathrm{d}\overline{Q}}{\mathrm{d}\overline{P}}\right| \mathscr{F}_t\right], \quad t \in T$$

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***Proof***: i) → ii): We can suppose that $\mathsf{D}Q$ has a cadlag modification, since $\mathfrak{F}$ is right continuous. If $\mathsf{D}Q$ is uniformly integrable, there is an $\mathscr{F}_\infty$-integrable r.v. $\mathsf{D}Q_\infty$ such that $\mathsf{D}Q_\infty = \lim_{t\to+\infty} \mathsf{D}Q_t$ almost surely and in $L^1$ and for every $t \in T$, $\mathsf{D}Q_t = \mathbb{E}\left[\mathsf{D}Q_\infty \mid \mathscr{F}_t\right]$ a.s. Thus for every $t \in T$ and $A \in \mathscr{F}_t$ $Q(A) = \mathbb{E}\left[1_A \mathsf{D}Q_t\right] = \mathbb{E}\left[1_A \mathsf{D}Q_\infty\right]$. Thus the mapping

$$\overline{\mathscr{F}}_\infty^P \ni A \longmapsto \overline{Q}(A) \stackrel{\text{def}}{=} \mathbb{E}\left[1_A \mathsf{D}Q_\infty\right] \in [0,1]$$

is an extension of $Q$ and $\overline{Q} \ll \overline{P}$. If $\nu$ is another extension of $Q$ on $\overline{\mathscr{F}}_\infty^P$, then the probability measures $\nu$ and $\overline{Q}$ agree on $\bigcup_{t\in T} \mathscr{F}_t$, whence $\nu = \overline{Q}$.

ii) →i) Let $\overline{Z}_\infty = \frac{\mathrm{d}\overline{Q}}{\mathrm{d}\overline{P}}$. We need to show that for every $t \in T$, $\mathsf{D}Q_t = \mathbb{E}\left[\overline{Z}_\infty \,\middle|\, \mathscr{F}_t\right]$ a.s. Indeed, if $t \in T$ and $A \in \mathscr{F}_t$, then $\mathbb{E}\left[1_A \mathbb{E}\left[\overline{Z}_\infty \,\middle|\, \mathscr{F}_t\right]\right] = \mathbb{E}\left[1_A \overline{Z}_\infty\right] = \overline{Q}(A) = Q(A) = \mathbb{E}\left[1_A \mathsf{D}Q_t\right]$. Thus $\{\mathsf{D}Q_t\}_{t\in T} \subseteq \left\{\mathbb{E}\left[\overline{Z}_\infty \,\middle|\, \mathscr{A}\right] \,\middle|\, \mathscr{A} \subseteq \overline{\mathscr{F}}_\infty^P\right\}$ and the latter is uniformly integrable. □

**4.7.5 Remark** We want to stress that $Q \lhd P$ does not imply that $Q$ can be extended to a measure on $\mathscr{F}_\infty$. And the finitely additive extension $\tilde{Q}$ proved earlier, may not even be unique or σ-additive (with the notable extension of the canonical process; but beware: only on $\mathscr{F}_\infty^W$). And the vanishing property of a finitely additive measure on the $P$-null sets does not lead in any way to absolute continuity. Actually all the probability measures $Q_t$, $t \in T$, may be restrictions of a probability measure $Q$ defined on $\mathscr{F}_\infty$ that it is not absolutely continuous with respect to $P$. Even more, without the hypotheses of the last proposition, $Q$ cannot be extended to any completion of $\mathscr{F}_\infty$.

**4.7.6 Lemma** If $X$ is a non-negative right-continuous supermartingale and

$$\tau = \inf\{t \geq 0 : X_t = 0 \text{ or } X_{t^-} = 0\}$$

Then, $X$ vanishes on $1_{[\tau,+\infty)}$.

***Proof***: First, $\tau$ is a stopping time: If $t \in T$, then

$$[\tau \leq t] = [X_t = 0] \cup \left[\inf_{q\in\mathbb{Q}, q\leq t} X_q = 0\right] \in \mathscr{F}_t$$

Observe also that for any $\omega \in \Omega$ such that $\tau(\omega) < +\infty$, and $\varepsilon > 0$, there is some $\tau(\omega) < q < \tau(\omega) + \varepsilon$ such that $X_q = 0$ or $X_{q^-} = 0$. In the second case we can find some $\tau(\omega) < q' < q < \tau(\omega) + \varepsilon$ such that $X_{q'} < \varepsilon$. In either case, we find some $\tau(\omega) \leq q < \tau(\omega) + \varepsilon$ such that $X_q \leq \varepsilon$. Right continuity of $X$ implies that $\mathrm{X}_{\tau(\omega)} = 0$.

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Then for any $r > 0$, $\tau \leq \tau + r$, while both are stopping times and we can apply optional stopping (we can use it for unbounded times in the case of non-negative supermartingales) to obtain $0 = X_\tau 1_{[\tau<+\infty]} \geq \mathbb{E}\left[\mathrm{X}_{\tau+r} 1_{[\tau<+\infty]} \,\middle|\, \mathscr{F}_\tau\right] \geq 0$. Whence $X_{\tau+r} = 0$ a.s. □

**4.7.7 Proposition** If $Q \lhd P$ then $\mathsf{D}Q > 0$ $Q$-a.s. and for any $\mathfrak{F}$-stopping time $\tau$,

$$Q(A \cap [\tau < +\infty]) = \mathbb{E}\left[1_A 1_{[\tau<+\infty]} \mathsf{D}Q_\tau\right], \quad A \in \mathscr{F}_\tau$$

If further, $Q \ll P$ then $Q(A) = \mathbb{E}\left[1_A \mathsf{D}Q_\tau\right]$, for every $A \in \mathscr{F}_\tau$.

***Proof***: Put $\tau = \inf\{t \geq 0 : \mathsf{D}Q_t = 0 \text{ or } \mathsf{D}Q_{t^-} = 0\}$ and by the previous lemma, $\mathsf{D}Q$ vanishes on $1_{[\tau,+\infty)}$. Thus for any $t \in T$, $\mathsf{D}Q_t|_{[\tau\leq t]} = 0$. And $Q([\tau < +\infty]) = \sup_n Q([\tau \leq n]) = \sup_n \mathbb{E}\left[1_{[\tau\leq n]} \mathsf{D}Q_n\right] = 0$. So $Q([\tau = +\infty]) = 1$, which proves our first claim.

If $Q \ll P$ then $\mathsf{D}Q$ is uniformly integrable and we can apply the optional stopping to obtain $\mathsf{D}Q_\tau = \mathbb{E}\left[\mathsf{D}Q_\infty \mid \mathscr{F}_\tau\right]$, or equivalently $\mathbb{E}\left[1_A \mathsf{D}Q_\tau\right] = \mathbb{E}\left[1_A \mathsf{D}Q_\infty\right] = Q(A)$, for every $A \in \mathscr{F}_\tau$.

If merely $Q \lhd P$ then we need to restrict to bounded stopping times; instead of $\tau$ we apply the optional stopping to $\tau \wedge n$, $n \in \mathbb{N}$ and we obtain

$$Q(A \cap [\tau \leq n]) = \mathbb{E}\left[1_{A\cap[\tau\leq n]} \mathsf{D}Q_n\right] = \mathbb{E}\left[1_{A\cap[\tau\leq n]} \mathsf{D}Q_{\tau\wedge n}\right] = \mathbb{E}\left[1_{A\cap[\tau\leq n]} \mathsf{D}Q_\tau\right], \quad A \in \mathscr{F}_\tau$$

Thus, if $A \in \mathscr{F}_\tau$,

$$\begin{aligned} Q(A \cap [\tau < +\infty]) &= \sup_n Q(A \cap [\tau \leq n]) = \sup_n \mathbb{E}\left[1_{A\cap[\tau\leq n]} \mathsf{D}Q_\tau\right] \\ &= \mathbb{E}\left[\sup_n 1_{A\cap[\tau\leq n]} \mathsf{D}Q_\tau\right] = \mathbb{E}\left[1_{A\cap[\tau<+\infty]} \mathsf{D}Q_\tau\right] \end{aligned}$$ □

**4.7.8 Remark** The equality $Q(A \cap [\tau < +\infty]) = \mathbb{E}\left[1_A 1_{[\tau<+\infty]} \mathsf{D}Q_\tau\right]$ for every $A \in \mathscr{F}_\tau$, translates equivalently

$$\int_{[\tau<+\infty]} f \, \mathrm{d}Q = \int_{[\tau<+\infty]} f \mathsf{D}Q_\tau \, \mathrm{d}P$$

for every $f : \Omega \longmapsto \mathbb{R}$, $\mathscr{F}_\tau$-measurable function such that integral exists.

**4.7.9 Proposition** Let $(\Omega, \mathscr{F}, P)$ be a measure space and $Q$ be a measure on $(\Omega, \mathscr{F}_\infty)$ such that $Q \lhd P$ and $\mathsf{D}Q$ is continuous. If $M$ is a continuous local martingale (for the measure P and the filtration $\mathfrak{F}$), then

$${}^Q M = M - \frac{1}{\mathsf{D}Q} \bullet \langle\langle M, \mathsf{D}Q \rangle\rangle$$

is a local martingale for the measure $Q$ and the filtration $\mathfrak{F}$.

***Proof***: Let $X$ be a cadlag process such that $X\mathsf{D}Q$ is a martingale for the measure $P$. Then $X$ is a martingale for the measure $Q$: Let $s < t \in T$ and $A \in \mathscr{F}_s$. Then $\mathbb{E}\left[1_A X_s \mathsf{D}_s\right] = \mathbb{E}\left[1_A X_t \mathsf{D}_t\right]$.

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But the latter equality translates to $\int_A X_s \, \mathrm{d}Q = \int_A X_t \, \mathrm{d}Q$, for $\mathsf{D}Q \, \mathrm{d}P = \mathrm{d}Q$. The same reasoning holds true also for local martingales. So let $(\tau_n)_n$ be a nondecreasing sequence of bounded stopping times such that $\tau_n \uparrow +\infty$ $P$-a.s. and $1_{[\tau_n>0]} X^{\tau_n} \mathsf{D}Q^{\tau_n}$ is a martingale. Observe that $\tau_n \uparrow +\infty$ $Q$-a.s. as well. Let $s < t \in T$ and $A \in \mathscr{F}_s$. Then

$$\begin{aligned}\int_A 1_{[\tau_n>0]} X_{\tau_n \wedge s} \, \mathrm{d}Q &= \int_A 1_{[\tau_n>0]} X_{\tau_n \wedge s} \mathsf{D}Q_{\tau_n \wedge s} \, \mathrm{d}P \\ &= \int_A 1_{[\tau_n>0]} X_{\tau_n \wedge t} \mathsf{D}Q_{\tau_n \wedge t} \, \mathrm{d}P \\ &= \int_A 1_{[\tau_n>0]} X_{\tau_n \wedge t} \, \mathrm{d}Q\end{aligned}$$

Define $\rho_n = \inf\left\{ t \geq 0 \,\middle|\, \mathsf{D}Q_t \leq \frac{1}{n} \right\}$; these are all stopping times converging $Q$-a.s. to $+\infty$. Observe that $({}^Q M)^{\rho_n}$ is the sum of a martingale and a bounded variation process, whence a semimartingale. So we can apply the Itô's formula (or the integration by parts rule),

$$\begin{aligned}({}^Q M \cdot \mathsf{D}Q)\big|_0^{\rho_n \wedge t} &= ({}^Q M \cdot \mathsf{D}Q)^{\rho_n}\big|_0^t \\ &= \int_0^{\rho_n \wedge t} {}^Q M_s \, \mathrm{d}(\mathsf{D}Q)_s + \int_0^{\rho_n \wedge t} \mathsf{D}Q_s \, \mathrm{d}({}^Q M)_s + \left\langle\!\!\left\langle {}^Q M, \mathsf{D}Q \right\rangle\!\!\right\rangle_t \\ &= \int_0^{\rho_n \wedge t} {}^Q M_s \, \mathrm{d}(\mathsf{D}Q)_s + \int_0^{\rho_n \wedge t} \mathsf{D}Q_s \, \mathrm{d}M_s - \langle\!\langle M, \mathsf{D}Q \rangle\!\rangle_t + \left\langle\!\!\left\langle {}^Q M, \mathsf{D}Q \right\rangle\!\!\right\rangle_t \\ &= \int_0^{\rho_n \wedge t} {}^Q M_s \, \mathrm{d}(\mathsf{D}Q)_s + \int_0^{\rho_n \wedge t} \mathsf{D}Q_s \, \mathrm{d}M_s\end{aligned}$$

This proves that $({}^Q M \mathsf{D}Q)^{\rho_n}$ is a local martingale. We may even multiply all the above equations with $1_{[\rho_n>0]}$ which is $\mathscr{F}_0$-measurable and thus commutes with the integration symbols, to obtain that $1_{[\rho_n>0]}({}^Q M \cdot \mathsf{D}Q)^{\rho_n}$ is a local martingale too (for the measure $P$). By the first part of the proof, $1_{[\rho_n>0]} \cdot {}^Q M^{\rho_n}$ is thus a local martingale for the measure $Q$. Use the definition of local martingale, then let $n \to +\infty$ to conclude that ${}^Q M$ is a local martingale, for the measure $Q$. □

**4.7.10 Remark** Girsanov theorm thus, says that if $X$ is a semimartingale for $P$, then so is for $Q$. And even more, since the bracket of two semimartingales $X, Y$ is a limit in measure, convergence in $P$ implies convergence in $Q$. Put ${}^Q X$ for ${}^Q M + A$, whenever $X = M + A$, $M \in \mathscr{M}_c$, $A \in \mathscr{BV}$. Then $\left\langle\!\!\left\langle {}^Q X, {}^Q Y \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle {}^Q X, Y \right\rangle\!\!\right\rangle = \langle\!\langle X, Y \rangle\!\rangle$.

Hence $L^2(M) = L^2({}^Q M)$, for $M \in \mathscr{M}_c^2$ and for $K \in L^2_{\mathrm{loc}}(M)$,

$$\int_0 K_s \, \mathrm{d}{}^Q M_s = \int_0 K_s \, \mathrm{d}M_s - \int_0 K_s \, \mathrm{d}(\mathsf{D}Q^{-1} \bullet \langle\!\langle M, \mathsf{D}Q \rangle\!\rangle)_s$$

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$$
\begin{aligned}
&= \int_0 K_s \, \mathrm{d}M_s - \int_0 \mathsf{D}Q_s^{-1} \, \mathrm{d}(K \bullet \langle\!\langle M, \mathsf{D}Q \rangle\!\rangle)_s \\
&= \int_0 K_s \, \mathrm{d}M_s - \int_0 \mathsf{D}Q_s^{-1} \, \mathrm{d} \left\langle\!\!\left\langle \int_0 K_s \, \mathrm{d}M_s, \mathsf{D}Q \right\rangle\!\!\right\rangle_s \\
&= {}^{Q}\left( \int_0 K_s \, \mathrm{d}M_s \right)
\end{aligned}
$$

**4.7.11 Proposition** If $Q \lhd P$ and $\mathsf{D}Q > 0$ $P$-a.s. (so that $P, Q$ are equivalent on each $\mathscr{F}_t$, $t \in T$) then $L = \ln(\mathsf{D}Q)_0 + \int_0 \mathsf{D}Q_s^{-1} \, \mathrm{d}(\mathsf{D}Q)_s$ is the unique continuous local martingale satisfying $\mathsf{D}Q = \mathsf{E}(L) = \exp(L - {}^1\!/_2 \langle\!\langle L, L \rangle\!\rangle)$.

In this case, for any continuous local martingale $M$,

$$
{}^{Q}M = M - \langle\!\langle M, L \rangle\!\rangle
$$

***Proof***: Use the Itô's formula for $\ln \mathsf{D}Q$:

$$
\begin{aligned}
\ln \mathsf{D}Q_t - \ln \mathsf{D}Q_0 &= \int_0^t \mathsf{D}Q_s^{-1} \, \mathrm{d}(\mathsf{D}Q)_s - \frac{1}{2} \int_0^t \mathsf{D}Q_s^{-2} \, \mathrm{d} \langle\!\langle \mathsf{D}Q, \mathsf{D}Q \rangle\!\rangle_s \\
&= \int_0^t \mathsf{D}Q_s^{-1} \, \mathrm{d}(\mathsf{D}Q)_s - \frac{1}{2} \left\langle\!\!\left\langle \int_0 \mathsf{D}Q_s^{-1} \, \mathrm{d}(\mathsf{D}Q)_s, \int_0 \mathsf{D}Q_s^{-1} \, \mathrm{d}(\mathsf{D}Q)_s \right\rangle\!\!\right\rangle_t
\end{aligned}
$$

which shows immediately that the martingale $L$ defined by $L_t = \ln \mathsf{D}Q_0 + \int_0^t \mathsf{D}Q_s^{-1} \, \mathrm{d}(\mathsf{D}Q)_s$, $t \in T$ satisfies $\mathsf{D}Q = \mathsf{E}(L)$.

If there is also a continuous local martingale $M$ satisfying $\mathsf{E}(M) = \mathsf{D}Q = \mathsf{E}(L)$, then $M - {}^1\!/_2 \langle\!\langle M, M \rangle\!\rangle = L - {}^1\!/_2 \langle\!\langle L, L \rangle\!\rangle$ which implies that $M - L \in \mathscr{BV}$ and $M_0 = L_0$. Thus $M = L$, and this proves uniqueness.

If $M$ is a continuous local martingale, then just use the defining equation of $L$, to obtain $\int_0^t (\mathsf{D}Q)_s^{-1} \, \mathrm{d} \langle\!\langle M, \mathsf{D}Q \rangle\!\rangle_s = \langle\!\langle M, L \rangle\!\rangle$ and ${}^{Q}M = M - \langle\!\langle M, L \rangle\!\rangle$. □

**4.7.12 Remark** Note that under the hypotheses of the previous proposition, we also obtain that, since $\left\langle\!\!\left\langle {}^{Q}L, {}^{Q}L \right\rangle\!\!\right\rangle = \langle\!\langle L, L \rangle\!\rangle$, $\mathsf{E}(-{}^{Q}L) = \exp\left(-L + \langle\!\langle L, L \rangle\!\rangle - {}^1\!/_2 \left\langle\!\!\left\langle {}^{Q}L, {}^{Q}L \right\rangle\!\!\right\rangle\right) = \mathsf{E}(-L)$. Thus for any $A \in \mathscr{F}_t$, $t \in T$,

$$
P(A) = \int_A \mathsf{E}(-{}^{Q}L_t) \, \mathrm{d}Q, \quad Q(A) = \int_A \mathsf{E}(L_t) \, \mathrm{d}P
$$

But we want to stress that these formulas are not necessarily valid on $\mathscr{F}_\infty$.

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**4.7.13 Corollary** Under the hypotheses of the previous proposition and if $B$ is a Brownian motion for the measure $P$, then ${}^{Q}B = B - \langle\langle B, L \rangle\rangle$ is a Brownian motion for the measure $Q$.

***Proof***: Use the Levy's characterization. □

**4.7.14 Example** Brownian motion with drift. Let $B$ be a Brownian motion defined on the canonical space along with its own filtration $\mathfrak{F}^B$, and let $b \in \mathbb{R}$. Put $\rho = \inf\{t \in T : B_t \geq b\}$ for the first hitting time of $B$ at the level $b$. We have seen that the density of $\tau$ is

$$f_\rho(r) = \frac{|b|}{\sqrt{2\pi r^3}} \exp\left(-\frac{b^2}{2r}\right), \quad r > 0$$

thus, the moment generating function of $\tau$ is

$$m_\rho(a) = \mathbb{E}\left[e^{-a\rho}\right] = \exp\left(-\frac{|b|}{\sqrt{2a}}\right), \quad a > 0$$

The process $\mathsf{E}^\mu(B) = \left(\exp(\mu B_t - \mu^2 t / 2)\right)_{t \in T}$ is a true martingale and we can apply the previous proposition to obtain that ${}^{Q}B_t = B_t - \langle\langle B, \mu B \rangle\rangle_t = B_t - \mu t$ is a Brownian motion for the measure $Q$ induced by the martingale $\mathsf{E}^\mu(B)$. This measures uniquely extends to the whole $\mathscr{F}_\infty^B$. For the new measure $Q$, the original Brownian motion, is seen as the term $+\mu t$ is added, which is, what we call, a Brownian motion with drift $\mu$.

We now wish to discover the density of $\rho$ under the measure $Q$. Fix any $t \in T$. Then $[\rho \leq t] \in \mathscr{F}_t \cap \mathscr{F}_\rho$ and

$$\begin{aligned}
Q([\rho \leq t]) = Q_t([\rho \leq t]) &= \mathbb{E}\left[1_{[\rho \leq t]} \mathsf{E}^\mu(B)_t\right] \\
&= \mathbb{E}\left[1_{[\rho \leq t]} \mathbb{E}\left[\mathsf{E}^\mu(B)_t \,\middle|\, \mathscr{F}_{\rho \wedge t}\right]\right] \\
&= \mathbb{E}\left[1_{[\rho \leq t]} \mathsf{E}^\mu(B)_{\rho \wedge t}\right] = \mathbb{E}\left[1_{[\rho \leq t]} \mathsf{E}^\mu(B)_\rho\right] \\
&= \int_\Omega 1_{[\rho \leq t]} \exp(\mu B_\rho - \mu^2 \rho / 2)\, \mathrm{d}P(\omega) \\
&= \int_0^t \exp(\mu b - \mu^2 r / 2) f_\rho(r)\, \mathrm{d}r \\
&= \int_0^t \frac{|b|}{\sqrt{2\pi r^3}} \exp\left(-\frac{(b - \mu r)^2}{2r}\right) \mathrm{d}r
\end{aligned}$$

which leads to the new density function for the measure $Q$:

$${}^{Q}f_\rho(t) = \frac{|b|}{\sqrt{2\pi r^3}} \exp\left(-\frac{(b - \mu r)^2}{2r}\right), \quad r > 0$$

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Further,

$$\begin{aligned} Q([\rho < +\infty]) &= \sup_{n\in\mathbb{N}} Q_n([\rho \le n]) \\ &= \sup_{n\in\mathbb{N}} \mathbb{E}\left[1_{[\rho\le n]}\, \mathsf{E}^{\mu}(\boldsymbol{B})_{\rho}\right] \\ &= \mathbb{E}\left[\mathsf{E}^{\mu}(\boldsymbol{B})_{\rho}\right] \\ &= \exp(\mu b - |\mu b|) = \begin{cases} 1 & \text{if } \mu b \ge 0 \\ \exp(-2\,|\mu b|) & \text{if } \mu b < 0 \end{cases} \end{aligned}$$

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