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ΜΕΤΑΠΤΥΧΙΑΚΟ ΣΤΑΤΙΣΤΙΚΗ
MSc IN STATISTICS

From Traditional Univariate to Multivariate Capability Indices: Recent Developments and Applications

By

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Από τους Μονομεταβλητές στους Πολυμεταβλητούς Δείκτες Ικανότητας: Πρόσφατες Εξελίξεις και Εφαρμογές

Της

ΕΥΦΗΜΙΑΣ ΠΑΠΑΦΩΤΙΟΥ

ΜΕΤΑΠΤΥΧΙΑΚΗ ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ
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To my parents and friends, for their constant encouragement and support.

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Vita

Effimia Papafotiou is an M.Sc. candidate in the Department of Statistics at the Athens University of Economics and Business (AUEB). She holds a B.Sc. in Statistics and Insurance from the University of Piraeus (UniPi). Her thesis investigates Process Capability Indices (PCIs) for manufacturing processes, synthesizing recent developments in univariate and multivariate indices, and examining crucial assumptions for capability analysis-normality, stability, and independence-under the supervision of Professor Psarakis Stylianos. This study combines literature synthesis and case analysis, emphasizing robust diagnostics and practical implementation. Beyond quality control, her research interests encompass data analysis and data science, with R serving as her primary analysis environment.

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Abstract

Process Capability Indices (PCIs) play a critical role in quality management by providing a quantitative measure of a process's capability to generate outputs within specified limits. The traditional indices, such as C_p, C_{pk}, and C_{pm}, are the most famous and widely used. This thesis examines the theoretical foundations of Process Capability Indices, presents the assumptions and limitations, and explores the univariate and multivariate approaches. Furthermore, this research includes fuzzy and robust methods. Practical applications in the univariate and multivariate cases are applied. More specifically, two univariate and one multivariate implementations were completed. On the one hand, a pharmaceutical case study was implemented, and Fuzzy PCIs were applied to air quality monitoring in the programming language R. On the other hand, an engineering case study was implemented, using PCA-based multivariate capability indices, in the same programming language. These examples highlight the real-world relevance of PCIs and the role of statistical software capability assessment. Consequently, this thesis discusses recent challenges and future directions toward developing adaptive methods for modern manufacturing environments.

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Περίληψη

Οι Δείκτες Ικανότητας Διεργασιών ($PCIs$) διαδραματίζουν εξαιρετικά σημαντικό ρόλο στη διαχείριση ποιότητας, καθώς προσφέρουν ένα ποσοτικό μέτρο της ικανότητας της διεργασίας να παράγει αποτελέσματα εντός των επιτρεπτών, καθορισμένων ορίων. Οι κλασικοί δείκτεσ δείκτες, όπως οι C_p, C_{pk}, και C_{pm}, αποτελούν τους πιο γνωστούς και ευρέως χρησιμοποιούμενους. Η παρούσα διατριβή εξετάζει τις θεωρητικές βάσεις των Δεικτών Ικανότητας Διεργασιών, παρουσιάζει τις παραδοχές και τους περιορισμούς τους και διερευνά μονομεταβλητές και πολυμεταβλητές προσεγγίσεις. Επιπλέον, η έρευνα περιλαμβάνει "Ασαφείς" ($Fuzzy$) και "Ανθετικές" ($Robust$) μεθόδους. Ακουλούθως, πραγματοποιούνται εφαρμογές σε μονομεταβλητές και πολυμεταβλητές περιπτώσεις, συμπεριλαμβανομένων δύο μονομεταβλητών και μίας πολυμεταβλητής εφαρμογής στο προγραμματιστικό λογισμικό της R. Συγκεκριμένα, πραγματοποιήθηκε μία μελέτη στον φαρμακευτικό τομέα, ενώ στη συνέχερια εφαρμόστηκαν Ασαφείς Δείκτες Ικανότητας στη παρακολούθηση ποιότητας αέρα. Παράλληλα, υλοποιήθηκε μία ακόμη μελέτη στον τομέα της μηχανικής χρησιμοποιώντας πολυμεταβλητούς δείκτες ικανότητας βασισμένους στην Ανάλυση Κύριων Συνιστωσών ($PCA-based$). Τα παραδείγματα αυτά αναδεικνύουν τη σημασία των Δεικτών Ικανότητας Διεργασιών στην πράξη, καθώς και τον καθοριστικό ρόλο του στατιστικού λογισμικού στην αξιολόγηση της ικανότητας. Τέλος, η διατριβή συζητά για τις πρόσφατες προκλήσεις και τις μελλοντικές κατευθύνσεις για την ανάπτυξη προσαρμοστικών μεθόδων κατάλληλων για σύγχρονα περιβάλλοντα παραγωγής.

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1. Chapter 1: Introduction

Consistent quality has long been the main aim in manufacturing and service industries. In the domain of global competition, organizations increasingly rely on statistical methods to ensure that processes operate within specified limits and meet customer requirements. Process Capability Analysis (PCA) plays a critical role by determining whether a process can consistently produce outputs that meet the predefined specification limits. This assessment is achieved through the use of Process Capability Indices (PCIs), which convert complex process behavior into interpretable measures that support decision-making and continuous improvement.

The classical indices, such as C_p, C_{pk}, and C_{pm}, are the basis of capability analysis, providing insights into the variability of the process, how centered the process is, and alignment with target values. Their simplicity and practical relevance have led to widespread use in quality management frameworks such as Six Sigma and ISO standards. However, these indices rely on key assumptions, such as the normal distribution of the data, the stability of the process, and the independence of the observations, which are often violated in real-world scenarios.

On the one hand, Univariate Capability Analysis focuses on a single quality characteristic. On the other hand, the Multivariate Capability Analysis focuses on multiple interrelated quality characteristics and is applied in modern production environments. This ability accounts for correlations among variables to provide a more comprehensive view of process performance. Recent research also addresses challenges such as non-normality, autocorrelation, and measurement errors, proposing advanced solutions including robust estimation techniques, non-parametric approaches, and fuzzy logic-based models.

This thesis includes seven chapters in total. Firstly, Chapter 2 presents the evolution of statistical process control and the principles of process capability analysis. Then, Chapter 3 introduces the theoretical foundations, the assumptions of traditional PCIs, and their implementation through a pharmaceutical example in R. Chapter 4 examines traditional and Advanced Univariate PCIs with an application in air quality monitoring, by using the Fuzzy PCIs. Chapter 5 introduces Multivariate Capability Indices (MPCIs), focusing on PCA-based approaches and their practical engineering applications in R. Consequently, Chapter 6 reviews recent studies and emerging trends, and Chapter 7 presents the conclusions and directions for future research.

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2. Chapter 2: Literature Review

2.1 Evolution of Statistical Process Control

In process control of the modern world, one of the most crucial techniques used is Statistical Process Control or Statistical Quality Control (SPC or SQC), which helps to improve processes before problems arise, ensuring efficiency, waste reduction, and improvement in product quality. It enables manufacturers to predict and avoid non-conformities, minimizing material, human, energy, and control costs. During the 20th century, according to Montgomery (2019), SQC constituted one of the greatest technological advances because it is a powerful tool in any process, helping us achieve stability through variability reduction.

The seven fundamental SQC methods include histograms, check sheets, Pareto charts, cause-and-effect diagrams, defect concentration diagrams, scatter diagrams, and control charts. In 1924, control charts, invented by W.A. Shewhart, whose work at Bell Laboratories laid the foundation for statistical process control, were especially useful for distinguishing between natural process variability (common causes) and assigning reasons for variability (special causes). Control charts do not play a crucial role only for real-time monitoring but also for diagnostic purposes and ongoing process improvement. They play an important role in lean manufacturing and Six Sigma processes.

After the aforementioned figures, Juran (1999) and Deming (1986) took over. More precisely, they emphasized the significance of process capability in Total Quality Management (TQM), pressing that nonstop process enhancement and quality control are essential for maintaining competitiveness in manufacturing diligence. Their study emphasized the need for statistical methodologies for reducing variability and guaranteeing constant client circumstances. Taguchi's Loss Function framework aligns with Process Capability Analysis (PCA) by evaluating deviations from target specifications and their influence on quality. Taguchi G. (2005) underlined that the aim of quality loss is not just to meet the specification, but also to reduce variation from the target value (T). Minor diversions can result in client unhappiness and higher expenses in the long run. His methodology has been widely used in robust design methodologies to optimize manufacturing processes.

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To further understand the evolution of the SPC, it is crucial to take into account its applications in various industries. During the early 20th century, the automotive industry emerged as an important area for using SPC concepts (AIAG, 2010). Henry Ford pioneered assembly line production procedures, which cleared the way for the integration of SPC techniques to preserve consistency and reduce waste. The aerospace sector eventually adopted it as a necessity for precision and reliability in manufacturing aircraft components became critical. In the middle of the 20th century, major companies, such as Boeing and Lockheed, had adopted SPC tools to assure quality standards in an industry where safety and precision were critical.

Experts in this domain, such as W. Edwards Deming (1986), played a significant role in establishing SPC approaches abroad. Deming's efforts in post-war Japan, particularly with Toyota, helped to reshape the Japanese manufacturing scene. His emphasis on eliminating variation, finding root causes of defects, and continuously improving procedures paved the way for Toyota's renowned Production System. This system enhanced quality while also influencing global production practices. Similarly, Joseph M. Juran (1988) contributed to SPC's global spread by emphasizing the importance of top management commitment and quality training techniques. Juran's support for the Quality Trilogy (Planning-Control-Improvement) supported the belief that statistical methods were required not only for the shop floor but throughout the entire organization.

SPC's applicability evolved along with the industries. At the end of the 20th century, the semiconductor industry adopted SPC to address the difficulty of generating precise and defect-free integrated circuits that meet design specifications while ensuring reliable and efficient operation with no performance issues. Because these semiconductor components are very small and require strict tolerances, the control of variation was essential. SPC offered suitable tools for reducing these circuits. The semiconductor industry designs and produces materials, such as silicon, which serve as the foundation for electronic devices, including computer processors, memory chips, and sensors.

In the modern era, SPC has expanded into industries such as pharmaceuticals, biotechnology, and food production, where regulatory compliance and product safety are critical. This growth might be attributed to more strict global standards and regulatory frameworks that necessitate the thorough monitoring of processes and products. The use of real-time monitoring systems has enabled manufacturers in these industries to discover deviations quickly, ensure consistency of the products, and achieve these rigorous quality standards. As a result, SPS has evolved into more than just a tool for manufacturing efficiency. It is also an essential component of risk management and compliance in industries where quality and safety are especially necessary.

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2.2 Process Capability Analysis: Principles and Applications

2.2.1 Theoretical Framework and Significance

Process capability analysis is an important method in quality management that evaluates a manufacturing process's capacity to create products within set specifications. It is a quantitative strategy to discover process variability, maintain consistency, and enhance product quality. Essential throughout the product cycle, from development to final manufacturing, and is commonly used in Six Sigma approaches to improve procedures and eliminate defects. Process capability is a process's capacity to consistently create outputs that fulfill standards. The approach is focused on assessing the consistency of process performance and quantifying critical-to-quality (CTQ) attributes. There are two forms of process variation to consider:

- **Instantaneous Variability**: The natural fluctuations of a process at a given time.
- **Variability Over Time**: The long-term fluctuations that can be observed across multiple cycles of production.

The **Six Sigma** (Define - Measure - Analyze - Improve - Control) **DMAIC** methodology relies heavily on determining process capacity. While it is largely used in the Analyze phase, it also makes substantial contributions to the Improve phase by identifying process variances and recommending corrective actions. Understanding the distribution of process data is critical to the accuracy of this assessment. To further assess process capability, one frequent method is to examine its statistical distribution. For a normal distribution, process capability is assessed using the upper and lower natural tolerance limits, which are mathematically defined as:

- Upper Natural Tolerance Limit (**UNTL**): $\mu + 3\sigma$
- Lower Natural Tolerance Limit (**LNTL**): $\mu - 3\sigma$

These limits encapsulate 99.73% of the process variability, implying that only 0.27% of the output is expected to deviate beyond these boundaries. However, when the process output distribution is non-normal, the percentage exceeding the $mu \pm 3\sigma$ range may significantly differ, necessitating alternative statistical adjustments or transformation techniques to maintain analytical accuracy.

Consequently, PCA is a structured approach that enables a precise estimation of the ability of the process to meet specifications. This estimation is typically represented as a probability distribution with a defined shape, mean, and standard deviation, ensuring a quantitative basis for continuous quality improvement.

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2.2.2 Essential Statistical Methods for Assessing Process Performance

To successfully assess process capability, numerous statistical approaches are used to examine data distribution, stability, and overall performance. These tools provide vital insights into process behavior, allowing manufacturers to maintain consistent quality standards and reduce faults. Organizations can use approaches like histograms, probability plots, control charts, and experimental designs to discover process differences, predict possible problems, and make necessary adjustments.

The following statistical tools are essential for evaluating process capability:

1. **Histogram Analysis**: 'A histogram is a visual depiction of data distribution'.That was first proposed by Karl Pearson, a British mathematician and biostatistician, in 1895. Montgomery suggests using a dataset with at least 100 observations to assess process capabilities. The data is separated into equal-width bins to prevent excessive granularity or loss of detail. The optimal number of bins can be determined using a variety of techniques, including Freedman-Diaconis' rule, Sturges' rule, and the Square-root Rule. The histogram provides an instant visual representation of the data's shape, center, and variability. In addition, by superimposing specification limitations (LSL and USL) and target mean (T), manufacturers can analyze the process's capacity to achieve quality standards.

2. **Probability Plot Analysis**: Wilk (1968) and Gnanadesikan (2001) introduced the probability plot, which arranges data in ascending order and plots cumulative frequency. If the process follows a normal distribution, the points fall roughly along a straight line inside the confidence range. This method evaluates the distribution's shape, center, and spread, which aids in determining the proportion of nonconforming units. Cumulative frequency can be determined using the Median Rank approach, the Herd-Johnson method, the Kaplan-Meier method, or the Modified Kaplan-Meier method.

3. **Control Chart Analysis**: As it was mentioned in Chapter 2.1, Control charts, first developed by W.A. Shewhart, are effective tools for tracking process stability and variability over time. Montgomery considers them the primary technique for analyzing process capability, preferring variable control charts (such as range, standard deviation, and mean control charts) over attribute control charts. Variable control charts provide deeper insights by distinguishing between natural process variability and assignable causes. However, he emphasizes that capability assessment should only be undertaken if the process is stable, as unstable processes yield unreliable estimates. Control charts, thus, serve as both a diagnostic and preventive tool in statistical quality control.

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4. **Design of Experiments**: Ronald Fisher (1935) pioneered the Design of Experiments (DOE), which offers an active approach to process optimization as opposed to passive monitoring methods. DOE systematically manipulates input factors to determine their impact on output, ultimately establishing optimal process conditions. Unlike other statistical techniques for monitoring process stability, DOE allows producers to proactively modify processes by identifying the most relevant elements influencing product quality. It is frequently used in experimental settings to reduce variability, improve process resilience, and assess various sources of variance in manufacturing contexts.

2.3 Process Capability Indices

Process Capability Indices (PCIs) are vital statistical tools used in manufacturing and quality engineering to assess whether a process is capable of producing products that meet customer specifications and tolerance limits. They provide quantitative measures that compare the output of the process with the specified limits, facilitating decision-making for process improvements and quality control.

2.3.1 Historical Development and Industrial Application

The challenge of variation has been a persistent issue in manufacturing since the Industrial Revolution. Even when similar processes are controlled under identical conditions, the output variable remains inescapable, as no two manufactured parts are the same. Traditionally, industries addressed this challenge by defining strict specification limits for key product characteristics, ensuring that individual products met predefined standards. However, with the evolution of quality management, the focus shifted from assessing individual product capability to evaluating the entire production process.

Since the 1980s, the 'famous' structured framework, known as PCA, has been established to systematically determine whether a process can produce output that meets specification limits. This framework utilizes Process Capability Indices, which compare actual process performance with required tolerances. Companies such as Motorola, IBM, ABB, and SKF have successfully integrated process capability analysis into their quality management systems, using PCIs to identify areas for improvement, optimize production processes, and maintain consistency in product quality.

The adoption of PCIs initially began in the United States and soon expanded to industries requiring high precision, such as automated manufacturing, semiconductor production, and IC assembly. These industries adopted the concept of these indices to meet stringent customer specifications and maintain high levels of process control. The increasing reliance on PCIs is evident in supplier certification manuals and quality

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assurance guidelines published by major industry groups, such as Ford, the Automotive Industries Action Group (AIAG), and Chrysler's Tooling and Equipment Supplier Quality Assurance (TESQA).

2.3.2 Importance and Methodology

The utilization of PCIs simplifies the evaluation of manufacturing processes by linking the required accuracy with actual process variability. The standard procedure for conducting a process capability analysis involves several critical steps:

1. **Selection of a Quality Characteristic**: Correctly reflects the effectiveness of the manufacturing process while also aligning with client expectations or important product qualities.

2. **Measuring System Analysis**: Required to ensure the measurement instruments and methods are accurate and reliable. This often involves conducting a Gage Repeatability and Reproducibility (Gage R&R) study to confirm the measurement system's adequacy (AIAG, 2010). McNeese and Klein, Persijn and Nuland, and Pearn and Liao have all noted that the measuring system used has a considerable impact on the calculated process capability.

3. **Data Collection**: Gathering sufficient data over an appropriate time frame to capture all sources of process variability, including equipment fluctuations, operator differences, and environmental factors (Montgomery, 2019). To achieve statistical reliability, at least 25 subgroups of 4-5 observations each are required.

4. **Data Verification**: Normality assessment is critical since many capacity indices require normally distributed data. Histograms, probability plots, and statistical tests such as the Anderson-Darling, Kolmogorov-Smirnov, and Shapiro-Wilk tests can all be used to assess this. Furthermore, data independence must be assured, as some processes (for example, chemical or biological processes) may display autocorrelation, necessitating the use of specific statistical approaches. Control charts must also be used to analyze process stability; if the process is unstable, corrective actions must be implemented before capability analysis.

5. **Calculation of Capability Indices**. Computing appropriate PCIs such as C_p, C_{pk}, and C_{pm}, and comparing these values against industry benchmarks to determine process capability (Juran&Godfrey, 1999).

   Graphical tools, including histograms and box plots, complement the calculated indices by providing visual insights into data distribution and potential deviations, enhancing the overall assessment of process capability.

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2.3.3 Univariate vs. Multivariate Process Capability Indices

Traditionally, PCIs have been applied to single quality characteristics, known as univariate PCIs. They evaluate the capability related to one specific attribute independently, offering detailed insights into individual aspects of process performance (Juran & Godfrey, 1999). However, modern manufacturing processes often involve multiple interrelated quality characteristics. Evaluating each characteristic independently using univariate PCIs may lead to incomplete or misleading conclusions because it ignores the potential correlations between variables (Pan & Lee, 2010). For instance, in complex products like automotive components or electronic devices, multiple dimensions or performance attributes must meet specifications simultaneously. A univariate analysis might indicate acceptable capability for individual characteristics while failing to detect issues arising from their interactions.

To address this challenge, multivariate PCIs have been developed. These indices consider all relevant quality characteristics collectively, providing a comprehensive assessment of process capability (Mason & Young, 2002). Multivariate PCIs account for the covariance structure among variables, offering several advantages:

- **Holistic Evaluation**: Reflects the combined effect of all critical characteristics in product quality.
- **Consideration of Correlation**: Incorporates the relationships between variables, which can significantly impact process performance.
- **Enhanced Decision-Making**: Facilitates more accurate identification of process issues and improvement opportunities.

Methods for calculating multivariate PCIs include using statistical techniques like Hotelling's T^2 statistic and constructing multivariate control charts. These methods require sophisticated statistical analysis and sufficient multivariate data (Taam et al., 1993).

Before closing this subchapter, an issue not treated analytically in the next chapters is the case of unknown variance in both univariate and multivariate settings and its effect on index computation. More specifically:

1. **Univariate Case**: Univariate process capability indices (PCIs) compare the engineering specification width with the spread of the process. In their definitions, Kane (1986) and Palmer & Tsui (1999) highlight that the spread is the *true* standard deviation $\boldsymbol{\sigma}$. However, in practice, $\boldsymbol{\sigma} = \textit{unknown}$ and is replaced by the sample standard deviation $\boldsymbol{s}$ calculated from the data. The resulting figures are therefore *estimates* rather than *constants*. This substitution means that capability values vary from sample to sample, even when the process is stable, because sample standard

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deviation $\boldsymbol{s}$ and sample mean $\bar{X}$ fluctuate with sampling (Palmer & Tsui, 1999). For example, if two engineers observe the same stable process, they can get different results if they use different small samples, simply because $\boldsymbol{s}$ and $\bar{X}$ are not going to be the same across the samples.

The effect shows up first in the Potential Capability Index C_p and the Process Performance Index C_{pk}. On the one hand, it is known that C_p asks how many "process widths" fit inside the specification width (USL-LSL). When the standard deviation $\boldsymbol{\sigma}$ is replaced by the sample standard deviation $\boldsymbol{s}$, the estimate of the Potential Capability Index takes the form of $\hat{C}_p = \frac{USL-LSL}{6s}$. The relationship between $\hat{C}_p$ and $\boldsymbol{s}$ is inversely proportional. More specifically, if $\boldsymbol{s}$ is overestimated, the capability looks worse than it really is, and if $\boldsymbol{s}$ is underestimated, the capability looks better than it truly is (Kane, 1986). According to Palmer & Tsui (1999), the sample size matters because, when there are few observations, $\boldsymbol{s}$ can be noisier and $\hat{C}_p$ can swing more than expected.

On the other hand, the Process Performance Index C_{pk} includes the effect of off-centering by comparing the mean to each specification side in units of $\mathbf{3}\boldsymbol{s}$ and taking the smaller value. Since the estimate $\hat{C}_{pk}$ uses both the sample mean $\bar{X}$ and the sample standard deviation $\boldsymbol{s}$, it changes more from sample to sample than the $\hat{C}_p$, which uses only $\boldsymbol{s}$ (Kane, 1986; Palmer & Tsui, 1999). Also, for the One-Sided Specification Indices, C_{pu} and C_{pl}, behave similarly with the same sensitivity to how well $\boldsymbol{s}$ is estimated.

Furthermore, the Taguchi Capability Index C_{pm} and the Process Capability Target-Based Index C_{pmk} penalize deviations from a chosen target $\boldsymbol{T}$. Their denominators combine the spread with the squared distance between the sample mean $\bar{X}$ and $\boldsymbol{T}$. This makes them highly informative when being "on target" matters, but it also increases sensitivity to estimation noise in the mean. Even when the process spread is small, a slight estimated shift of the mean can pull these indices down. If the mean really has shifted, that "drop" is exactly what we want to see. But with small samples, part of the drop might just be sampling noise (Palmer & Tsui, 1999). For this reason, they suggested taking into consideration a set of indices and not just a single number. For example, it is better to report an index that shows only spread ($\hat{C}_p$) alongside another one that reflects centering or target loss ($\hat{C}_{pk}$ or $\hat{C}_{pm}$).

2. **Multivariate Case**: In multivariate capability analysis, a covariance matrix $\boldsymbol{\Sigma}$, which in real-world processes is unknown and must be estimated from the data, summarizes the joint spread and the correlation of the quality characteristics. Xekalaki & Perakis (2002) rely on Principal Component Analysis (PCA) from the sample covariance to carry out the capability calculation. Their vision was to transform a correlated, multi-characteristic problem into several principal components or one-

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dimensional problems, then evaluate a known univariate capability index on each of those dimensions, and finally combine the results from each component into a single multivariate capability index (MPCI). Because PCA is built from the estimated covariance, the final index naturally reflects the covariance uncertainty without requiring a separate theoretical correction.

When the covariance matrix is unknown, important changes can be spotted. **Everything can be estimated from the data**, typically from the sample covariance $\boldsymbol{S}$. This happens through two steps. The first step is to replace the process mean and the covariance matrix with their versions from the sample, and the second step is to run the PCA on this sample covariance, so both directions (loadings) and importance (eigenvalues) of the principal components come from the observed data. Also, **specifications must be projected into the PCA space**. Each principal component is a weighted combination of the original characteristics, so its effective lower and upper specification limits, and of course, any target, are the same weighted combination of the original limits. After this projection, every component that remains has its own one-dimensional specification interval.

Extending this line of reasoning, **aggregation or the way to combine the components matters**. If every principal component carried the same information, a simple combination (geometric-mean style) of per-component indices would be sufficient. In practice, as Xekalaki & Perakis (2002) highlight, the first components usually capture much more variation than the later ones. Therefore, they propose weighted MPCIs that give more influence to components with larger eigenvalues, so the directions that explain more of the process variability have a bigger influence on the final measure. Finally, when many industrial characteristics have only one specification limit, only upper or only lower, the authors suggest providing **PCA-based MPCIs tailored to unilateral specs**, ensuring the method remains meaningful after the PCA rotation.

Following Xekalaki & Perakis (2002), proposed the following way of calculating the MPCIs in the case of unknown covariance. The calculation begins entirely from the data. Firstly, collect $\boldsymbol{n}$ observations on $\boldsymbol{p}$ quality characteristics together with each characteristic's LSL/USL, and a target, if applicable. Because of the problem that the true process mean and covariance are unknown, the solution that appears is the estimation of them. Thus, the computation of the sample mean vector and the sample covariance is necessary. The next step is to run PCA on the sample covariance to obtain loadings (define new, uncorrelated directions or axes, and express one vector per component) and eigenvalues (they quantify how much variation each axis explains). Also, need to map the specifications into the PCA space. More analytically, for each principal component, it is necessary to create

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its effective LSL/USL (and target) by taking the same linear combination of the original limits as specified by the loading vector of each component. After that step, every component has its own one-dimensional specification interval.

Continuing in the same vein, it is obligatory to decide which of the principal components (PCs) should be kept. The PCs that will remain are those that carry meaningful variation, as the paper recommends, and denote this number by $\boldsymbol{k}$. In the next step, each retained PC must be treated as a single quality characteristic. The sample mean and the sample standard deviation for each PC must be computed from its scores, and its mapped specification limits must be used. Then, with these quantities, it is necessary to evaluate a standard univariate capability index for the component, such as C_p, C_{pk}, C_{pm}, or C_{pmk}. After the computation of a univariate capability value for each of the $\boldsymbol{k}$ retained PCs, it is obligatory to combine all of them to get an overall multivariate capability index (MPCI). This can happen without the use of weights by merging the per-component indices, by using a geometric mean, which gives MC_p, MC_{pk}, MC_{pm}, or MC_{pmk} depending on the index used. An alternative way is the use of a weighted combination so that components with larger eigenvalues can contribute more to the final number. In practice, it is important to set weights proportional to the eigenvalues and normalize them to sum to one. This leads to a single, balanced MPCI that reflects both performance in each principal direction and the relative importance of those directions.

2.3.4 Impact of Non-Normal Data on Process Capability Analysis

Statistical Process Control techniques and PCIs often rely on the assumption that the quality characteristics follow the normal distribution. Control charts, process capability calculations, and hypothesis tests typically require this assumption to be valid for accurate interpretations (Montgomery, 2019).

Before applying any SPC techniques, it is essential to assess the normality of the data. Statistical tests, such as the Shapiro-Wilk test and graphical tools, such as Q-Q plots, can contribute to the validation of the normality assumption (Shapiro & Wilk, 1965).

However, if the normality assumption is violated, significant deviations from normality can adversely affect SPC tools, such as control charts and even process capability indices. Regarding the control charts, non-normal data can lead to incorrect control limits, increasing the likelihood of Type I or "false alarm" and Type II or "missed detection" (Yourstone & Zimmer, 1992). On the other hand, PCIs calculated under the assumption of normality may not accurately reflect the true process capability when the data do not follow a normal distribution. This misrepresentation can lead to inappropriate quality decisions (Wright, 1995).

To mitigate the impact of non-normality, the following approaches can be employed:

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1. **Data Transformation**: Applying mathematical transformations to the data, with the aim of approximate normality, is a common practice. For this purpose, the **Box-Cox** transformation is widely used (Box & Cox, 1964). Additionally, the **Johnson** transformation offers a flexible method to transform various types of non-normal data into normal (Johnson, 1949).

   The last transformation system was the known "Johnson Transformation System". This transformation contains a system of families of distributions. More specifically, three families of distributions (SL, SU, SB) can be spotted that can model a wide range of data shapes. By fitting a Johnson distribution to the data, a transformation function can be derived to normalize the data (Slifker & Shapiro, 1980).

   Thus, the transformation involves the identification of the appropriate family of the data, the estimation of the parameters by using methods such as the method of moments. Also, the application of the transformation for the normalization of the data and the conduct of standard SPC analyses on the transformed data.

2. **Non Parametric Methods**: When the transformations are not appropriate or effective, non-parametric SPC methods that do not assume a specific data distribution can be used. These include control charts based on ranks or signs, which are robust against normality (Chakraborti et al., 2001).

3. **Alternative Distributions**: Fitting the data with a different theoretical distribution that represents the observed data, such as Weibull or Log-Normal Distribution, allows the customization of SPC techniques and capability analysis to be suitable for that distribution (Meeker & Escobar, 1998).

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3. Chapter 3: Fundamentals of PCIs

3.1 Basic Statistical Concepts and Definitions

Process Capability is a key component of modern quality control, demonstrating the ability of a process to produce items that meet predetermined specifications or client expectations. The notion focuses on how well a process can deliver results within the expected range despite inherent variability. It is the foundation of quantitative quality analysis, especially in Six Sigma methodologies and ISO-certified environments. In simple terms, capability analysis assesses to fit between the "voice" of the process (natural variation) and the "voice" of the customer (tolerance range).

3.1.1 An Overview of Process Capability

In 1986, Kane was among the first who introduce the concept of PCIs, such as C_p and C_{pk}, as a structured approach for evaluating and comparing process capability. He understood that mathematical indices could quantify how well a process meets its specifications. This concept was transformative. Rather than depending exclusively on visual inspections or pass/fail criteria, quality practitioners could now employ quantitative measures to evaluate process performance and lead improvement efforts.

Later, Kotz and Johnson (1999, 2002) expanded Kane's work by emphasizing the distinction between process capability and process performance. They broadened the theoretical foundation of these indices by investigating interrelationships, mathematical boundaries, and advanced generalizations. This investigation aimed to address asymmetric tolerances and not on-target performances. Also, Palmer and Tsui (1999) extended these discussions further to multivariate, non-normal situations or robust versions of PCIs, providing a unified framework for their classification and application.

A few years before the statements and the theories of the scientists above, Cheng (1994) presented his theory by providing extensive insights into their practical use. He accomplished that by emphasizing the importance of sample-based estimation in real-world applications, as well as the limitations of classical indices under uncertainty or instability.

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Thus, the overarching purpose of process capability analysis is to establish not only if a process can produce within specifications but also how efficient, centered, and consistent it is. It also functions as a diagnostic tool, promoting continuous development and reduction in cost.

3.1.2 Core Definitions and Key Parameters

A thorough understanding of the fundamental words that support everything is a necessary tool for capability analysis. Various core components are appropriate for building the backbone of process capability evaluations. The most well-known are the following:

- **Process Mean (μ)**: The central tendency of the process data.
- **Process Standard Deviation (σ)**: A standard deviation-based measure of process spread or dispersion.
- **Specification Limits**: There are two specification limits. The Upper (USL) and the Lower (LSL) Specification Limits. These limitations represent the expected range of process outcomes.
- **Tolerance Range**: Equal to the difference between the Upper and the Lower Specification Limit (USL-LSL). A higher tolerance suggests more flexibility, but customer expectations may not always permit it.
- **Target Value (T)**: The desired outcome is frequently represented by the midpoint between USL and LSL.
- **Process Spread**: In a normally distributed process is defined by 6σ. This is the basis for calculating capability indices, such as C_p, that measure the ratio of specification width to process width.

The factors above are used to calculate capability indices such as C_p, C_{pk}, and C_{pm} to determine how well a process can satisfy its specification restrictions. A crucial point is a full understanding of the interaction of the elements above. If a process is consistently off-target and has a low standard deviation, then the result is the process's poor performance, resulting in a low C_{pk} or C_{pm}, despite a high C_p.

Kotz and Johnson (1999, 2002) stated that comprehending these metrics in isolation is insufficient because the best result is given by the interaction between the process mean, variation, and specification limits. This interaction aims to determine the full significance or true value of the process capability indices. Moreover, C_p evaluates the gap between specification limits and actual process change, whereas C_{pk} takes into account any shift in the process mean.

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3.1.3 Types and Sources of Variation

Variation is an unavoidable element of any manufacturing process, but knowing its causes is crucial for capability analysis. This is the reason why there is no exact similarity between products, even when manufactured under the same conditions. However, not all variations are created equally. In the literature of quality engineering, variation is distinguished into two main categories. The first is based on time, and the second is based on source. More specifically:

A. Time-Based Variation This type of variation is currently used when a decision needs to be made between short-term (C_p and C_{pk}) and long-term (P_p and P_{pk}) capability indices. The short-term and long-term variations are going to be presented below as follows:

- **Short-term** (within-subgroup) variation refers to natural fluctuations that occur consistently over a short period, such as within a subgroup or sampling period. This type of variation is typically used to estimate the process standard deviation, σ, while assuming stability and normality. Furthermore, represents natural process noise under stable conditions (common cause variation, in-control process, reliability of measurement system), and it is used to compute indices C_p and C_{pk}. According to Kane (1986) and Montgomery (1991), these indices represent a process's optimal performance, provided it is statistically controlled.

- **Long-term** (between-subgroup) variation refers to bigger changes that occur over time, for example, differences between days of production, work shifts, or material batches. Considers broader influences such as raw materials, operators, machine wear, and environmental changes. This overall standard deviation captures long-term variation, and it is utilized in indices like P_p and P_{pk} to assess process performance under real-world conditions. As Palmer and Tsui (1999) and Anis (2008) stated in later studies from Kane and Montgomery, using short-term data alone can misrepresent a process's true capability, particularly if long-term disturbances exist. This happens because short-term data might make the process appear more consistent than it is, especially if some problems or changes occur over a long time.

  Also, Anis (2008) have proposed modified or alternative estimating approaches that capture broader process behavior across time, like pooled or combined or overall variances (the technique of estimation of variance of several populations when the variance of each population is assumed to be the same, even when the means of the populations may differ), robust estimators, and Bayesian models.

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B. Source-Based Variation

- **Common Cause** variation: These are the normal irregularities incorporated into a process. They are statistically predictable, constant, and acceptable as long as they stay within the specification limits. When a process presents only common causes of variation, it is considered to be in statistical control, and traditional indices (C_p, C_{pk}) are suitable instruments for measuring capability.

  Montgomery (1991) stated that: *"It is meaningless to compute process capability for a process that is not in control, because the process performance is unpredictable"*. Identifying and establishing this stable state is critical before any capability analysis. Through this phrase, he emphasizes that process capability indices are based on a state of control, and using them when the process faces instability could lead to incorrect and potentially misleading results.

- **Special Cause** variation: These variations are caused by definable, typically disruptive events, such as machine breakdowns, operator errors, and material defects. However, these events do not occur during regular process operation. This presence indicates the instability of the process and the unpredictability of the process's behavior. When exceptional causes exist, any calculated capability index is invalid in statistical terms because it incorporates abnormal sources of variation that mislead the process's actual performance.

  Gunter (1989a-d) cautioned that estimating capability indices in the presence of special cases is one of the most common mistakes in quality analysis. He highlighted that this might provide a false sense of security by making a process appear capable when, in reality, it does not regularly produce within the specifications. Moreover, Gunter noted that making choices based on such inaccurate indices can lead to serious and inevitable problems, such as unnecessary process changes.

  Similarly, Steiner et al. (1997) discovered that special cause variation "harms" the estimation of standard deviation and process averages, resulting in incorrect and exaggerated PCI values. Their research demonstrated that process stability is essential for meaningful capability analysis. They claim that quality practitioners should always utilize control charts to identify and eliminate special causes before evaluating capability.

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Apart from the above assumptions, there is a case of cooperation between them (time-based and source-based variation). The following table shows how the two classifications work together to guide capability analysis, despite the fact that these two classifications focus on different aspects.

Category	**Focus**	**How they apply?**
Time-Based	When	Decides which of the PCIs (Cp or Pp) should be used
Source-Based	Why	Determines whether PCIs should be used at all

Table 3.1: Category, Focus, and Applications of Process Capability Indices

For example, a process may have low short-term variation, which means a high value of C_p, yet be affected by frequent external shocks (special causes), reducing long-term capability, which means a low value of P_{pk}. Understanding both dimensions enables practitioners to carry out valid and strategically useful process evaluations (Montgomery, 1991; Gunter, 1989; Steiner et al., 1997; Anis, 2008).

3.1.4 Relationships between Target Values, Specification Limits, and Process Capability

Specification Limits, known as the Upper and the Lower Specification Limits (USL and LSL, respectively), are predefined boundaries within which a process output must fall to be regarded as acceptable. These boundaries are defined by the customer requirements, engineering design criteria, or standard practices in the industry. However, judging a process solely on its capacity to produce within these specification limits can lead to incomplete or misleading assessments of quality.

A process may continuously produce values within specification, yet those values may be far from the desired target (T). This problem may not cause immediate rejection, but it could lead to decreased functionality, consumer unhappiness, or higher long-term costs. This is where process capability indices take action, particularly those that take into account centering and targeting. Indices, such as C_p, C_{pk}, and C_{pm}, provide varying amounts of insight into the process. C_p analyzes potential capability, assuming that the process is centered. C_{pk} adjusts for shifts in the mean, and C_{pm} goes a step further by including the target value (T) and penalizing any deviation from it.

Kotz and Johnson (1999) suggested that C_{pm} is a more comprehensive measure of quality, particularly in target-oriented manufacturing environments. This index was partially inspired by the Taguchi Loss Function, which contends that any deviation from the target, even within specification, results in quality loss. Unlike traditional perspectives that identify outputs as acceptable or defective based on fixed limits, Taguchi's philosophy encourages the concept of continuous quality degradation as one that moves away from the target.

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This idea seems especially relevant in precision-driven industries, such as aerospace, semiconductor, and pharmaceuticals, where even small deviations from the T can result in functional issues. According to Palmer and Tsui (1999), capability analysis is increasingly focusing on the target, with indices such as C_{pm} being favored for industries with strict tolerances and product performance reliability.

Another major application of the indices is to promote continuous improvement and strategic process modifications. Quality practitioners can identify specific performance concerns by comparing different indices. More specifically:

- A high value of C_p and a low value of C_{pk} indicate the process has a permissible spread but is not on-center.
- Even when C_p and C_{pk} are acceptable, a low value of C_{pm} indicates that the process is not hitting the target, and it is characterized as "non-satisfactory" for the customer.

The fundamental advantage of capability indices is that they simplify complex process data into a single and understandable number. The most common industry rule states that $C_p \geq 1.33$, which indicates acceptable process capability, while assuming normality and centering. This standard reflects a process that does not vary too much and stays well within the specification limits, making it reliable to meet the expectations of the industry. Kane (1986) implied that this threshold corresponds to a process that produces fewer than 64 defects per million opportunities (DPMO) under optimal conditions. A few years later, this criterion was supported by later authors, such as Porter and Oakland (1991) and Kotz and Johnson (1999).

Nevertheless, when the process is not perfectly centered, the threshold above can not be held, and it is not sufficient. This is why a lot of companies require both C_{pk} and C_{pm} to exceed 1.33 or even 1.67, for quality characteristics. Thus, for the indices C_p, C_{pk}, and C_{pm}, the following applies:

- $C_p = C_{pk} > 1.33$: Te process is **capable** and **centered**.
- $C_p > 1.33$, $C_{pk} < 1.33$: The process **spread is acceptable** but **it is not on-center**.
- $C_{pk} > 1.33, C_{pm} < 1.33$: The process is **centered between specifications**, but **not close to the target value** T.

Additionally, it is crucial to note that there are relationships between the traditional PCIs. According to the research of Kotz and Johnson (1999, 2002) and Palmer and Tsui (1999), the results are the following:

1. If $C_p = C_{pk} = C_{pm}$, then the process is perfectly **centered** and **on-target**.
2. If $C_p = C_{pk} > C_{pm}$, then the process is **centered** and **not on-target**.
3. If $C_p > C_{pk} > C_{pm}$, then the process is **not centered** and **off-target**.

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3.2 Classical Process Capability Indices and Mathematical Formulations

These important tools have changed dramatically since their inception as a way of quantifying the relationship between process performance and specification limits. Their role goes beyond simple quality review, impacting decision-making in product manufacturing, development, and tasks based on quality assurance. This section covers the basic categories of PCIs, the rationale for their development, and their strengths and weaknesses, using key literature.

The classical indices C_p, C_{pk}, and C_{pm} serve as the foundation of PCIs. In 1986, Kane defined the C_p index as a fundamental ratio of specification width to process spread, assuming normality and centering processes. When the process is 'not on track', which means that the process is not on target, this index turns out to be overly optimistic. Then, C_{pk}, also discussed by Kane and expanded by Kotz and Johnson in 1999, addresses this limitation by considering the process mean's deviation from the center. Moreover, the Taguchi Capability Index, known as C_{pm}, penalizes 'divergence' or deviation from a specified target (T), and it is particularly useful in precision-critical industries, according to Kotz and Johnson, and Palmer and Tsui in 1999.

3.2.1 The Potential Capability Index C_p

According to Kane(1986), C_p or **Potential Capability Index** is the easiest and one of the earliest creations in the domain of process capability. Determines if a process has sufficient spread or precision to meet specification limits, assuming that it is properly centered. Also, describes the "potential" capability of the process by comparing the process's spread, which is measured by $6\sigma = 6$ standard deviations, to the width of the specification limits. Also, this index assumes perfect centering and measures the spread of the process in comparison to the acceptable spread.

The mathematical formulation of C_p is the following:

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma} \tag{3.2.1}$$

Where:

USL, LSL = the Upper and Lower Specification Limits

σ = the standard deviation of the process

3.2.2 The Process Performance Index C_{pk}

C_{pk} or **Process Performance Index** evaluates both spread and process centering and demonstrates the "actual" capability by accounting for process mean deviation. Adjusts for off-centering of the process mean by calculating the distance between the mean and the closest specification limit. This index suggests that process spread and centering should be combined. It is especially useful when process outputs are not symmetrically distributed around the target because it emphasizes the impact of process mean shifts on quality performance.

The mathematical formulation of C_{pk} is the following:

$$C_{pk} = \min\left(\frac{\text{USL} - \mu}{3\sigma}, \frac{\mu - \text{LSL}}{3\sigma}\right) = \min\{C_{pu}, C_{pl}\} \tag{3.2.2}$$

3.2.3 The One-sided Capability Indices C_{pu}, C_{pl}

These indices are the **Upper** (C_{pu}) and **Lower** (C_{pl}) **Process Capability Indices**. According to Kotz and Johnson (1999):

- C_{pu} focuses only on how close the upper specification limit is to the process mean, which is useful when the lower limit is not crucial or does not exist.
- C_{pl} determines how near the process mean is to the lower specification limit. It works with C_{pu} when asymmetry on one-sided limitations is important.

Their mathematical formulations are the following:

$$C_{pu} = \frac{\text{USL} - \mu}{3\sigma} \tag{3.2.4.1}$$

$$C_{pl} = \frac{\mu - \text{LSL}}{3\sigma} \tag{3.2.4.2}$$

3.2.4 The Taguchi Capability Index C_{pm}

C_{pm} or **Taguchi Capability Index** goes even further, comparing the process mean to the target value and penalizing an off-target performance. It demonstrates targeting and variation. This index includes the deviation from the target value (T) (Chan et al., 1988). It is especially useful in processes where remaining close to the target is critical, as it incorporates both process variability and the distance of the process mean from the desired target (Kotz and Johnson, 1999).

The mathematical formulation of C_{pm} is the following:

$$C_{pm} = \frac{\text{USL} - \text{LSL}}{6\sqrt{\sigma^2 + (\mu - T)^2}} \tag{3.2.3}$$

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3.2.5 Performance Indices P_p, P_{pk}, and P_{pm}

The indices, such as P_p, P_{pk}, and P_{pm}, are used for long-term capability, instead of all the previous capabilities, such as C_p, C_{pk}, and C_{pm}, which are used for short-term capability. The performance indices use the overall standard deviation (s) instead of the within-subgroup variation (σ). These indices are mentioned only in this Chapter, but will not be analyzed further.

Furthermore, they are critical for determining how the process performs over time, taking into account all sources of variation. According to Cheng (1994), in unstable and high-variation situations, P_{pk} can be lower than C_{pk}. A high C_{pk} but low P_{pk} indicates that the process may appear good in the short term, but the same condition does not hold in the long term.

The mathematical formulations of the performance indices are the following:

1. **The Process Performance Index P_p**

$$P_p = \frac{\text{USL} - \text{LSL}}{6s} \tag{3.2.6.1}$$

According to Montgomery (2019) and Anis (2008), the P_p index is the long-term equivalent of the Potential Process Index (C_p), employing the overall standard deviation rather than within-subgroup variability. It reflects observed variation over time, including environmental conditions, shifts, tool wear, and differences in operation. Kotz and Johnson (2002) suggest that P_p is a more realistic metric for analyzing sustained performance across long production runs, while C_p represents potential capability under perfect conditions.

2. **The Long-Term Performance Index P_{pk}**

$$P_{pk} = \min\left(\frac{\text{USL} - \bar{X}}{3s}, \frac{\bar{X} - \text{LSL}}{3s}\right) \tag{3.2.6.2}$$

Montgomery (2019) and Cheng (1994) suggest that, similarly to the index C_{pk}, P_{pk} adjusts the P_p value based on the process's mean proximity to the nearest specification limit, but with the total standard deviation. This index is useful for evaluating product quality over time, especially in supplier audits or customer contracts, when actual field performance is more important than short-term capabilities. According to Steiner et al. (1997), a considerable gap between C_{pk} and P_{pk} indicates that short-term control is misleading, and the process meets long-term shifts or drifts.

3. **The Long-Term Capability with Targeting P_{pm}**

$$P_{pm} = \frac{\text{USL} - \text{LSL}}{6\sqrt{s^2 + (\bar{X} - T)^2}} \qquad (3.2.6.3)$$

Using Taguchi's Loss Function, P_{pm} penalizes both variation and deviation from a desired target value T. Cheng (1994) elaborates on this index, emphasizing its usefulness in industries where precision targeting is critical, such as electronics, aerospace, and pharmaceuticals. Similarly to C_{pm}, but under long-term terms, P_{pm} provides a more conservative estimate of quality by integrating spread, centering, and deviation from the target. According to Kotz and Johnson (1999), when stability is uncertain, this long-term capability index provides a more accurate representation of what customers experience across time.

Where:

$\bar{X}$ = the process mean

σ = the overall standard deviation

T = the target value

3.3 Assumptions Underlying Traditional Capability Indices

The validity of process capability indices is based on several critical assumptions. These assumptions are the following:

1. **Normality** distribution of the data. Most traditional capability indices, such as C_p and C_{pk}, assume a normal distribution for process data. This assumption is crucial because these indices relate process spread (measured by standard deviation) to specification limits using the properties of the normal curve. Specifically, $\sim 99.73\%$ of the data in a normal distribution lies within $\pm 3\sigma$ of the mean.

   However, many real-world processes generate non-normal data due to skewness (for example, right-skewed distributions due to tool wear), multimodality (for example, from mixed production batches), or heavy tails, where extreme values often appear. In such instances, applying indices based on normal distribution may underestimate or overestimate actual process capability. A right-skewed distribution may appear capable via C_p, but result in a significant defect rate on the upper specification side.

   To address this, Tang and Than (1999) and Anis (2008) examined a variety of solutions. One of them was the **transformations in data** that approximate

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normality. These transformations are accomplished through techniques such as Box-Cox and Johnson transformations. Another one referred to **non-parametric or quantile-based (percentile) indices** that operate under no specific distribution and use observed percentiles instead of theoretical assumptions. They are beneficial when Q-Q plots, histograms, or any other visual tool show the appearance of normality or not. Furthermore, **alternative distributions**, such as Weibull or Log-normal, were fitted properly with the aim of computing capability.

2. **Stability** of the process. The second, and perhaps most crucial, assumption is that the process must be statistically stable, or in control, before the capability's evaluation. To verify control status, the first step is the use of control charts, such as $\bar{X}$–R or $\bar{X}$–S charts. Stability indicates that only common cause variation exists, and the process runs consistently across time. Only once stability is proven can standard deviation estimates and capability indices be accurately interpreted.

   Montgomery (1991) and Gunter (1989c) stated that if particular reasons of variation exist, the process becomes unpredictable, and any capability estimate becomes useless or invalid. This happens because of variation. In such circumstances, variation is not inherent in the process but caused by external or assignable elements that may be eliminated or corrected. Porter and Oakland (1991) additionally claimed that applying PCIs in unstable processes is wrong because it leads to biased results, since these indices represent temporary disruptions rather than long-term capability.

3. **Independence** that the observations possess. Traditional PCIs assume that observations are statistically independent, meaning no data point is influenced by the previous one. This assumption is frequently violated in real-world processes, including autocorrelation, particularly in time-series data or batch processes. There are several tools to detect dependence, such as runs tests to check patterns, Autocorrelation function plots (ACF), and control charts for individual data points(moving average charts).

   If the existence of dependence is clear, then adjusted control limits or capability methods should be applied. For instance, when autocorrelation is present, charts like the EWMA (Exponentially Weighted Moving Average) chart and the CUSUM (Cumulative Sum) chart can be extremely useful because they account for serial autocorrelation and identify small shifts more efficiently than the classical Shewhart charts. Regarding the capability methods, the most suitable are non-normal capability indices, robust methods, or quantile-based indices.

   Additionally, in some cases, model-based approaches, such as time-series models, may be necessary. These models, such as ARIMA (AutoRegressive Integrated Moving Average), are used to remove time-based patterns and apply capability analysis on

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the residuals, which should behave like independent data.

Cogollo Flórez et al. (2017) observed that neglecting autocorrelation might result in an underestimated standard deviation, inflating capability indices, and making a process more capable than it is. Similarly, Gunter (1989a) highlighted that the meaning of noticing and understanding the relation between data points is an important part because it could lead to incorrect results about whether a process is stable and how capable it is.

4. **Valid Measurement System**: Capability analysis gives the best performance if the data comes from an accurate, consistent, and reliable measurement system every time that it is used. A weak or poor measurement system can create errors (variation), which might hide serious problems and make the process look worse.

   Issues in the measurement system can create various problems. One of them is the low values in the indices C_p and P_p, which are caused by the artificially high variation. Also, it is natural to have incorrect standard deviation estimates. Due to noisy signals, masked process shifts are created. This means that if too much noise is added to the measurement system, real changes in the process will be hidden, which will make it harder to see if something is going wrong.

   To guarantee a reliable measurement system, a Gage R&R (Repeatability and Reproducibility) study should be conducted. According to Montgomery (2019) and AIAG (2010) recommendations, the measurement system should account for no more than 10% of the total observed variation for the data to be valid.

Consequently, violating any of the assumptions above can lead to unfortunate events, such as misinterpretation of capability indices, poor process improvement decisions, and invalid evaluations of suppliers or product quality. According to Borroni et al. (2010) and Kotz and Johnson (1999), traditional process capability indices are highly sensitive to these assumptions and should be treated with caution when they are not completely met. Through the use of additional diagnostic tools and visual data examination, the final results are more accurate, and unnecessary mistakes can be avoided. This procedure happens when data conditions are uncertain or the assumptions are weak.

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3.4 Process Capability Analysis in Pharmaceutical Domain with qcc package in R: A Univariate Case on Tablet Weight Uniformity

In the pharmaceutical domain, tablets are solid oral dosage forms that patients take by mouth. These tablets are produced by compressing a mixture of API (Active Pharmaceutical Ingredient) and excipients (inactive substances) into a solid form. To confirm that the medicine is safe, each tablet must have a suitable amount of API. One way of verification is through the **tablet's weight**. This happens because, when the API is evenly mixed, it leads to a strong relationship between the weight of the tablet and its API content.

Furthermore, if the weights of the tablets vary, patients may receive a small or a large amount of this drug, which can lead to the failure of the treatment. The solution to this problem is to maintain weight uniformity within specified limits, as defined by regulatory agencies such as the European Medicines Agency (EMA) and the U.S. Food and Drug Administration (FDA) (EMA, 2014; ICH Q6A, 1999).

The purpose of this example is to evaluate whether the pharmaceutical manufacturing process is capable of producing tablets within the specification limits. One way to examine whether a process meets the quality requirements is through the known PCA or Process Capability Analysis. For this reason, a simulated dataset of 5000 tablet weights was generated from the Normal Distribution, with mean (μ) equal to 50 mg and standard deviation (σ) equal to 1.2 mg. From Figure 3.1, it is evident from the bell curve shape that the tablet weights data follow a normal distribution.

The mean value corresponds to the target ($T = 50$) tablet weight or the nominal dose provided on the product label, and the standard deviation represents a realistic level of variability in a well-controlled pharmaceutical process. Additionally, the Control Limits were set at UCL $= \bar{x} + 3\sigma =$55 mg and LLC $= \bar{x} - 3\sigma =$45 mg. These limits are applied to the early process validation.

The next step is the famous Process Capability Analysis (PCA), which is a common way to evaluate whether a process meets the quality requirements that are mentioned above. This procedure uses the Process Capability Indices (PCIs) to confirm if a process can consistently produce items within the Specification Limits and be close to the target. The PCIs that are used in this case were: The Process Capability Index C_p, the Process Performance Index C_{pk}, the One-Sided Capability Indices C_{pu}, C_{pl}, and the Taguchi Capability Index C_{pm}, which were mentioned analytically in Subchapter 3.2.

To compute all these that are mentioned above, the `qcc` package in R was used. According to R Documentation and Scrucca (2004), this package is designed for statistical process monitoring and quality control charting. Moreover, includes a variety of Control

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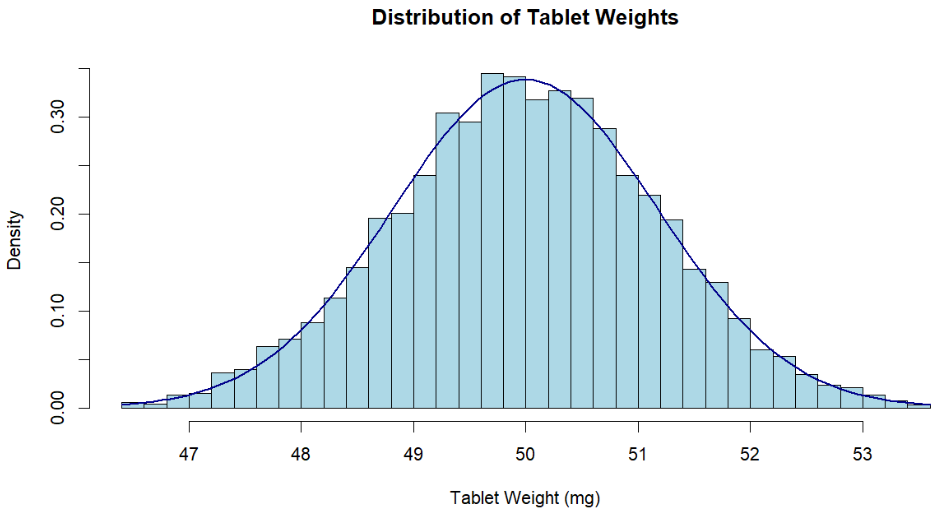

Figure 3.1: Histogram of the distribution of Tablet Weights data

Charts (Shewhart, CUSUM, EWMA), calculates the aforementioned capability indices, and visualizes quality performances.

In this example, before the calculation of the PCIs, a $\bar{X}$ **control chart** (`xbar.one`) was used to check the **stability** of the process, through the function `qcc()`. This chart is appropriate for individual measurements, which is common when dealing with continuous process data, such as tablet weights. Also, it is based on the 3σ rule, which means that for a Normal distribution, approximately 99.73% of the observations fall within the specification limits. This leads to a significance level equal to $\alpha = 0.0027$, because it detects significant shifts in the process and reduces false signals, unlike the most common significance level ($\alpha = 0.05$).

Before proceeding with the main analysis of this example, it is crucial to acknowledge the presence of outliers that were removed from the initial dataset. The original dataset contained 5000 observations. Following the removal of the outliers, 4986 observations remained for analysis. They were removed repeatedly to achieve stability in the dataset, as PCIs should only be calculated when the process shows common cause or normal variation. In Figure 3.2, no outliers were present, which confirms the stability of the process.

By observing Figure 3.2, it is obvious that there are no points that fall beyond the Control Limits, indicating that there is no alarm and confirming the absence of special cause variation. The process seems to be capable and well-centered. To be more specific, the process mean equals 49.9945 mg, which is very close to the target($T = 50$), and the standard deviation equals 1.1929 mg, which indicates a low level of variation relative to the tolerance range (± 10 mg), indicating that the process variation is well within acceptable limits. Additionally, the LCL equals 46.4157 mg, and the UCL equals 53.5732 mg. Thus, the process is considered statistically stable, which permits us to proceed to the calculation of the indices C_p, C_{pk}, C_{pl}, C_{pu}, and C_{pm}.

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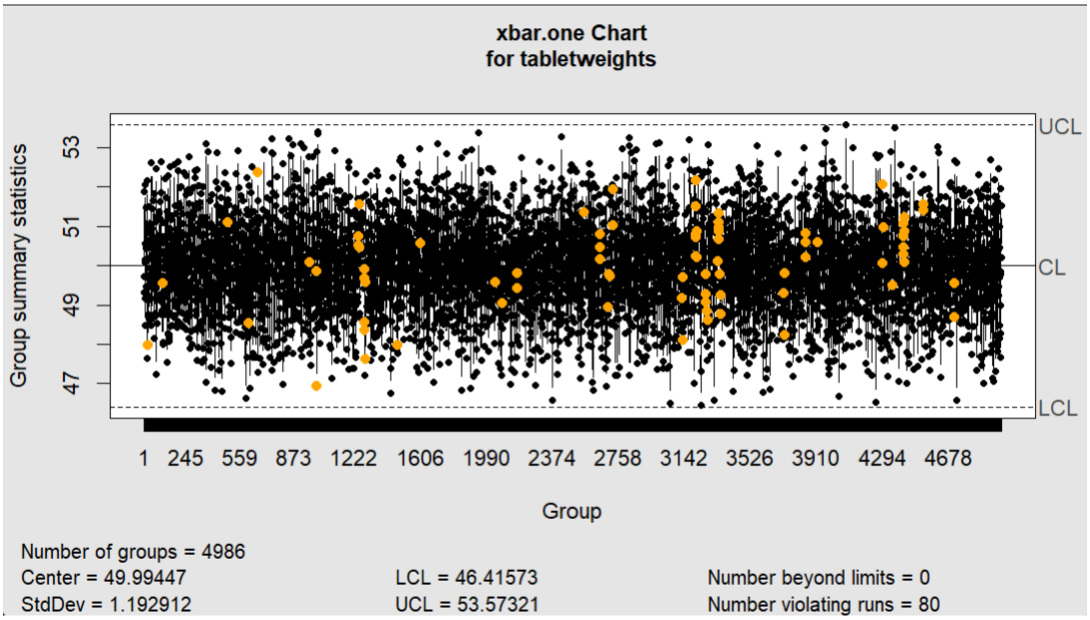

Figure 3.2: $\bar{X}$ Control Chart for Process Stability of Tablet Weights data

The last step of this analysis is the calculation of the aforementioned indices. By using the function `process.capability()` from the `qcc` package in R, all the indices were calculated successfully, and they are presented in the following table.

Cp	Cpk	Cpu	Cpl	Cpm
1.397141	1.395595	1.398687	1.395595	1.397126

Table 3.2: The Process Capability Indices (PCIs)

The results of Table 3.2 are going to be analyzed as follows:

- Process Capability Index C_p= 1.397141

  This index measures the potential capability of the process, under the assumption that it is centered. It is true that $C_p \geq 1.33$, which indicates a capable process (Montgomery, 2019; Bothe, 2001). Furthermore, C_p has a value that is close to 1.40, which means that the weights of the tablet do not vary compared to the permitted range of the control limits. Thus, the tablets stay within the limits because of their consistency in weight.

- Process Performance Index C_{pk}= 1.395595

  This index accounts for both variation and centering. It is true that $C_{pk} \geq 1.33$, which indicates that the actual process capability meets the quality requirements. Also, $C_{pk} \approx C_p$, indicating that the process creates tablets very close to the target value of 50 mg. Thus, the tablets are evenly spread around the ideal weight.

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- Taguchi Capability Index C_{pm}= 1.397126

  This index penalizes deviation from the target value. It is obvious that $C_{pm} \approx C_p$, which indicates that the process is very close to the target($T = 50$ mg), indicating minimal deviation from the nominal dose or target.

- One-Sided Capability Indices C_{pu}= 1.398687, C_{pl}= 1.395595

  These indices measure how close the process is to the Upper and Lower limits, respectively. From the values of the one-sided indices, the conclusion is that they are almost equal. These values also indicate symmetry around the target. Thus, the tablets are not leading toward being too heavy or too light.

4. Chapter 4: Univariate Process Capability Analysis

4.1 Understanding Univariate PCIs

Univariate Process Capability Analysis is the foundation of statistical quality control. This case specializes in the evaluation of a process based on a single quality parameter, such as fill weight, dimension (weight, diameter, and length), physical property (viscosity, tensile strength), and a measure of performance(pressure or speed). The primary challenge is whether the process produces suitable results or items that meet predefined specification limits (USL and LSL) by the customer's expectations, design requirements, and regulatory bodies. In contrast to multivariate PCIs, which deal with interrelated variables, univariate indices focus just on one variable at a time, making them the most widely employed form of capability analysis in industrial practice.

According to de-Felipe and Benedito (2017), univariate capability analysis is not only widely applied, but it also serves as the foundation for more complicated multivariate approaches. Because numerous critical product characteristics are tested and monitored individually, the univariate case offers a straightforward and interpretable way for assessing and controlling quality.

The significance of process capability indices stems from their ability to transform statistical properties of the process data into clear and usable metrics. When they are employed properly, they provide a brief yet effective overview of how a process performs, assisting engineers, quality managers, and decision-makers in determining whether intervention or improvement is required.

In a univariate situation, there is an assumption that is based on monitoring only one critical-to-quality variable. This simplification allows for easier interpretation, visualization, and benchmarking. Indices, such as C_p, C_{pk}, and C_{pm}, ensure quality consistency in healthcare, research laboratories, manufacturing, and service systems.

The three most famous indices in univariate capability analysis are C_p, C_{pk}, and C_{pm}, where each has a particular role in interpreting process behavior. The formulas behind these indices were discussed in Chapter 3, but their conceptual value is stressed in this chapter. More specifically:

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- C_p assumes that the process is precisely centered
- C_{pk} penalizes it for being off-center
- C_{pm} imposes an additional penalty if it is not on-target

Firstly, C_p, which was introduced in the mid-20th century, is also known as the Potential Capability Index. The width of the process spread, measured as six standard deviations ($\pm 3\sigma$), is compared to the width of the tolerance band between Upper and Lower specification limits (USL-LSL). However, the C_p index does not take into account how correctly the process is centered between these limits. To overcome this problem, the creation of the Process Performance Index (C_{pk}) was the solution. This index fixes the deviation of the process mean from the specification center. If the process is off-center, then C_{pk} will be lower than C_p, even if there is small variability. This modification makes C_{pk} a more accurate indicator of actual performance.

While C_p and C_{pk} evaluate variability and centering, they do not consider what happens with the target value (T), which is typically useful in pharmaceuticals, electronics, and precision machining industries. The result was the invention of the Taguchi Capability Index (C_{pm}), which was introduced by Chan et al. (1988) and later examined by Denniston (2006), who proposed the use of this index when customer satisfaction relies on not only being in-spec, but also on being on-target. The Taguchi Loss Function supports this notion, claiming that the quality loss increases when a process deviates from the target, if it remains within specification limits.

Furthermore, Bill Denniston (2006) stated that C_{pm} should be emphasized in circumstances when the goal is to adhere to ideal values rather than simply conform to specifications. His findings are consistent with the growing interest in performance-based quality metrics, particularly where quality loss, either financial or functional, is sensitive to target deviations. Also, his work on C_{pm} demonstrates that conformance to specifications does not always correlate with customer happiness or cost-efficiency. Processes with low variability but bias from the target may appear feasible under C_p and C_{pk}, but may still result in product waste, rework, or warranty costs.

However, as Denninston (2006) demonstrated, high capability values do not always bring happy results for the customers. A process that consistently delivers values at the edge of a specification range may nonetheless cause performance issues, even if C_p or C_{pk} appears acceptable. This is when the indices C_{pm} and C_{pmk} take action. They take into account how far the process is from the ideal target value, which presents how small shifts from the target can create quality problems, as the Taguchi Loss Function states. This idea is especially important in industries such as aerospace, medical devices, and precision machining, where small deviations can have major product-level consequences.

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When the center of the specification differs from the desired target, it is clear that target-oriented capability measures are required. While C_p and C_{pk} can describe the process's consistency and centering, they do not say much about whether a process produces outputs that meet the customers' expectations. For example, a process may be well within specification but regularly below or above the ideal value. In such cases, product performance and customer satisfaction may suffer, even if traditional PCIs indicate acceptable or high capability.

This gap inspired the invention of C_{pm} and its variants, such as C_{pmk}, which combines the process variability of centeredness and distance from the target into a single index. According to Zimmer et al.(2001), these indices provide a more complete picture of the process performance, especially in regulated industries, such as automotive, food & beverage, energy, etc., where tolerance conformance and target accuracy are critical.

It is also crucial to note that these indices are statistical estimates generated by samples. Novoa and Artiles-Leon (2008) showed that $\hat{C}_{pk}$, the estimator of C_{pk}, follows a skewed distribution that depends on process variance, mean, and sample size. Similarly, Zimmer et al. (2001) underlined that: To ensure reliable confidence intervals, the estimator of the index C_{pm} ($\hat{C}_{pm}$) requires large sample sizes. These findings suggest that the engineers should examine the uncertainty surrounding the estimated index, particularly when making quality decisions in regulated or high-risk contexts.

In Six Sigma environments, the use of the index C_{pm} aligns with the principle of minimizing variation while achieving performance excellence. As Chen et al. (2009) point out that target-driven indices are extremely useful in Six Sigma projects to guarantee that improvements address both spread and aim. Thus, C_{pm} is more than an advanced index. This index symbolizes a shift in quality philosophy: one that values meeting expectations precisely rather than simply passing conformance tests.

Process Capability Indices in the Six Sigma domain are more than just analytical tools. They act as critical performance benchmarks. A key goal of Six Sigma is to reduce process variation and defects to less than 3.4 defects per million opportunities (DPMO). This level of performance corresponds to a capability index of approximately $C_{pk} = 2.0$ for a centered process with a 6σ spread. In the majority of industrial applications, a short-term benchmark of $C_{pk} \geq 1.33$ translates to around 63 DPMO, which is typically acceptable for most product characteristics. According to Pyzdek and Keller (2014), the threshold of $C_{pk} \geq 1.33$ serves as a quality gate, distinguishing between processes that barely meet requirements and those that consistently perform at higher reliability. In other words, it assists companies in determining whether a process is acceptable or statistically strong enough to provide consistent and predictable quality across time.

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Univariate indices, such as C_p, C_{pk}, and C_{pm}, are crucial in assessing baseline performance, identifying improvement needs, and monitoring long-term process control within the DMAIC structure of Six Sigma. During the second phase, or "Measure" phase, indices such as C_{pk} and C_{pm} contribute to how well the current process meets performance targets and specifications. In the last step of the DMAIC structure, the "Control" phase, these indices are constantly monitored to ensure that gains obtained through improvement efforts are maintained. As Pyzdek and Keller underlined, these indices are more than just statistics. They serve as decision-making tools for root cause investigation, improvement prioritization, and overall process alignment with customer expectations. Because of the strong connection between capability indices and business goals, univariate process capability analysis is an essential element of Six Sigma success.

Another significant issue in the univariate case is the common assumption that individual data points are independent of one another. In practice, however, this assumption is frequently violated, especially in processes requiring continuous production lines, batch operations, or chemical and thermal systems, where autocorrelation arises naturally. Autocorrelation indicates that the value of one measurement is influenced by the value of the previous ones, resulting in a pattern or dependency in the data. Lovelace et al. (2009) investigated how autocorrelated data can affect the estimation of univariate capability indices such as C_p and C_{pk}. Their findings demonstrate that autocorrelation can falsely reduce or increase the estimated process standard deviation, which is the basis of the calculation of these indices. This deformation can lead to inaccurate capability assessments. For instance, a high value of C_{pk} may indicate a strong performance, but the process may be drifting or cycling in a way of violates statistical control.

This issue is especially important in the univariate setting because the indices are based on a single measured characteristic. When the variable's data is autocorrelated, the resulting index may no longer accurately reflect the process's true capability or stability. For example, a filling machine may look capable of producing consistent short-term results, but over time, due to tool wear or gradual changes, the average fill level might slowly drift. If the gradual changes are autocorrelated, a basic C_p and C_{pk} calculation from raw data may still appear adequate, even if the process is no longer stable. This raises the risk of covering up real problems, which is why practitioners should evaluate independence before depending on univariate PCIs for decision-making.

Modern quality control requires more sophisticated tools to handle asymmetric tolerances, target drift, and multiple performance characteristics, while univariate capability analysis still relies on C_p, C_{pk}, and C_{pm}. Through the next subchapters, advanced and derived indices are going to be introduced. This will provide a more complete picture of process behavior under real-world conditions.

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4.2 A Development of Univariate PCIs

The traditional univariate capability indices are useful for analyzing basic process stability and performance, but they may not be suitable in more complicated or nuanced production environments. Many real-world processes are influenced by non-ideal elements, such as asymmetric tolerances, short production runs, inaccurate target values, or even vague quality criteria. These settings necessitate "updated" capability indices that extend or adapt the core metrics to better represent performance in non-standard conditions.

This subchapter introduces and discusses a series of advanced univariate indices, which were created in response to these problems. Indices like these are the following:

- The robust index C_{pmk} addresses deviation from target and asymmetry in specification limits.
- The metric C_{pp} which specializes in short-run or small-lot manufacturing environments.
- The yield-based capability index S_{pk} is directly related to the proportion of conforming items.
- The Fuzzy Capability Indices (FPCIs) permit uncertainty in process measurements and specifications, which makes them especially helpful in early-stage development and service-oriented businesses.

Exploring these indices provides us with tools not just for improving quality evaluations but also for tailoring capability analysis to a wider range of industrial scenarios. Each of the indices that will be analyzed below addresses unique limitations or assumptions of traditional indices, providing additional insights into process performance, conformance, and client satisfaction.

4.2.1 The Process Capability Index C_{pmk}

The C_{pmk} index is a creative generalization that addresses the limitations of the previous known indices. This index penalizes deviations from both the specification center and the nominal target value, unlike other metrics (C_p, C_{pk}, and C_{pm}) that just consider process spread or centering. This dual sensitivity enables a more comprehensive evaluation of process performance, particularly in industries where precision and alignment with functional targets are essential.

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Pearn, Lin, and Chang (2003) were the first who present the index C_{pmk}. Also, this index was included in Kotz's (2002) research about the capability indices. The inspiration for its development originated from practical industrial needs. More specifically, it was recognized that processes may exhibit acceptable variability while producing outputs that systematically shift away from important functional targets. In these cases, classical or traditional indices may falsely imply high capability, covering actual quality losses.

The structure of C_{pmk} illustrates its dual role: it adjusts the measure of process capability for both natural variation and systematic bias from the target T. This makes it especially useful in sectors like the semiconductor manufacturing industry, aerospace, and medical device production, where even small deviations from optimal settings can result in major issues in reliability.

The mathematical formulation of C_{pmk} is the following:

$$C_{pmk} = \min\left\{\frac{\mathrm{USL}-\mu}{3\sqrt{\sigma^2+(\mu-T)^2}}, \frac{\mu-\mathrm{LSL}}{3\sqrt{\sigma^2+(\mu-T)^2}}\right\} = \frac{d-|\mu-M|}{3\sqrt{\sigma^2+(\mu-T)^2}} \tag{4.2.1}$$

Where:

μ = the process mean

σ = the standard deviation of the process

T = the target or nominal value

M = the midpoint of the specification interval

d = the half-length of the specification interval

USL **and** LSL = the Upper and the Lower Specification Limits, respectively

Furthermore, the estimated version of the index C_{pmk}, symbolized as $\hat{C}_{pmk}$, is obtained from sample data and has specific statistical properties that should be considered during capability analysis. Zimmer et al. (2001) and Pearn et al. (2003) discovered that C_{pmk} has a slight negative bias, particularly when it is calculated from samples that have a small size. This bias causes a constant underestimation of true process capability, which can cause unfortunate events in the progress of quality engineers if not properly accounted for. Moreover, the distribution of $\hat{C}_{pmk}$ is often right-skewed when the sample sizes are small. However, when the sample size is larger than 100, which means that it contains more than 100 observations, it tends to approach the normal distribution. Because of the skewness in small samples, the approximations of standard confidence intervals are incorrect, based on normality assumptions.

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As a result, recent techniques propose using bootstrap methods or large-sample adjusted formulas to construct more accurate confidence intervals. Recognizing these statistical features is critical because capability assessments based on limited data may overstate or underestimate the true process performance, possibly leading to wrong and costly operational decisions.

Also, it is important to emphasize the following:

1. A **high** C_{pmk} value implies that the process outputs are tightly clustered around the nominal target while staying within the specification limits. This combination provides both precision and high levels of customer satisfaction.

2. A **low** C_{pmk} value indicates process variability, lack of centering, or significant deviation from the target, which indicates deeper issues within the process.

The C_{pmk} has notable advantages over previous indices, such as C_p, C_{pk}, and C_{pm}, as it addresses many sources of process efficiency at once. Specifically, while C_p measures natural variability assuming perfect centering and C_{pk} accounts for mean-off centeredness, only C_{pmk} penalizes both deviation from the center and the target (T) at the same time. All the indices above are responsible for the measurement of the natural variability, but only C_{pk} and C_{pmk} handle an off-center mean, and only C_{pm} and C_{pmk} penalize deviation from the target. Thus, C_{pmk} encompasses both penalties simultaneously, making it extremely useful for modern quality management environments. In these environments, achieving excellence means accurately adhering to optimal target values.

4.2.2 The Process Yield Index S_{pk}

The Process Yield Capability Index was first established by Boyles (1994). After a few years, Lee, Hung, and Pearn (2002) completed further studies based on the S_{pk}. It provides a step forward in capability analysis by directly relating process capability to expected yield, making statistical capability analysis more naturally related to production and business goals.

Unlike C_{pk}, which focuses on spread and centering, S_{pk} converts performance into a percentage of acceptable products. It estimates the probability that a randomly selected unit will meet specifications, making it especially appealing in high-volume and cost-sensitive manufacturing businesses.

The mathematical formulation of S_{pk} is the following:

$$S_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{\text{USL}-\mu}{\sigma}\right) + \frac{1}{2}\Phi\left(\frac{\mu-\text{LSL}}{\sigma}\right)\right\} \tag{4.2.2}$$

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Where:

$d = \frac{\textbf{USL}-\textbf{LSL}}{2}$ = the half-width of the specification range

$\Phi(\cdot)$ = the cumulative distribution function of the standard normal distribution $N(0,1)$.

What is Yield in Process Capability Analysis?

In the context of capability analysis, and as it is mentioned in the definition of p_0, yield is the proportion of conforming units produced by a process that fall within the defined specification limits without requiring rework, adjustment, or rejection. It is a simple yet valuable measure of the process capacity to consistently produce conforming products. Kotz (2002) defines yield as the probability that a product falls within the Upper and the Lower Specification limits (USL and LSL, respectively). More specifically, in a mathematical way, the probability is expressed as follows:

$$\text{Yield} = P(\text{LSL} \leq X \leq \text{USL})$$

This approach emphasizes that yield is the area under the probability density function between the specification limits. In simple words, yield answers the fundamental question of the industry: *"What proportion of produced units are good and do not require inspection, rework, or scrap?"*. Thus, understanding and maximizing process yield remains an important goal in quality engineering and operational excellence programs.

Also, it is crucial to mention the difference between high and low yield rates, as follows:

- **High yield rates** recommend that most products meet specifications on the first pass
- **Low yield rates** show that variability or centering issues are the root of errors

According to Pearn et al. (2003) and Lee et al. (2002), higher capability indices, such as C_{pk}, C_{pmk}, and S_{pk}, indicate a higher expected yield due to more tightly clustered process outputs. The index S_{pk} estimates process yield without assuming the exact normality conditions, making it extremely helpful in real-world, non-ideal industrial environments.

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The adjustment using Φ^{-1} converts yield information into a capability measure compatible with the conventional Six Sigma way of thinking. This provides a mechanism to connect probabilistic yield estimates with standardized capability indices. The estimated version of S_{pk}, symbolized as $\hat{S}_{pk}$, has various statistical properties. The sampling distribution of $\hat{S}_{pk}$ tends to be approximately normal for moderate to large sample sizes, for example, when n is greater than 50 (n > 50). But for very small sample sizes, the sampling distribution shows a left-skewed character. Pearn and Chuang (2004) discovered that the variance of $\hat{S}_{pk}$ is smaller than the variance of traditional estimators, such as

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$\hat{C}_{pk}$, meaning that the Process Yield Index S_{pk} offers more reliable and stable results. Additionally, S_{pk} is slightly conservative, which means that it tends to underestimate the real or true yield when it is derived from small datasets. This might be useful in high-risk industries where conservative estimates are desired.

4.2.3 The Process Incapability Index C_{pp}

The Process Incapability Index C_{pp} was developed to solve issues faced in short-run production environments and processes, when limited data are available. This index, which was introduced by the researchers Pearn, Ko, and Wang (2002) and Lin and Pearn (2005), measures a process's incapability rather than its capability. This shift in perspective makes C_{pp} particularly useful for practical applications where constant and long-run data are not available.

Unlike classical indices (C_p and C_{pk}), which assume long-term process behavior, the C_{pp} index is extremely useful in the domains that are going to be presented above. More specifically, in:

1. Short production runs. For example, some types of short production runs are aerospace prototypes and custom manufacturing

2. Cases that require early capability assessments before the beginning of mass production

3. Data collection scenarios that require a huge cost and a lot of time

Moreover, further modifications of C_{pp} have modified the index to include the impact of the location of the process mean, similar to the adjustment that was made in the Process Performance Index C_{pk}.

Statistically, the estimated version of C_{pp}, symbolized as $\hat{C}_{pp}$, follows a right-skewed distribution when the sample size is small. However, as the sample size is getting larger, the distribution eventually approaches normality. According to Lin et al. (2003), and under autocorrelated conditions, $\hat{C}_{pp}$ tends to show a slight bias. But this bias can be toned down through resampling or suitable analytical modifications. Moreover, compared to conventional PCIs, C_{pp} shows a greater robustness when it is applied to small sample sizes, making it a more reliable choice in the first stages of process evaluation.

When this index is compared to the traditional indices, various advantages come into view. Unlike C_p and C_{pk}, C_{pp} is designed for short-run evaluations, focusing directly on early-stage incapability and remaining less sensitive to immediate process shifts. C_{pp} is an important extension of the traditional capability indices, allowing organizations to make careful quality decisions despite time, volume, or cost constraints.

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The mathematical formulation of C_{pp} is the following:

$$C_{pp} = \frac{d}{3\sigma} \tag{4.2.3}$$

Where:

$d = \frac{\mathbf{USL-LSL}}{2}$ = the half-width of the specification range

σ = the standard deviation of the process

4.2.4 The Fuzzy Process Capability Indices (FPCIs)

Fuzzy Process Capability Indices or FPCIs are commonly represented as $\tilde{C}_p$, $\tilde{C}_{pk}$, $\tilde{C}_{pm}$, where the tilde denotes that the parameters, such as process mean, standard deviation, and specification limits are handled as fuzzy numbers, which are mathematical representations of imprecise or uncertain values and appear commonly as single points and not as ranges or membership functions. These symbols indicate that the indices are computed with fuzzy arithmetic and logic-based estimations rather than traditional statistical approaches.

Abdolshah (2012) highlighted that typical PCIs fail to capture the qualitative nature of human choices when precise specification limits are hard to define. In such cases, fuzzy PCIs increase the flexibility of capability analysis by modeling data with insufficient information using phrases as "approximately acceptable" or "almost defective". These phrases are translated into fuzzy numbers, and fuzzy arithmetic is used to generate indices expressing process quality in vagueness.

A few years later, Kaya and Çolak (2020) conducted a comprehensive literature review based on the history of FPCIs, from early conceptual models to hybrid methods. On the one hand, early conceptual models are defined as the first models that used fuzzy logic for the description of process performance when precise numbers were not available and presented a way to use fuzzy thinking to face the problem of uncertainty, but they provide helpful estimates of quality. On the other hand, hybrid models work as a bridge for fuzzy logic and other advanced methods, such as neural networks or machine learning techniques (decision trees).

Kaya and Çolak also emphasize that fuzzy PCIs are particularly effective in sectors such as medical diagnostics, service quality evaluation, and precision manufacturing, where human ambiguity, perception, and subjectivity play a significant role. Their research reveals that as quality becomes more data-driven, FPCIs provide an additional way for assessing capability when traditional statistical assumptions do not hold.

Consequently, fuzzy indices go beyond traditional definitions by embracing fuzzy set theory, which allows processes to be evaluated under uncertain conditions. According to the aforementioned researchers, FPCIs treat tolerances and process parameters as fuzzy

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numbers. This formulation is especially effective for early-stage development, customer specifications, and subjective quality assessments.

In many real-world and practical scenarios, exact specification limits and process parameters cannot be precisely defined due to insufficient data, expert opinion, or inherent ambiguity. Traditional PCIs assume that certain characteristics, such as the process mean, standard deviation, and specification limits, are known and fixed. However, in cases where this assumption is violated, Fuzzy Process Capability Indices or FPCIs have been developed to deal with imprecision, based on Zadeh's (1965) fuzzy set theory ideas.

FPCIs model input parameters as fuzzy numbers rather than precise values and often use membership functions as triangular or trapezoidal. These fuzzy shapes are both understandable and mathematically tractable, allowing quality engineers to define unknown quantities or parameters as ranges with varied degrees of possibility. For example, an upper specification limit may not be precisely 200 but rather close to it. The most likely value is going to be 200, but nearby values, such as 197 or 203, are still going to be considered as possible values, though with less confidence.

These fuzzy indices change the basic role of the famous traditional indices by permitting to fuzzy process mean $\tilde{\mu}$, process variation $\tilde{\sigma}$, and specification limits $\tilde{\mathrm{LSL}}$, $\tilde{\mathrm{USL}}$ to be imprecise. This allows for performance analysis in instances such as:

- Quality is characterized using phrases, such as "smooth" or "acceptable", rather than exact numbers.
- Tolerances are flexible and customizable, and acceptable limits vary depending on the situation, product type, and customer needs.
- Data collection is subjective, with measurements based on human observation and opinion rather than instruments alone.
- Variability can be caused by human judgment or sensor inaccuracies, leading to inconsistent evaluation results.

More specifically:

1. **Fuzzy C_p - Potential Capability under Uncertainty** is the fuzzy version of the Potential Process Capability Index C_p and is responsible for the measurement of the process's potential capability while accounting for uncertainty in inputs, assuming that it is centered.

   This index generates a fuzzy number as output, which requires particular interpretation techniques. One method is to use a-cuts, which provide confidence-based intervals indicating where the index falls at different certainty levels. Defuzzication is another method, which, like the centroid method, simplifies the fuzzy result to a

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single representative value for simple comparison. Finally, fuzzy ranking techniques allow you to compare various fuzzy indices based on their shape, peak, or average behavior.

The mathematical formulation of $\tilde{C}_p$ is the following:

$$\tilde{C}_p = \frac{\tilde{\text{USL}} - \tilde{\text{LSL}}}{6\tilde{\sigma}} \tag{4.2.4.1}$$

2. **Fuzzy C_{pk} - Actual Capability with Mean Shift** considers both centering and process spread when the process mean ($\tilde{\mu}$) is uncertain. The following formula captures asymmetric proximity to specification limits via "soft" thresholds. It is especially beneficial when mean shifts are qualitatively understood but exactly quantified, for example, "slightly above target". Also, fuzzy C_{pk} better captures prediction uncertainty in operator-driven or semi-automated controls.

   The mathematical formulation of $\tilde{C}_{pk}$ is the following:

$$\tilde{C}_{pk} = \min\left(\frac{\tilde{\text{USL}} - \tilde{\mu}}{3\tilde{\sigma}}, \frac{\tilde{\mu} - \tilde{\text{LSL}}}{3\tilde{\sigma}}\right) \tag{4.2.4.2}$$

3. **Fuzzy C_{pm} - Capability Penalizing Deviations from Target** measures variability and deviation from a target, making it ideal for precision-oriented processes. This index penalizes processes that deviate from the target, even if they fall within the fuzzy specification limits.

   The mathematical formulation of $\tilde{C}_{pm}$ is the following:

$$\tilde{C}_{pm} = \frac{\tilde{\text{USL}} - \tilde{\text{LSL}}}{6\sqrt{\tilde{\sigma}^2 + (\tilde{\mu} - T)^2}} \tag{4.2.4.3}$$

   where T is the crisp target value, which is precisely known.

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4.3 Practical Applications in Univariate Capability Analysis

Univariate Process Capability Indices have been extended beyond their traditional role in manufacturing environments to satisfy the needs of modern industries and service sectors. This section investigates how PCIs have evolved to deal with real-world challenges, such as fuzzy information, customer satisfaction, service quality, environmental evaluation, and sector-specific adaptations. Emerging domains such as AI-driven estimations, fuzzy logic applications, and material studies in steel and concrete evaluation demonstrate that Univariate PCIs are not statistical tools, but flexible approaches or methodologies that evolve in response to technological and societal needs.

Over the years, numerous industry-specific studies have taken place. These studies are going to be presented analytically below. More specifically:

About the Traditional Indices

- In the **Steel** industry, where quality characteristics such as tensile strength, hardness, and elongation are particularly strict, Sung, Chen, and Go (2002) suggested a unique analytical method for the estimation of process capability. The capability analysis took into account material-specific factors, including mechanical deformation and chemical composition variability, rather than relying simply on the traditional indices C_p and C_{pk}.

  Their solution employs a probabilistic-deterministic hybrid model to represent steel quality measures. The first key feature of this methodology includes the adjustment of the process spread to account for both statistical and expected metallurgical variability. Another key feature was the use of weighted averages when multiple correlated material properties influence the final performance. The last key allows for dynamic recalibration of control limits based on the grade of the steel.

  This modification increases the applicability of PCIs for metallurgical processes, where properties are frequently interdependent rather than independent, as expected in basic PCIs models.

- **Chloride ion penetration into concrete** has a significant impact on reinforcing corrosion and structural durability. In their publication, Sung, Chen, and Li (2005) suggested a modified capability analysis approach to evaluate this chemical degradation mechanism.

  Rather than assuming a normally distributed dataset, they realized that the chloride ion concentrations often follow a positively skewed distribution due to the nature of diffusion processes. To support this, their approach includes various innovations.

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One of the innovations was the application of the log-normal transformation to approximate the normal distribution in the chloride concentration data.

After this transformation, the next step was the calculation of the traditional indices (C_p and C_{pk}). On the one hand, the Potential Process Capability Index (C_p) was utilized to examine the total spread of chloride concentration values relative to predetermined critical limits. On the other hand, the Process Capability Performance Index (C_{pk}) was used to determine the degree of centering, which refers to how closely the average chloride concentration remains within the acceptable safety thresholds.

Instead of conventional symmetric specification limits, customized thresholds were applied, where the critical chloride content acted as a one-sided or traditional specification limit, representing the asymmetric risk of corrosion. Sung et al. (2005) presented a more accurate and meaningful assessment of process performance for concrete durability, which is especially significant for infrastructure quality assurance.

- In **Agricultural and Soil Sciences**, da Silva and Key (2004) developed an interesting link between process capability analysis and the concept of Least Limiting Water Range (LLWR), which is a solid quality metric that describes optimal root growth circumstances. They applied the PCI technique to evaluate two things. Firstly, "How often a soil's physical properties, such as bulk density, porosity, and air-water balance, fall within the LLLWR?" and secondly, "How do agricultural methods influence the soil's capability to support plant growth without compaction or aeration difficulties?".

  The basic keys of this method were the treatment of soil physical qualities as process outputs, the definition of the LLLWR boundaries as specification limits, and the calculation of a capability index that represents soil health. Thus, the PCI concepts were initially created for industrial manufacturing and are being applied in environmental and agricultural applications, exhibiting their great adaptability.

About the Process Capability Index C_{pmk}

According to Pearn et al. (2003) and Zimmer et al. (2001), C_{pmk} is a beneficial tool in businesses where both specification center and target value must be respected simultaneously. Practical applications of C_{pmk} can be spotted in **Semiconductor Manufacturing**, where voltage targets must be satisfied strictly. Also, in **Medical Device Fabrication**, the assurance of catheters, stents, and surgical equipment must remain centered and around safety thresholds. Moreover, in **Aerospace Production**, where serious operational risks can be caused by even minor shifts from the target.

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According to Pearn et al. (2003), organizations must prioritize the way that a sample is taken. A sample size that contains more than 100 observations is often recommended to reduce estimator bias and improve the reliability of confidence intervals.

About the Process Yield Index S_{pk}

Lee et al. (2002) and Pearn and Chuang (2004) established the index S_{pk} as a yield process capability measure, aiming to link process performance to customer expectations. Practical applications of S_{pk} can be spotted in **Semiconductor Fabrication Lines**, where it contributes to the estimation of real-world die yields, critical for cost-effective wafer production. Also, in the **Automotive Industry**, where hundreds of small parts must meet suitable specifications. Furthermore, in the **High-Volume Electronics Manufacturing**, such as smartphone components, can result in significant financial losses due to small errors across millions of units.

An important note is that a slight increase in S_{pk} relates to higher yield, increased customer satisfaction, but lower scrap rates. Industries that employ the index S_{pk} often use automated statistical software to monitor it because of its computational simplicity, once the process mean and the standard deviation are known. Pearn and Chuang (2004) stated that the S_{pk} index identifies issues in the process from the early stages. When this index starts to drop, it generally signals that the process is shifting away from the target or that its variability is increasing. This happens even if traditional indices, such as C_p and C_{pk}, still offer acceptable values and remain stable.

About the Process Incapability Index C_{pp}

The Process Incapability Index C_{pp} is mostly used in short-run production environments. Lin and Pearn (2005) and Pearn, Ko, and Wang (2002) state that the index C_{pp} is commonly used in various domains. The first is the **Custom Aerospace Production**, where one or a few prototypes must meet tight tolerances without an expansive collection of data. The second domain refers to **Defense and Military Component Fabrication**, where short-run productions necessitate an early and trustworthy capability assessment. The third domain is the **Precision Tooling for Automotive R&D**, where fast validation of prototype molds is critical.

Additionally, this index enables capability assessments without making unrealistic assumptions about long-term stability. Engineers typically utilize Bootstrap confidence intervals to account for small-sample uncertainty when implementing C_{pp}, following Lin et al. (2003) guidelines. Implementing this index is often connected to early-stage Six Sigma projects, allowing for go-no-go or green-light judgments before mass production investments. An important note about the Incapability Index is that an early detection of prototype or small-batch process limitations can prevent costly full-scale failures.

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About the Fuzzy Process Capability Indices

- Kaya and Kahraman (2009) developed fuzzy robust capability indices specifically for assessing **environmental risk**, such as **air pollution levels**. Traditional PCIs require tight specification limits and stable distributions, but environmental measurements often include uncertain thresholds, such as "acceptable" particulate matter levels, and due to environmental sensors, fluctuating measurement errors are caused.

  In contrast with the classical indices, the robust fuzzy PCI framework supports a variety of conditions. The first condition supports that pollution measures (for example, PM2.5 concentration) are modeled as fuzzy numbers. Another condition says that fuzzy set theory and robust statistics are used to handle uncertainty and outliers at the same time. The last condition produces fuzzy PCIs that show not only if the air is clean, but also with what degree of confidence that judgment can be made. Thus, this approach is important for risk-based regulatory compliance because it provides a more realistic assessment under uncertainty, particularly when the exact environmental control is difficult.

- Chen and Yang (2000) introduced the **Service Performance Index (SPI)** to address the problem of defining the capability in service settings. SPI uses fuzzy membership functions to express qualitative performance attributes, such as "excellence time of response" (for example, < 3 minutes), "acceptable response time" (for example, 3-5 minutes), or "unacceptable" (for example, > 5 minutes).

  Instead of strict specification limits, SPI accepts customer expectations and allows for variation in service outcomes. Also, SPI uses fuzzy set theory to translate subjective performance assessments into actionable capability metrics. Finally, this research shows how process capability thinking can be applied to **non-manufacturing** domains such as **healthcare, tech support, and banking**, where service delivery is fundamentally fuzzy and driven by customers.

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4.4 Air Quality Monitoring through FuzzyNumbers Package in R

The following study introduces a process capability analysis of an air quality monitoring system, which focuses on PM2.5 concentration levels, through fuzzy logic. PM2.5, or particulate matter smaller than 2.5 microns, serves as a crucial indicator of air pollution and is connected with the appearance of respiratory, cardiovascular, and more serious health issues. Safety, environmental frameworks, and smart countries, such as Japan, the USA, and South Korea, are conducting continuous and strict monitoring of PM2.5 to ensure alignment with regulatory guidelines.

The following study aims to evaluate the performance of a PM2.5 monitoring system by using the fuzzy capability indices to account for uncertainty in measurement and specification parameters. For conducting this research and simulating a realistic scenario, 5000 daily sensor readings of PM2.5 were generated from a normal distribution, with $\mu = 36\mu\text{g/m}^3$ and $\sigma = 5\mu\text{g/m}^3$, in the famous significance level of $\alpha = 5\%$. The following histogram was plotted to ensure the assumption of normality:

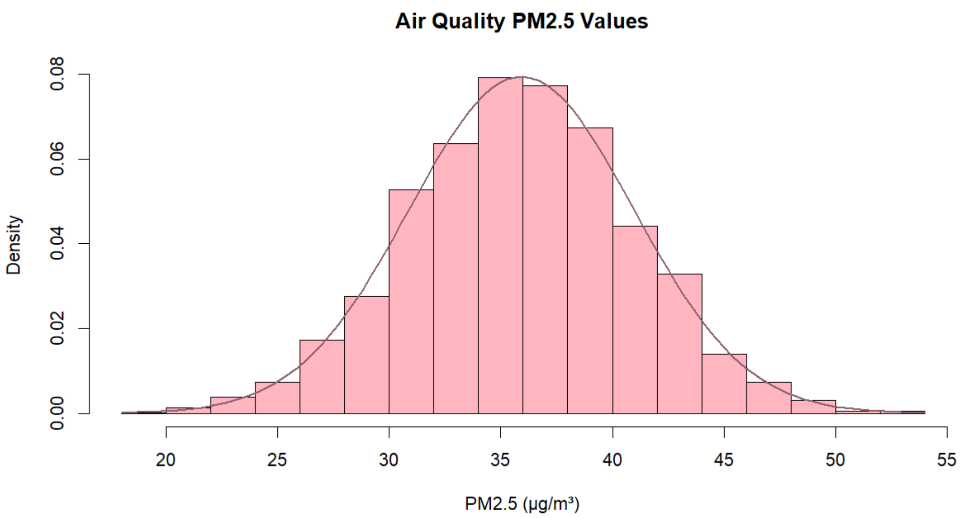

Figure 4.1: Histogram of the distribution of PM2.5 data

Figure 4.2 shows that the distribution of PM2.5 data is Normal because of the known bell-shaped curve. Also, the Shapiro test for normality was performed, where the result was a p-value equal to 0.744. This p-value is larger than the significance level, which means that the Null Hypothesis is not rejected and the assumption of normality is accepted.

The choices of the values of mean and standard deviation are based on real-world applications. The mean value was chosen to simulate a realistic urban air quality, where the mean exceeds the suitable amount or ideal target of 35 $\mu\text{g/m}^3$. As for the standard deviation, 5 $\mu\text{g/m}^3$ represents typical, daily variability in PM2.5 levels observed in real-

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world systems. Various conditions, such as traffic, heating systems, weather, and industrial activity, influence the levels of fluctuations.

To proceed deeply into this example, the `FuzzyNumbers` package in R was necessary to perform the fuzzy logic properly. This package provides tools for the creation of fuzzy numbers, such as triangular, trapezoidal, and piecewise linear fuzzy numbers. Also, it defines fuzzy numbers, calculates α-cuts, and plots fuzzy sets. It is beneficial when uncertainty and imprecision are spotted in data and parameters, as it permits these uncertainties to be modeled and analyzed mathematically using fuzzy set theory.

The first step was to estimate the mean and standard deviation. More specifically, the estimated mean (central tendency of the data) and standard deviation (level of variability in daily PM2.5 readings) of the simulated PM2.5 dataset are equal to 35.928 μg/m^3 and 5.0247 μg/m^3, respectively. This step forms the basis for the definition of fuzzy numbers that incorporate a small range of uncertainty. The transition from crisp to fuzzy values allows a more realistic model of the imprecision, especially in environmental measurements.

For the next step of this analysis, the function `TriangularFuzzyNumber(a, m, b)` plays a crucial role. Where:

- a = lower bound or minimum possible value
- m = most likely value
- b = upper bound or maximum possible value

This function is used to represent imprecise values and define triangular fuzzy numbers. It follows a triangular-shaped membership function, where the lower and upper bounds represent lower confidence (membership = 0) and the middle value represents full confidence (membership = 1). As it was mentioned before in 4.2.4, a fuzzy number permits a range of possible values, with one centered value considered the most likely. However, this is not possible with crisp numbers, which represent one single value.

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I) Estimation of Fuzzy Mean and Standard Deviation

By using the command that is mentioned above, the fuzzy mean and standard deviation were computed. The fuzzy mean is presented as a trapezoidal fuzzy number with support $[34.928, 36.928]$ and core $[35.928, 35.928]$. The margin of ± 1 $\mu g/m^3$ was chosen due to typical precision tolerance supported by the manufacturers of PM2.5 sensors or in air quality readings. The support means that the true mean (35.928) is plausibly within this range, and the core represents the most likely or exact estimate of the process mean. Because of the triangular fuzzy number, the core is the peak of the triangle or a single point. The process mean, based on the uncertainty, is around $35.93 \mu g/m^3$, but it might be as low as 34.93 or as high as 36.93.

The fuzzy standard deviation is presented as a trapezoidal fuzzy number with support $[4.5247, 5.5247]$ and core $[5.0247, 5.0247]$. The margin of ± 0.5 $\mu g/m^3$ was chosen due to the difficulty of estimating the variability. The support presents the range of possible values for the standard deviation, and the core is the exact estimate of the most likely variability of the data, which is equal to the initial variability. The estimate of the standard deviation is around 5.02, which is not perfectly exact. But, it could fall between a range of 4.52 and 5.52.

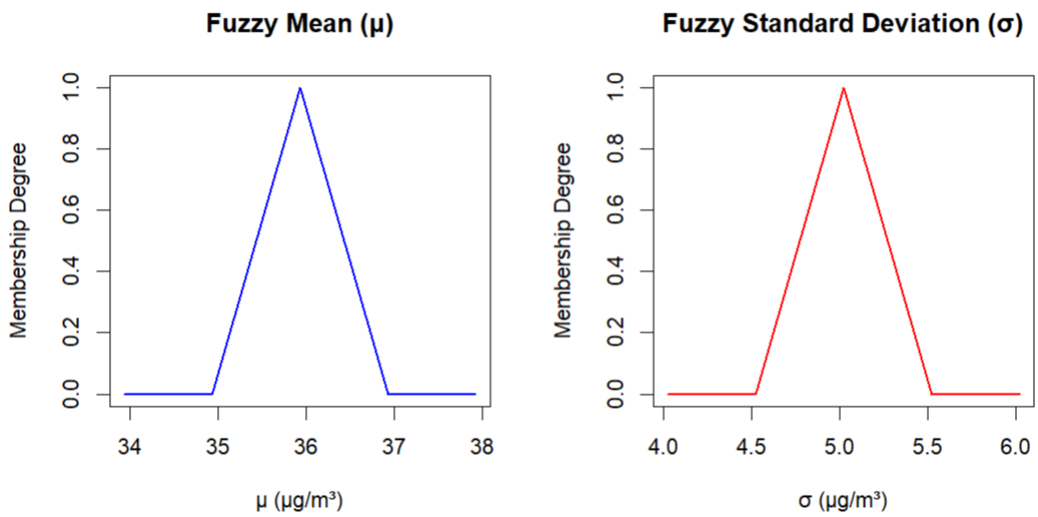

Figure 4.2: Triangular fuzzy-shaped representation of mean and standard deviation. Values closer to the core or central peak of the triangle are more credible, whereas those toward the edges indicate more uncertainty.

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II) Estimation of Fuzzy Specification Limits and Target

Except for the estimation of the parameters above, the fuzzy specification limits, LSL and USL, and the target may also be estimated. They were estimated through the use of the function `TriangularFuzzyNumber( )`. To deal with real-world imprecision, each value was expressed as a triangular fuzzy number with a support range of ± 1 $\mu g/m^3$. More analytically, based on the WHO Global Air Quality Guidelines (WHO, 2021) and the Air Sensor Guidebook U.S. Environmental Protection Agency (U.S. EPA, 2021), the Lower Specification Limit (LSL) is defined as the triangular fuzzy number (14, 15, 16), where the 15 $\mu g/m^3$ represents the core or the centered value (annual average PM2.5 exposure limit), but it may range between 14 and 16 $\mu g/m^3$, which values allow for minor uncertainty or interpretation, due to environmemtal conditions and regional variation in air quality guidelines. The Upper Specification Limit (USL) is defined as the triangular fuzzy number (54, 55, 56), where the 55 $\mu g/m^3$ represents the core value, closely aligning with India's National Ambient Air Quality Standard (NAAQS) for PM2.5, which sets a 24-hour threshold close to 60. Also, there is a range of variation or the support between the values 54 to 56 $\mu g/m^3$. Finally, the target (T) is defined as the triangular fuzzy number (14, 15, 16), where the core value for PM2.5 is 35 $\mu g/m^3$, with a small tolerance band ranging from 34 to 36 $\mu g/m^3$, to reflect to real-world situations.

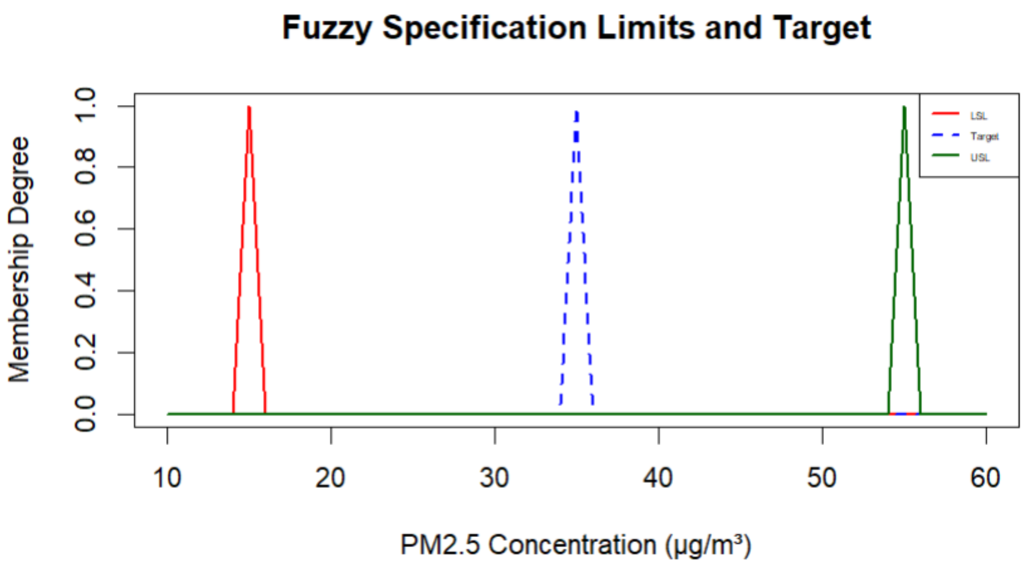

Figure 4.3: Visual representation of the fuzzy specification limits and target for PM2.5 concentrations. A red line represents the Fuzzy LSL, a dotted blue line represents the Fuzzy Target, and a green line represents the Fuzzy USL.

The figure above represents the specification limits and target values as triangular fuzzy numbers, which model uncertainty around air quality thresholds. The x-axis represents the PM2.5 concentrations in $\mu g/m^3$ and the y-axis represents the degree of membership,

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which fluctuates from 0 (no confidence) to 1 (max confidence). As mentioned above, each parameter has a **core** or the most likely value and a **support** or a range of values that it could take. Fuzzy LSL (14, 15, 16) represents the lower boundary below which all the PM2.5 concentrations are extremely clean. In countries such as the European Union (European Environment Agency, 2023), the annual average PM2.5 limits are often ≤ 15 $\mu g/m^3$. The fuzzy USL (54, 55, 56) is considered the maximum acceptable value of PM2.5, before the air quality becomes dangerous (EPA). The fuzzy target (34, 35, 36) represents the typical or common PM2.5 value that a system wishes to achieve. This value is not chosen randomly. But it is based on the air quality guidelines, which are given by the Central Pollution Control Board (CPCB, 2023).

III) Definition of α-cuts and Computation of Index Bounds

The aim of this part is the computation of fuzzy capability indices C_p, C_{pk}, and C_{pm}. Until now, this research has been based on uncertain quantities, such as the fuzzy mean, standard deviation, specification limits, and target. This leads to a question: *"How is it possible to analyze fuzzy values across different levels of confidence?"*.

Therefore, the first step is the use of the method α-cuts. This method permits converting each triangular fuzzy number into a specific interval at each level of α, which ranges from 0 to 1 and in steps of 0.1. This represents degrees of certainty about fuzzy estimates in ascending order. The full range of possible values or the maximum uncertainty is achieved when $\alpha = 0$, while the core or the most likely value happens when $\alpha = 1$. For each α, through the application of the `alphacut( )` function to each of the parameters (mean, standard deviation, USL, LSL, and target), the result is specific numeric intervals [lower, upper], which represent the degree of certainty connected with the proper level of α. These intervals are useful for the construction of the lower and upper bounds of the indices using their standard formulas, adjusted to work with internal values.

α-cuts	Cp Lower	Cp Upper	Cpk Lower	Cpk Upper	Cpm Lower	Cpm Upper
0.0	1.2067	1.4734	1.0300	1.5418	1.1900	1.4433
0.1	1.2177	1.4573	1.0516	1.5208	1.2006	1.4282
0.2	1.2289	1.4415	1.0736	1.4900	1.2113	1.4134
0.3	1.2404	1.4261	1.0960	1.4598	1.2223	1.3988
0.4	1.2520	1.4110	1.1188	1.4300	1.2334	1.3846
0.5	1.2639	1.3962	1.1420	1.4013	1.2448	1.3706
0.6	1.2760	1.3818	1.1657	1.3729	1.2563	1.3569
0.7	1.2883	1.3676	1.1899	1.3452	1.2681	1.3435
0.8	1.3009	1.3537	1.2145	1.3180	1.2801	1.3303
0.9	1.3137	1.3400	1.2396	1.2913	1.2923	1.3174
1.0	1.3268	1.3268	1.2652	1.2652	1.3047	1.3047

Table 4.1: C_p, C_{pk}, and C_{pm} at different α or degree of confidence levels

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Before the interpretation of the table above, it is important to highlight the definitions of the classical indices, which are extremely useful in the domain of quality control to assess how well a process performs within specification limits. More specifically, the index C_p or Process Capability Index measures the potential capability of a process, assuming it is perfectly centered, and compares the spread or variability of the process to the width of the permitted limits. The index C_{pk} or Process Performance Index measures the potential capability of the process by considering how centered the process is (reflects both spread and any shift away from the target). Finally, C_{pm} or Taguchi Capability Index not only includes spread and centering but also the deviation from the desired target value.

Thus, the results in Table 4.1 show the evaluation of the traditional fuzzy indices (C_p, C_{pk}, and C_{pm}) across the different levels of α, which range from 0.0 to 1.0. As α increases, the range between the lower and upper bounds becomes smaller, showing that the uncertainty around the indices decreases and the capability indices tend to reflect a more capable process (Parchami et al., 2012; Wang and Lee, 2002). Moreover, lower α values, 0.1-0.4, can lead to worst-case risk assessments, while higher values, 0.8-1.0, support more confident and less conservative evaluations.

Firstly, at the **level of maximum uncertainty or fuzziness** ($\alpha = 0$), the index C_p ranges from 1.2067 to 1.4734, the index C_{pk} ranges from 1.0300 to 1.5418, and the index C_{pm} ranges from 1.1900 to 1.4433. This indicates that the ranges are wide at this level. Also, at $\alpha = 0.5$, the middle level of confidence, the index C_p ranges from 1.2639 to 1.3962, the index C_{pk} ranges from 1.1420 to 1.4013, and the index C_{pm} ranges from 1.2448 to 1.3706. It is obvious that the ranges are not so wide at this level, but the indices might be considered acceptable, although still influenced by uncertainty. However, as α increases and approaches the **level of 1.0**, it is obvious that the indices converge to single point values, the **core** values of the fuzzy indices ($C_p = 1.3268$, $C_{pk} = 1.2652$, and $C_{pm} = 1.3047$). This suggests more stability and reliability under high confidence.

IV) Interpretation of the Triangular Fuzzy-Shaped Indices

The following figure shows the fuzzy capability indices C_p, C_{pk}, and C_{pm} across the different α-cuts, to visualize how uncertainty affects process evaluation. All the visual representations have a triangular shape. The first triangle represents the index C_p, the second the C_{pk}, and the third the C_{pm}. Each shaded area (blue, green, and purple) corresponds to the interval range of the respective index at various confidence levels. After the observance of the plots, at the first level ($\alpha = 0$), there is maximum uncertainty, the intervals are widest, reflecting the full possible variation due to imprecision in the initial parameters, such as mean, standard deviation, and specification limits. As α moves to the higher levels, the intervals narrow and converge towards crisp values (values at level 1.0), which shows greater confidence in the measurements and reduces uncertainty.

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The **first or blue** triangle of the fuzzy index C_p presents a range from approximately 1.22 to 1.47 across all different levels of α, suggesting a process with strong, potential capability, even when uncertainty is present. This means that the spread of PM2.5 data fits well within the specification limits, even if the process is centered. The **second or green** triangle of the fuzzy index C_{pk} ranges from around 1.03 to 1.54 and indicates that the centering of the process, or how close the average value is to the middle of the specification range, is more affected by uncertainty. However, all the values remain within the acceptable range, indicating that the process is not entirely off-target. The **last or purple** triangle of the fuzzy index C_{pm} ranges from 1.19 to 1.44, which means that the process remains close to the proper target under uncertainty. This happens because this index takes into account both variability and deviation from the target.

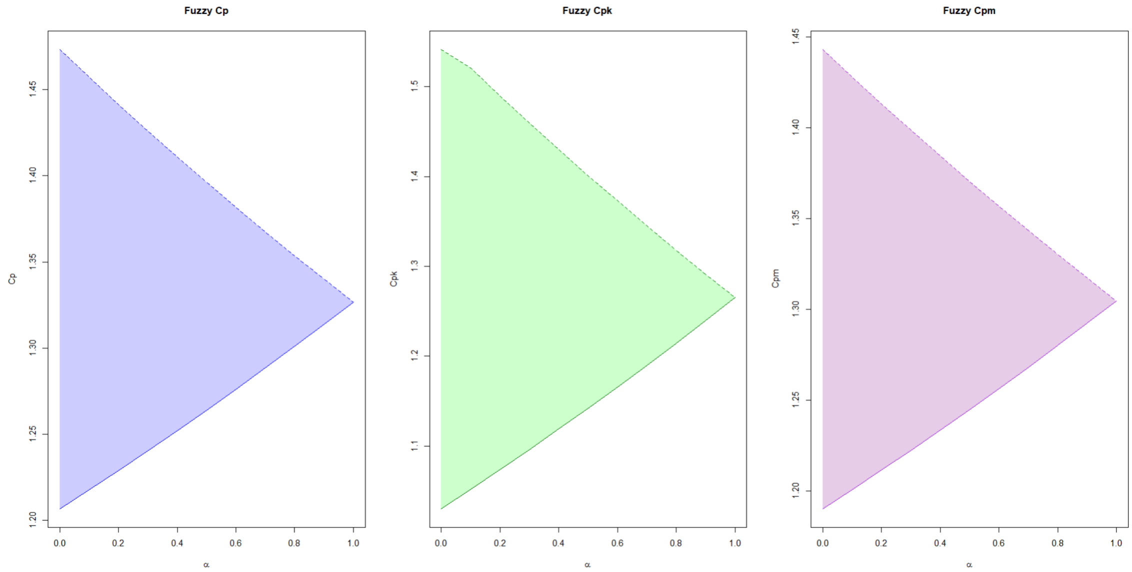

Figure 4.4: Triangular Fuzzy-Shaped C_p, C_{pk} and C_{pm} Indices. The blue triangle represents the fuzzy C_p, the green represents the fuzzy C_{pk}, and the purple represents the C_{pm}.

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5. Chapter 5: Multivariate Process Capability Analysis

5.1 An Introduction to Multivariate Case

As modern production processes become more complicated, multivariate process capability analysis has become increasingly important. In such contexts, multiple interrelated quality characteristics must be regulated simultaneously, and univariate indices are insufficient to provide a thorough evaluation. This has resulted in the creation and the introduction of Multivariate Process Capability Indices (MPCIs), which take into account both the marginal behavior of individual variables and their dependency structure.

The multivariate scenario presents some unique challenges that do not exist in the univariate case. These include the representation of multivariate specification regions, the handling of inter-variable correlations, and the effect of dimensionality in estimation accuracy. MPCIs must account for the joint behavior of the quality variables, which is commonly accomplished using multivariate statistics, such as Mahalanobis distance, covariance matrices, and ellipsoidal confidence regions.

MPCIs originated in the late 1980s and the early 1990s. During this time, quality control research shifted toward multivariate frameworks as industrial processes became more interconnected and relied on multiple control characteristics. Influential early contributions, such as Hotelling's T^2 statistic from multivariate control chart theory, laid the conceptual foundation for multidimensional process monitoring. Chan, Cheng, and Spiring (1991) made a foundational contribution by proposing a multivariate generalization of the famous univariate index C_p. Their approach identified the multivariate process region using generalized variance and compared it to the volume of the specification range.

Since the 1990s, numerous MPCI formulations have appeared to handle various aspects of multivariate process control. Probability-based indices estimate the joint likelihood that all characteristics meet specifications, while geometric indices quantify the volume of the specification region relative to the process region, which is frequently represented as an ellipsoid. Recent studies have introduced indices designed for specific conditions, such as bivariate non-normality (Castagliola & Garcia Castellanos, 2008), multiple independent or correlated characteristics (Pearn & Wu, 2006), and small sample

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or short-run production (Wang, 2005). These advancements have had an especially large impact on high-precision industries, such as electronics, biomedical manufacturing, and aerospace, where multivariate relationships have a significant impact on product quality.

This chapter provides a systematic examination of MPCIs, beginning with the assumptions that underpin their use, continuing through traditional formulations, and finishing with advanced models designed for more practical application in difficult real-world scenarios

5.2 Assumptions Underlying Multivariate Capability Analysis

Assumptions are fundamental to any statistical model, but the assumption of **multivariate normality (MVN)** is especially important in the context of multivariate process capability analysis. This assumption underpins several commonly used MPCIs, influencing how specification regions are defined and probabilities of conformance are computed (Wang et al., 2000) and (Pearn et al., 2007).

An important note that needs to be explained is the definition of MVN or multivariate normality. Multivariate normality means that any linear combination of the quality variables also follows a normal distribution. It ensures that a mean vector and a covariance matrix are completely sufficient to explain the joint distribution of all the process variables. This simplifies the calculation of multivariate analogs of univariate indices- generalizations that extend univariate measures (C_p and C_{pk}), and account for multiple, interdependent quality characteristics. These multivariate analogs include not only the spread and the centering of individual variables but also their covariances, and allow the use of Mahalanobis' distance as a generalized measure of deviation (Mardia, 1970).

Several assumptions, either implicit or explicit, are involved in the construction and interpretation of MPCIs. The first assumption refers to **Stationarity**, where the process will remain stable over time, with a constant mean and covariance structure. The second assumption is the **Independence of Observations**, where data points are expected to be independent of one another, a condition that is often violated in autocorrelated industrial processes. Furthermore, **Linearity and Additivity** is another assumption that implies that many models assume or can approximate a linear relationship between the variables. Finally, the specification limits, the tolerance region, and the distance metric type depend on the assumptions about the distribution and the data structure. This is the final assumption, known as **Correct Model Specification**.

Violations of the assumptions above, especially MVN, might result in inaccurate interpretations of process performance. For example, if the joint distribution is skewed or multimodal, ellipsoidal tolerance areas may not accurately represent the true process spread,

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leading to misleading capability estimates (Ahmand et al., 2009). Therefore, except for the confirmation of MVN through statistical diagnostics, such as Q-Q plots, Mahalanobis' distance, and Mardia's tests, it is also crucial to determine the presence of multivariate outliers, to examine the homogeneity of variances and covariances across subgroups, and investigate the marginal distributions and relationships among the variables.

In reality, when the MVN assumption is violated, approaches such as robust estimators (estimators less sensitive to outliers, such as Minimum Covariance Determinant), empirical tolerance regions, or transformations such as the Johnson or the Box-Cox should be applied (Castagliola & Garcia Castellanos, 2008). These methods enable the creation of MPCIs that are more resistant to deviations from ideal statistical behavior, hence improving the validity of capability analyses in realistic manufacturing settings.

Moreover, multivariate capability analysis is often based on the assumption that process data follow a multivariate normal distribution (MVN). This assumption simplifies the mathematical treatment of process variation because the MVN framework models specification and tolerance areas as ellipsoids. The Mahalanobis distance, which is a measure from the mean adjusted for covariance, is supported by MVN, making it easier to estimate the proportion of output falling within specifications.

In mathematical terms, a random vector $\mathbf{X} = (X_1, X_2, \ldots, X_m)^T$ follows a multivariate normal distribution if the probability density function is the following:

$$f_X(x) = \frac{1}{(2\pi)^{m/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$

Where:

μ = the mean vector $(m \times 1)$

Σ = the covariance matrix $(m \times m)$

$|\Sigma|$ = the determinant of Σ

m = the number of quality characteristics

Also, under the multivariate normality (MVN), a natural tolerance region is an ellipsoid, which is defined as follows:

$$(x-\mu)^T\Sigma^{-1}(x-\mu) \leq c^2$$

Where: c^2 is determined by the desired coverage probability. For example, for 3σ-limits, equal to 99.73%.

However, in real industrial contexts, this assumption is frequently violated. Data may show skewness, kurtosis, multimodality, or other characteristics that differ from the MVN model. Under these conditions, using traditional or classical indices may result in major misestimation of process capability, often overestimating or underestimating it. There are several statistical tools and ways to detect violations of MVN. More specifically:

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- **Multivariate Q-Q plots**: They are suitable for comparing the quantiles of observed Mahalanobis distances to those of a Chi-squared distribution.
- **Chi-Squared Plots**: They plot Mahalanobis distances squared against Chi-squared quantiles.
- **Mardia's Skewness and Kurtosis Test**: Using the data's third and fourth statistical moments, they determine whether the shape of a multivariate dataset deviates from the bell-shaped pattern expected in a normal distribution by checking for skewness (asymmetry) and kurtosis (heaviness of tails).

5.3 Multivariate Process Capability Indices (MPCIs)

5.3.1 Volume-Based Indices MC_p, MC_{pk} and MC_{pm}

Volume-based indices were among the first attempts to apply univariate capability concepts to multivariate processes. The role of these indices is to evaluate how well the variation in multiple quality characteristics remains within acceptable limits by comparing the overall "space" defined by the specifications to the space covered by the actual process variability, using a multivariate generalization of spread based on volume. More specifically:

1. One of the most well-known indices in this category is the Multivariate Capability Index MC_p. It allows you to estimate how much of the specification space is occupied by process variation in the multivariate setting. It accomplishes this by comparing the volume described by the specification limits to the volume that represents the actual spread of the process, as represented by the covariance matrix. The covariance matrix describes how each pair of quality characteristics varies together, and its determinant is used to estimate the process's overall multivariate spread, also known as generalized variance.

   The mathematical formulation of MC_p, under the assumption of multivariate normality, is the following:

$$MC_p = \left(\frac{|\mathbf{U} - \mathbf{L}|}{6\sqrt{\det(\Sigma)}} \right)^{1/m} \tag{5.3.1.1}$$

   Where:

   $|\mathbf{U} - \mathbf{L}|$ = the range vector across m quality characteristics

   Σ = the covariance matrix

   $\det(\Sigma)$ = the generalized variance

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m = the number of variables

The formulation above is based on the work of Chan, Cheng, and Spiring (1991), who expanded the univariate concept of process spread into a multivariate context by employing the determinant of the covariance matrix to account for multivariate variability. The determinant reflects how much data scatters in multiple dimensions and extends the concept of variance to higher dimensions. This makes MC_p an effective summary of multivariate process capability under the multivariate normality assumption.

Wang et al. (2000) investigated this index further, comparing it with other indices and emphasizing its sensitivity to correlations among variables. As the correlation among variables increases, the generalized variance ($\det(\Sigma)$) decreases, which leads to a higher value of MC_p and a false interpretation of process capability. Thus, while this index is theoretically valid, it is essential to consider the shape and orientation of the process distribution while interpreting its value.

Furthermore, one essential feature of MC_p is that it assumes that the process is centered within the specification limits. It does not penalize if the mean deviates from the center, as long as the overall variation of the process lies within the limits. To remedy this issue, the centered version (MC_{pk}) is often employed.

2. The following formula of MC_{pk} generalizes the univariate index C_{pk} by evaluating the worst-case standardized distance between the process mean and the closest limit for each variable. Although it is simple and interpretable, it implies independence among characteristics and does not explicitly account for multivariate correlation.

   Each term in the formulation represents how close the mean of a particular characteristic is to its closest specification limit, as scaled by process variability. Taking the minimum in the equation guarantees that the index reflects the worst-performing dimension, resulting in a cautious yet useful estimate of multivariate capability.

   The mathematical formulation of MC_{pk} is the following:

$$MC_{pk} = \min\left\{\frac{\mu_i - L_i}{3\sigma_i}, \frac{U_i - \mu_i}{3\sigma_i}\right\} \qquad (5.3.1.2)$$

   Where:

μ_i = the mean of the i-th quality characteristic

L_i, U_i = the Lower and the Upper Specification limits for the i-th quality characteristic

σ_i = the standard deviation of the i-th quality characteristic.

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3. Taam et al. (1993) introduced the multivariate index MC_{pm} for the evaluation of the variability and the centering of the process of a process, under the assumption of the multivariate normality. Santos Fernández and Scagliarini (2012) highlighted that MC_{pm} is defined as the ratio between two volumes. On the one hand, the volume of a modified tolerance and ellipsoidal region R_1, which is centered at the target values and represents the specification limits. On the other hand, the volume of a reference process and ellipsoidal region R_2, which includes 99.73% of the actual data of the multivariate process distribution.

   The mathematical formulation of this Multivariate Capability Index is the following:

$$MC_{pm} = \frac{\text{Vol.(modified tolerance region)}}{\text{Vol.}\left\{\mathbf{X} : (\mathbf{X}-\boldsymbol{\mu})^{\top}\boldsymbol{\Sigma}_T^{-1}(\mathbf{X}-\boldsymbol{\mu}) \leq K(m)\right\}} \tag{5.3.1.3}$$

   Where:

   Vol.(modified tolerance region) or Vol.($\mathcal{R}_1$) equals to:

$$\frac{2\pi^{\nu/2}\prod_{i=1}^{\nu} a_i}{\nu\Gamma\left(\frac{\nu}{2}\right)}$$

   If we suppose that a_i $(i = 1, 2, \ldots, \nu)$ represent the lengths of the semi-axes

   $\mathbf{X}$ represents the vector of measurements $(\nu \times 1)$

   $\boldsymbol{\mu}$ represents the process mean vector

   $\mathbf{T}$ represents the vector of target values

   $\boldsymbol{\Sigma}_T$ represents the mean square error matrix concerning the target vector $\mathbf{T}$, which equals to:

$$\boldsymbol{\Sigma}_T = \mathbb{E}\left[(\mathbf{X}-\mathbf{T})(\mathbf{X}-\mathbf{T})^{\top}\right]$$

   $K(m)$ is the 99.73-th percentile from the chi-squared distribution with ν degrees of freedom

   Also, the mathematical formulation of the index MC_{pm} can be transformed and expressed differently, by using the volume of the reference process region R_2 and the volume of another region, called R_3, which is the "ideal" ellipsoidal region representing the process variability, assuming that the process mean is exactly on target.

   Firstly, the mathematical formulation that connects these two volumes is the following:

$$\text{Vol.}(R_2) = \text{Vol.}(R_3) \times \left[1 + (\boldsymbol{\mu}-\mathbf{T})^{\top}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-\mathbf{T})\right]^{1/2}$$

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Then, the transformed formulation of MC_{pm} can be written as follows:

$$MC_{pm} = \frac{\text{Vol.}(R_1)}{\text{Vol.}(R_3)} \times \frac{1}{\left[1 + (\boldsymbol{\mu} - \mathbf{T})^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - \mathbf{T})\right]^{1/2}} = \frac{C_p}{D} \tag{5.3.1.4}$$

Where:

C_p compares the tolerance region to the process variability, under the assumption that the process is perfectly centered

D measures deviation from the target

5.3.2 Probability-Based Indices MC_pM

Probability-based multivariate capability indices estimate the likelihood that a process will produce outputs that meet all of the specification limits at once. Probability-based indices, as opposed to volume-based indices, which rely on geometric ratios, directly quantify conformance through statistical properties.

The following formula of these indices represents the joint probability of all quality characteristics falling within the specification (Lower and Upper) limits. It is commonly estimated under the condition of multivariate normality, with the Mahalanobis distance defining an ellipsoidal zone around the process mean. Also, the Mahalanobis distance is expressed as follows:

$$D^2 = (\mathbf{X} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{X} - \boldsymbol{\mu})$$

Where:

$\mathbf{X}$ = the vector of the observed characteristics

$\boldsymbol{\mu}$ = the process's mean vector

Σ = the covariance matrix

Under multivariate normality, the squared Mahalanobis distance:

$$D^2 \sim \chi^2_m$$

The condition above makes it possible to determine how much of the process distribution falls within a specific tolerance region, such as a 99.73% confidence ellipsoid.

Thus, the general mathematical expression of these indices is the following:

$$MC_{pM} = P(L_i \leq X_i \leq U_i) \quad \text{for all } i = 1, \ldots, m \tag{5.3.2}$$

Probability-based indices have the advantage of accounting for both individual variability and inter-variable correlation, resulting in a more comprehensive view of process

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capability. However, one significant weakness is their sensitivity to the multivariate normality assumption. If the process data does not follow the normal distribution, the probability estimates may be incorrect.

Wang et al. (2000) and Pearn et al. (2007) formalized and evaluated these indices, demostrating their usefulness and discussing how to build confidence intervals for these estimates. Probability-based MPCIs are particularly effective in high-reliability businesses, where even small deviations from the specifications can cause serious problems.

5.3.3 Geometric Indices MC_p2

Geometric indices provide an easy, visual method for evaluating multivariate process capability by comparing the shape and size of specification and process regions. These methods are especially useful in bivariate situations, where both regions can be presented as ellipses. Castagliola and Garcia Castellanos (2005) proposed the bivariate capability index, which is widely used as a geometric index. It is defined as the ratio between the area of the specification ellipse and the area of the process ellipse.

The mathematical formulation of $\mathbf{MC}_p2$ is the following:

$$MC_p2 = \frac{ab}{\sqrt{\lambda_1 \lambda_2}} \tag{5.3.3}$$

Where:

a, b = the semi-axes of the specification ellipse, which are equal to half the range between the Upper and the Lower Specification Limits for each variable

λ_1, λ_2 = the eigenvalues of the Covariance Matrix of the process (Σ), which represent the spread in the principal component directions.

The formulation (5.3.3) represents the size and orientation of the process distribution concerning the specification region. The orientation of the process ellipse is defined by the eigenvectors of the covariance matrix, which define the principal axes of the ellipse. If the process and specification ellipses are misaligned in direction, even if their sizes are similar, it may suggest insufficient process capability due to directional variability. The index MC_p2 will be greater than 1 ($MC_p2 > 1$) if the specification ellipse completely contains the process ellipse, which indicates a capable process (Castagliola and Garcia Castellanos, 2005).

Also, geometric indices are extremely useful for exploratory analysis and provide obvious visual interpretations, especially when working with two variables. They naturally incorporate the covariance structure and can reveal problems, such as misalignment or disproportionate variability. On the other hand, these indices are more difficult to extend to higher dimensions. Because defining and interpreting multidimensional ellipsoids is

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mathematically hard, volume-based or probability-based approaches are more typically used.

Despite this disadvantage, geometric MPCIs are nevertheless effective in processes with few variables or where visual assessment is critical, such as mechanical tolerance analysis or optical inspection.

5.4 Practical Applications in Multivariate Capability Analysis

Multivariate Process Capability Indices (MPCIs) are rapidly being used in industry because of their ability to monitor, diagnose, and improve complex manufacturing and service systems with multiple correlated quality attributes. The following applications from academic literature demonstrate the practical applications of MPCIs in various scenarios. As mentioned in Subchapter 5.3, there are various indices, such as volume-based. probability-based, geometric, non-normal, or statistical inference indices. The choice of index is based on the type of data, the process structure, and the underlying distributional assumptions.

Chen, Hsu, and Wu (2006) and Cheng, Huang, and Li (2001) used MPCIs to evaluate processes with multiple stages. **Volume-based indices**, such as MC_p, were utilized at each stage to assess how effectively the process stayed within the specification limits. These stage-level indices were then used to determine the overall capability of the final product. This is especially important in semiconductor and electronics manufacturing because each step of the process affects the overall quality. Using MC_p in **multi-process environments** provides insights into how variability accumulates across the different stages. When a step performs poorly, engineers can isolate and improve it before it harms downstream operations. This integration of stage-wise MPCIs aids in both quality assurance and continuous improvement, ensuring that the final product meets complex multivariate specifications effectively.

Cheng and Wang (2006) demonstrated how to efficiently include MPCIs into the famous Six Sigma **MAIC** (Measure- Analyze- Improve- Control) context. Their case study was based on TFT-LCD, or Thin-Film Transistor Liquid Crystal Display, panel production. This is a type of high-resolution, flat-screen display used in monitors, televisions, and smartphones. During this research, the investigators used **Probability-based indices**, such as MC_pM, to monitor process performance across multiple correlated characteristics. The manufacturing process for TFT-LCD panels is complex and sensitive to even small variations in the quality characteristics. By collecting capability index data at each stage and evaluating it over time, they were able to detect the sources of variation and quality loss. This understanding offered them the chance to apply the suitable corrections. This

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indicates how probability-based multivariate indices can not only quantify quality, but also act as a diagnostic tool to lead process improvement initiatives into high-precision but at the same time complex production systems.

Chen (2001) demonstrated that MPCIs can be applied in non-manufacturing systems, specifically in **administrative and service** processes. In these situations, quality is frequently evaluated using variables such as response time, wait time, or customer satisfaction metrics, many of which are non-normally distributed. **Geometric** indices in the bivariate situation enable service managers to determine whether a process performs within acceptable performance ranges. These methods are especially useful for determining which service parameters are poorly centered or highly variable, enabling more data-driven service optimization efforts.

Wang (2005) offered a solution for processes with only a small amount of data, such as **pilot production lines, prototyping, specialized high-mix, and low-volume manufacturing**. In such instances, the typical MPCIs based on large sample assumptions become unreliable. Wang introduced empirical correction methods and Bayesian estimates to enable the implementation of multivariate capability indices even with limited data. These techniques, when combined with **robust Non-normal indices**, such as BC_{pk}, allow for meaningful capability evaluations in the face of uncertainty. This enables informed decision-making early in the product development cycle, when accurate insights are required despite insufficient measurements.

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5.5 An Engineering Case Study using PCA-Based Multivariate Capability Indices through the MPCI Package in R

In Chapter 5.5, a practical example of the Multivariate Process Capability Indices (MPCIs) is going to be presented in the field of engineering quality control. The purpose of this study is to demonstrate how statistical methods can be applied to evaluate the capability of a manufacturing process, including multiple interrelated quality characteristics. More specifically, this example is inspired by the domain of **mechanical engineering**, with a particular focus on the **production of engine pistons**.

Heywood (1988) and Stone & Ball (2004) mentioned that engine pistons are critical parts of internal combustion engines. They move up and down inside the cylinder and contribute to the transition of energy from burning fuel into mechanical motion that powers the engine. Their performance and durability are based on extremely strict specifications related to physical properties, such as diameter, length, weight, hardness, and wall thickness. These characteristics are dependent, and they often exhibit statistical correlation due to shared manufacturing processes, material properties, and design constraints. Thus, evaluating each characteristic separately may omit critical multivariate interactions that influence the overall quality.

In this example, the five variables or the structural properties of the engine pistons will be analyzed as follows:

- Diameter(mm) is the piston's fit within the cylinder
- Length (mm) is the piston's range of motion
- Weight (g) impacts inertia and contributes to the dynamic balance of the engine
- Hardness, measured in Rockwell HRC, is important for resistance to wear and thermal stress (Heywood, 1988)
- Wall Thickness (mm) plays a key role in both heat dissipation and the structural strength of the component (Cengel & Boles, 2014)

To conduct this analysis, two packages were used. The first one was the `MASS` and the second one was the `MPCI`. According to the R Documentation and the purpose of this example, the `MASS` package was used to simulate correlated quality characteristics of engine pistons using the `mvnorm()` function, which generates data from a multivariate normal distribution. Furthermore, the `MPCI` package serves as the basis of this analysis, as it is responsible for the performance of the multivariate capability analysis and the

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computation of multivariate process capability indices based on PCA, as proposed by Wang and Chen (1998), Wang (2005), and Xekalaki and Perakis (2002).

A dataset of 5,000 observations (n = 5,000) was simulated to represent the production output of the engine pistons. Santos Fernández and Scagliarini (2012) proposed that a **Multivariate Normal** distribution was assumed for the vector of quality characteristics. This assumption is used to reflect the widely accepted statistical model for quality variables in PCA. Also, this type of distribution is suitable because of its ability to represent the correlations among variables through the covariance matrix

As mentioned above, five quality characteristics take part in this analysis. Let X be the vector of $\nu = 5$ quality parameters corresponding to Diameter, Length, Weight, Hardness, and Wall Thickness. This concludes to:

$$X \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

Where:

$\boldsymbol{\mu}$: The mean vector equals the engineering target values or the target vector ($\mu = T$). The reason for the equality between these two vectors is the desired well-centered process, which establishes process variation as the primary factor under evaluation rather than bias or systematic error. Thus:

$$\boldsymbol{\mu} = \begin{bmatrix} 75 & 120 & 350 & 10 & 25 \end{bmatrix}^{\mathrm{T}}$$

$\boldsymbol{\Sigma}$: The covariance matrix of the distribution, which defines the variances and covariances among the variables. This matrix captures the assumption that many of the characteristics are correlated due to manufacturing and designing rules. Thus:

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0.05 & 0.01 & 0.10 & 0.02 & 0.03 \\ 0.01 & 0.08 & 0.20 & 0.01 & 0.02 \\ 0.10 & 0.20 & 1.00 & 0.05 & 0.10 \\ 0.02 & 0.01 & 0.05 & 0.50 & 0.10 \\ 0.03 & 0.02 & 0.10 & 0.10 & 0.60 \end{bmatrix}$$

Also, it is necessary to define the specification limits. In this case, they can be defined as *vectors* and can be used because they are the most suitable Upper and Lower Specification Limits for each of the variables. These tolerances were selected to reflect real-world engineering requirements for the engine pistons. For example, Diameter (1^{st} component) has very tight tolerances to ensure proper fit, but other characteristics, such as Hardness and Wall Thickness (4^{th} and 5^{th} components), have slightly wider ranges. More specifically:

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$$\mathbf{LSL}^\top = [LSL_1, \dots, LSL_5]^\top = \begin{bmatrix} 74.5 & 119 & 345 & 8 & 20 \end{bmatrix},$$
$$\mathbf{USL}^\top = [USL_1, \dots, USL_5]^\top = \begin{bmatrix} 75.5 & 121 & 355 & 12 & 30 \end{bmatrix}.$$

The next step is the computation of the Multivariate Capability Indices MC_p, MC_{pk}, MC_{pm}, and MC_{pmk} based on Process Capability Analysis (PCA). The majority of multivariate processes often contain correlated quality characteristics, which happens in this engineering example. Then, PCA is applied to transform the original, correlated variables(Diameter, Length, Weight, Hardness, and Wall Thickness) into a set of uncorrelated principal components. Each principal component is a new axis that captures variability in the data. This leads to the reduction of dimensionality while preserving most of the original variability.

PCA was originally proposed by Wang and Chen (1998) with the following formula:

$$\mathbf{\Sigma} = \mathbf{UDU}^\top$$

Where:

$\mathbf{\Sigma}$ = the spectral decomposition of the covariance matrix of the distribution

$\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_\nu)$ is the matrix of eigenvectors of the covariance matrix $\mathbf{\Sigma}$, which shows the direction of the principal components

$\mathbf{D} = \mathbf{diag}(\lambda_1, \lambda_2, \dots, \lambda_\nu)$ is the diagonal matrix of eigenvalues, which explains the variance by each principal component

As mentioned above, three PCA-based methods were used in this analysis for constructing MPCIs. These are the following:

1. **Wang and Chen (1998)** suggested a way where the original correlated variables are transformed through Process Capability Analysis (PCA). Then, the process capability is evaluated using a subset m of PCs ($m < n$), which denotes the number of principal components used for the evaluation of the capability in the process. The Multivariate Capability Index MC_p has the following form:

$$MC_p = \left(\prod_{i=1}^{m} C_{p,PC_i} \right)^{\frac{1}{m}} \tag{I}$$

where:

$$C_{p,PC_i} = \frac{USL_{PC_i} - LSL_{PC_i}}{6\sigma_{PC_i}}, \qquad \sigma_{PC_i} = \sqrt{\lambda_i}$$

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From the (I), it is observable that for the construction of the multivariate index, a univariate capability measure C_{p,PC_i} for the i-th PC component was used.

Similarly, the multivariate indices $MC_{pk}, MC_{pm}, MC_{pmk}$ are computed through the $C_{pk,PC_i}, C_{pm,PC_i}, C_{pmk,PC_i}$ respectively for $i = 1, 2, ..., m$. If the indices indicate values greater than 1, then the process can be considered capable.

2. **Wang (2005)** proposed a different method for short-run capability assessment using a weighted geometric mean. The eigenvalues of the diagonal matrix $\mathbf{D}$ (λ_i) represent the weights of each of the PCs. The Multivariate Capability Index MWC_p has the following form:

$$MWC_p = \left(\prod_{i=1}^{m} C_{p,PC_i}^{\lambda_i} \right)^{\frac{1}{\sum_{i=1}^{m} \lambda_i}} \tag{II}$$

Similar to the Wang and Chen (1998) method, the same applies to this method for the computation of the indices $MWC_{pk}, MWC_{pm}, MWC_{pmk}$. If the indices indicate values greater than 1, then the process can be considered capable.

3. **Xekalaki and Perakis (2002)** highlighted the fact that different principal components explain different proportions of the total variability. This happened through the introduction of weighted capability indices. The capability index for each principal component is weighted by its corresponding eigenvalue. This method provides more weight to the important PCs or the ones that have large eigenvalues, which means that they have a bigger impact on the overall capability measure. The Multivariate Capability Index MXC_p has the following form:

$$MXC_p = \frac{\sum_{i=1}^{m} \lambda_i C_{p,PC_i}}{\sum_{i=1}^{m} \lambda_i} \tag{III}$$

The remaining indices $MXC_{pk}, MXC_{pm}, MXC_{pmk}$ are also computed following the same procedure. Once again, if the indices indicate values greater than 1, then the process can be considered capable.

Each of the aforementioned methods uses PCA to contribute to the reduction of dimensionality and the calculation of capability indices. An important decision to make in PCA-based indices is to determine the number of principal components to maintain, since this affects the results. According to the Documentation of R and Santos Fernández and Scagliarini (2012), the MPCI package offers the following five decision methods. The **Method 1- Percentage Method** is responsible for the selection of PCs that capture at least 80% of the total variability. The **Method 2- Average Method** is responsible for keeping PCs with eigenvalues above the average. The **Method 3- Scree Plot Method**, which uses the scree diagram to choose principal components. The **Method 4- Bartlett's Test** is responsible for the selection of the most significant principal components through

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statistical testing. The **Method 5- Anderson's Test** selects the most significant PCs through Anderson's approach.

However, in this example, Methods 1,2,3,4 were used. For each method, the calculation of the four indices MC_p, MC_{pk}, MC_{pm}, and MC_{pmk} completed. This computation was performed by using Method 1 for Wang and Chen (1998), Method 2 for Wang (2005), Method 4 for Xekalaki and Perakis (2002), and Method 3 for the visual selection of components. Also, in this analysis, the significance level or the default value equals $\alpha = 0.05$, based on the recommendations of the MPCI package (Santos Fernández and Scagliarini, 2012).

Before we proceed to the main analysis, it is important to make an important note. Although the fact that the `MPCI` package in R automatically determines how many principal components will be included according to the specified method, the PCA summary is completed through the function `summary(princomp(enginepistons))`. Also, PCA was performed to explain the proportion of the variability explained by each of the five components that take part in this analysis. The results can be observed in Table 5.1 below:

Components	**Diameter**	**Length**	**Weight**	**Hardness**	**Wall Thickness**
Standard deviation	1.0511	0.7891	0.6653	0.2206	0.1694
Proportion of Variance	0.4916	0.2771	0.1969	0.0216	0.0128
Cumulative Proportion	0.4916	0.7687	0.9656	0.9872	1.0000

Table 5.1: The Importance of Principal Components (PCs)

According to this table, Principal Component 1 (Diameter) explains 49.16% of the total variability, Principal Component 2 (Length) adds 27.71% with a cumulative proportion of 76.87%, Principal Component 3 (Weight) adds 19.69% with a cumulative proportion of 96.56%, Principal Component 4 (Hardness) adds 2.16% with a cumulative proportion of 98.72%, and Principal Component 5 (Wall Thickness) adds 1.28% with a cumulative proportion of 100%. These mean that the first three components (Diameter, Length, Weight) accounted for approximately 96.56% of the total variation, indicating that the most variability is concentrated in PC1, PC2, and PC3. However, for the last two principal components (Hardness and Wall Thickness), their contribution is indeed very small, which means that they might not be necessary for the majority of the methods.

1. *Wang and Chen (1998) - PCA Method 1 (Percentage Method)*

   For the calculation of the following multivariate capability indices using the MPCI package, we set `index="wang"` with the default Method 1. As it was mentioned above, from the PCA summary (Table 5.1), the first three PCs accounted for approximately 96.56% of the total variation. This meets the criterion of Method 1 about the selection of the smallest number of components that collectively capture at least 80% of the total variability (96.56% > 80%).

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MCp	MCpk	MCpm	MCpmk
1.0514	1.0463	1.0512	1.0461

Table 5.2: Multivariate Capability Indices by Wang and Chen (1998) - PCA Method 1 (Percentage Method)

By observing Table 5.2, it is obvious that all the values of the indices MC_p, MC_{pk}, MC_{pm}, and MC_{pmk} were greater than 1, which indicates that the process is capable but close to the threshold. By using the first three principal components (Diameter, Length, Weight), the specified engineering requirements are satisfied. Method 1 was a suitable choice because it avoids the noise from the less influential PCs. Consequently, it is the default method implemented in the MPCI package, making it an acceptable starting point for the multivariate capability analysis.

2. *Wang (2005) - PCA Method 2 (Average Method)*

   For the calculation of the following multivariate capability indices using the MPCI package, we set `index="wangw"` with the default Method 2. In this case, the selection of PCs is different. Method 2 keeps only the principal components with eigenvalues that are greater than the average eigenvalue. From the PCA summary (Table 5.1), only the first two PCs (Diameter and Length) met this criterion because they explain 76.87% of the total variation.

MCp	MCpk	MCpm	MCpmk
1.8534	1.8458	1.8529	1.8453

Table 5.3: Multivariate Capability Indices by Wang (2005) - PCA Method 2 (Average Method)

By observing Table 5.3, it is obvious that all the values are significantly higher than 1, indicating a highly capable process. This approach is effective because it takes into account the most influential quality dimensions for performance and cost control, but ignores less relevant sources of variability or smaller contributions, such as the last two principal components (Hardness and Wall Thickness).

3. *Xekalaki & Perakis (2002) - Scree Plot (PCA Method 3)*

   For the calculation of the following multivariate capability indices using the MPCI package, we set `index="xeke"` with the default Method 3. The number of PCs will be established using insights derived from the Scree Plot. According to Cattell (1966) and Rencher (2002), a scree plot is a visualization tool that presents the eigenvalues of the PCs in descending order. The main purpose of this plot is to find the textbf "elbow". This is a specific point that detects the transition from the components that explain significant variation to those that add additional noise.

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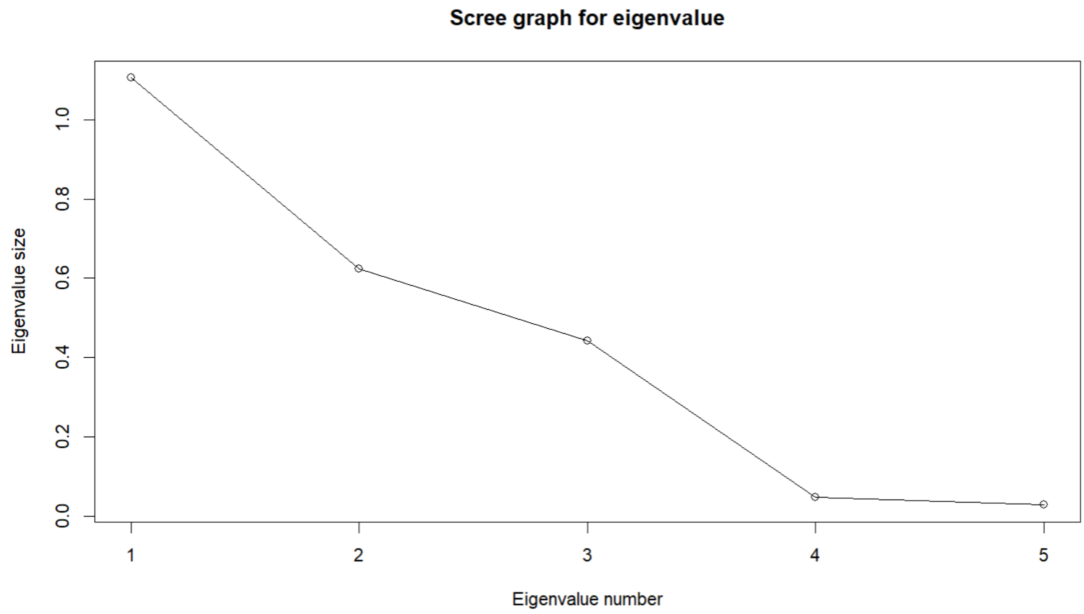

Figure 5.1: Scree Plot of the eigenvalues of all the principal components. After the PC2 (Length), there is an elbow, which suggests the use of only two principal components.

Figure 5.5 shows the eigenvalues of the five principal components that take part in this example (Diameter, Length, Weight, Hardness, Wall Thickness). As it was mentioned in the PCA summary (Table 5.1), the PC1 or Diameter accounts for the largest amount of variation (49.16%), the PC2 or Length adds a significantly smaller amount of variation (27.71%), and the PC3 or Weight adds an even smaller amount of variation (19.69%). The cumulative proportion equals 96.56%. The last two components, PC4 (Hardness) and PC5 (Wall Thickness), contribute a very small amount of variation (2.16% and 1.28% respectively).

After the PC2, the slope of the curve flattens noticeably, meaning that the other three principal components (PC3, PC4, PC5) present a small contribution to the variance. This is the point that we are looking for, the "elbow". Based on the elbow criterion, the suitable number of principal components that need to be retained is two (`npc=2`). This happens because Diameter and Length explain together 76.87% of the total variability.

MCp	MCpk	MCpm	MCpmk
1.8986	1.8788	1.8632	1.8554

Table 5.4: Multivariate Capability Indices by Xekalaki & Perakis (2002) - Scree Plot (PCA Method 3)

By observing Table 5.4, it is obvious that all the values are significantly higher than 1, indicating a highly capable process.

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4. *Xekalaki & Perakis (2002) - PCA Method 4 (Bartlett's Test)*

For the calculation of the following multivariate capability indices through the MPCI package, we set `index="xeke"` using the default Method 4. Xekalaki & Perakis (2002) introduce the use of statistical tests for the choice of how many PCs will contribute to the capability computation, based on Bartlett's Test. Bartlett's Test is a statistical procedure that identifies if smaller eigenvalues can be considered zero, at a significance level of $\alpha = 0.05$. According to Rencher (2002), if the test determines that some eigenvalues are practically zero, the corresponding components need to be excluded because of their slight contribution to the variability in the data.

From the PCA summary (Table 5.1), it is obvious that the first four principal components(Diameter, Length, Weight, Hardness) met the criterion. This is true because they capture 98.72% of the total variability. The reason for not keeping the last PC (Hard Thickness) is that it explains only 1.28% of the variance. The eigenvalue of the fifth component is not statistically significant from zero.

MCp	**MCpk**	**MCpm**	**MCpmk**
1.5280	1.5217	1.5276	1.5213

Table 5.5: Multivariate Capability Indices by Xekalaki & Perakis (2002) - PCA Method 4 (Bartlett's Test)

By observing Table 5.5, it is obvious that all the values are greater than 1, indicating a capable process. Compared to the Percentage and Average Method, this approach contains more components, and the values of the multivariate capability indices are slightly lower than the values of the indices of the Average Method. By including more components, this approach includes minor sources of variation, leading to a more conservative and statistically estimate of capability.

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6. Chapter 6: Recent Studies in Process Capability Indices

6.1 Recent Studies in Univariate Case

Modern statistical and analytical techniques have been applied in recent studies in the univariate case of process capability analysis. In Chapter 6.1, two real case studies are going to be presented. The focus of the first one is to examine the air quality from PM10 concentrations from the combination of process capability analysis and environmental risk. The focus of the second one is to improve product lifetime performance by using fuzzy logic under the exponential distribution.

- **"Air Quality through Process Capability and Risk Assessment"**

  Air pollution, particularly **PM10 or particulate matter**, which denotes extremely small airborne particles of 10 micrometers or less, is a significant danger to human health and environmental quality. Moreover, these particles are found in dust, pollen, and car emissions and can be absorbed into the lungs. Heating systems, industrial operations, and automotive emissions are only a few of the many sources of PM10. Because of its enormous health effects, controlling and assessing particulate matter concentrations has become the priority in environmental monitoring. In Turkey, PM10 levels are routinely recorded, but traditional evaluation methods focused solely on exceeding permitted limits fail to capture the consistency and the risk associated with pollution.

  Dagsuyu (2020) offered a novel approach to solve this gap by merging the Process Capability Index or C_p with environmental risk assessment techniques. This study applies this index, a quality control technique extensively employed in manufacturing, to environmental evaluation. This integration seeks to provide a more complete picture of how measurement stations operate in terms of keeping PM10 concentrations under "safe" limits.

  The following study proposes an integrated technique that evaluates air quality by combining statistical quality control strategies with environmental risk assessment,

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with a focus on PM10 concentrations. This methodology starts with the **Fine-Kinney** (1976) risk analysis method, which is used to determine the severity of the different concentration levels of PM10. For example, the different concentration levels ranged from the lowest or least harmful ("Noticeable") to the highest or the most harmful ("Catastrophe"). PM10 measurement ranges are assigned to a severity value S_i, $i = 1, 3, 7, 15, 40, 100$ to represent their potential impact on health and environment. Higher PM10 levels correspond to higher severity scores, according to predifen groupings.

The next step of this methodology is the calculation of the exposure probability, or the **Probability of PM10 values** falling within each of the six measurement categories. The mathematical formulation of the probability P_i is the following:

$$P_i = \frac{n_i}{N} \qquad i = 1, 2, 3, 4, 5, 6 \tag{I}$$

Where:

n_i = the number of obervations that belong to category i

N = the overall number of measurements that happened during the year

The **Environmental Effect**(EE_i) is the result of the multiplication of the probabilities of the PM10 values (P_i) by the severity values (S_i). The EE_i quantifies the impact of each PM10 range, taking into account both frequency and severity. The Environmental Effect is presented as follows:

$$EE_i = P_i \times S_i \qquad i = 1, 2, 3, 4, 5, 6 \tag{II}$$

The third step is the presentation of the **Environmental Quality Ratio (EOR)**. The EOR is used to assess a measurement station's compliance with air quality regulations. This value indicates how much of the station's pollution effect comes from PM10 values within the safe and acceptable range, which is between 0 to 50 μg/m^3. In other words, this ratio indicates how frequently the station records good air quality in comparison to all other measured pollution levels. A higher EOR shows that the station is doing a better job of remaining under permitted air quality limits. The EOR is computed as follows:

$$EOR = \frac{EE_1}{\sum_{i=1}^{6} EE_i} \tag{III}$$

Where:

EE_1 = the environmental effect from the desired 0-50 μg/m^3 PM10 range

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$\sum_{i=1}^{6} EE_i$ = the overall environmental effect from all the PM10 categories

Unfortunately, EOR does not address the consistency of the PM10 data, while it accurately measures compliance in terms of severity and frequency. For this reason, the **Process Capability Index** (C_p) is used to assess the ability of the process to stay within specification limits. Also, this index measures how closely the process is clustered around the target range. A high C_p value shows that the data are always within acceptable limits. However, a low C_p value indicates the existence of a high amount of variability. As it is known from Chapter 3, the mathematical formulation of C_p is the following:

$$C_p = \frac{\text{Spec. Width}}{\text{Process Spread}} = \frac{\text{Allowable Process Spread}}{\text{Actual Process Spread}} = \frac{USL - LSL}{6\sigma} \tag{IV}$$

The final component of this research is the **Air Capability Value (ACV)**, which is represented as a composite performance measure to integrate the knowledge from EOR (III) and C_p (IV). This metric measures both the statistical control of the PM10 measurement process and its adherence to environmental standards. A higher ACV value implies that the air quality often meets the required limits and that the measurements are steady and well-contained within those limits. The Air Capability Value is presented as follows:

$$ACV = C_p \times EOR \tag{V}$$

In 2017, data from multiple Turkish cities were analyzed to validate the model. PM10 values were measured daily. Ankara, according to the results, was labeled as a "polluted" city, compared to "non-polluted" or clean cities (Bitlis, Yozgat, Artvin, Kırıkkale, Kırşehir, and Eskişehir). Although many stations in Ankara reported similar minimum and maximum PM10 readings, their C_p values varied due to differences in the spread of the data. This demonstrated that evaluating stations solely on exceedance counts or averages can obscure the underlying process instability. Some stations appeared to have lower C_p values due to the high variability, even though they had acceptable air quality levels.

In Ankara, eight quality monitoring stations were analyzed. While PM10 levels generally belonged in the "poor" category, their C_p values varied significantly. This variation revealed differences in the consistency and stability of pollution measurements among stations, even when average pollution levels were almost the same. Some stations had higher variability, resulting in lower C_p scores and worse control over PM10 concentrations. This demonstrated the limitations of using a single metric. The EOR can provide a more accurate picture of a station's performance

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by taking into consideration the severity and frequency of exceedances, rather than just the average pollution level or C_p score. The Air Capability Value (ACV) provides a more realistic assessment of each station's environmental performance by incorporating both EOR and C_p.

This methodology was also employed in a group of cleaner cities, such as Bitlis, Yozgat, Artvin, Kırıkkale, Kırşehir, and Eskişehir. Interestingly, even in cities where PM10 concentrations remained within the acceptable limits, few monitoring stations had C_p values below the required threshold. This was related to data inconsistencies and fluctuations in the data, demonstrating that low pollution levels do not ensure a stable process. In Artvin and Kırşehir, low C_p values indicated measurement instability, despite the low PM10 values. The introduction of ACV allowed the study to better distinguish between stations that only met but exceeded air quality standards. In Eskişehir and Yozgat, a combination of high EOR and moderate C_p scores led to strong ACV outcomes, indicating environmental compliance and stable measurement processes. Bitlis and Kırıkkale show that even with similar average pollution levels, differences in process control and measurement reliability can still exist.

Consequently, this study integrates PCIs with probabilistic and severity-based measures to present a comprehensive picture of air quality performance. The suggested Air Capability Value (ACV) metric is especially good at discriminating between stations that occasionally satisfy air quality standards and those that continuously maintain safe PM10 levels. Furthermore, this study emphasizes the need to examine pollution data's frequency, severity, and stability to avoid depending exclusively on regulatory thresholds or single indicators. Although several stations suffered from missing data, the overall framework provides a more practical approach to environmental monitoring. The research of Dagsuyu (2020) extends the use of the Process Capability Index C_p beyond industrial quality control, highlighting the importance of this index in public health and environmental policy.

- **"Lifetime Performance of Products: A result by the combination of Fuzzy Testing Model and Exponential Distribution"**

This research describes the step-by-step creation of the Fuzzy Lifetime Performance Index proposed by Chen and Chang (2020). Their approach permits the evaluation of how reliable the products are in situations where there is uncertainty in the data. The methodology blends classical statistical techniques and fuzzy logic to create a more adaptable and realistic decision-making tool.

Product quality and durability are crucial points in today's production environment. However, one of the most important features is the product lifetime, which refers to

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the amount of time a product can function before failure. This means that product lifetime is a quality characteristic that depends on time. Also, the product lifetime can be considered as an "LTB" or "larger-the-better" quality characteristic. By using the **Lifetime Performance Index**, denoted as C_L, it is possible to determine whether a product meets the specified lifetime standards. The C_L index denotes whether the average product life is beyond a minimum threshold (L). Thus, this index is denoted as follows:

$$C_L = \frac{\lambda - L}{\lambda} = 1 - \frac{L}{\lambda} \tag{6.1.1}$$

Where:

λ denotes the true mean product lifetime

L , as it was mentioned above, denotes the minimum threshold for product lifetime. Additionally, L units of time represent the Lower Specification Limit (LSL).

Based on the previous information, there is a relationship between the components L and λ. On the one hand, if the mean product lifetime is larger than the minimum threshold, then it is true that $C_L \geq 0$. On the other hand, when the mean product lifetime is less than the minimum threshold, then $C_L < 0$. But, when the mean product lifetime becomes extremely large, then the Lifetime Performance Index approaches the value 1.

This Lifetime Performance Index does not usually follow the normal path or the normal distribution. The same thing happens for other quality characteristics, such as voltage, durability, and viscosity. More specifically, most of the time, product lifetime follows other kinds of distributions. These distributions can be Exponential, Generalized Exponential, Weibull, or Gamma. In this case, the Exponential distribution will be the starting point of this analysis.

Let's suppose that X is a random variable that represents the product lifetime. As it was mentioned before, the product lifetime follows an exponential distribution with a mean equal to λ. Then:

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda}, \quad x > 0,\, \lambda > 0$$

$$F(x) = 1 - e^{-x/\lambda}, \quad x > 0,\, \lambda > 0$$

The first equation represents the Probability Density Function and the second represents the Cumulative Distribution Function of the product lifetime X. It is obvious from the equations above that the mean and the standard deviation are the same and equal to λ. Furthermore, it is easy to define the failure rate r(x) of a

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product:

$$r(x) = \frac{f(x)}{1 - F(x)} = \frac{1}{\lambda} = r$$

Due to the unknown parameter λ, the C_L index can be estimated based on the sample data. Imagine a set of random samples $X_1, \ldots, X_j, \ldots, X_n$ of size n. Then, the estimate of the Lifetime Performance Index or C_L^* can be expressed as follows:

$$C_L^* = 1 - \frac{(n-1)L}{V(Z)} = 1 - \frac{(n-1)L}{\sum_{j=1}^{n} X_j} \tag{6.1.2}$$

Where:

$V(Z) = \sum_{j=1}^{n} X_j$ = the overall observed lifetimes in the sample.
$Z = (X_1, \ldots, X_j, \ldots, X_n)'$ = a vector or list of all the individuals observed lifetimes

Based on the expression of the estimate of the Lifetime Performance Index above, it is necessary to mention the following important information:

1. $V(Z) \sim \text{Gamma}(n, \lambda)$
2. $X_j \sim \text{Gamma}(1, \lambda)$

The next step is a presentation of a hypothesis test at a specific significance level a. Through this test, it is visible whether the lifetime performance of a product meets a specified standard ω. This component represents the lowest acceptable level of performance, defined by the manufacturer. The null hypothesis represents the fact that the product lifetime equals or exceeds the lowest acceptable level of performance. The alternative represents the disability of the product to meet the lowest level. Thus, the hypothesis test is the following:

$$H_0 : C_L \geq \omega$$

$$H_1 : C_L < \omega$$

By combining the definitions of the Lifetime Performance Index and its estimate, it is easy to define $W \sim \text{Gamma}(n, 1)$:

$$W = \frac{V(Z)}{\lambda} = (n-1) \cdot \frac{1 - C_L}{1 - C_L^*}$$

Another important step is the definition of the critical value C_0, which is defined as follows:

$$C_0 = 1 - \frac{(n-1)(1-\omega)}{\text{Gamma}_\alpha(n, 1)}$$

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Where $Gamma_{\alpha}(n, 1)$ is the lower quantile of the Gamma distribution with parameters n and 1.

If $C_L^* \leq C_0$, then the Null hypothesis: $H_0 : C_L \geq \omega$ is going to be rejected. The rejection of the Null hypothesis means that the product lifetime performance does not meet the manufacturer's defined restrictions ($C_L < \omega$). If $C_L^* > C_0$, then the Null hypothesis will not be rejected. This acceptance means that the product lifetime performance meets the manufacturer's defined restrictions ($C_L \geq \omega$).

Following the methodology, the authors introduced fuzzy logic as a new and elevated path for the evaluation of the lifetime performance of products. The use of fuzzy logic is crucial because the working, real-life data is often uncertain or imprecise. This different logic is responsible for the "management" of the uncertainty, by allowing performance not to be evaluated by an absolute outcome. For example, instead of saying that a product "meets" or "does not meet" a quality standard, fuzzy logic implies that the product satisfies the requirement by about 70%. As it was visible above, through traditional methods and hypothesis tests, the result is either the rejection or the acceptance of a product based on a specific test value.

The question is, "How can it represent uncertainty in the Lifetime Performance Index C_L?". The answer was given using Buckley's method. This method was initially introduced by James J. Buckley (2005) and appears analytically in *Fuzzy Statistics: Hypothesis Testing with Fuzzy Data, published by Springer*. More specifically, it turns confidence intervals into fuzzy numbers by linking different confidence levels with membership values or degrees of belief. A higher confidence level, for example 95%, indicates a strong degree of membership, while a lower confidence level indicates less certainty. This creates a fuzzy set that captures the uncertainty in statistical results.

Chen and Chang (2020) apply Buckley's method to convert the C_L^* into a triangular fuzzy number. This transformation offers a clearer way to represent uncertainty and allow for a more cautious evaluation of how reliable a product is. This number makes it easier to visualize the possible results and how confident someone can feel in each one. Also, it is a practical and useful tool for engineers and quality managers due to its ability to present not just whether a product can meet the standard, but, under the uncertainty condition, how close or far the product is from reaching that goal. Thus, the triangular fuzzy-shaped number to represent uncertainty in the estimate of C_L will be the following:

$$\tilde{C}_L^* = \left(C_L'^{L}, C_L'^{M}, C_L'^{R}\right)$$

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Where:

$C_L'^L = 1 - (1 - C_L^*) \cdot \frac{\mathbf{Gamma}_{0.995}(n,1)}{\mathbf{Gamma}_{0.5}(n,1)}$: The left bound or the lowest likely value

$C_L'^M = C_L^*$: The middle value and most plausible estimate

$C_L'^R = 1 - (1 - C_L^*) \cdot \frac{\mathbf{Gamma}_{0.005}(n,1)}{\mathbf{Gamma}_{0.5}(n,1)}$: The right bound or the highest likely value and the most optimistic scenario

To test the accuracy of the methodology above, Chen and Chang (2020) provided a case study. In this case study, 24 air-conditioning units were analyzed. The aim was to observe if the lifetime performance exceeded the required level $\omega = 0.8$, with a minimum lifetime threshold L= 21. From the data, the critical value or the test threshold was C_0= 0,7074 and the estimate of the Lifetime Performance Index was $C_L^* = 0.7167$. $C_L^* > C_0$, which means that the Null Hypothesis $H_0 : C_L \geq \omega$ will not be rejected, and the product lifetime performance meets the manufacturer's defined restrictions. By using fuzzy logic, the triangular fuzzy-shaped number of C_L^*, will be:

$$\tilde{C}_L^* = (0.5393, 0.7167, 0.8413)$$

Where, the middle value ($C_L'^M$)equals to 0.7167, the left bound ($C_L'^L$) equals to 0.5393, and the right bound ($C_L'^R$) equals to 0.8413. On the one hand, the middle value or the center of the triangle indicated that the performance was accepted. On the other hand, the left bound was below the test threshold C_0, which causes an issue by showing that the product might not meet the requirement. Thus, the fuzzy set proposed rejection.

Consequently, this study is focused on the improvement of the traditional index C_L, which is typically used to evaluate whether a product's durability meets the minimum standards defined by manufacturers. Usually, this index relies on binomial results, for example, "success" or "failure", and uses classical hypothesis testing. However, most of the time, it is possible to deal with small data or data where uncertainty and imprecision are common components. As it was presented above, Chan and Cheng (2020) presented a fuzzy version of the Lifetime Performance Index, $\tilde{C}_L^*$, which is capable of handling uncertainty successfully and making it more applicable to real-world manufacturing scenarios. The authors highlighted that while the classical estimate exceeded the acceptance limit, the fuzzy version presented potential risk by showing that the left bound was below the test threshold.

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6.2 Recent Studies in Multivariate Case

The analysis of the Multivariate PCIs will be developed in the domain of **Supplier Selection**. This domain is critical for maintaining product quality, profitability, and overall supply chain efficiency. In Chapter 6.2, three recent improvements and studies that have included the Process Capability Indices in the frameworks of supplier evaluation will be presented, increasing the objectivity and precision of selection processes.

- **"The Development of a Fuzzy Green Supplier Model through the use of Six Sigma Quality Indices"**

  Since the 1980s, Taiwan's electronics industry has grown into a highly advanced and competitive manufacturing sector. Taiwanese enterprises are among the world's leading manufacturers of semiconductors, LCDs, LEDs, and other critical electronic components. These organizations are one of the most significant participants in the global supply chain, acting as Original Equipment Manufacturers (OEMs) or Original Design Manufacturers (ODMs) for numerous high-tech products. However, this success comes with its own set of challenges. To remain competitive, manufacturers must eliminate waste, rework, and maintain a stable level of quality. A critical method to accomplish this is to select the right suppliers, particularly those who regularly produce components of premium quality and support sustainability goals.

  As the pressure in the environmental field increases, particularly in reaction to global warming, companies are expected to lessen their environmental footprint while expanding. The 4R philosophy (Reduce, Reuse, Recycle, and Recovery) is a widely recognised method. All of these four areas are immediately affected by the improvement of quality. By reducing defects and process waste, there is an increase in product availability, a decrease in the waste of components, and an extension of maintenance cycles. At the same time, higher-quality components are more likely to be reused or recycled, and a longer product life equals less need for complete replacement or recovery.

  Selecting the proper suppliers is critical for remaining competitive because the quality of materials and components directly impacts the final product. Supplier quality is increasingly evaluated through the Process Capability Indices (PCIs), such as C_p, C_{pk}, and C_{pm}, rather than sales performance. When paired with fuzzy logic, advanced indices, such as Q_{pkh}, improve decision-making in uncertain situations. This enables businesses to reduce risk, increase quality, and strengthen supply chains.

  Within this context, Chen, Wang, and Than (2019) present a green supplier evaluation model that promotes both industrial performance and environmental responsibility. The **Six Sigma Quality Index (SSQI)**, presented as Q_{pkh}, is used in

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this approach to evaluate how capable the processes of suppliers are in producing products within specifications. More specifically, the index Q_{pkh} measures both how centered and consistent the supplier's process is over time. The authors highlight that this index is ideal for industrial applications as it combines process yield and quality level in a single measure.

In statistical terms, the process of a supplier h follows the normal distribution:

$$X \sim \mathcal{N}(\mu_h, \sigma_h^2), \text{ where } \mu_h \text{ is the process mean, and } \sigma_h \text{is the standard deviation}$$

Thus, the mathematical formulation of the Six Sigma Quality Indices of supplier h is the following:

$$Q_{pkh} = \frac{1 - |\delta_h|}{\gamma_h} + 1.5 \tag{6.3.1.1}$$

Where:

$\delta_h = \frac{(\mu_h - T)}{d}$ is the normalized deviation from the target

$\gamma_h = \frac{\sigma_h}{d}$ is the normalized process spread

$T = \frac{\textbf{USL+LSL}}{2}$

$d = \frac{\textbf{USL}-\textbf{LSL}}{2}$

Chen, Wang, and Tan used confidence intervals around the SSQI or Q_{pkh} to overcome uncertainties caused by small sample sizes, material variation, and measurement error. These intervals present a more reliable perspective of supplier capabilities by estimating the range within which the true process performance is more likely to fall.

To improve the concept of decision-making, the authors applied Buckley's technique to fuzzy numbers. Buckley's (2005) approach converts statistical confidence intervals into fuzzy numbers, indicating how likely each value is in the range. The value in the middle of the interval has full membership, and the values towards the edges have lower membership. This method contributes to decision-making when the exact number is unknown. Also, this approach allows for comparison based on the degree of confidence that one supplier outperforms another, rather than relying on fixed thresholds.

Consequently, even with a small number of data points, the model can compare suppliers based on the range of their confidence intervals. Traditional approaches in this case study were unsuitable for determining which supplier had the advantage since the results were too close. However, the model implies that the data follows a normal distribution and has symmetric tolerances. To broaden its applicability in real-world settings, future studies should account for data that are not normally

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distributed or have asymmetric tolerances. Overall, this model is a practical and adaptable tool that helps companies select better suppliers by combining fuzzy logic with Six Sigma.

- **"The Development of a Quality-Based Supplier Selection Model from the perspective of a Buying Company"**

  As the competition globally increases, enterprises must develop stronger supply chain strategies to ensure long-term quality parts. One critical aspect of this is the proper selection of suppliers who can regularly offer high-quality parts. To deal with this, Chen, Chung, and Chang (2021) created a supplier selection model that considers each supplier's process quality. This model uses the **Process Capability Index** S_{pk}, which was presented in Chapter 4 analytically. This index is based on how much of a supplier's output meets the specified quality standards. The applied methodology is developed from the buying company's perspective and provides a clear, data-based method for comparing suppliers and making better decisions.

  In manufacturing, process yield is an important measure of process quality and operational efficiency. On the one hand, higher process yield implies reduced costs, stability in performance, and a decrease in defects. Traditional PCIs, such as C_{pk} and C_{pm}, are often used to assess quality. However, the disadvantage of these indices is that they only provide and are connected to certain assumptions (normal distribution of the data and symmetric tolerance limits). The index S_{pk} has a direct association with process yield, making it a more accurate and meaningful tool for supplier evaluation. It enables manufacturers to turn quality measurements into an accurate estimate of how much output meets the proper specifications.

  In real-world applications, the most important components (the process mean and the standard deviation) are usually unknown. As a result, S_{pk} must be estimated using sample data, introducing uncertainty and increasing the risk of making wrong decisions. The authors address this by applying a Lower Confidence Limit (LCL) to the estimated Process Yield Index. This boundary offers a trustworthy minimum estimate of the process capability, allowing buyers to make more secure decisions about supplier performance.

  Furthermore, there are two types of statistical error when there is a need for the estimation of values for decision making. These two types of error are the famous Type I and Type II. Type I error, also known as producer's risk, occurs when a capable supplier is falsely rejected. On the contrary, the Type II error or customer's risk occurs when a low-quality supplier is mistakenly accepted. Using the LCL as a benchmark, the model decreases the likelihood of both errors, aiding organizations in avoiding mistakes in supplier selection.

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This model also contains the famous Process Capability Analysis Chart (PCAC), which is an important visual decision-making tool. This chart enables buyers to rapidly determine whether a supplier meets the acceptable quality level and detect any issues caused by a lack of poor centering (mean shift) or process precision (variation). To ensure the validity of the model's LCL values, the researchers employed Monte Carlo simulation to evaluate and confirm how stable the results are under various sampling circumstances.

Chen, Chung, and Chang (2021) applied two real-world scenarios to demonstrate the effectiveness of their model. The first case study included **the selection of a supplier for a 17-tooth motorcycle sprocket**. The significant quality attribute was the keyway range, which was the $[4.45mm, 4.55mm]$. Four suppliers took part in this study. They were evaluated by using sample data, their estimated S_{pk}, and their LCLs were plotted in the PCAC. Only one of the four suppliers succeeded in meeting the required quality level (4σ). Unfortunately, the other three suppliers failed to meet the required quality level due to excessive variation or an average value that was too distant from the target, or both. The authors suggested that these suppliers need to proceed with improvements, such as improving the training of the staff, adjusting machines, and replacing the old equipment.

In the second case study, the model was used **to evaluate gear suppliers for machine tools, with an emphasis on the internal diameter of an 18-tooth gear**. The acceptable range was $[13.984mm, 14.016mm]$, and the target value was 14.000 mm. Three suppliers took part in this study. Two of the three suppliers successfully met the standard. More specifically, Supplier 2 was the better long-term partner due to its lower process variation and consistent product quality. However, only one of them did not meet them, because of the large mean shift.

Consequently, this research presents a practical and statistically valid strategy for supplier selection. As it was mentioned above, this model uses the yield-based or process yield index S_{pk} and LCL to evaluate suppliers fairly and with accuracy, even with only sample data available. Moreover, the PCAC is responsible for decision-making and identifies areas for supplier improvement. This strategy assists manufacturers in selecting high-quality suppliers and in making stronger, long-term supplier relationships.

- **"A Two-Phase Selection Framework: Production Costs of Suppliers and Quality Requirements of Buyers"**

  In today's competitive global market, selecting the proper suppliers is critical for improving the quality, reducing the costs, and developing long-term business relationships. To achieve this goal, Yang and Chen (2019) developed a two-phase

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supplier selection methodology that assesses suppliers not just on quality performance but also on production costs. This integrated model intends to assist consumers in making smart selections that strike a balance between quality assurance and operational efficiency.

A large number of products possess various quality characteristics, such as stability, performance tolerances, and each may have a different type of measurement. The different types of measurement are:

1. STB or "smaller-the-better"
2. LTB or "larger-the-better"
3. NTB or "nominal-the-best"

Also, under the assumption of normality, it is true that:

1. For the STB and LTB, which include unilateral specification cases,

$$|\mu - LSL| = d - 1.5\sigma \quad \text{or} \quad |\mu - USL| = d - 1.5\sigma$$

2. For the NTB, which includes bilateral specification cases,

$$|\mu - T| = 1.5\sigma$$

Where:

$d = \frac{USL-LSL}{2}$ = the half-width of the specification range
$T = \frac{USL+LSL}{2}$ = the target value

Traditional methods, such as Multi-Criteria-Decision-Making (MCDM) approaches, sometimes rely on subjective input and cannot always accurately capture the true quality performance of manufacturing processes. In contrast, this methodology focuses on objective quality evaluation through the use of Six Sigma metrics and a systematic decision-making process.

The first phase of the method introduces the **SSQCAC or Six Sigma Supplier Quality Capability Analysis Chart**. This tool analyzes suppliers by computing their UCL or Upper Confidence Limit of a combined SSQI index called Q_{Plh}. This index is a combination of the following components:

Q_{pu} = Unilateral Specification SSQI for STB. Also, Q_{pu} is the development of the index C_{pu} or Unilateral Specification PCI for STB. The mathematical formulation of this index is the following:

$$Q_{pu} = \frac{USL - \mu + 1.5\sigma}{\sigma} \tag{6.3.2.1}$$

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Q_{pl} = Unilateral Specification SSQI for LTB. Also, Q_{pl} is the development of the index C_{pl} or Unilateral Specification PCI for LTB. The mathematical formulation of this index is the following:

$$Q_{pl} = \frac{\mu - LSL + 1.5\sigma}{\sigma} \tag{6.3.2.2}$$

Q_{pk} = Bilateral Specification SSQI for NTB. Also, Q_{pk} is the development of the index C_{pk} or Bilateral Specification PCI for NTB. The mathematical formulation of this index is the following:

$$Q_{pk} = \min(Q_{pu}, Q_{pl}) = \min\left(\frac{USL - \mu + 1.5\sigma}{\sigma}, \frac{\mu - LSL + 1.5\sigma}{\sigma}\right) = \frac{d - |\mu - T| + 1.5\sigma}{\sigma} \tag{6.3.2.3}$$

$\gamma = \frac{\sigma}{d}$ is the accuracy index

$\delta = \frac{\mu - T}{d}$ is the precision index

Also, the UCL controls for sampling variability, increasing the evaluation's reliability. If the supplier's performance does not meet the quality threshold, which is between the minimum (3σ) and the target (6σ), then this supplier will not be taken into consideration.

As for the SSQI Q_{Plh}... Let's suppose that the quality characteristic X_h follows a normal distribution $X_h \sim \mathcal{N}(\mu_{X_h}, \sigma^2_{X_h})$. The definition of the standardized distance is:

$$Y_h = \frac{X_h - T_h}{d_h}$$

The components T_h and d_h are based on the type of the quality characteristic, such as STB, LTB, and NTB: More specifically:

- For STB: $T_h = 0, \quad d_h = USL_h$
- For LTB: $T_h = 2LSL_h, \quad d_h = LSL_h$
- For NTB: $T_h = \frac{USL_h + LSL_h}{2}, \quad d_h = \frac{USL_h - LSL_h}{2}$

Thus, the mathematical formulation of Q_{Plh} is the following:

$$Q_{Plh} = \begin{cases} \frac{1-\delta_h}{\gamma_h} + 1.5, & \text{for STB} \\ \frac{1+\delta_h}{\gamma_h} + 1.5, & \text{for LTB} \\ \min(Q_{pu_h}, Q_{pl_h}), & \text{for NTB} \end{cases} \tag{6.3.2.4}$$

In the second phase, the surviving suppliers are compared through the Euclidean Distance method, which ranks them according to how close their performance is

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to the buyer's desired quality level. This allows decision-makers to objectively determine the most capable supplier from those who passed the initial screening.

To illustrate this concept, the authors created a real-world case study of a Company named F. Company F needed to buy shaft couplings, which were required to meet precise quality standards. The product had one STB-type (the surface roughness) and three NTB-type quality characteristics (the diameter, the distance between two holes, and the inside diameter). Data was gathered from eight possible suppliers, where each one of them provided 30 samples per quality characteristic.

During the first phase, and after the calculation of the UCLs for each supplier, two out of eight potential suppliers were excluded since their quality levels were outside the permitted range of the SSQCAC. That happened because of their diameter measurements and surface roughness. In the second phase, the six remaining suppliers were compared using the Euclidean Distance method. According to this analysis, only one supplier (Supplier 2) was selected as the top-performing supplier, due to its consistency in quality performance.

This study also compared this methodology to other well-known PCI and MCDM techniques. Unlike such approaches, Yang and Chen's method takes into account multiple quality types, accounting for sample errors and providing clear rankings. Additionally, it ensures early elimination of the weak suppliers, which reduces computation time and complexity. Through this research, the authors found that their approach is successful, easy to implement, and promotes long-term supplier bonds. They indicated that future improvements may include the management of non-normal data, the involvement of fuzzy logic, and the evaluation of delivery performance as an additional element in supplier selection.

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7. Chapter 7: Conclusions

This thesis aimed to investigate the theoretical foundations, the assumptions, and the applications of the Process Capability Indices both in univariate and multivariate contexts, and to review recent studies and developments that enhance their applicability in modern manufacturing. The indices of Chapter 3, such as C_p, C_{pk}, and C_{pm}, are widely used due to their ability to be simple and applicable at the same time. However, their validity depends on critical assumptions, including normality, stability, and independence, which are often violated in real-world scenarios. To address these limitations, researchers introduced advanced Univariate Capability Indices, such as C_{pmk}, S_{pk}, and Fuzzy PCIs, which are presented analytically in Chapter 4, along with Multivariate Capability Indices (Chapter 5) that account for correlations among the different quality characteristics.

Also, various techniques were applied, such as PCA-based methods and probability-based approaches, to present a more understandable view of process capability. This thesis also addresses challenges associated with the non-normal condition, autocorrelation, and measurement errors, and examines solutions, such as transformations, non-parametric methods, and robust estimation. Recent case studies and practical applications were conducted in R, with the `qcc`, `FuzzyNymbers`, and `MPCI` packages to present simple real-world scenarios.

Data-driven adoption of new technologies and methodologies transforms quality management practices. Manufacturers may improve process monitoring and maintenance by leveraging digital tools and increasing connectivity in production systems, as well as gain more accurate and timely insights into production trends. This has resulted in faster adjustments, fewer defects, and an overall boost in efficiency.

On the one hand, White et al. (2021) proposed a set of indices for a more comprehensive assessment of process health, highlighting the necessity of combining traditional metrics with new indicators that reflect broader production conditions. These indices include measures for detecting long-term trends, assessing the stability of various quality parameters at the same time, and ensuring that process adjustments are consistent with both immediate and strategic quality objectives. By incorporating such indices into the PCI framework, manufacturers can receive a more precise and forward-looking perspective of their production capabilities.

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In contrast, Manzoor et al. (2021) explain how digital technologies in Architecture, Engineering, and Construction (AEC) have improved data-driven decision-making and performance assessments. Similarly, applying these digital techniques to PCIs allows for real-time monitoring and a more comprehensive understanding of process variation. Manufacturers can use interconnected systems to more effectively identify patterns and correlations between quality attributes, resulting in continuous improvement in manufacturing processes and better overall quality outcomes. By incorporating PCIs into Quality 4.0 frameworks, manufacturers may continuously refine their metrics to reflect both immediate production conditions and long-term performance goals.

Furthermore, the Industrial Internet of Things (IIoT) establishes a solid foundation for real-time PCI monitoring. Connected devices and sensors on production lines generate continuous data streams that feed directly into PCI calculations. This real-time insight enables quick modifications to the process variables, ensuring that quality standards are maintained without a delay. As a result, manufacturers achieve greater efficiency, less waste, and more consistent results and outputs.

Traditional Univariate PCIs are frequently insufficient for modern, complex production environments where multiple quality characteristics interact. The future of these indices will include multivariate techniques that take these relationships into account. Adaptive PCIs, which adapt dynamically based on real-time conditions, provide a more complete picture of the context of process capability.

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Appendix

R Code for Univariate Case – Tablet Weights

```
###################################
### CHAPTER 3 - UNIVARIATE CASE ###
###       TABLET WEIGHTS        ###
###################################


#install.packages("qcc")
library(qcc)

set.seed(123)
tabletweights <- rnorm(5000, mean = 50, sd = 1.2)


hist(tabletweights,
     main = "Distribution of Tablet Weights",
     xlab = "Tablet Weight (mg)",
     col = "lightblue",
     border = "black",
     breaks = 40,
     freq = FALSE)


curve(dnorm(x, mean = mean(tabletweights), sd = sd(tabletweights)),
      col = "darkblue", lwd = 2, add = TRUE)


LSL <- 45
USL <- 55
target <- 50

repeat {
  qcc_obj <- qcc(tabletweights, type = "xbar.one", plot = FALSE)

  limits <- qcc_obj$limits
  LCL <- limits[1]
```

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```
  UCL <- limits[2]


  outliers <- which(tabletweights < LCL | tabletweights > UCL)

  if (length(outliers) == 0) {
    break
  } else {
    tabletweights <- tabletweights[-outliers]
  }
}


length(tabletweights)

qcc_final <- qcc(tabletweights, type = "xbar.one")
capability_final <- process.capability(qcc_final,
                                       spec.limits = c(LSL, USL),
                                       target = target)

print(capability_final)
```

Listing 7.1: R code for Univariate Analysis (Tablet Weights analysis) in Chapter 3

R Code for Fuzzy PCIs – PM2.5

```
###################################
### CHAPTER 4 - UNIVARIATE CASE ###
###     FUZZY PCIs - PM2.5      ###
###################################



#install.packages("FuzzyNumbers")
library(FuzzyNumbers)

set.seed(42)
PM25 <- rnorm(5000, mean = 36, sd = 5)

hist(PM25, breaks = 15, col = "lightpink",
     main = "Air Quality PM2.5 Values",
     xlab = "PM2.5 (ug/m^3)")
curve(dnorm(x, mean = mean(PM25), sd = sd(PM25)),
      col = "pink4", lwd = 2, add = TRUE)

# Shapiro Test for Normality
shapiro.test(PM25) # p = 0.744 > 0.05
```

 https://doi.org/10.26219/heal.aueb.9439

```

# Estimate fuzzy process parameters
estimated_mean <- mean(PM25)
estimated_sd <- sd(PM25)

# Define fuzzy numbers for mean and sd
fuzzy_mean <- TriangularFuzzyNumber(estimated_mean - 1,
                                     estimated_mean,
                                     estimated_mean + 1)
fuzzy_sd <- TriangularFuzzyNumber(estimated_sd - 0.5,
                                   estimated_sd,
                                   estimated_sd + 0.5)


par(mfrow = c(1, 2))
plot(fuzzy_mean, col = "blue", lwd = 2,
     xlim = c(estimated_mean - 2, estimated_mean + 2),
     main = "Fuzzy Mean (mu)",
     ylab = "Membership Degree",
     xlab = "mu (ug/m^3)")
plot(fuzzy_sd, col = "red", lwd = 2,
     xlim = c(estimated_sd - 1, estimated_sd + 1),
     main = "Fuzzy SD (sigma)",
     ylab = "Membership Degree",
     xlab = "sigma (ug/m^3)")

# Define fuzzy specification limits and target
fuzzy_LSL <- TriangularFuzzyNumber(14, 15, 16)
fuzzy_USL <- TriangularFuzzyNumber(54, 55, 56)
fuzzy_Target <- TriangularFuzzyNumber(34, 35, 36)

# Plot fuzzy LSL, Target, USL
plot(fuzzy_LSL, col = "red", lwd = 2,
     xlim = c(10, 60), ylim = c(0, 1),
     xlab = "PM2.5 Concentration (ug/m^3)",
     ylab = "Membership Degree",
     main = "Fuzzy Specification Limits")
plot(fuzzy_Target, col = "blue", lwd = 2, add = TRUE, lty = 2)
plot(fuzzy_USL, col = "darkgreen", lwd = 2, add = TRUE)
legend("topright", legend = c("LSL", "Target", "USL"),
       col = c("red", "blue", "darkgreen"),
       lty = c(1, 2, 1), lwd = 2, cex = 0.35)

# Alpha-cuts and index bounds
alpha_levels <- seq(0, 1, 0.1)
results <- data.frame()

```

https://doi.org/10.26219/heal.aueb.9439

```
for (alpha in alpha_levels) {
  mu_a <- alphacut(fuzzy_mean, alpha)
  sigma_a <- alphacut(fuzzy_sd, alpha)
  target_a <- alphacut(fuzzy_Target, alpha)
  LSL_a <- alphacut(fuzzy_LSL, alpha)
  USL_a <- alphacut(fuzzy_USL, alpha)

  cp_L <- (USL_a[1] - LSL_a[1]) / (6 * sigma_a[2])
  cp_U <- (USL_a[2] - LSL_a[2]) / (6 * sigma_a[1])

  cpk_L <- min(USL_a[1] - mu_a[2], mu_a[1] - LSL_a[1]) / (3 * sigma_a
     [2])
  cpk_U <- min(USL_a[2] - mu_a[1], mu_a[2] - LSL_a[2]) / (3 * sigma_a
     [1])

  cpm_L <- (USL_a[1] - LSL_a[1]) /
           (6 * sqrt(sigma_a[2]^2 + (mu_a[2] - target_a[2])^2))
  cpm_U <- (USL_a[2] - LSL_a[2]) /
           (6 * sqrt(sigma_a[1]^2 + (mu_a[1] - target_a[1])^2))

  results <- rbind(results, data.frame(
    Alpha = alpha,
    Cp_Lower = cp_L, Cp_Upper = cp_U,
    Cpk_Lower = cpk_L, Cpk_Upper = cpk_U,
    Cpm_Lower = cpm_L, Cpm_Upper = cpm_U
  ))
}

print(results)

# Plot fuzzy indices
par(mfrow = c(1, 3))

plot(results$Alpha, results$Cp_Lower, type = 'l', col = 'blue',
     ylim = range(results$Cp_Lower, results$Cp_Upper),
     ylab = "Cp", xlab = expression(alpha), main = "Fuzzy Cp")
lines(results$Alpha, results$Cp_Upper, col = 'blue', lty = 2)
polygon(c(results$Alpha, rev(results$Alpha)),
        c(results$Cp_Lower, rev(results$Cp_Upper)),
        col = rgb(0, 0, 1, 0.2), border = NA)

plot(results$Alpha, results$Cpk_Lower, type = 'l', col = 'darkgreen',
     ylim = range(results$Cpk_Lower, results$Cpk_Upper),
     ylab = "Cpk", xlab = expression(alpha), main = "Fuzzy Cpk")
lines(results$Alpha, results$Cpk_Upper, col = 'darkgreen', lty = 2)
polygon(c(results$Alpha, rev(results$Alpha)),
        c(results$Cpk_Lower, rev(results$Cpk_Upper)),
```

```r
        col = rgb(0, 1, 0, 0.2), border = NA)

plot(results$Alpha, results$Cpm_Lower, type = 'l', col = 'purple',
     ylim = range(results$Cpm_Lower, results$Cpm_Upper),
     ylab = "Cpm", xlab = expression(alpha), main = "Fuzzy Cpm")
lines(results$Alpha, results$Cpm_Upper, col = 'purple', lty = 2)
polygon(c(results$Alpha, rev(results$Alpha)),
        c(results$Cpm_Lower, rev(results$Cpm_Upper)),
        col = rgb(0.5, 0, 0.5, 0.2), border = NA)
```

Listing 7.2: R code for Univariate Analysis (Fuzzy PCIs - PM2.5 values) in Chapter 4

R Code for Multivariate PCIs – Engine Pistons

```r
#######################################################
###       CHAPTER 5 - UNIVARIATE CASE               ###
###   MULTIVARIATE PCIs (PCA-BASED METHODS)         ###
###            ENGINE PISTONS                       ###
#######################################################


#install.packages("MASS")
#install.packages("MPCI")
library(MASS)
library(MPCI)


# Dataset of Engine Pistons contains Diameter (mm), Length (mm), Weight (g), Hardness (HRC), Wall Thickness (mm)
Target <- c(75, 120, 350, 10, 25)
LSL <- c(74.5, 119, 345, 8, 20)
USL <- c(75.5, 121, 355, 12, 30)
alpha <- 0.05


set.seed(2025)
n <- 5000
mu <- Target
cov_matrix <- matrix(c(
  0.05, 0.01, 0.1, 0.02, 0.03,
  0.01, 0.08, 0.2, 0.01, 0.02,
  0.1,  0.2,  1.0, 0.05, 0.1,
  0.02, 0.01, 0.05, 0.5, 0.1,
  0.03, 0.02, 0.1, 0.1, 0.6
), ncol = 5)

```

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```r
enginepistons <- mvrnorm(n = n, mu = mu, Sigma = cov_matrix)
colnames(enginepistons) <- c("Diameter", "Length", "Weight", "Hardness", "WallThickness")
View(enginepistons)


###---Application of MPCI Methods


# PCA summary to decide the number of components
summary(princomp(enginepistons))

## Wang & Chen (1998) - PCA Method 1
result_wang <- mpci(index = "wang", enginepistons, LSL, USL, Target, Method = 1)
result_wang

## Wang (2005) - PCA Method 2
result_wangw <- mpci(index = "wangw", enginepistons, LSL, USL, Target, Method = 2)
result_wangw

## Xekalaki & Perakis (2002) - PCA Method 4 (Bartlett's Test)
result_xeke <- mpci(index = "xeke", enginepistons, LSL, USL, Target, Method = 4)
result_xeke

## Xekalaki & Perakis (2002) - Scree Plot (Method = 3)
scree_plot <- mpci(index = "xeke", enginepistons, LSL, USL, Target, Method = 3)
```

Listing 7.3: R code for Multivariate Analysis (Engine Pistons) in Chapter 5